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On a Proof of Pandora's Rule in Optimal Stopping Problem

by

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Weitzman は 1979 年に *Econometrica* [9] に「Optimal Search for the Best Alternative」という論文を発表しました。私はこれに関連した論文とし 1989 年度に「Alternative Proof of Pandora's Rule in Optimal Search for the Best Alternative」という標題でＤＰ-No.404 を出しました。ところがその後 Weitzman の論文における主要な定理の証明に決定的な誤りのあることが判明しました。そこで、今回、この誤りの内容を詳細に検討し、正しい証明をを与え（この部分はＤＰ-No.404 と同じ）、さらにその後に得られた若干の成果も含め、標題も「On a Proof of Pandora's Rule in Optimal Stopping Problem」と改め、ＤＰ-No.404 とは独立したものとしてこのＤＰを出すことにしました。ご高覧いただければ幸いです。

最適停止問題におけるパンドラのルールの証明について

生田誠三

－ 1990/10/18 －

M.L. Weitzman は最適停止問題に関する次のような一般モデルを定式化した。いまここに N 個の箱がある。箱 $i = 1,2,...,N$ には未知の利得 x_i が隠されている。c_i 円を支払うと箱 i を開けることができ、t_i 時間後にその中の未知の利得は既知となる。箱 i に隠されている未知の利得はある与えられた分布 $F_i(x_i)$ に従うものとする。さて箱は一つづつ開けていくものとし、もうそれ以上開けることを停止した場合、それまでに既知となった利得のうちで最大なものを受け取って過程は終了する。ここで問題は、このようにして受け取った利得の現在価値からそれまでに箱を開けるために支払った費用の現在価値の総計を引いたものの期待値を最大にすることである。この期待値を最大にする最適戦略は次の二つの決定規則から成る。方程式 $-c_i + \beta_i \int \max\{w,y\} dF_i(w) - y = 0$ ($\beta_i = e^{-\alpha t_i}$, αは瞬間割引率）の解を z_i とし、これを箱 i の留保価格と呼ぶ。**選択規則**：箱を開けるときは、まだ開けていない箱のうちで最大の留保価格を持つ箱を開けよ。**停止規則**：それまでに開けられた箱の利得（すでに既知となっている）の最大値が残りのどの箱の留保価格よりも大きければ、その最大の利得を受け取って過程を終了し、さもなくば次の箱を開けよ。彼はこの決定規則を「パンドラのルール」と名付けた。

彼のこの結論は幸いなことに結果的には正しいが、残念ながらその証明に重大な誤りのあることが判明した（彼の結論が正しいというのは全くの偶然に過ぎない）。そのことを指摘した上で一つの正しい証明を与えることが本研究の主要な目的である。さらに本論文では、このモデルの興味ある変形として (1) 計画期間が有限の場合と (2) N 個の箱のうち高々 M 個までしか開けることが許されていない場合の二つについても検討する。

ON A PROOF OF PANDORA'S RULE IN OPTIMAL STOPPING PROBLEM

By Seizo Ikuta

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Abstract

M.L. Weitzman formulated the following general model of a broad class of optimal stopping problems. Suppose there are N boxes. In box $i = 1,2,\dots,$ N, an unknown reward x_i is contained. Paying c_i dollars, you can open box i and know the unknown reward in it after time periods t_i. Here assume that the unknown reward in box i follows a known distribution function $F_i(x_i)$. Boxes are opened one by one, and if you stop to open further boxes after having opened some boxes, you can accept the maximum rewards revealed so far. Here the objective is to maximize the expectation of the present worth of the maximum reward accepted less the total of the present worth of the costs paid to open boxes so far. The optimal decision rule, maximizing the expectation, consists of the following two decision rules. Let a solution of the equation $-c_i + \beta_i \int \max\{w,y\} dF_i(w) - y = 0$ ($\beta_i = e^{-\alpha t_i}$; α is an instant discount factor) be denoted by z_i, called a reservation price of box i, where α is a discount factor. *Selection Rule* : If a box is to be opened, it should be that closed box with the highest reservation price. *Stopping Rule* : Terminate the search if the maximum sampled reward is greater than or equal to the reservation price of every closed box, otherwise open the next box. He named a pair of the rules *Pandora's Rule.* Although eventually having turned out to be correct, unfortunately it has been founded that <u>there exists a serious mistake in his proof</u>; it is a mere coincident that his conclusion was correct. It is a main purpose of this paper to point out the mistake and provide a correct proof. In addition, as interesting variations of the model, we also examine the following two cases: (1) A planning horizon is finite and (2) At most M boxes can be opened.

ON A PROOF OF PANDORA'S RULE IN OPTIMAL STOPPING PROBLEM

By Seizo Ikuta

0. INTRODUCTION

M.L. Weitzman [9] formulated a general model of a broad class of economic search problems. In his definition of the model, he used a fixed number of different closed boxes, each containing an unknown reward. This reward was revealed after paying a search cost to open the box and waiting a certain length of time. The time lag, search cost, and the degree of uncertainty of reward varies from box to box. The searcher must accept the reward in a box among them. If he decides not to open a box further, the search process terminates with the maximum reward sampled to that point being gained. If all the boxes were opened, the maximum of the (known) rewards in them must be accepted.

It is obvious that, the more boxes that were opened, the larger the possibility of obtaining an offer of higher value potential, but a larger total cost could also incur. The question therefore arises as to what order the boxes should be opened and when to stop so as to maximize the *expected present discounted value*, i.e., the expectation of the present worth of the maximum reward gained after termination of the search less the present worth of the total search costs incurred in opening boxes to that point in time.

Weitzman produced the optimal decision strategy for the model, which he called *Pandora's Rule* : it involves opening the boxes in the order of their *reservation prices*, easily computed for each box, and stopping the search accepting the maximum reward obtained to that point if it is greater than or equal to the highest reservation price within the boxes remaining.

Unfortunately a serious mistake was founded in his discussions through which Pandora's Rule was derived. Although the conclusion that he obtained turns out to be correct eventually, it is only a mere coincidence. The main purpose of this paper is to examine in detail the content of the mistakes as well as to provide a correct proof. Furthermore, we will attempt to apply the way of our proof to two kinds of variations of his model; one is with a finite planning horizon, and the other is with only opening M of the given N boxes being allowed.

1. MODEL

The model formulated by Weitzman is as follows. Suppose there are N closed boxes. Each box i, $1 \leqq i \leqq N$, contains an unknown reward x_i. Only the following three items of information are available concerning each box before the search is commenced.

1. the probability distribution function $F_i(x_i)$ of a reward x_i in box i, which is assumed to be continuous in the current paper,

2. the search cost c_i incurred in opening box i and finding out reward in it, and

3. the time lag t_i, after opening box i when the reward in it becomes known.

Both cost and reward are measured in the same monetary value and are continuously discounted by the instant discount rate α. By being continuously discounted, we mean that one unit of monetary value after time periods t is equivalent to $e^{-\alpha t}$ at that time. Below let $\beta_i = e^{-\alpha t_i}$. An initial reward y_0 is available before starting the search, and the option to collect the y_0 and abandon, from the outset, entering into the search process is permitted. If the search is terminated after opening some or all boxes, the maximum rewards discovered to that point, including the initial reward, is the gain.

The search process proceeds, for instance, as follows. First, you must decide either to accept the initial reward y_0 and quit the search process or to open one of the N boxes. If you decide to open, let us say, box 4 by paying the cost c_4, the reward x_4 in it can be revealed after a time lag t_4 when you must decide again either to accept the maximum reward $y_1 = \max\{y_0, x_4\}$ or to open a further one of the $N-1$ boxes remaining. Now, if the search is terminated after the opening of two boxes, say box 4 and box 2 in that order, the *realized* present discounted value obtained can be expressed as $-c_4 - c_2 e^{-\alpha t_4} + \max\{y_1, x_2\} e^{-\alpha(t_4+t_2)}$.

The objective of the search problem is to maximize the *expected* present discounted value.

