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MULTI-SYMMETRIC STRUCTURES AND NON-EXPECTED UTILITY

by

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ABSTRACT

This paper develops multi-symmetric structures and their numerical representations which generalize the bisymmetric structure in additive conjoint measurement. Then we discuss applications of those structures to axiomatic characterizations of non-expected utility theories.

1. INTRODUCTION

The main purpose in this paper is to introduce multi-symmetric structures and then develop their numerical representations. The multi-symmetric structures, which generalize the bisymmetric structures first introduced by Pfanzagl (1959), postulate three primitives: a nonempty set A, a binary relation $\precsim^*$ on A, and an n-ary operation ω that maps a subset of A $\times \ldots \times$ A (n times), denoted A^n, into A. Denote an element of A^n by $a_1 \ldots a_n$ in place of $(a_1, \ldots, a_n)$. A weak representational form of the multi-symmetric structures yields that there exist real numbers, μ and $\lambda_i \geq 0$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i > 0$, and real valued function f on A such that for all $a, b, a_1, \ldots, a_n \in A$,

$$a \precsim^* b \text{ iff } f(a) \leq f(b),$$

and if $a_1 \precsim^* \ldots \precsim^* a_n$, then

$$f(\omega(a_1 \ldots a_n)) = \sum_{i=1}^{n} \lambda_i f(a_i) + \mu.$$

Recall that the bisymmetric structures provide that when ω is a binary operation defined on $A \times A$, $f(\omega(ab)) = \nu f(a) + \lambda f(b) + \mu$ for all $a, b \in A$ and some constants, $\nu > 0$, $\lambda > 0$, and μ (see Krantz, et al. (1971, Chapter 6)).

Recent developments in non-expected utility theories motivate our interest in the multi-symmetric structures. They are Schmeidler-Gilboa's generalizations of subjective expected utility, and Quiggin's anticipated utility, which generalizes von Neumann-Morgenstern expected utility.

To accommodate Ellsberg-type violation of additive probability measures, Schmeidler (1984, 1989) and Gilboa (1987) adopted Choquet integration originally developed by Choquet (1955) as a representational form, called Choquet expected utility representation. Schmeidler's axiomatization is based on the idea of lottery acts (functions from the states of the world into probability distributions on the consequence space) introdued by Anscombe and Aumann (1963). Gilboa used Savage's basic formulation (see Savage (1954) and Fishburn (1970)), so that the sets of states is continuously divisible. Recently, Wakker (1989a,b) developed topological approaches when the set of states is finite, and Nakamura (1990a) developed an algebraic approach in that case.

Quiggin (1982) proposed anticipated utility to accommodate Allais-type violations of von Neumann-Morgenstern expected utility. His representational form is based on expectations of gambles with respect to their distorted (de)cummulative distributions. Because of this, Quiggin's utility is sometimes called rank-dependent utility. Several axiomatic and descriptive developments of the anticipated utility have been achieved. They include Chew (1984), Chew, Karni and Safra (1987), Chew and Epstein (1989), Hilton (1988), Quiggin (1985), Segal (1989), Yaari (1987) and others. Those theories presume that the underlying consequence space is the real line. Recently, Nakamura (1990b) provided an axiomatic characterization when the consequence space is arbitrary.

Wakker (1989c) discussed the equivalence of Choquet expected utility and the anticipated utility. This paper discusses an application of the multi-symmetric structures to argue that a common algebraic feature provides an axiomatic characterization for both Choquet expected utility and the anticipated utility.

The paper is organized as follows. Section 2 introduces weak and strong forms of the multi-symmetric structures, and then presents two representation theorems. Section 3 considers multiple binary operations on A, and provides a joint representation of those operations. Section 4 applies multi-symmetric structures to characterize the Choquet expected utility and the anticipated utility. Section 5 provides the proofs of the representation theorems of the multi-symmetric structures.

2. MULTI-SYMMETRIC STRUCTURES

Let $n > 1$ be an integer and $N = \{1,\ldots, n\}$. We start with three primitives: a nonempty set A, a binary relation $\precsim^*$ on A, and an n-ary operation ω that maps a subset of A^n into A. Let $A^n_+ = \{a_1\ldots a_n \in A^n\colon a_1 \precsim^* \ldots \precsim^* a_n$ and $a_i \in A$ for $i \in N\}$. Let $\sim^*$ and $\prec^*$ be defined in the usual way: $a \sim^* b$ iff $a \precsim^* b$ and $b \precsim^* a$; $a \prec^* b$ iff $a \precsim^* b$ and not($b \precsim^* a$). When B and C are subsets of A, $B \precsim^* C$ means that $a \precsim^* b$ for all $a \in B$ and all $b \in C$. We shall denote a singleton set {a} as a.

For $k \in N$, define $\omega_k(ab) = \omega(a_1\ldots a_n)$ when $a_i = a$ for $i = 1,\ldots,k$, and $a_i = b$ for $i = k+1,\ldots,n$. Note that $\omega_n(ab) = \omega(a\ldots a)$ when $k = n$. We say that k is <u>left-inessential</u> if for all $a,b,c \in A$, $\omega_k(ac) \sim^* \omega_k(bc)$ whenever $a \precsim^* b \precsim^* c$, and <u>right-inessential</u> if for all $a,b,c \in A$, $\omega_k(ab) \sim^* \omega_k(ac)$ whenever $a \precsim^* b \precsim^* c$. Note that n is right-inessential. When k is not left (right)-inessential, we say that k is <u>left</u> (<u>right</u>)-<u>essential</u>. When k is left- and right-essential, we say that k is *essential*. If there is an essential $k \in N$, then $<^*$ is not empty, so that $a <^* b$ for some $a,b \in A$.

Let K be any set of consecutive integers. For an essential k, we define a *standard sequence* as a set $\{a_i: a_i \in A, i \in K\}$ for which there exist $a,b \in A$ such that not($a \sim^* b$), either $\{a, b\} \precsim^* \{a_i\}$ and $\omega_k(aa_i) \sim^* \omega_k(ba_{i+1})$ for all $i,i+1 \in K$, or $\{a_i\} \precsim^* \{a, b\}$ and $\omega_k(a_i a) \sim^* \omega_k(a_{i+1}b)$ for all $i,i+1 \in K$.

In what follows, we shall define weak and strong forms of multi-symmetric structures, which depend on domains of ω, and present their numerical representations. The weak form, which restricts the domain of ω to A^n_+, is given as follows.

DEFINITION 1. *The quadruple* $\langle A, \precsim^*, \omega, A^n_+\rangle$ *is a* weak multi-symmetric structure *iff* ω *is defined on* A^n_+, *and for all* $a,b,c,d,a_i,b_i \in A$, $i \in N$, *and all* $k \in N$, *the following seven axioms hold*:

A1. $\precsim^*$ *is a weak order*.

A2. *some* $j \in N$ *is essential*.

A3. *if* $\{a, b\} \precsim^* c$ *and* $\omega_k(ac) \precsim^* d \precsim^* \omega_k(bc)$, *then* $d \sim^* \omega_k(ec)$ *for some* $e \in A$ *with* $e \precsim^* c$;

if $c \precsim^* \{a, b\}$ *and* $\omega_k(ca) \precsim^* d \precsim^* \omega_k(cb)$, *then* $d \sim^* \omega_k(ce)$ *for some* $e \in A$ *with* $c \precsim^* e$.

A4. *Every strictly bounded standard sequence is finite*.

A5. *if* k *is left-essential and* $\{a, b\} \precsim^* c$, *then*

$$a \precsim^* b \text{ iff } \omega_k(ac) \precsim^* \omega_k(bc);$$

if k *is right-essential and* $c \precsim^* \{a, b\}$, *then*

$$a \precsim^* b \text{ iff } \omega_k(ca) \precsim^* \omega_k(cb).$$

A6. *if* $a_1 \precsim^* \ldots \precsim^* a_n$, $b_1 \precsim^* \ldots \precsim^* b_n$, *and* $a_i \precsim^* b_i$ *for* $i \in N$, *then* $\omega(a_1 \ldots a_n) \precsim^* \omega(b_1 \ldots b_n)$.