2. PANDORA'S RULE

For any real number y, define

$$(2.1) \qquad K_i(y) = -c_i + \beta_i \int_{-\infty}^{\infty} \max\{w, y\}\, dF_i(w) - y, \quad 1 \leq i \leq N,$$

and let

$$(2.2) \qquad z_i = \sup\{y \mid K_i(y) > 0\},$$

called the *reservation price* of box i, where $K_i(y) > (\leq)\ 0$ for $y < (\geq)\ z_i$ because $K_i(y)$ is nonincreasing in y for all i due to

$$(2.3) \qquad dK_i(y)/dy = \beta_i F_i(y) - 1 \leq 0$$

for all i and all y. Throughout the paper, without loss in generality, let

$$(2.4) \qquad z_1 \geq z_2 \geq \cdots \geq z_N.$$

Now, *Pandora's Rule*, the optimal decision strategy attaining the maximum of an expected present discounted value, consists of the following two rules:

Selection Rule: *If a box is to be opened, it should be that closed box with the highest reservation price,*

Stopping Rule : *The search is to be terminated whenever the maximum sampled reward is greater than or equal to the reservation price of every closed box.*

The selection rule is, in other words, to open boxes in order of their reservation prices. Although the stopping rule was defined by Weitzman as "… the maximum sampled reward exceeds the reservation price of every closed box," for convenience in later discussions, we shall here define it as the above (the optimal choice between stopping and continuing becomes indifferent at the reservation price).

3. MISTAKES IN WEITZMAN'S PROOF AND MY PROOF

3.1. Functional Equation

Let us denote a set of the N boxes available by $S = \{1,2,\dots,N\}$ and any subset of S by L. Define $v(L,y)$ as the maximum expected present discounted value starting with boxes L and a maximum reward y so far. Then, from the principle of optimality in dynamic programming,

$$(3.1) \qquad v(L,y) = \max\{y, \max_{i \in L} \{-c_i + \beta_i \int_{-\infty}^{\infty} v(L_i, \max\{y,w\})\, dF_i(w)\}\}$$

where

$$(3.2) \qquad v(\Phi,y) = y,$$

$$(3.3) \qquad \beta_i = e^{-\alpha t_i},$$

$$(3.4) \qquad L_i = L - \{i\}.$$

The first term y and second term $\max_{i \in L}\{\cdot\}$ on the right hand side of (3.1) are, respectively, the value if stopping and the maximum expected present discounted value if continuing. For convenience of later discussions, let us rewrite (3.1) as follows:

$$(3.5) \qquad v(L,y) = y + \max\{0, \max_{i \in L} K_i(L,y)\}$$

where

$$(3.6) \qquad K_i(L,y) = -c_i + \int_{-\infty}^{\infty} v(L_i, \max\{y,w\})\, dG_i(w) - y,$$

$$(3.7) \qquad G_i(w) = \beta_i F_i(w).$$

3.2. *On a Weitzman's Proof*

From (3.5) we have $v(\{k\},y) = y + \max\{0, K_k(y)\}$ for every $k \in S$, so if $y < z_k$, opening box k is optimal due to $K_k(y) > 0$, or else stopping with accepting the maximum reward y is optimal due to $K_k(y) \leq 0$, implying that Pandora's Rule is optimal when starting with a closed box, that is, $m = 1$.

Assuming, as an *induction hypothesis,* that Pandora's Rule is optimal with any m closed boxes remaining and any maximum reward y, we shall start with any $m+1$ closed boxes $L = \{i_1, i_2, \dots, i_{m+1}\}$ and any maximum reward y where *let j, h, and r be boxes with, respectively, biggest, second biggest, and third biggest reservation prices in L.* Then, by O_0 we shall denote the expected present discounted value when opening no box; clearly $O_0 = y$.

i. If $y \geqq z_j$, since $\max\{y,x_k\} \geqq z_j \geqq z_h$ for any x_k, the expected present discounted value from opening any box $k \in L$ becomes $O_k = -c_k + \beta_k \int \max\{y,x_k\} dF_k(x_k)$ from the induction hypothesis; therefore, $O_k - O_0 = K_k(y) \leqq 0$ because of $y \geqq z_j \geqq z_k$, so not opening any box, or stopping with accepting the current maximum reward y is optimal.
ii. If $z_j > y$, the expected present discounted value from opening box j and then stopping becomes $O_j = -c_j + \beta_j \int \max\{y,x_j\} dF_j(x_j)$. Therefore, $O_j - O_0 = K_j(y) > 0$, implying that opening no box does not become optimal. Here a question as to which box should be opened arises.

The above discussions that were made by Weitzman are all correct, but the subsequent discussions involves the following serious mistake. Roughly speaking, defining the expected present discounted value of opening box j (any box $k \neq j$) with from now on proceeding by Pandora's Rule as A (B), he attempted to give the answer to the question by showing $A - B > 0$. However, he interpreted the definition of A by mistake as follows:

"*Consider the following alternative. Open box j first. Let h be a box with second biggest reservation price in the collection of m+1 closed box;* $h \in L-\{j\}$, $z_h = \max_{i \in L-\{j\}} z_i$. *If* $x_j \geqq z_h$, *terminate.* (1)*Otherwise, open box k next.* (2)*From then on proceed by Pandora's Rule. Let the expected present discounted value of this alternative policy be A.*" (pp.652)

First the underlined sentence (2) has to follow the sentence "*Open box j first.*" Next the underlined sentence (1) is not correct for the following reason. Suppose $z_h > x_j$. Then if $y \geqq z_h$, it is optimal to terminate due to $\max\{y,x_j\} \geqq z_h$, and if $z_h > y$, then $z_h > \max\{y,x_j\}$, so not *(any) box k* but *box h* with second biggest reservation price has to be opened next because we have to follow the induction hypothesis after having opened box j. Accordingly, the above quotation must be rewritten as follows:

"*Consider the following alternative. Open box j first.* (2)*From then on proceed by Pandora's Rule. Let h be a box with second biggest reservation price in the collection of m+1 closed box;* $h \in L-\{j\}$, $z_h = \max_{i \in L-\{j\}} z_i$. *If* $x_j \geqq z_h$, *terminate.* (1)*Suppose* $z_h > x_j$. *Then, if* $y \geqq z_h$, *terminate, otherwise open box h next. Let the expected present discounted value of this alternative policy be A.*"

Now the A and B that were defined above can be expressed as follows where let $z_h > y$, hence $z_j > y$ and let $k \neq j$ and $k \neq h$ for B.

$$(3.8)\quad A = -c_j+\beta_j\{P_r(x_j \geqq z_h)E(\max\{y,x_j\}|x_j \geqq z_h) \\ +P_r(z_h > x_j)(-c_h+\beta_h E(v(L-\{j\}-\{h\},\ \max\{y,x_j,x_h\})|z_h > x_j))\}$$

$$(3.9)\quad B = -c_k+\beta_k\{Pr(x_k \geqq z_j)E(x_k|x_k \geqq z_j) \\ +Pr(z_j > x_k)(-c_j+\beta_j E(v(L-\{k\}-\{j\},\ \max\{y,x_k,x_j\})|z_j > x_k))\}$$

which are expressed in a Weitzman's fashion as follows, respectively,

$$(3.10)\quad A = -c_j+\beta_j\{P_r(x_j \geqq z_j)E(x_j|x_j \geqq z_j) \\ +P_r(z_j > x_j \geqq z_h)E(\max\{y,x_j\}|z_j > x_j \geqq z_h) \\ +P_r(z_h > x_j)(-c_h+\beta_h[Pr(x_h \geqq z_r)E(\max\{y,x_j,x_h\}|z_h > x_j, x_h \geqq z_r) \\ +Pr(z_r > x_h)E(v(L-\{j\}-\{h\},\ \max\{y,x_j,x_h\})|z_h > x_j, z_r > x_h)])\}$$

$$(3.11)\quad B = -c_k+\beta_k\{\Pr(x_k\geq z_j)E(x_k|x_k\geq z_j)$$
$$+\Pr(z_j>x_k\geq z_h)(-c_j$$
$$+\beta_j[\Pr(x_j\geq z_j)E(x_j|x_j\geq z_j)$$
$$+\Pr(z_j>x_j\geq z_h)E(\max\{y,x_k,x_j\}|z_j>x_k\geq z_h,\ z_j>x_j\geq z_h)$$
$$+\Pr(z_h>x_j)E(\max\{y,x_k\}|z_j>x_k\geq z_h)])$$
$$+\Pr(z_h>x_k)(-c_j+\beta_j[\Pr(x_j\geq z_j)E(x_j|x_j\geq z_j)$$
$$+\Pr(z_j>x_j\geq z_h)E(\max(y,x_j)|z_j>x_j\geq z_r)$$
$$+\Pr(z_h>x_j)E(v(L-\{k\}-\{j\},$$
$$\max\{y,x_k,x_j\})|z_h>x_k,z_h>x_j))])\}$$