A7. *if* $a_1 \precsim^* \ldots \precsim^* a_n$, $b_1 \precsim^* \ldots \precsim^* b_n$, *and* $a_i \precsim^* b_i$ *for* $i \in N$, *then* $\omega_k(\omega(a_1 \ldots a_n)\omega(b_1 \ldots b_n)) \sim^* \omega(\omega_k(a_1 b_1) \ldots \omega_k(a_n b_n))$.

A weak order means by definition that $\precsim^*$ is complete and transitive. A2 is an essentiality axiom which implies that $<^*$ is not empty. A3 is a restricted solvability axiom. Since A3 applies to all $k \in N$, we note that when $k = n$, if $\omega(a \ldots a) \precsim^* c \precsim^* \omega(b \ldots b)$, then $c \sim^* \omega(d \ldots d)$ for some $d \in A$. A4 is an Archimedean axiom. A5 and A6 are monotonicity axioms. A6 is not required when $n = 2$, since A1, A2, and A5 imply A6 in that case. A7 is a weak multi-symmetry axiom that generalizes the bisymmetry axiom. When $n = 2$, we say that $\langle A, \precsim^*, \omega, A_+^2 \rangle$ is a *weak bisymmetric structure*.

The numerical representation of the weak multi-symmetric structure is given as follows.

THEOREM 1. *If* $\langle A, \precsim^*, \omega, A_+^n \rangle$ *is a weak multi-symmetric structure, then there exist real numbers* μ *and* $\lambda_i \geq 0$ *for* $i \in N$ *with* $\sum_{i=1}^n \lambda_i > 0$, *and real valued function* f *on* A *such that for all* $a, b, a_1, \ldots, a_n \in A$,

(1) $a \precsim^* b$ *iff* $f(a) \leq f(b)$,

(2) *if* $a_1 \precsim^* \ldots \precsim^* a_n$, *then* $f(\omega(a_1 \ldots a_n)) = \sum_{i=1}^n \lambda_i f(a_i) + \mu$,

(3) if μ', $\lambda_i' \geq 0$ *for* $i \in N$, *and* f' *on* A *satisfy the representations* (1) *and* (2), *then there exist constants* $\alpha > 0$ *and* β *such that*

$$f' = \alpha f + \beta,$$

$$\lambda_i' = \lambda_i \text{ for } i \in N,$$

$$\mu' = \alpha\mu + \beta(1 - \sum_{i=1}^{n}\lambda_i).$$

We note that if ω is <u>idempotent</u>, i.e., for all $a \in A$, $\omega(a \ldots a) \sim^* a$, then the representation (2) in Theorem 1 is reduced to

$$f(\omega(a_1 \ldots a_n)) = \sum_{i=1}^{n} \lambda_i f(a_i),$$

where $\sum_{i=1}^{n}\lambda_i = 1$. When $n = 2$, the proof of the theorem will be given in the next section. For the general case, the proof is deferred to Section 5. The strong form of the multi-symmetric structures, which applies ω to A^n, is given by

DEFINITION 2. *The quadruple* $\langle A, \precsim^*, \omega, A^n\rangle$ *is a* multi-symmetric structure *iff* ω *is defined on* A^n, *and* $\precsim^*$ *satisfies* A1-A4 *and the following three axioms*: *for all* $a,b,c,a_i,b_i \in A$, $i \in N$, *and all* $k \in N$,

A5*. *if* k *is left-essential*, *then* $a \precsim^* b$ *iff* $\omega_k(ac) \precsim^* \omega_k(bc)$;

if k *is right-essential*, *then* $a \precsim^* b$ *iff* $\omega_k(ca) \precsim^* \omega_k(cb)$.

A6*. *if* $a_i \precsim^* b_i$ *for all* $i \in N$, *then* $\omega(a_1 \ldots a_n) \precsim^* \omega(b_1 \ldots b_n)$.

A7*. $\omega_k(\omega(a_1 \ldots a_n)\omega(b_1 \ldots b_n)) \sim^* \omega(\omega_k(a_1b_1) \ldots \omega_k(a_nb_n))$.

A5*-A7* are strong versions of A5-A7, respectively. When $n = 2$, A7* is the bisymmetry axiom. The following theorem gives the numerical representation of the multi-symmetric structure. The proof of the theorem will be given in Section 5.

THEOREM 2. _If_ $\langle A, \precsim^*, \omega, A^n \rangle$ _is a multi-symmetric structure, then there exist real numbers_ μ _and_ $\lambda_i \geq 0$ _for_ $i \in N$ _with_ $\sum_{i=1}^{n} \lambda_i > 0$, _and real valued function_ f _on_ A _such that_ (1) _and_ (3) _in Theorem 1 hold, and for all_ $a_1, \ldots, a_n \in A$, $f(\omega(a_1 \ldots a_n)) = \sum_{i=1}^{n} \lambda_i f(a_i) + \mu$.

3. A JOINT REPRESENTATION

This section concerns weak bisymmetric structures and their numerical representations. Consider two binary operations, ω and σ, on A. We say that ω and σ are _weakly isometric_ if for all $a,b,c,d \in A$ with $a \precsim^* \{b, c\} \precsim^* d$,

$$\omega(\sigma(ab)\sigma(cd)) \sim^* \sigma(\omega(ac)\omega(bd)).$$

The weak isometry condition is a generalized version of the isometry-relation in Pfanzagl (1959). A joint representation of ω and σ, which generalizes Theorem 2 in Pfanzagl (1959), is given as follows.

PROPOSITION 1. _Suppose that_ $\langle A, \precsim^*, \omega, A_+^2 \rangle$ _and_ $\langle A, \precsim^*, \sigma, A_+^2 \rangle$ _are weak bisymmetric structures. If_ ω _and_ σ _are weakly isometric, then there exist real numbers_ $\alpha_i > 0$, $\beta_i > 0$, _and_ γ_i _for_ $i = 1, 2$, _and real valued function_ f _on_ A _such that for all_ $a,b \in A$,

(1) $a \precsim^* b$ _iff_ $f(a) \leq f(b)$,

(2) _if_ $a \precsim^* b$, _then_

$$f(\omega(ab)) = \alpha_1 f(a) + \beta_1 f(b) + \gamma_1,$$
$$f(\sigma(ab)) = \alpha_2 f(a) + \beta_2 f(b) + \gamma_2.$$

(3) $\gamma_1(1 - \alpha_2 - \beta_2) = \gamma_2(1 - \alpha_1 - \beta_1)$.

(4) if α_i', β_i', γ_i' for $i = 1, 2$, and f' on A satisfy the representations (1) and (2), then there exist constants $\lambda > 0$ and μ such that $f' = \lambda f + \mu$, $\alpha_i' = \alpha_i$, $\beta_i' = \beta_i$, and $\gamma_i' = \lambda\gamma_i + \mu(1 - \alpha_i - \beta_i)$ for $i = 1, 2$.

First we shall prove a lemma, that provides the representation of Theorem 1 when $\langle A, \precsim^*, \omega, A_+^2\rangle$ is a weak bisymmetric structure, and then give the proof of Proposition 1. For $a,b \in A$ with $a \precsim^* b$, define

$$A^* = \{x \in A\colon c <^* x <^* d \text{ for some } c,d \in A\},$$
$$A_a = \{x \in A\colon a \precsim^* x\},$$
$$A^a = \{x \in A\colon x \precsim^* a\},$$
$$A_a^b = \{x \in A\colon a \precsim^* x \precsim^* b\},$$
$$A_{max} = \{x \in A\colon x <^* c \text{ for no } c \in A\},$$
$$A_{min} = \{x \in A\colon c <^* x \text{ for no } c \in A\}.$$

If $\langle A, \precsim^*, \omega, A_+^2\rangle$ is a weak bisymmetric structure, then A^* is nonempty, since $<^*$ is nonempty and dense, i.e., $a <^* c <^* b$ for some $c \in A$ if $a <^* b$. A_{min} and A_{max} may be empty.