Expressing the A and B by using the notations defined by Weitzman yields, respectively,

$$(3.12)\quad A = -c_j+\beta_j\{\pi_j w_j+\lambda_j\tilde{v}_j+(1-\pi_j-\lambda_j)(-c_h+\beta_h[\pi_h d^*+(1-\pi_h)\Phi_a])$$

$$(3.13)\quad B = -c_k+\beta_k\{\pi_k w_k+\lambda_k(-c_j+\beta_j[\pi_j w_j+\lambda_j d+(1-\pi_j-\lambda_j)\tilde{v}_k)])$$
$$+\ (1-\pi_k-\lambda_k)(-c_j+\beta_j[\pi_j w_j+\lambda_j\tilde{v}_j+(1-\pi_j-\lambda_j)\Phi_b])\}$$

where

$$\pi_j = \Pr(x_j\geq z_j),\qquad \pi_k = \Pr(x_k\geq z_j),\qquad \pi_h = \Pr(x_h\geq z_r),$$
$$w_j = E(x_j|x_j\geq z_j),\qquad w_k = E(x_k|x_k\geq z_j),$$
$$\lambda_j = \Pr(z_j>x_j\geq z_h),\qquad \lambda_k = \Pr(z_j>x_k\geq z_h)$$
$$\tilde{v}_j = E(\max\{y,x_j\}|z_j>x_j\geq z_h),\qquad \tilde{v}_k = E(\max\{y,x_k\}|z_j>x_k\geq z_h),$$
$$d = E(\max\{y,x_k,x_j\}|z_j>x_k\geq z_h,\ z_j>x_j\geq z_h),\quad d^* = E(\max\{y,x_j,x_h\}|z_h>x_j,x_h\geq z_r),$$
$$\Phi_a = E(v(L-\{j\}-\{h\},\ \max\{y,x_j,x_h\})|z_h>x_j,z_r>x_h)),$$
$$\Phi_b = E(v(L-\{k\}-\{j\},\ \max\{y,x_k,x_j\})|z_h>x_k,z_h>x_j)),$$

in which Φ_b is the same as Φ defined by him (pp.653), but Φ_a can not always be equal to the Φ. It is clear that (3.10) (or (3.12)) is different from A in pp. 653 as well as that the terms related to Φ_a and Φ_b can not be eliminated by taking the difference of A and B as in (20) of pp. 653. Here it goes without saying that the elimination of the terms related to $v(\cdot)$ ($\Psi(\cdot)$ in his definition) in (20) was caused by the mistake in the underlined parts of the above quotation.

3.3. *My Proof*

Without loss in generality, we shall let here $z_1 \geq z_2 \geq \cdots \geq z_N$. Then in order to verify Pandora's Rule, it would suffice to prove the following theorem where k represents the smallest element in a given subset L of S and $\mathcal{L}_m$ denotes the family of subsets $L = \{i_1,i_2,\cdots,i_m\}$, $i_1 < i_2 < \cdots < i_m$, consisting of m elements in S, $1 \leq m \leq N$.

THEOREM 1. *For any* $L \in \mathcal{L}_m$, $1 \leq m \leq N$,

(a) $v(L,y) = y + \max\{0, K_k(L,y)\}$, $k = i_1$ (= min L), *for all* y,

(b) $K_k(L,y)$ *is nonincreasing in* y *with* $K_k(L,z_k) = 0$ *where* $K_k(L,y) >(\leq) 0$ *for* $y <(\geq) z_k$,

Proof: In addition to the two assertions (a) and (b), we shall also prove herein the following supplementary statement:

(c) $K_{i_1}(\{i_1,i_2,\cdots,i_m\},y) = K_{i_1}(\{i_1,i_2,\cdots,i_{m-1}\},y)$ *for* $y \geq z_{i_m}$, $2 \leq m \leq M$,

which is necessary to prove (a) and (b). First, clearly

(3.14) $\quad v(\{1\},y) = y + \max\{0, K_1(\{1\},y)\}$,

(3.15) $\quad K_1(\{1\},y) = K_1(y)$,

implying that both (a) and (b) hold true for $\{1\}$, hence also for any $L \in \mathscr{L}_1$. Next, arranging (3.6) for $\{1,2\}$ by substituting (3.14) for $\{2\}$ instead of $\{1\}$ yields

(3.16) $$K_1(\{1,2\},y) = K_1(y) + \int_{-\infty}^{\infty}\max\{0, K_2(\max\{y,w\})\}\,dG_1(w).$$

If $y \geq z_2$, then since $\max\{y,w\} \geq z_2$ for all w, $K_1(\{1,2\},y) = K_1(y) = K_1(\{1\},y)$; therefore, (c) holds for $\{1,2\}$, hence also for any $L \in \mathscr{L}_2$.

Below, assuming (a) to (c) for any $L \in \mathscr{L}_n$, $1 \leq n \leq m-1$, we shall prove that these are also true for any $L \in \mathscr{L}_m$, say $L = \{1,2,\ldots,m\}$.

First, let us prove (b). It is clear from the induction hypothesis that

(3.17) $\quad v(L_1,y) = y + \max\{0, K_2(L_1,y)\}$,

(3.18) $\quad v(L_i,y) = y + \max\{0, K_i(L_i,y)\}, \quad 2 \leq i \leq m.$

Arranging (3.6) by substituting (3.17) and (3.18) yields

(3.19) $$K_1(L,y) = K_1(y) + \int_{-\infty}^{\infty}\max\{0, K_2(L_1,\max\{y,w\})\}\,dG_1(w),$$

(3.20) $$K_i(L,y) = K_i(y) + \int_{-\infty}^{\infty}\max\{0, K_i(L_i,\max\{y,w\})\}\,dG_i(w), \quad 2 \leq i \leq m.$$

It can be easily seen from (3.19) and the induction hypothesis that (b) holds for the L, hence also for any $L \in \mathscr{L}_m$.

Next, we shall prove (c). Suppose $y \geq z_m$. Then, since $K_2(\{2,3,\cdots,m\},y) = K_2(\{2,3,\cdots,m-1\},y)$ from the induction hypothesis, we have

(3.21) $$\begin{aligned} K_1(\{1,2,\cdots,m\},y) &= K_1(y) + \int_{-\infty}^{\infty}\max\{0, K_2(\{2,3,\cdots,m\},\max\{y,w\})\}\,dG_1(w) \\ &= K_1(y) + \int_{-\infty}^{\infty}\max\{0, K_2(\{2,3,\cdots,m-1\},\max\{y,w\})\}\,dG_1(w) \\ &= K_1(\{1,2,\cdots,m-1\},y). \end{aligned}$$

Thus, (c) holds for $\{1,2,\cdots,m\}$, hence also for any $L \in \mathscr{L}_m$.

Finally, let us prove (a). If $y \geq z_1$, then $K_i(L,y) = K_i(y) \leq 0$ for $1 \leq i \leq N$ from (3.19) and (3.20). Hence, (a) becomes true for $y \geq z_1$. Below, let us prove that this is also true for $y < z_1$. Here note that, for any y $(-\infty < y < +\infty)$, (3.19) and (3.20) can be expressed as, respectively, (3.22) $$K_1(L,y) = K_1(y) + (K_2(L_1,y)G_1(y) + \int_y^{z_2} K_2(L_1,w)\,dG_1(w))I(y<z_2),$$

(3.23) $$K_i(L,y) = K_i(y) + (K_i(L_i,y)G_i(y) + \int_y^{z_i} K_i(L_i,w)\,dG_i(w))I(y<z_i), \quad 2 \leqq i \leqq m$$

where I(S) is an indicator function; that is, if a given statement S is true, then I(S) = 1, or else I(S) = 0. Then, differentiating the above two with respect to y produces, respectively,

(3.24) $$dK_1(L,y)/dy = G_1(y)(1+dK_2(L_1,y)/dy\,I(y<z_2)) - 1,$$

(3.25) $$dK_i(L,y)/dy = G_i(y)(1+dK_i(L_i,y)/dy\,I(y<z_i)) - 1, \quad 2 \leqq i \leqq m.$$

Now define

(3.26) $$\Gamma_k(y) = \prod_{i=1}^{k} G_i(y).$$

Then, recursively expressing $dK_1(L,y)/dy$ for $L \in \mathscr{L}_n$ in the order of $n = 1,2,\cdots,m$ by starting with $dK_1(L,y)/dy = G_1(y) - 1$ for $L = \{1\}$ ($\in \mathscr{L}_1$) produces