LEMMA 1. Suppose that $\langle A, \precsim^*, \omega, A_+^2\rangle$ is a weak bisymmetric structure. Then there exist real numbers $\alpha > 0$, $\beta > 0$, γ, and real valued function f on A such that for all $a,b \in A$,

(1) $a \precsim^* b$ iff $f(a) \le f(b)$.

(2) if $a \precsim^* b$, then $f(\omega(ab)) = \alpha f(a) + \beta f(b) + \gamma$.

(3) <u>if</u> α', β', γ', <u>and</u> f' <u>satisfy the representations</u> (1) <u>and</u> (2), <u>then there exist constants</u> $\lambda > 0$ <u>and</u> μ <u>such that</u> $f' = \lambda f + \mu$, $\alpha' = \alpha$, $\beta' = \beta$, <u>and</u> $\gamma' = \lambda\gamma + \mu(1 - \alpha - \beta)$.

Proof. Suppose that $\langle A, \precsim^*, \omega, A_+^2\rangle$ is a weak bisymmetric structure. Although Lemma 2 in Nakamura (1990a) is established in the different framework, it is easy to see that that lemma is still valid in the present context, so that there are two real valued functions ϕ and ψ on A such that for all $a,b,c,d \in A$ with $a \precsim^* b$ and $c \precsim^* d$,

$$\omega(ab) \precsim^* \omega(cd) \text{ iff } \phi(a) + \psi(b) \leq \phi(c) + \psi(d).$$

Moreover, if ϕ' and ψ' satisfy the representation also, then there exist constants $\nu > 0$, μ_1, and μ_2 such that

$$\phi' = \nu\phi + \mu_1 \text{ on } A^* \cup A_{min},$$
$$\psi' = \nu\psi + \mu_2 \text{ on } A^* \cup A_{max}.$$

Then a similar proof of Lemma 3 in Nakamura (1990a) and Theorem 10 in Krantz, et al. (1971, Chapter 6) applies to obtain that there are constants $\alpha > 0$, $\beta > 0$, and γ such that for all $a,b \in A^*$ with $a \precsim^* b$,

$$\phi(\omega(ab)) = \alpha\phi(a) + \beta\phi(b) + \gamma.$$

Since ϕ is order preserving, i.e., for all $a,b \in A$, $a \precsim^* b$ iff $\phi(a) \leq \phi(b)$, the uniqueness of ϕ and ψ implies that $\alpha\psi = \beta\phi$ on A^*.

We shall define

$$f(a) = \phi(a) \quad \text{for } a \in A \cup A_{min}$$
$$= \frac{\alpha}{\beta}\,\psi(a) \text{ for } a \in A_{max}.$$

It easily follows that for all $a,b \in A$, $a \lesssim^* b$ iff $f(a) \leq f(b)$. We are to show that for all $a,b \in A$, if $a \lesssim^* b$, then

$$f(\omega(ab)) = \alpha f(a) + \beta f(b) + \gamma.$$

The uniqueness part (3) of the lemma easily follows (see the proof of Theorem 10 in Krantz, et al. (1971, Chapte 6)). When $a,b \in A^*$ and $a \lesssim^* b$, the desired result follows from the preceding paragraph. Thus we have two cases to examine: either $a \in A_{min}$, $b \in A^*$ or $a \in A^*$, $b \in A_{max}$; $a,b \in A_{min} \cup A_{max}$.

CASE 1(either $a \in A_{min}$, $b \in A^*$ or $a \in A^*$, $b \in A_{max}$). Suppose that $a \in A_{min}$ and $b \in A^*$. Then $a <^* b$. When $a \in A^*$ and $b \in A_{max}$, the proof is similar. First assume that $\omega(bb) \sim^* \omega(ac)$ for some $c \in A^*$. Then by the definition of f,

$$\omega(bb) \sim^* \omega(ac) \text{ iff } \phi(b) + \psi(b) = \phi(a) + \psi(c)$$
$$\text{iff } \alpha f(b) + \beta f(b) = \alpha f(a) + \beta f(c),$$

so $\beta f(c) = (\alpha+\beta)f(b) - \alpha f(a)$. By A5, $\omega(aa) <^* \omega(ab) <^* \omega(ac) <^* \omega(bc) <^* \omega(cc)$ and $\omega(cc) \in A^*$. Then by A7, we obtain

$$\omega(\omega(ab)\omega(cc)) \sim^* \omega(\omega(ac)\omega(bc))$$

$$\text{iff } \alpha f(\omega(ab)) + \beta f(\omega(cc)) = \alpha f(\omega(ac)) + \beta f(\omega(bc)).$$

Noting that $f(\omega(ac)) = f(\omega(bb))$ and $\beta f(c) = (\alpha+\beta)f(b) - \alpha f(a)$, rearrangement of the above equation gives

$$\begin{aligned}\alpha f(\omega(ab)) &= \alpha f(\omega(bb)) + \beta\{f(\omega(bc)) - f(\omega(cc))\} \\ &= \alpha\{(\alpha+\beta)f(b) + \gamma\} + \alpha\beta\{f(b) - f(c)\} \\ &= \alpha\{(\alpha+2\beta)f(b) - \beta f(c) + \gamma\} \\ &= \alpha\{\alpha f(a) + \beta f(b) + \gamma\}.\end{aligned}$$

Hence the conclusion obtains.

Next assume that not($\omega(bb) \sim^* \omega(ac)$) for all $c \in A^*$. Since $\omega(aa) <^* \omega(ab) <^* \omega(bb)$, A3 implies that $\omega(ab) \sim^* \omega(dd)$ for some $d \in A$. By A5, $a <^* d <^* b$, so the preceding paragraph gives $f(\omega(ad)) = \alpha f(a) + \beta f(d) + \gamma$. Since $\omega(ad)$, $\omega(bb)$, $\omega(ab)$, and $\omega(db)$ are in A^*, A7 implies

$$\omega(\omega(ad)\omega(bb)) \sim^* \omega(\omega(ab)\omega(db))$$
$$\text{iff } \alpha f(\omega(ad)) + \beta f(\omega(bb)) = \alpha f(\omega(ab)) + \beta f(\omega(db)),$$

which is rearranged to give

$$\begin{aligned}\alpha f(\omega(ab)) &= \alpha f(\omega(ad)) + \beta\{f(\omega(bb)) - f(\omega(db))\} \\ &= \alpha\{\alpha f(a) + \beta f(d) + \gamma\} + \alpha\beta\{f(b) - f(d)\} \\ &= \alpha\{\alpha f(a) + \beta f(b) + \gamma\}.\end{aligned}$$

Hence the desired result obtains.

CASE 2($a,b \in A_{min} \cup A_{max}$). First assume $a,b \in A_{min}$. When $a,b \in A_{max}$, the proof is similar. Let $\{a, b\} <^* c$ for $c \in A^*$. Then Case 1 and A7 give

$$\omega(\omega(ab)\omega(cc)) \sim^* \omega(\omega(ac)\omega(bc))$$
$$\text{iff } \alpha f(\omega(ab)) + \beta f(\omega(cc)) = \alpha f(\omega(ac)) + \beta f(\omega(bc)),$$

which is rearranged to give the desired result by applying Case 1.

Next assume that $a \in A_{min}$ and $b \in A_{max}$. Let $a <^* c <^* b$ for $c \in A^*$. Then Case 1 and A7 give

$$\omega(\omega(ac)\omega(ab)) \sim^* \omega(\omega(aa)\omega(cb))$$
$$\text{iff } \alpha f(\omega(ac)) + \beta f(\omega(ab)) = \alpha f(\omega(aa)) + \beta f(\omega(cb)),$$

which is rearranged to give the desired result by applying Case 1 and the preceding paragraph. [Q.E.D.]

Proof of Proposition 1. Suppose that $\langle A, \precsim^*, \omega, A_+^2\rangle$ and $\langle A, \precsim^*, \sigma, A_+^2\rangle$ are weak bisymmetric structures and that ω and σ are weakly isometric. We note that $\langle A^*, \precsim^*, \omega, A_+^{*2}\rangle$ and $\langle A, \precsim^*, \sigma, A_+^{*2}\rangle$ are also weak bisymmetric structures, since by A5, $\omega(ab)$ and $\sigma(ab)$ are in A^* if $a \precsim^* b$ and $a,b \in A^*$. Lemma 1 implies that there are constants $\alpha_k > 0$, $\beta_k > 0$, and γ_k, and real valued functions f_k on A for k = 1, 2, such that for all $a,b \in A$ and k = 1, 2,

$$a \precsim^* b \text{ iff } f_k(a) \le f_k(b),$$

and if $a \precsim^* b$, then

$$f_1(\omega(ab)) = \alpha_1 f_1(a) + \beta_1 f_1(b) + \gamma_1;$$
$$f_2(\sigma(ab)) = \alpha_2 f_2(a) + \beta_2 f_1(b) + \gamma_2.$$

In what follows, we shall prove that for all $a,b \in A$, if $a \precsim^* b$, then for a constant γ,

$$f_1(\sigma(ab)) = \alpha_2 f_1(a) + \beta_2 f_1(b) + \gamma.$$

Thus by Lemma 1(3), there exist constants $\alpha > 0$ and β such that $f_1 = \alpha f_2 + \beta$ and $\gamma = \alpha\gamma_2 + \beta(1 - \alpha_2 - \beta_2)$. By the weak isometry condition, $\sigma(\omega(aa)\omega(aa)) \sim^* \omega(\sigma(aa)\sigma(aa))$. Thus we obtain

$$\begin{aligned} & f_1(\sigma(\omega(aa)\omega(aa))) = f_1(\omega(\sigma(aa)\sigma(aa))) \\ & \Rightarrow (\alpha_2+\beta_2)f_1(\omega(aa)) + \gamma = (\alpha_1+\beta_1)f_1(\sigma(aa)) + \gamma_1 \\ & \Rightarrow (\alpha_2+\beta_2)\{(\alpha_1+\beta_1)f_1(a) + \gamma_1\} + \gamma = (\alpha_1+\beta_1)\{(\alpha_2+\beta_2)f_1(a) + \gamma\} + \gamma_1 \\ & \Rightarrow (1 - \alpha_1 - \beta_1)\gamma = (1 - \alpha_2 - \beta_2)\gamma_1. \end{aligned}$$

Hence, letting $f = f_2$, $\alpha = 1$, and $\beta = 0$, we obtain (1), (2), and (3). (4) easily follows from Lemma 1(3).

Given $a,b \in A^*$ with $a <^* b$, define a binary relation $\precsim^{ab}$ on $A^a \times A^b_a$ as follows: for all ordered pairs $xy, zw \in A^a \times A^b_a$,

$$xy \precsim^{ab} zw \text{ iff } \omega(\sigma(xa)\sigma(yb)) \precsim^* \omega(\sigma(za)\sigma(wb)).$$

Since by A5, $\{\omega(xy), \omega(zw)\} \precsim^* \omega(ab)$, it follows from A1, A5 and the weak isometry condition that

$$\omega(\sigma(xa)\sigma(yb)) \precsim^* \omega(\sigma(za)\sigma(wb))$$
$$\text{iff } \sigma(\omega(xy)\omega(ab)) \precsim^* \sigma(\omega(zw)\omega(ab))$$
$$\text{iff } \omega(xy) \precsim^* \omega(zw).$$

Hence Lemma 1 implies that for all $xy, zw \in A^a \times A_a^b$,

$$xy \precsim^{ab} zw \text{ iff } \alpha_1 f_1(\sigma(xa)) + \beta_1 f_1(\sigma(yb)) \le \alpha_1 f_1(\sigma(za)) + \beta_1 f_1(\sigma(wb))$$
$$\text{iff } \alpha_1 f_1(x) + \beta_1 f_1(y) \le \alpha_1 f_1(z) + \beta_1 f_1(w)$$

Since a and b are arbitrary in A^*, it easily follows from the uniqueness of additive representations that there exist constant λ, and real valued functions g on A^* such that for all $x \in A^* \cup A_{min}$ and all $y \in A^*$ with $x \precsim^* y$,

$$f_1(\sigma(xy)) = \lambda f_1(x) + g(y).$$

Similarly, define $\precsim_{ab}$ on $A_a^b \times A_b$ as

$$xy \precsim_{ab} zw \text{ iff } \omega(\sigma(ax)\sigma(by)) \precsim^* \omega(\sigma(az)\sigma(bw)),$$

so by Lemma 1, for all $xy, zw \in A_a^b \times A_b$,

$$xy \precsim_{ab} zw \text{ iff } \alpha_1 f_1(\sigma(ax)) + \beta_1 f_1(\sigma(by)) \le \alpha_1 f_1(\sigma(az)) + \beta_1 f_1(\sigma(bw))$$
$$\text{iff } \alpha_1 f_1(x) + \beta_1 f_1(y) \le \alpha_1 f_1(z) + \beta_1 f_1(w).$$

Hence, for some constants λ' and real valued function g' on A^* such that for all $x \in A^*$ and all $y \in A^* \cup A_{max}$ with $x \precsim^* y$,

$$f_1(\sigma(xy)) = \lambda' f_1(y) + g'(x).$$

It follows from the preceding two paragraphs that for all $a, b \in A^*$ with $a \lesssim^* b$, there exist a constant γ such that

$$g'(a) - \lambda f_1(a) = g(b) - \lambda' f_1(b) = \gamma,$$

so $f_1(\sigma(ab)) = \lambda f_1(a) + \lambda' f_1(b) + \gamma$. By Lemma 1(3), $\lambda = \alpha_2$ and $\lambda' = \beta_2$, so that for all $a, b \in A^*$,

$$f_1(\sigma(ab)) = \alpha_2 f_1(a) + \beta_2 f_1(b) + \gamma.$$

It remains to show that if $a \lesssim^* b$, and a or b is in $A_{min} \cup A_{max}$, then $f_1(\sigma(ab)) = \alpha_2 f_1(a) + \beta_2 f_1(b) + \gamma$. First assume that $a \in A_{min}$ and $b \in A^*$. When $a \in A^*$ and $b \in A_{max}$, the proof is similar. Then by the weak isometry condition, $\sigma(\omega(ab)\omega(bb)) \sim^* \omega(\sigma(ab)\sigma(bb))$. Since $\omega(ab)$, $\sigma(ab)$, $\omega(bb)$, and $\sigma(bb)$ are in A^*, we obtain

$$\alpha_2 f_1(\omega(ab)) + \beta_2 f_1(\omega(bb)) + \gamma = \alpha_1 f_1(\sigma(ab)) + \beta_1 f_1(\sigma(bb)) + \gamma_1,$$

which is rearranged to give

$$\begin{aligned}
\alpha_1 f_1(\sigma(ab)) &= \alpha_2 f_1(\omega(ab)) + \beta_2 f_1(\omega(bb)) - \beta_1 f_1(\sigma(bb)) + \gamma - \gamma_1 \\
&= \alpha_2\{\alpha_1 f_1(a) + \beta_1 f_1(b) + \gamma_1\} + \beta_2\{(\alpha_1+\beta_1) f_1(b) + \gamma_1\} \\
&\quad - \beta_1\{(\alpha_2+\beta_2) f_1(b) + \gamma\} + \gamma - \gamma_1 \\
&= \alpha_1\{\alpha_2 f_1(a) + \beta_2 f_1(b)\} + \gamma(1-\beta_1) - \gamma_1(1-\alpha_2-\beta_2).
\end{aligned}$$

Therefore, since $(1-\alpha_1-\beta_1)\gamma = (1-\alpha_2-\alpha_2)\gamma_1$, the deisired result obtains.