(3.27) $$dK_1(L,y)/dy = \Gamma_1(y)I(z_2 \leqq y)+\sum_{k=2}^{m-1}\Gamma_k(y)I(z_{k+1} \leqq y<z_k)+\Gamma_m(y)I(y<z_m)-1.$$

Here remember that we assumed $y < z_1$, hence $I(y<z_1) = 1$. Then, for L_i, $3 \leqq i \leqq m-1$, (3.27) can be expressed as

$$dK_1(L_i,y)/dy = \Gamma_1(y)I(z_2 \leqq y) + \sum_{k=2}^{i-2}\Gamma_k(y)I(z_{k+1} \leqq y<z_k) + \Gamma_{i-1}(y)I(z_{i+1} \leqq y<z_{i-1})$$
$$+ \sum_{k=i+1}^{m-1}\Gamma_k(y)G_i(y)^{-1}I(z_{k+1} \leqq y<z_k) + \Gamma_m(y)G_i(y)^{-1}I(y<z_m) - 1.$$

Arranging the above by substituting $\Gamma_{i-1}(y)I(z_{i+1} \leqq y<z_{i-1}) = \Gamma_i(y)G_i(y)^{-1}I(z_{i+1} \leqq y<z_i) + \Gamma_{i-1}(y)I(z_i \leqq y<z_{i-1})$ leads to

(3.28) $$dK_1(L_i,y)/dy = \Gamma_1(y)I(z_2 \leqq y) + \sum_{k=2}^{i-1}\Gamma_k(y)I(z_{k+1} \leqq y<z_k)$$
$$+ \sum_{k=i}^{m-1}\Gamma_k(y)G_i(y)^{-1}I(z_{k+1} \leqq y<z_k) + \Gamma_m(y)G_i(y)^{-1}I(y<z_m) - 1.$$

It is easy to check that, if $i = 2$ ($i = m$), then the second (third) term in the right hand side of (3.28) is eliminated. Arranging the right hand side of (3.25) by substituting (3.28) produces

(3.29) $$dK_i(L,y)/dy = G_i(y)\Gamma_1(y)I(z_2 \leqq y) + G_i(y)\sum_{k=2}^{i-1}\Gamma_k(y)I(z_{k+1} \leqq y<z_k)$$
$$+ \sum_{k=i}^{m-1}\Gamma_k(y)I(z_{k+1} \leqq y<z_k) + \Gamma_m(y)I(y<z_m) - 1, \quad 2 \leqq i \leqq m.$$

From (3.27) and (3.29), we have

$$(3.30)\qquad dK_1(L,y)/dy - dK_i(L,y)/dy$$
$$= (1-G_1(y))\Big(\Gamma_1(y)I(z_2 \leq y) + \sum_{k=2}^{i-1}\Gamma_k(y)I(z_{k+1} \leq y < z_k)\Big),\quad 2 \leq i \leq m.$$

Evidently the right hand side of the above expression is nonnegative as well as equal to 0 on $y < z_i$, implying that the difference $K_1(L,y) - K_i(L,y)$ is constant on $y < z_i$ and nondecreasing on $z_i \leq y < z_1$. Consequently, in order to prove (a), it suffices to show $K_1(L,z_i) - K_i(L,z_i) \geqq 0$ for $2 \leqq i \leqq m$. Repeatedly applying the induction hypothesis (c) by regarding $\max\{z_i,w\} \geqq z_i \geqq z_{i+1} \geqq \cdots \geqq z_{m-1} \geqq z_m$ for all w yields

$$(3.31)\qquad K_2(\{2,3,\cdots,m\},\max\{z_i,w\}) = K_2(\{2,3,\cdots,i-1\},\max\{z_i,w\}),$$

$$(3.32)\qquad K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},z_i) = K_i(\{1,2,\cdots,i-1\},z_i),$$

from which

$$(3.33)\qquad K_1(\{1,2,\cdots,m\},z_i) = K_1(z_i) + \int_{-\infty}^{\infty}\max\{0,\ K_2(\{2,3,\cdots,m\},\max\{z_i,w\})\}\,dG_1(w)$$
$$= K_1(z_i) + \int_{-\infty}^{\infty}\max\{0,\ K_2(\{2,3,\cdots,i-1\},\max\{z_i,w\})\}\,dG_1(w)$$
$$= K_1(\{1,2,\cdots,i-1\},z_i),$$

$$(3.34)\qquad K_i(\{1,2,\cdots,m\},z_i) = \int_{-\infty}^{\infty}\max\{0,\ K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},\max\{z_i,w\})\}\,dG_i(w)$$
$$\leqq \beta_i\max\{0,\ K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},z_i)\}$$
$$\leqq \max\{0,\ K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},z_i)\}$$
$$= K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},z_i)$$
$$= K_i(\{1,2,\cdots,i-1\},z_i),\quad 2 \leqq i \leqq m.$$

Thus, $K_1(\{1,2,\cdots,m\},z_i) - K_i(\{1,2,\cdots,m\},z_i) \geqq 0$, implying that (a) holds for all $y < z_1$ and for $\{1,2,\cdots,m\}$, hence also for all $L \in \mathscr{L}_m$. Consequently, it follows that (a) holds for all $L \in \mathscr{L}_m$ and all y. Q.E.D.

4. On Pandora's Rule

One of the most peculiar points of Pandora's Rule is that it depends *only* on the set of reservation prices of the given N boxes, $\{z_1,z_2,\cdots,z_N\}$. The following interpretation, although a little intuitive, will make this peculiarity more striking.

Consider N problems, each with only one box i, $i = 1,2,\ldots,N$, and with any initial value y. The *relative* expected present discounted value from opening box i instead of accepting the initial value y is then $\{-c_i+\beta_i\int\max\{y,w\}\,dF_i(w)\} - y = K_i(y)$. Then our intuition will tend to lead us to feelings that a box with larger relative expected present discounted value for *all* y has more potential for gain if opened. Therefore, it will follow that if $K_1(y) \geqq K_2(y) \geqq \cdots \geqq K_N(y)$ for *all* y, since we get $z_1 \geqq z_2 \geqq \cdots \geqq z_N$ due to the monotonicity of $K_i(y)$ in y for all i, a box with a higher reservation price will present more gain if opened, and hence boxes are to be opened in order of their reservation prices (i.e., Pandora's Rule becomes optimal). Fortunately, Pandora's Rule asserts truth in the above interpretation. However, *even* in a case that there exists a y for which the *relative* expected present discounted value from opening a box with lower reservation price first is greater than that of opening a box with higher reservation price first, Pandora's Rule still remains optimal for *all* y.

As such a case, we can take the following example. Consider two boxes the rewards of which follow the uniform distribution on $[0,1]$, and let $c_1 = 0.26$ and $\beta_1 = 1.00$ for box 1 and $c_2 = 0.10$ and $\beta_2 = 0.70$ for box 2. Then the reservation prices for the two boxes become, respectively, $z_1 = 1-\sqrt{0.52} = 0.27888...$ and $z_2 = (1-\sqrt{0.65})/0.7 = 0.27682...$, and furthermore $K_1(0) = 0.24$ and $K_2(0) = 0.25$, respectively. Accordingly, since $k_2(0) > K_1(0)$ and $z_1 > z_2$ in the case, it follows that, in spite of $z_1 > z_2$, there exists at least one pair (y',y''), $0 < y' < y'' < z_2$, such as $K_2(y') > K_1(y') > 0$ and $K_1(y'') > K_2(y'') > 0$ (See Figure 1), hence $v(\{2\},y') > v(\{1\},y')$ and $v(\{1\},y'') > v(\{2\},y'')$.

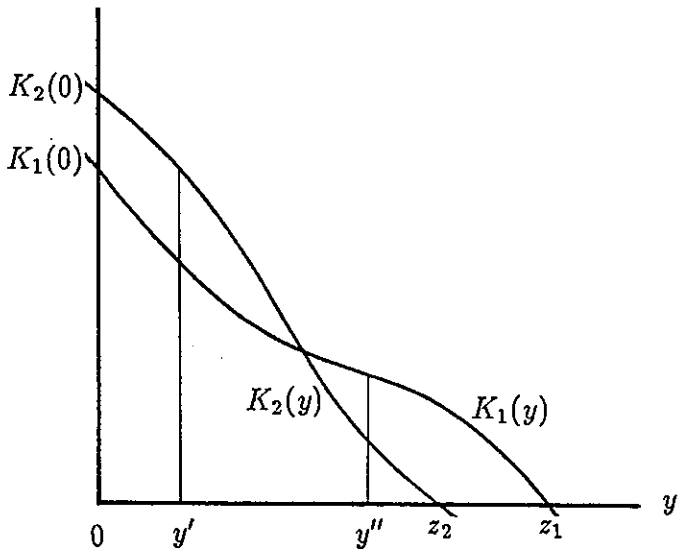

Figure 1

5. TWO VARIATIONS

Weitzman emphasized in his paper the necessity for studies of different variations of his model in order to make it more realistic. Here we shall examine two out of them.