Next assume that $a,b \in A_{min}$. When $a,b \in A_{max}$, the proof is similar. For $c \in A^*$, the weak isometry condition implies that $\sigma(\omega(ac)\omega(bc)) \sim^* \omega(\sigma(ab)\sigma(cc))$. Since $\omega(ac)$ and $\omega(bc)$ are in A^*, we obtain

$$\alpha_2 f_1(\omega(ac)) + \beta_2 f_1(\omega(bc)) + \gamma = \alpha_1 f_1(\sigma(ab)) + \beta_1 f_1(\sigma(cc)) + \gamma_1,$$

which is similarly rearranged to give the desired result.

Last assume that $a \in A_{min}$ and $b \in A_{max}$. Then $a <^* b$, so by A5, $\omega(ab)$ and $\sigma(ab)$ are in A^*. By the weak isometry condition, $\sigma(\omega(ab)\omega(bb)) \sim^* \omega(\sigma(ab)\sigma(bb))$, so we obtain

$$\alpha_2 f_1(\omega(ab)) + \beta_2 f_1(\omega(bb)) + \gamma = \alpha_1 f_1(\sigma(ab)) + \beta_1 f_1(\sigma(bb)) + \gamma_1.$$

Thus a similar rearrangement in the preceding two paragraphs gives the desired result. [Q.E.D.]

4. AN APPLICATION TO NON-EXPECTED UTILITY

This section concerns an application of the weak multi-symmetric structures to the Choquet expected utility in decisions under uncertainty and the anticipated utility in decisions under risk. We argue that a common algebraic feature provides axiomatic characterizations for both utility representations. To do this, we shall first provide a general algebraic framework to derive a joint representation of multiple operaions on A.

Let a binary relation $<^o$ on a set Λ be a strict partial order, i.e., it is irreflexive and transitive. We say that α^* is maximal in Λ when $\alpha <^o \alpha^*$ for all $\alpha \in \Lambda$ with $\alpha \neq \alpha^*$, so that if a maximal element exists in Λ, then it is unique. A finite increasing sequence in Λ is denoted by $\pi = [\alpha_1, \ldots, \alpha_n]$, where n is a positive integer, and $\alpha_1 <^o \ldots <^o \alpha_n$ for $\alpha_i \in \Lambda$, $i = 1, \ldots, n$. The number of elements in π is denoted by $|\pi|$. Let Π be a set of finite increasing sequences in Λ such that $\Pi \supseteq \{\pi: \pi = [\alpha], \alpha \in \Lambda\}$. Let $\Omega(\Pi)$ be the set of all $(|\pi|+1)$-ary operations ω_π on A contingent upon each $\pi \in \Pi$. We shall denote $\omega_\pi = \omega_\alpha$ when $\pi = [\alpha]$, and $\omega_\pi^k(ab) = \omega_\pi(a_1 \ldots a_n)$ when $a_i = a$ for $i = 1, \ldots, k$ and $a_i = b$ for $i = k+1, \ldots, n$. We say that $\alpha \in \Lambda$ is left-inessential if for all $a,b,c \in A$, $\omega_\alpha(ac) \sim^* \omega_\alpha(bc)$ whenever $a \precsim^* b \precsim^* c$, and right-inessential if for all $a,b,c \in A$, $\omega_\alpha(ab) \sim^* \omega_\alpha(ac)$ whenever $a \precsim^* b \precsim^* c$, and that π is essential if some α in π is neither left-inessential nor right-inessential.

The following proposition gives a joint representation of multiple operations in $\Omega(\Pi)$.

PROPOSITION 2. <u>Suppose that</u> Λ <u>has a maximal element</u> α^* <u>and at least one essential element</u>. <u>If the following five conditions hold</u>: <u>for all</u> $\omega_\pi \in \Omega(\Pi)$, <u>all</u> $\alpha, \beta \in \Lambda$, <u>and all</u> $a,b \in A$ <u>with</u> $a \precsim^* b$,

B1. <u>if</u> π <u>is essential and</u> $n = |\pi| + 1$, <u>then</u> $\langle A, \precsim^*, \omega_\pi, A_+^n \rangle$ <u>is a weak multi-symmetric structure</u>,

B2. ω_α <u>and</u> ω_β <u>are weakly isometric</u>,

B3. <u>if</u> $\alpha <^o \beta$, <u>then</u> $\omega_\beta(ab) \precsim^* \omega_\alpha(ab)$,

B4. <u>if</u> $\pi = [\alpha_1, \ldots, \alpha_n]$, <u>then</u> $\omega_\pi^k(ab) \sim^* \omega_{\alpha_k}(ab)$ <u>for</u> $k = 1, \ldots, n$, B5. α^* _is right-inessential_,

then there exist real valued functions, u _on_ A _and_ $\tau \geq 0$ _on_ Λ^*, _and constant_ μ, _such that for all_ $a,b \in A$, _all_ $\alpha,\beta \in \Lambda^*$ _and all_ $\omega_\pi \in \Omega(\Pi)$,

(1) $a \lesssim^* b$ _iff_ $u(a) \leq u(b)$.

(2) _if_ $\alpha <^0 \beta$, _then_ $\tau(\alpha) \leq \tau(\beta)$.

(3) _if_ $\pi = [\alpha_1, \ldots, \alpha_{n-1}]$, $\alpha_n = \alpha^*$, _and_ $a_1 \lesssim^* \ldots \lesssim^* a_n$, _then_

$$u(\omega_\pi(a_1 \ldots a_n)) = \tau(\alpha_1)u(a_1) + \sum_{i=2}^{n}(\tau(\alpha_i) - \tau(\alpha_{i-1}))u(a_i) + \mu.$$

(4) _if_ u', τ', _and_ μ' _satisfy the representations_ (1), (2), _and_ (3), _then there exist constants_ $\lambda > 0$ _and_ ν _such that_ $u' = \lambda u + \nu$, $\tau' = \tau$, _and_ $\mu' = \lambda\mu + \nu(1-\tau(\alpha^*))$.

The proof of the Proposition 2 will be given at the end of the section. In what follows, we discuss that B1-B5 in Proposition 2 provide a common algebraic feature for axiomatic characterizations of both Choquet expected utility and the anticipated utility representations. Let P be a set of decision alternatives. In decisions under uncertainty, we interpret P as the set of all acts that are functions from the set S of states into the consequence space X. Subsets of S are called events. Let $2^S = \{A: A \subseteq S\}$. In decisions under risk, P is interpreted as the set of all probability distributions over X. Let I be the closed unit interval. Then we take

$$\Lambda = 2^S \text{ and } <^0 = \subset \text{ (in decisions under uncertainty)}$$

$$\Lambda = I \text{ and } <^0 = < \text{ (in decisions under risk).}$$

Thus π is interpreted as an finite increasing sequences of events or probability numbers in each case.

For $\pi = [\alpha_1, \ldots, \alpha_{n-1}]$, we shall denote a decision alternative in P by $\rho_\pi(x_1 \ldots x_n)$. In act-interpretation, $\rho_\pi(x_1 \ldots x_n)$ is an act such that $x_i \in X$ obtains if $s \in \alpha_i \setminus \alpha_{i-1}$ occurs, where α_0 = empty set and $\alpha_n = S$. For $x \in X$ and all $\pi \in \Pi$, a constant act is an act $\rho_\pi(x \ldots x)$ that always yields x. In probability-interpretation, $\rho_\pi(x_1 \ldots x_n)$ is a probability distribution such that $x_i \in X$ obtains with probability $\alpha_i - \alpha_{i-1}$, where $\alpha_0 = 0$ and $\alpha_n = 1$. For $x \in X$ and all $\pi \in \Pi$, a one-point measure is a probability distribution $\rho_\pi(x \ldots x)$ that yields x with probability 1.

Let $\precsim$ be a binary preference relation on P. The indifference relation $\sim$ is defined as follows: for $\rho, \rho' \in P$, $\rho \sim \rho'$ iff $\rho \precsim \rho'$ and $\rho' \precsim \rho$. Each $x \in X$ is identified with a constant act or a one-point measure in each interpretation. For the time being, for each $\rho_\pi(x_1 \ldots x_n)$ in P, we assume that there exists a certainty equivalent $x \in X$ such that $x \sim \rho_\pi(x_1 \ldots x_n)$. Denote a certainty equivalent of $\rho_\pi(x_1 \ldots x_n)$ by $\omega_\pi(x_1 \ldots x_n)$. Since $x = \rho_\pi(x \ldots x)$ in both interpretations, $\omega_\pi(x \ldots x) \sim x$, so that ω_π is idempotent.

The following eight axioms, that are understood to apply to all $\alpha \in \Lambda$, all $\pi \in \Pi$, and all $x, y, z, w, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, are restatements of B1-B5 in terms of decision altenatives in P.

C1. $\precsim$ on P is a weak order.

C2. some $\gamma \in \Lambda$ is essential.

C3. if $\{x, y\} \precsim z$ and $\rho_\alpha(xz) \precsim w \precsim \rho_\alpha(yz)$, then $w \sim \rho_\alpha(z'z)$ for some $z' \in X$ with $z' \precsim z$;

if $z \precsim \{x, y\}$ and $\rho_\alpha(zx) \precsim w \precsim \rho_\alpha(zy)$, then $w \sim \rho_\alpha(zz')$ for some $z' \in X$ with $z \precsim z'$.

C4. every strictly bounded standard sequence is finite.

C5. if α is left-essential and $\{x, y\} \precsim z$, then

$$x \precsim y \text{ iff } \rho_\alpha(xz) \precsim \rho_\alpha(yz);$$

if α is right-essential and $z \precsim \{x, y\}$, then

$$x \precsim y \text{ iff } \rho_\alpha(zx) \precsim \rho_\alpha(zy).$$

C6. if $x_1 \precsim \ldots \precsim x_n$, $y_1 \precsim \ldots \precsim y_n$, and $x_i \precsim y_i$ for $i = 1, \ldots, n$, then $\rho_\pi(x_1 \ldots x_n) \precsim \rho_\pi(y_1 \ldots y_n)$.

C7. if $x_1 \precsim \ldots \precsim x_n$, $y_1 \precsim \ldots \precsim y_n$, and $x_i \precsim y_i$ for $i = 1, \ldots, n$, then $\rho_\alpha(\omega_\pi(x_1 \ldots x_n)\omega_\pi(y_1 \ldots y_n)) \sim \rho_\pi(\omega_\alpha(x_1 y_1) \ldots \omega_\alpha(x_n y_n))$.

C8. if $\alpha <^o \beta$ and $x \precsim y$, then $\rho_\beta(xy) \precsim \rho_\alpha(xy)$.

C1-C7 correspond to A1-A7 in Definition 1, respectively, where C7 is slightly modified to imply B2. C8 is a restatement of B3. By the definitions of decision alternatives in both interpretations, B4 and B5 hold, so we do not include them as axioms. We note by C3, C6 and idempotency of ω_π that $\omega_\pi(x_1 \ldots x_n)$ exists for each $\rho_\pi(x_1 \ldots x_n)$. Therefore, C1-C8 provide the representations (1)-(4) in Proposition 2 with $\mu = 0$, which are Choquet expected utility representation in act-interpretation, and anticipated utility representation in probability-interpretation.

Proof of Proposition 2. Suppose that Λ has at least one essential element, α^* is a maximal element in Λ, and B1-B5 hold. Then for all essential $\alpha \in \Lambda$, B1 implies that $\langle A, \precsim^*, \omega_\alpha, A_+^2 \rangle$ is a weak bisymmetric structure. It follows from B2 and Proposition 1 that for each essential α, there exist real numbers $\lambda_1(\alpha) > 0$, $\lambda_2(\alpha) > 0$, and $\nu(\alpha)$, and real valued function u on A such that for all a,b∈A,

$$a \precsim^* b \text{ iff } u(a) \leq u(b),$$

and if $a \precsim^* b$, then

$$u(\omega_\alpha(ab)) = \lambda_1(\alpha)u(a) + \lambda_2(\alpha)u(b) + \nu(\alpha).$$

Therefore, we obtain (1).

Since $\omega_\pi^k(aa) = \omega_\pi^j(aa)$, Λ has a maximal element, and $\precsim^*$ is a weak order, B4 implies that $\omega_\alpha(aa) \sim^* \omega_\beta(aa)$ for all $\alpha,\beta\in\Lambda$. Thus for all essential $\alpha,\beta\in\Lambda$, $(\lambda_1(\alpha) + \lambda_2(\alpha))u(a) + \nu(\alpha) = (\lambda_1(\beta) + \lambda_2(\beta))u(a) + \nu(\beta)$, which is rearranged to give $(\lambda_1(\alpha) + \lambda_2(\alpha) - \lambda_1(\beta) - \lambda_2(\beta))u(a) = \nu(\beta) - \nu(\alpha)$. Since a is arbitrary in A and u is not constant, we obtain that $\lambda_1(\alpha) + \lambda_2(\alpha) = \lambda_1(\beta) + \lambda_2(\beta)$ and $\nu(\alpha) = \nu(\beta)$ for all essential $\alpha,\beta\in\Lambda$.

Let $\lambda = \lambda_1(\alpha) + \lambda_2(\alpha)$ and $\mu = \nu(\alpha)$ for some essential $\alpha\in\Lambda$. Since Λ has at least one essential element, $<^*$ is not empty. If $a <^* b$, and α is left and right-inessential, then by definition, $\omega_\alpha(ab) \sim^* \omega_\alpha(bb)$ and $\omega_\alpha(aa) \sim^* \omega_\alpha(ab)$, so $\omega_\alpha(aa) \sim^* \omega_\alpha(bb)$. By A5, $\omega_\alpha(aa) <^* \omega_\alpha(bb)$, a contradiction. Therefore, α cannot be left and right-inessential. Thus we define a real valued function τ on Λ as follows.

$$\tau(\alpha) = \begin{cases} \lambda_1(\alpha) & \text{if } \alpha \text{ is essential,} \\ 0 & \text{if } \alpha \text{ is left-inessential,} \\ \lambda & \text{if } \alpha \text{ is right-inessential.} \end{cases}$$

First we are to show that for all $\alpha \in \Lambda$, if $a \precsim^* b$, then

$$u(\omega_\alpha(ab)) = \tau(\alpha)u(a) + (\tau(\alpha^*) - \tau(\alpha))u(b) + \mu,$$

so that (2) follows from (1) and B3. Since α^* is right-inessential, if α is essential, then this easily follows from the preceding paragraphs and the definition of τ. If α is left-inessential and $a \precsim^* b$, then $\omega_\alpha(ab) \sim^* \omega_\alpha(bb)$. Since $\omega_\alpha(bb) \sim^* \omega_\beta(bb)$ for an essential β, and α^* is right-inessential, we obtain that

$$\begin{aligned} u(\omega_\alpha(ab)) &= u(\omega_\beta(bb)) \\ &= \tau(\alpha^*)u(b) + \mu \\ &= \tau(\alpha)u(a) + (\tau(\alpha^*) - \tau(\alpha))u(b) + \mu. \end{aligned}$$

If α is right-inessential, a similar proof gives the desired result.