5.1. Finite Planning Horizon

So far, it has been postulated by implication that a planning horizon is infinite. This is not a realistic assumption. Here, we shall assume it is finite; therefore, not all the boxes can always be opened. For example, consider a problem with three boxes as $t_1+t_2+t_3 > t_1+t_3 > t_1+t_2 > t_1 > t_2+t_3 > t_3 > t_2$. The subsets of the boxes that can be opened when the planning horizon is T becomes as follows:

$$
\begin{array}{rll}
\{1,2,3\},\ \{1,3\},\ \{1,2\},\ \{2,3\},\ \{1\},\ \{3\},\ \{2\},\ \Phi & \text{if} & T \geqq t_1+t_2+t_3, \\
\{1,3\},\ \{1,2\},\ \{2,3\},\ \{1\},\ \{3\},\ \{2\},\ \Phi & \text{if} & t_1+t_2+t_3 > T \geqq t_1+t_3, \\
\{1,2\},\ \{2,3\},\ \{1\},\ \{3\},\ \{2\},\ \Phi & \text{if} & t_1+t_3 > T \geqq t_1+t_2, \\
\{2,3\},\ \{1\},\ \{3\},\ \{2\},\ \Phi & \text{if} & t_1+t_2 > T \geqq t_1, \\
\{2,3\},\qquad \{3\},\ \{2\},\ \Phi & \text{if} & t_1 > T \geqq t_2+t_3, \\
\{3\},\ \{2\},\ \Phi & \text{if} & t_2+t_3 > T \geqq t_3, \\
\{2\},\ \Phi & \text{if} & t_3 > T \geqq t_2, \\
\Phi & \text{if} & t_2 > T \geqq 0,
\end{array}
$$

It is sure that Pandora's Rule is optimal for any given subset. The question to be answered here is which is the best subset of boxes to be opened. Fortunately, we can prove the following theorem, almost obvious.

THEOREM 2. *If (2.4) is satisfied, then* $v(L,y) \geqq v(M,y)$ *for any subset* M *of any* $L\ (\subset S)$ *and any* y.

Proof: See Appendix A for a strict proof. Q.E.D.

By using the theorem, the above list of subsets can be reduced to

$$
\begin{array}{lll}
\{1,2,3\} & \text{if} & T \geqq t_1+t_2+t_3, \\
\{1,3\},\ \{1,2\},\ \{2,3\} & \text{if} & t_1+t_2+t_3 > T \geqq t_1+t_3, \\
\{1,2\},\ \{2,3\} & \text{if} & t_1+t_3 > T \geqq t_1+t_2, \\
\{2,3\},\ \{1\} & \text{if} & t_1+t_2 > T \geqq t_1, \\
\{2,3\} & \text{if} & t_1 > T \geqq t_2+t_3, \\
\{3\},\ \{2\} & \text{if} & t_2+t_3 > T \geqq t_3, \\
\{2\} & \text{if} & t_3 > T \geqq t_2, \\
\Phi & \text{if} & t_2 > T \geqq 0,
\end{array}
$$

Now we demonstrated in Section 4 that $v(\{1\},y)$ is not always greater than or equal to $v(\{2\},y)$ for all y; in other word, starting with a box having smaller reservation price may yield greater expected present discounted value than starting with a box having bigger reservation price. However, if $\beta_1F_1(w) \leqq \beta_2F_2(w)$ is satisfied in addition to $z_1 \geqq z_2$, we can verify $v(\{1\},y) \geqq v(\{2\},y)$ for all y. More generally, the following theorem holds:

THEOREM 3. *Suppose that both (2.4) and*

(5.1) $\beta_1F_1(w) \leqq \beta_2F_2(w) \leqq \cdots \leqq \beta_NF_N(w)$ *for all* w

are satisfied. Then if $(i_1,i_2,\cdots,i_m) \leqq (k_1,k_2,\cdots,k_m)^{\dagger}$, $1 \leqq m \leqq N$, *we have* $\max\{0,\ K_{I_1}(\{i_1,i_2,\cdots,i_m\},y)\} \geqq \max\{0,\ K_{k_1}(\{k_1,k_2,\cdots,k_m\},y)\}$ *for all* y, *hence* $v(\{i_1,i_2,\cdots,i_m\},y) \geqq v(\{k_1,k_2,\cdots,k_m\},y)$ *for all* y.

Proof: See Appendix B. Q.E.D.

It is easy to show that there exist a case satisfying both (2.4) and (5.1) (See Appendix C). If the conditions in the theorem are satisfied, the list of subsets remaining after having applied Theorem 2 can be further reduced to

$$\begin{array}{lll}
\{1,2,3\} & \text{if} & T \geq t_1+t_2+t_3, \\
\{1,2\} & \text{if} & t_1+t_2+t_3 > T \geq t_1+t_3, \\
\{1,2\} & \text{if} & t_1+t_3 > T \geq t_1+t_2, \\
\{2,3\},\ \{1\} & \text{if} & t_1+t_2 > T \geq t_1, \\
\{2,3\} & \text{if} & t_1 > T \geq t_2+t_3, \\
\{2\} & \text{if} & t_2+t_3 > T \geq t_3, \\
\{2\} & \text{if} & t_3 > T \geq t_2, \\
\Phi & \text{if} & t_2 > T \geq 0,
\end{array}$$

Case of $F_1(W) = F_2(W) = \cdots = F_N(W)$: The conditions in Theorem 3 may be said to be very severe ones; however, if we confines the problem to cases that the distribution function of reward is independent of box, the theorem will provides an efficient procedure to reduce the list of subsets. Now suppose the boxes have already been renumbered in a decreasing order of reservation prices. Then the theorem can be applied to any subset of boxes such that β_i is nondecreasing in i; for example, if the β_i, the function of i, is depicted as shown in Figure 2, then, as examples of such subset, we can take $\{\beta_1,\beta_6\}$, $\{\beta_2,\beta_{10}\}$, $\{\beta_3,\beta_{10},\beta_{14}\}$, $\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5\}$, $\{\beta_8,\beta_9,\beta_{10},\beta_{11},\beta_{12},\beta_{13}\}$, $\{\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_{10},\beta_{11},\beta_{12},\beta_{13}\}$, and so on. In the case, if {1} and {6} remain in the list, then {6} can be eliminated, and if {1,2,3}, {5,10,12}, and {2,4,13} remain in the list, then only {1,2,3} remains after having applied Theorem 3.

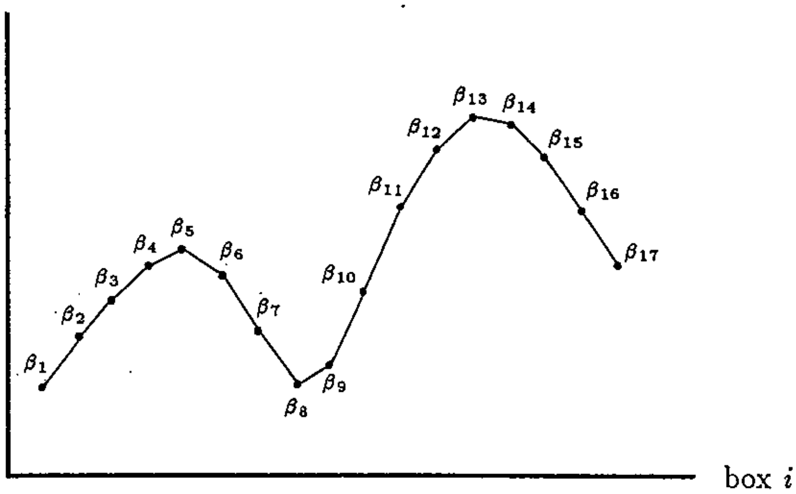

Figure 2

† Both $(i_1, i_2 \cdots, i_m)$ and $(k_1, k_2, \cdots, k_m)$ are vectors.