Next we show (3). Suppose that $\pi = [\alpha_1, \ldots, \alpha_{n-1}]$ is essential. Then by B1, $\langle A, \precsim^*, \omega_\pi, A_+^n \rangle$ is a weak multi-symmetric structure. Thus by Theorem 1, there exist real numbers $\mu(\pi)$, and $\lambda_i(\pi) \geq 0$ for $i = 1, \ldots, n$ with $\sum_{i=1}^n \lambda_i(\pi) > 0$, and real valued function f_π on A such that for all $a, b \in A$,

$$a \precsim^* b \text{ iff } f_\pi(a) \leq f_\pi(b),$$

and if $a_1 \precsim^* \ldots \precsim^* a_n$, then

$$f_\pi(\omega_\pi(a_1 \ldots a_n)) = \sum_{i=1}^n \lambda_i(\pi) f_\pi(a_i) + \mu(\pi).$$

By B4, if α_k is essential and $a \precsim^* b$, then

$$f_\pi(\omega_{\alpha_k}(ab)) = f_\pi(\omega_\pi^k(ab))$$
$$= (\sum_{i=1}^{k} \lambda_i(\pi))f_\pi(a) + (\sum_{i=k+1}^{n} \lambda_i(\pi))f_\pi(b) + \nu(\pi).$$

It follows from Proposition 1(4) and the preceding paragraphs that there exist constants $\eta > 0$ and η' such that $u = \eta f_\pi + \eta'$, $\tau(\alpha_k) = \sum_{i=1}^k \lambda_i(\pi)$, $\tau(\alpha^*) = \sum_{i=1}^n \lambda_i(\pi)$, and $\mu = \eta\nu(\pi) + \eta'(1-\tau(\alpha^*))$. Let $\eta = 1$ and $\eta' = 0$, so $f_\pi = u$ and $\nu(\pi) = \mu$. If α_k is either left-inessential or right-inessential, then $\sum_{i=1}^k \lambda_i(\pi) = 0$ or $\sum_{i=k+1}^n \lambda_i(\pi) = 0$, respectively. Therefore, letting $\alpha_n = \alpha^*$, $\tau(\alpha_k) = \sum_{i=1}^k \lambda_i(\pi)$ for all $k = 1, \ldots, n$. Solving those equations with respect to $\lambda_i(\pi)$ for $i = 1, \ldots, n$, we obtain that $\lambda_1(\pi) = \tau(\alpha_1)$ and $\lambda_i(\pi) = \tau(\alpha_i) - \tau(\alpha_{i-1})$ for $i = 2, \ldots, n$. Hence we obtain (3). The uniqueness part (4) of the proposition easily follows from Proposition 1(4). [Q.E.D.]

5. PROOFS OF THE THEOREMS

This section provides the proofs of Theorems 1 and 2 in Section 2.

Proof of Theorem 1. Suppose that $\langle A, \precsim^*, \omega, A_+^n \rangle$ is a weak multi-symmetric strucuture. Let $N = \{1, \ldots, n\}$. Then by A2, there is an essential $k \in N$. If we regard ω_k as a binary operation on A_+^2, then for an essential $k \in N$, $\langle A, \precsim^*, \omega_k, A_+^2 \rangle$ is a weak bisymmetric structure. By A7, ω_k and ω_j are weakly isometric. If $k < j$ and $a \precsim^* b$, then by A6, $\omega_j(ab) \precsim^* \omega_k(ab)$. Therefore, if we take $\Lambda = N$, $<^o = <$ on N, $\Pi = \{\pi : \pi = [i],$

$i \in N\}$, and $\Omega = \{\omega_k : k \in N\}$, then n is a maximal element in Λ, so the hypotheses of Proposition 2 hold. It is easy to see from the proof of Proposition 2 that the representation in Proposition 2 is valid in that case without applying the result of the present theorem. Hence, there exist real numbers μ and $\alpha_k \geq 0$ for $k \in N$ with $\alpha_n > 0$, and real valued function f on A such that $\alpha_1 \leq \ldots \leq \alpha_n$, and for all $k \in N$ and all $a, b \in A$,

$$a \precsim^* b \text{ iff } f(a) \leq f(b),$$

and if $a \precsim^* b$, then

$$f(\omega_k(ab)) = \alpha_k f(a) + (\alpha_n - \alpha_k) f(b) + \mu.$$

Suppose that $M = \{i_1, \ldots, i_m\} \subseteq N$, $0 < m < n$, and $i_0 = 0 < i_1 < \ldots < i_m < n$. We shall denote $\omega_M(a_1 \ldots a_{m+1}) = \omega(b_1 \ldots b_n)$ when for $j \in N$ and $k = 1, \ldots, m$, $b_j = a_k$ if $i_{k-1} < j \leq i_k$; $b_j = a_{m+1}$ if $i_m < j$. Let $\alpha_0 = 0$. We are to show by induction that for $m = 1, \ldots, n-1$, if $a_1 \precsim^* \ldots \precsim^* a_{m+1}$, then

$$f(\omega_M(a_1 \ldots a_{m+1})) = \sum_{k=1}^{m} (\alpha_{i_k} - \alpha_{i_{k-1}}) f(a_k) + (\alpha_n - \alpha_{i_m}) f(a_{m+1}) + \mu.$$

Thus letting $\lambda_i = \alpha_i - \alpha_{i-1}$ for $i \in N$, we obtain (2). (3) easily follows from Proposition 1(4).

Since the case for $m = 1$ has already been established, we shall assume $m > 1$. Suppose that the conclusion is true for $m < k$. Let $m = k$. If $a_i \sim^* a_j$ for some $1 \leq i < j \leq k+1$, then the desired result easily follows from A1, A6, and the hypothesis of induction. Thus we shall assume that $a_1 <^* \ldots <^* a_{k+1}$.

The following three cases cover all possibilities.

CASE 1. $\omega_i(a_1a_1) \sim^* \omega_i(aa_2)$ and $\omega_i(a_{k+1}a_{k+1}) \sim^* \omega_i(a_kb)$ for some $a,b\in A$ with $a \precsim^* a_2$ and $a_k \precsim^* b$, and some $i\in N$.

CASE 2. either $\omega_i(a_1a_1) \sim^* \omega_i(aa_2)$ for some $a\in A$ with $a \precsim^* a_2$ and some right-essential $i\in N$ or $\omega_j(a_{k+1}a_{k+1}) \sim^* \omega_j(a_kb)$ for some $b\in A$ with $a_k \precsim^* b$, and some left-essential $j\in N$.

CASE 3. neither $\omega_i(a_1a_1) \sim^* \omega_i(aa_2)$ nor $\omega_j(a_{k+1}a_{k+1}) \sim^* \omega_j(a_kb)$ for all $a,b\in A$ with $a \precsim^* a_2$ and $a_k \precsim^* b$, all right-essential $i\in N$, and all left-essential $j\in N$.

Let $M' = \{i_1,\ldots, i_{k-1}\}$ and $M'' = \{i_2,\ldots, i_k\}$.