5.2. Only Opening M of the N Boxes

This section examines a problem of only opening M of the given N boxes, $N > M \geq 1$; we shall refer to the problem as an *(N,M)-problem.* Let $v_m(L,y)$ be the maximum expected present discounted value from starting with n boxes $L = \{i_1, i_2, \cdots, i_n\}$ ($\in S$) and the maximum reward y so far, provided that only m in L are allowed to be opened ((n,m)-problem). Then

$$(5.2) \qquad v_m(L,y) = y + \max\{0, \max_{j \in L}\{-c_j + \int_{-\infty}^{\infty} v_{m-1}(L_j, \max\{y,w\})\, dG_j - y\}\}, \quad m = 1,2,\ldots,n$$

where $v_0(\Phi,y) = y$, hence

$$(5.3) \qquad v_1(L,y) = y + \max\{0, \max_{j \in L} K_j(y)\}, \quad \text{(n,1)-problem}$$

It is of course that once given the list of m boxes to be opened, Pandora's Rule becomes the optimal decision strategy for the list. Then the question here arises as to whether or not (n,m)-problem can always be solved by finding the best (attaining the maximum expected present discounted value) out of all the possible lists of m boxes to be opened. It is clear from the example of Section 4 that, in (n,1)-problem, the maximum with respect to j in the right hand side of (5.3) is not always attained by box 1 with biggest reservation price for all y. Accordingly, the answer to the above question is negative for (n,1)-problem; that is, Pandora's Rule does not always become optimal for (n,1)-problem. More generally, we can show such a case as

$$(5.4) \qquad v_m(L,y) > \max_{\{i_1, i_2, \cdots, i_m\} \in L} v(\{i_1, i_2, \cdots, i_m\}, y)$$

for a certain y (See Appendix E); it goes without saying that Pandora's Rule does not become optimal in such a case. Here it is of course that the inequality

$$(5.5) \qquad v_m(L,y) \geq \max_{\{i_1, i_2, \cdots, i_m\} \in L} v(\{i_1, i_2, \cdots, i_m\}, y)$$

always holds.

THEOREM 4. *Let* $L = \{i_1, i_2, \cdots, i_n\}$. *Then*

(a). $v_1(L,y) = \max_{j \in L} v(\{j\}, y)$ *for all* y,

(b) *If (2.4) and (5.1) are satisfied, then, for all* y,

$$v_m(L,y) = \max_{\{i_1, i_2, \cdots, i_m\} \subset L} v(\{i_1, i_2, \cdots, i_m\}, y) = v(\{1,2,\cdots,m\}, y), \quad m = 1,2,\cdots,n-1$$

Proof: See Appendix D. Q.E.D.

The above theorem indicates that there exists such a case that inequality (5.5) holds with equality sign, i.e.,

$$(5.6) \qquad v_m(L,y) = \max_{\{i_1, i_2, \cdots, i_m\} \in L} v(\{i_1, i_2, \cdots, i_m\}, y)$$

as well as that it does not always means that Pandora's Rule becomes optimal whenever (5.6) holds.

6. SOME LIMITATIONS AND FUTURE STUDIES

Pointing out that "*… certain other aspects of the optimal search problem have been abstracted away. Many of the underlying assumptions of the current formulation are unrealistic …,*" Weitzman suggested the necessity of studying subjects in which the following provisions would be taken into account: (1) adaptive learning about probability distributions, (2) parallel search activity, (3) risk aversion, (4) incomplete or no recall, (5) collecting some reward before the search is terminated, (6) randomly generated new opportunities, (7) a binding time horizon, (8) uncertain search costs or search time, (9) opening only M of the N boxes available, etc. In the current paper, we investigated two of these subjects, (7) and (9).

Furthermore, if the following are taken into consideration, the model becomes more realistic as well as persuasive: (10) a finite search budget (the total money available for the entire search activity is limited), (11) multiple offers can be accepted, (12) game theory, (13) shifting cost from one box to others, and so on.

For (10), we, first of all, should realize that rational investors are always intent on adding to their fortune through any business opportunity, and hence their concern is not only in the total profit from alternatives they decide to take but also in the interest from investing in any other investment opportunities the remaining budget after having allocated the presently available budget among them. It is true of course that, if a search cost is relatively small, then the effect of such an investment on the optimal decision will not be large enough to be worth discussion. However, when it is enormous, for example, as in a research and development problem, it could not longer be neglected practically. Now, at a glance, the problem seems to be the same as (7) in the sense that not all boxes can always be opened within a limited amount of search budget. When the investment of a remaining budget is considered however, the problem does have to be reformulated to one of maximizing the sum of the expected present discounted value and the remaining budget after stopping the search, causing it to assume an aspect quite different from in (7) in the mathematical treatment. In this case, we encounter the very troublesome question that a budget available at a time increases with time due to a given rate of interest; if box i is opened at a time when a remaining budget is x, then it increases or decreases to $(x - c_i)/\beta_i$ up to the time when the next box is ready to be opened; in other words, it increases if $x \geqq c_i/(1-\beta_i)$, or else decreases. This makes any mathematical treatment of the problem quite difficult and intractable. If it is assumed that either $\beta_1 = \beta_2 = \cdots = \beta_N = 1$ or the principal of the remaining budget is only utilized as a search cost, then similar discussions to those in Section 4 hold; these assumptions are practical enough if the assumed interest rate is sufficiently small or if the time lag of each box is sufficiently short.

For (12), Reinganum (1983) studied a variation in which the notion of Nash equilibrium was introduced, revealing that there exists a Nash equilibrium search policy with the same form of optimal search policy as Pandora's Rule.

For (13), the following fishing problem can be considered. Suppose several promising fishing grounds have been found in the North Atlantic Ocean by means of satellite observation and a skipper of a fishing fleet is considering which of these sites would prove best. The shifting costs from the home base of the fishing fleet to each fishing site and from these to the others are known. After conducting a trial in a site at a cost, the expected catch is revealed in a short time. Let the expected catch be assumed to be a random variable following a known distribution function, varying from site to site. Now suppose the skipper has known the expected catch in a certain site and abandoned moving to any other site for a further trial fishing. Then the alternatives open to him are two: (i) start full-scale fishing at the site and (ii) return to any one of these checked so far and start full-scale fishing there. It is possible that some of the promising fishing sites have moved or disappeared or that new promising site have appeared. These variations can be said to be those into which a search problem and shortest route problem are combined.

In the example that Weitzman cited in his paper, if considering that a cost incurs when changing from a technology (for production of some commodity) to another, depending on the potential of both, it can be theoretically reduced to the same problem as the above fishing problem.

The ideas underlying our method of proof for Pandora's Rule will become an useful tool in studying these different variations.

APPENDIX A

— Proof of Theorem 2 —

First consider a case of $L = \{i_1, i_2\}$. Then clearly,

$$\begin{aligned} v(\{i_1,i_2\},y) &= \max\{y, \max_{j=i_1,i_2} \{-c_j + \int_{-\infty}^{\infty} v(\{i_1,i_2\}-\{j\}, \max\{y,w\})\, dG_j\}\} \\ &\geqq \max\{y, \max_{j=i_1,i_2} \{-c_j + \int_{-\infty}^{\infty} \max\{y,w\}\, dG_j\}\} \\ &= \max_{j=i_1,i_2} \max\{y, -c_j + \int_{-\infty}^{\infty} \max\{y,w\}\, dG_j\} \\ &= \max_{j=i_1,i_2} v(\{j\},y) \geqq v(\{j\},y), \quad j = i_1, i_2. \end{aligned}$$

Therefore, the assertion in the theorem holds for any $L = \{i_1, i_2\}$. Suppose it holds for any L consisting of m boxes. Then, for any L consisting of $m+1$ boxes and any subset M of L, we have

$$\begin{aligned} v(L,y) &\geqq \max\{y, \max_{j\in M}\{-c_j + \int_{-\infty}^{\infty} v(L_j, \max\{y,w\})\, dG_j\}\} \\ &\geqq \max\{y, \max_{j\in M} \{-c_j + \int_{-\infty}^{\infty} v(M_j, \max\{y,w\})\, dG_j\}\} \qquad (M_j \subset L_j) \\ &= v(M,y). \quad \text{Q.E.D.} \end{aligned}$$

APPENDIX B

– *Proof of Theorem 3* –

For any $i < k$, $dK_i(y)/dy - dK_k(y)/dy = G_i(y) - G_k(y) \leqq 0$ for all y; hence, $K_i(y) - K_k(y)$ is nonincreasing in y. In addition, since $K_i(z_i) - K_k(z_i) = -K_k(z_i) \geqq 0$, it follows that $K_i(y) - K_k(y) \geqq 0$ for $y \leqq z_i$, hence $\max\{0,K_i(y)\} \geqq \max\{0,K_k(y)\}$ for $y \leqq z_i$. If $z_i < y$, $K_i(y) \leqq 0$ and $K_k(y) \leqq 0$, implying $\max\{0,K_i(y)\} = \max\{0,K_k(y)\} = 0$. Therefore, it follows that $\max\{0,K_i(y)\} \geqq \max\{0,K_k(y)\}$ for all y. Thus, the assertion in the theorem is true for $m = 1$. Assume it is true for $m = 1,2,\ldots,n-1$. Then, from the induction hypothesis,

$$(B.1)\qquad \begin{aligned} K_{k_1}(\{k_1,i_2,\cdots,i_n\},y) &= K_{k_1}(y) + \int_{-\infty}^{\infty}\max\{0,\ K_{i_2}(\{i_2,i_3,\cdots,i_n\},\max\{y,w\})\}\,dG_{k_1} \\ &\geqq K_{k_1}(y) + \int_{-\infty}^{\infty}\max\{0,\ K_{k_2}(\{k_2,k_3,\cdots,k_n\},\max\{y,w\})\}\,dG_{k_1} \\ &= K_{k_1}(\{k_1,k_2,\cdots,k_n\},y) \end{aligned}$$

for all y; hence, we get

$$(B.2)\qquad \max\{0,\ K_{k_1}(\{k_1,i_2,\cdots,i_n\},y)\} \geqq \max\{0,\ K_{k_1}(\{k_1,k_2,\cdots,k_n\},y)\}$$

for all y. Differentiating (B.1) and

$$(B.3)\qquad K_{i_1}(\{i_1,i_2,\cdots,i_n\},y) = K_{i_1}(y) + \int_{-\infty}^{\infty}\max\{0,\ K_{i_2}(\{i_2,i_3,\cdots,i_n\},\max\{y,w\})\,dG_{i_1}$$

with respect to y yields (See (3.24)), respectively,

$$(B.4)\qquad dK_{k_1}(\{k_1,i_2,\cdots,i_n\},y)/dy = G_{k_1}(y)(1+ dK_{i_2}(\{i_2,i_3,\cdots,i_n\},y)/dy I(y<z_{i_2})) - 1,$$

$$(B.5)\qquad dK_{i_1}(\{i_1,i_2,\cdots,i_n\},y)/dy = G_{i_1}(y)(1+ dK_{i_2}(\{i_2,i_3,\cdots,i_n\},y)/dy I(y<z_{i_2})) - 1,$$

from which

$$(B.6)\qquad \begin{aligned} &dK_{i_1}(\{i_1,i_2,\cdots,i_n\},y)/dy - dK_{k_1}(\{k_1,i_2,\cdots,i_n\},y)/dy \\ &\qquad = (G_{i_1}(y) - G_{k_1}(y))(1 + dK_{i_2}(\{i_2,i_3,\cdots,i_n\},y)/dy I(y<z_{i_2})). \end{aligned}$$

Then since (B.6) $\leqq 0$ for all y (See (3.27)), $K_{i_1}(\{i_1,i_2,\cdots,i_n\},y) - K_{k_1}(\{k_1,i_2,\cdots,i_n\},y)$ is nonincreasing in y. In addition, since $K_{i_1}(\{i_1,i_2,\cdots,i_n\},z_{i_1}) - K_{k_1}(\{k_1,i_2,\cdots,i_n\},z_{i_1}) = -K_{k_1}(\{k_1,i_2,\cdots,i_n\},z_{i_1}) \geqq 0$, $K_{i_1}(\{i_1,i_2,\cdots,i_n\},y) - K_{k_1}(\{k_1,i_2,\cdots,i_n\},y) \geqq 0$ for $y \leqq z_{i_1}$, hence $\max\{0,\ K_{i_1}(\{i_1,i_2,\cdots,i_n\},y)\} \geqq \max\{0,\ K_{k_1}(\{k_1,i_2,\cdots,i_n\},y)\}$ for $y \leqq z_{i_1}$. If $y > z_{i_1}$, since $K_{i_1}(\{i_1,i_2,\cdots,i_n\},y)\} \leqq 0$ and $K_{k_1}(\{k_1,i_2,\cdots,i_n\},y) \leqq 0$ (See Theorem 1(c)), we get $\max\{0,\ K_{i_1}(\{i_1,i_2,\cdots,i_n\},y)\} = \max\{0,\ K_{k_1}(\{k_1,i_2,\cdots,i_n\},y)\} = 0$. Accordingly, for all y, we obtain

$$(B.7)\qquad \begin{aligned} \max\{0,\ K_{i_1}(\{i_1,i_2,\cdots,i_n\},y)\} &\geqq \max\{0,\ K_{k_1}(\{k_1,i_2,\cdots,i_n\},y)\} \\ &\geqq \max\{0,\ K_{k_1}(\{k_1,k_2,\cdots,k_n\},y)\} \text{ from (B.2)} \qquad \text{Q.E.D.} \end{aligned}$$

APPENDIX C

–A case satisfying both (2.4) and (5.1) –

Here we shall show that there exists a case where both (2.4) and (5.1) are satisfied. Let a denote the set $\{c_1,c_2,\cdots,c_N,\beta_1,\beta_2,\cdots,\beta_N,F_1,F_2,\cdots,F_N\}$ and Z the set of all the possible $\mathbf{a}$'s satisfying (2.4). If (2.4) is not satisfied for a given $\mathbf{a}$, we can renumber, without loss in generality, the boxes in a decreasing order of reservation prices. Accordingly, it suffices to consider only $\mathbf{a} \in Z$. Now, since $K_i(y)$ is a decreasing function of y for all i, "$z_1 \geqq z_2 \geqq \cdots \geqq z_N$" is equivalent to "$K_i(z_i) = 0$ and $K_{i+1}(z_i) \leqq 0$, $i = 1,2,\cdots,N-1$" ; the latter event can be written as

$$\text{(C.1)} \qquad c_{i+1} \geqq c_i + \int_{-\infty}^{\infty} \max\{w,z_i\}\, dG_{i+1} - \int_{-\infty}^{\infty} \max\{w,z_i\}\, dG_i, \quad i = 1,2,\cdots,N-1$$

where notice that the r.h.s. of (C.1) does not depend on c_{i+1}. First, suppose any set $\mathbf{b} = \{c_2,\cdots,c_N,\beta_1,\beta_2,\cdots,\beta_N,F_1,F_2,\cdots,F_N\}$. Then, we can take a sufficiently large c_2 to satisfy (C.1) for $i = 1$, the r.h.s. of which depends only on $\{c_1,\beta_1,\beta_2,F_1,F_2\}$. Similarly, we can take a sufficiently large c_3 to satisfy (C.1) for $i = 2$, the r.h.s. of which depends only on $\{c_1,c_2,\beta_1,\beta_2,\beta_3,F_1,F_2,F_3\}$. Like this, we can sequentially take c_2, c_3, $\cdots$, c_N satisfying (C.1) starting with the given $\mathbf{b}$. It is clear that the $\mathbf{a}$ resulting from adding the sequence $\{c_2, c_3, \cdots, c_N\}$ to $\mathbf{b}$ belongs to Z, implying that the $\mathbf{a}$ satisfies (2.4) and (5.1).

APPENDIX D

– Proof of Theorem 4 –

(a). $v_1(L,y) = y + \max\{0, \max_{j\in L} K_j(y)\} = \max_{j\in L}\{y + \max\{0, \max K_j(y)\}\} = \max_{j\in L} v(\{j\},y)$.

(b). For $m = 1$, the assertion is obviously true from (a) and Theorem 3. Suppose it is true for $m-1$. Then we have

$$v_m(L,y) = y+\max\{0, \max_{j\in L}\{-c_j+\int_{-\infty}^{+\infty} v_{m-1}(L_j,\max\{y,w\})\, dG_j-y\}\}$$

$$= y+\max\{0, \max_{j\in L}\{-c_j+\int_{-\infty}^{+\infty} \max_{\{i_1,i_2,\cdots,i_{m-1}\}\subset L_j} v(\{i_1,i_2,\cdots i_{m-1}\},\max\{y,w\})\, dG_j-y\}\} = (*).$$

Noticing that, in general, $\int \max_{i\in K}\{f_i(x)\}\, dx = \max_{i\in K} \int f_i(x)\, dx = \int f_1(x)\, dx$ if $f_1(x) \geqq f_i(x)$ for all x and for all $i \in K$, we have from Theorem 3

$$(*) = y+\max\{0, \max_{j\in L}\{-c_j+\max_{\{i_1,i_2,\cdots,i_{m-1}\}\subset L_j}\int_{-\infty}^{+\infty} v(\{i_1,i_2,\cdots i_{m-1}\},\max\{y,w\})\, dG_j-y\}\}$$

$$= y+\max\{0, \max_{j\in L}\max_{\{i_1,i_2,\cdots,i_{m-1}\}\subset L_j}\{-c_j+\int_{-\infty}^{+\infty} v(\{i_1,i_2,\cdots i_{m-1}\},\max\{y,w\})\, dG_j-y\}\}$$

$$= y+\max\{0, \max_{\{i_1,i_2,\cdots,i_m\}\subset L}\max_{j\in\{i_1,i_2,\cdots,i_m\}}\{-c_j+\int_{-\infty}^{+\infty} v(\{i_1,i_2,\cdots i_m\}-\{j\},\max\{y,w\})\, dG_j-y\}\} = (**)$$

where that the last two expressions are equal can be explained as follows. For simplicity, let $A(\{i_1,i_2,\cdots,i_{m-1}\},j) = -c_j+\int v(\{i_1,i_2,\cdots,i_{m-1}\},\max\{y,w\})\, dG_j-y$. Consider, for example, a case of $L = \{1,2,3,4\}$ and $m = 3$. Then

$$\max_{j\in\{1,2,3,4\}}\max_{\{i_1,i_2\}\subset L_j} A(\{i_1,i_2\},j)$$

$$= \max\{A(\{2,3\},1),\ A(\{2,4\},1),\ A(\{3,4\},1)$$
$$A(\{1,3\},2),\ A(\{1,4\},2),\ A(\{3,4\},2)$$
$$A(\{1,2\},3),\ A(\{1,4\},3),\ A(\{2,4\},3)$$
$$A(\{1,2\},4),\ A(\{1,3\},4),\ A(\{2,3\},4)\}$$

$$= \max\{A(\{2,3\},1),\ A(\{1,3\},2),\ A(\{1,2\},3)$$
$$A(\{2,4\},1),\ A(\{1,4\},2),\ A(\{1,2\},4)$$
$$A(\{3,4\},1),\ A(\{1,4\},3),\ A(\{1,3\},4)$$
$$A(\{3,4\},2),\ A(\{2,4\},3),\ A(\{2,3\},4)\}$$

$$= \max\{A(\{1,2,3\}-\{1\},1),\ A(\{1,2,3\}-\{2\},2),\ A(\{1,2,3\}-\{3\},3)$$
$$A(\{1,2,4\}-\{1\},1),\ A(\{1,2,4\}-\{2\},2),\ A(\{1,2,4\}-\{4\},4)$$
$$A(\{1,3,4\}-\{1\},1),\ A(\{1,3,4\}-\{3\},3),\ A(\{1,3,4\}-\{4\},4)$$
$$A(\{2,3,4\}-\{2\},2),\ A(\{2,3,4\}-\{3\},3),\ A(\{2,3,4\}-\{4\},4)$$

$$= \max_{\{i_1,i_2,i_3\}\subset L}\ \max_{j\in\{i_1,i_2,i_3\}} A(\{i_1,i_2\}-\{j\},j)$$

It is of course that the above relation holds also in general; its explanation will become quite tedious, so we shall omit it. Then

$$(**)\ = \max_{\{i_1,i_2,\cdots,i_m\}\subset L}\{y+\max\{0,\ \max_{j\in\{i_1,i_2,\cdots,i_m\}}\{-c_j+\int_{-\infty}^{+\infty} v(\{i_1,i_2,\cdots i_m\}-\{j\},\max\{y,w\})\,dG_j-y\}\}$$

$$= \max_{\{i_1,i_2,\cdots,i_m\}\subset L} v(\{i_1,i_2,\cdots i_m\},y)$$

$$= v(\{1,2,,\cdots m\},y) \quad \text{from Theorem 3.} \quad \text{Q.E.D.}$$

APPENDIX E

— An example where (5.4) holds —

Consider (3,2)-problem with $L = \{1,2,3\}$ where the reward in each box follows the same uniform distribution function on $[0,1]$ and where $0.5\beta_i - c_i \geqq 0$, $i = 1,2,3$. Then

$$v_2(\{1,2,3\},0)$$
$$= \max\{0, -c_1+\int_{-\infty}^{+\infty} v_1(\{2,3\},w)\,dG_1,\ -c_2+\int_{-\infty}^{\infty} v_1(\{1,3\},w)\,dG_2,\ -c_3+\int_{-\infty}^{\infty} v_1(\{1,2\},w)\,dG_3\}$$

Now $-c_i + \int v_1(\{1,2,3\}-\{i\},w)\,dG_i \geqq -c_i + \int w\,dG_i = 0.5\beta_i - c_i \geqq 0$ for $i = 1,2,3$. From this and Theorem 4(a) the above expression becomes

$$v_2(\{1,2,3\},0)$$
$$= \max\{-c_1+\int_{-\infty}^{+\infty} v_1(\{2,3\},w)\,dG_1,\ -c_2+\int_{-\infty}^{\infty} v_1(\{1,3\},w)\,dG_2,\ -c_3+\int_{-\infty}^{\infty} v_1(\{1,2\},w)\,dG_3\}$$
$$= \max\{-c_1+\int_{-\infty}^{+\infty} \max\{v(\{2\},w),v(\{3\},w)\}\,dG_1,\ -c_2+\int_{-\infty}^{\infty} \max\{v(\{1\},w),v(\{3\},w)\}\,dG_2,$$
$$-c_3+\int_{-\infty}^{+\infty} \max\{v(\{1\},w),v(\{2\},w)\}\,dG_3\} = (*)$$

As having already shown in Section 4, for $i < j$ with $i, j = 1,2,3$, there may exists a pair of $(w'_{i,j}, w''_{i,j})$, $0 < w'_{i,j} < w''_{i,j} < z_j$, such as $v(\{j\},w'_{i,j}) > v(\{i\},w'_{i,j})$ and $v(\{i\},w''_{i,j}) > v(\{j\},w''_{i,j})$. Hence,

$$
\begin{aligned}
(*) > \max\{ & -c_1 + \max\{ \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_1, \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_1\}, \\
& -c_2 + \max\{ \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_2, \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_2\}, \\
& -c_3 + \max\{ \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_3, \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_3\}\}
\end{aligned}
$$

$$
\begin{aligned}
= \max\{ & -c_1 + \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_1, \ -c_1 + \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_1, \\
& -c_2 + \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_2, \ -c_2 + \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_2, \\
& -c_3 + \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_3, \ -c_3 + \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_3\}
\end{aligned}
$$

$$
\begin{aligned}
= \max\{ & -c_1 + \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_1, \ -c_2 + \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_2, \\
& -c_1 + \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_1, \ -c_3 + \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_3, \\
& -c_2 + \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_2, \ -c_3 + \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_3\} = (**)
\end{aligned}
$$

Since the six expressions in the $\max\{-,-,-,-,-,-\}$ are all nonnegative, (**) can be expressed as $\max\{0,-,-,-,-,-,-\}$, and furthermore, this can be written

$$
\begin{aligned}
(**) = \max\{ & \max\{0, \ -c_1 + \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_1, \ -c_2 + \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_2\}, \\
& \max\{0, \ -c_1 + \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_1, \ -c_3 + \int_{-\infty}^{+\infty} v(\{1\},w)\,dG_3\}, \\
& \max\{0, \ -c_2 + \int_{-\infty}^{+\infty} v(\{3\},w)\,dG_2, \ -c_3 + \int_{-\infty}^{+\infty} v(\{2\},w)\,dG_3\}\}
\end{aligned}
$$

$$= \max\{v(\{1,2\},0),\ v(\{1,3\},0),\ v(\{2,3\},0)\}.$$

Thus, we have eventually $v_2(\{1,2,3\},0) > \max\{v(\{1,2\},0),\ v(\{1,3\},0),\ v(\{2,3\},0)\}$. Now, since $v(L,y)$ is a continuous function of y, for any sufficiently small $y > 0$, we also have $v_2(\{1,2,3\},y) > \max\{v(\{1,2\},y),\ v(\{1,3\},y),\ v(\{2,3\},y)\}$.

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