CASE 1. Suppose that $\omega_i(a_1a_1) \sim^* \omega_i(aa_2)$ and $\omega_i(a_{k+1}a_{k+1}) \sim^* \omega_i(a_kb)$ for some $a,b\in A$ with $a \precsim^* a_2$ and $a_k \precsim^* b$, and some $i\in N$. Since $a_1 <^* \ldots <^* a_{k+1}$, and $\omega_i(aa) = \omega_j(aa)$ for all $a\in A$ and an essential $j\in N$, A1 and A5 imply that $\omega_i(a_1a_1) <^* \ldots <^* \omega_i(a_{k+1}a_{k+1})$. Then by A6 and A7, we obtain

$$\begin{aligned}\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_{k+1}a_{k+1})) &\sim^* \omega_M(\omega_i(aa_2)\omega_i(a_2a_2)\ldots\omega_i(a_ka_k)\omega_i(a_kb)) \\ &\sim^* \omega_i(\omega_M(aa_2\ldots a_ka_k)\omega_M(a_2a_2a_3\ldots a_kb)) \\ &\sim^* \omega_i(\omega_{M'}(aa_2\ldots a_k)\omega_{M''}(a_2\ldots a_kb)).\end{aligned}$$

Thus the hypothesis of induction implies

$$\begin{aligned}&f(\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_{k+1}a_{k+1}))) \\ &\quad = f(\omega_i(\omega_{M'}(aa_2\ldots a_k)\omega_{M''}(a_2\ldots a_kb))) \\ &\quad = \alpha_i f(\omega_{M'}(aa_2\ldots a_k)) + (\alpha_n-\alpha_i)f(\omega_{M''}(a_2\ldots a_kb)) + \mu\end{aligned}$$

$$= \alpha_i\{\alpha_{i_1} f(a) + \sum_{j=2}^{k-1}(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + (\alpha_n-\alpha_{i_{k-1}})f(a_k) + \mu\}$$
$$+ (\alpha_n-\alpha_i)\{\alpha_{i_2} f(a_2) + \sum_{j=3}^{k}(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + (\alpha_n-\alpha_{i_k})f(b) + \mu\} + \mu.$$

Since $\omega_i(a_1a_1) \sim^* \omega_i(aa_2)$ and $\omega_i(a_{k+1}a_{k+1}) \sim^* \omega_i(a_kb)$, we note that $\alpha_i f(a) = \alpha_n f(a_1) - (\alpha_n-\alpha_i)f(a_2)$ and $(\alpha_n-\alpha_i)f(b) = \alpha_n f(a_{k+1}) - \alpha_i f(a_k)$. Substituting the expressions of $\alpha_i f(a)$ and $(\alpha_n-\alpha_i)f(b)$ for the above equation, rearrangement gives

$$f(\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_{k+1}a_{k+1})))$$
$$= \alpha_n\{\sum_{j=1}^{k}(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + (\alpha_n-\alpha_{i_k})f(a_{k+1}) + \mu\} + \mu.$$

On the other hand, A7 gives

$$\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_{k+1}a_{k+1})) \sim^* \omega_i(\omega_M(a_1\ldots a_{k+1})\omega_M(a_1\ldots a_{k+1})),$$

so we obtain

$$f(\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_{k+1}a_{k+1}))) = \alpha_n f(\omega_M(a_1\ldots a_{k+1})) + \mu.$$

Since $\alpha_n > 0$, togehter with the last equation in the preceding paragraph, this gives the desired result.

CASE 2. Suppose that $\omega_i(a_1a_1) \sim^* \omega_i(aa_2)$ for some $a\in A$ with $a \precsim^* a_2$, and some right-essential $i\in N$. The proof is similar for the other case. Since $a_k <^* a_{k+1}$, A5 and A6 imply that $\omega_i(a_ka_k) <^* \omega_i(a_ka_{k+1}) \precsim^*$

$\omega_i(a_{k+1}a_{k+1})$. A3 implies that $\omega_i(a_k a_{k+1}) \sim^* \omega_i(bb)$ for some $b \in A$, so $\omega_i(a_k a_k) <^* \omega_i(bb)$. Thus A6 and A7 imply

$$\begin{aligned}\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_ka_k)\omega_i(bb)) &\sim^* \omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_ka_k)\omega_i(a_ka_{k+1}))\\ &\sim^* \omega_i(\omega_M(a_1\ldots a_ka_k)\omega_M(a_1\ldots a_{k+1}))\\ &\sim^* \omega_i(\omega_{M'}(a_1\ldots a_k)\omega_M(a_1\ldots a_{k+1})).\end{aligned}$$

Then we obtain

$$\begin{aligned}&f(\omega_M(a_1a_1)\ldots\omega_i(a_ka_k)\omega_i(bb))\\ &\qquad = \alpha_i f(\omega_{M'}(a_1\ldots a_k)) + (\alpha_n-\alpha_i)f(\omega_M(a_1\ldots a_{k+1})) + \mu,\end{aligned}$$

which is rearranged to give

$$\begin{aligned}&(\alpha_n-\alpha_i)f(\omega_M(a_1\ldots a_{k+1}))\\ &\qquad = f(\omega_M(\omega_i(a_1a_1)\ldots\omega_i(a_ka_k)\omega_i(bb)) - \alpha_i f(\omega_{M'}(a_1\ldots a_k)) - \mu.\end{aligned}$$

Since $\omega_i(a_ka_k) <^* \omega_i(bb)$, A5 and A6 imply that $a_k <^* b$. Therefore, the hypothesis of Case 1 holds for $a_1, \ldots, a_k$, and b. Note that $\alpha_n f(b) = \alpha_i f(a_k) + (\alpha_n-\alpha_i)f(a_{k+1})$, since $\omega_i(bb) \sim^* \omega_i(a_ka_{k+1})$. Then by Case 1,

$$\begin{aligned}&f(\omega_M(a_1a_1)\ldots\omega_i(a_ka_k)\omega_i(bb))\\ &\quad = \alpha_n\{\Sigma_{j=1}^k(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + (\alpha_n-\alpha_{i_k})f(b) + \mu\} + \mu\\ &\quad = \alpha_n\{\Sigma_{j=1}^k(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + \mu\}\end{aligned}$$

$$+ (\alpha_n-\alpha_{i_k})\{\alpha_i f(a_k) + (\alpha_n-\alpha_i)f(a_{k+1})\} + \mu.$$

By the hypothesis of induction,

$$f(\omega_{M'}(a_1 \ldots a_k)) = \textstyle\sum_{j=1}^{k-1}(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + (\alpha_n-\alpha_{i_{k-1}})f(a_k) + \mu.$$

Substituting the above two equations for the last equation in the preceding paragraph, rearrangement gives

$$(\alpha_n-\alpha_i)f(\omega_M(a_1 \ldots a_{k+1})) = (\alpha_n-\alpha_i)\{\textstyle\sum_{j=1}^{k}(\alpha_{i_j}-\alpha_{i_{j-1}})f(a_j) + (\alpha_n-\alpha_{i_k})f(a_{k+1}) + \mu\}.$$

Since i is right-essential, $\alpha_i < \alpha_n$. Hence the desired result obtains.

CASE 3. Let i be essential. Since $a_1 <^* a_2$, A5 implies that $\omega_i(a_1a_1) <^* \omega_i(a_1a_2) <^* \omega_i(a_2a_2)$. Thus by A3, $\omega_i(a_1a_2) \sim^* \omega_i(aa)$ for some $a \in A$, so $\omega_i(a_1a_1) <^* \omega_i(aa) <^* \omega_i(a_2a_2)$. Therefore, $a_1 <^* a <^* a_2$, since i is essential. The hypothesis of Case 2 holds for a, $a_2, \ldots, a_{k+1}$. Hence a similar analysis of Case 2 gives the desired result. [Q.E.D.]

Proof of Theorem 2. Suppose that $\langle A, \precsim^*, \omega, A^n\rangle$ is a multi-symmetric structure. Since $\precsim^*$ satisfies A1-A7 in Definition 1, let μ, $\lambda_i \geq 0$ for $i \in N$ and f on A be obtained by Theorem 1 with $\alpha_k = \sum_{i=1}^{k}\lambda_i$ for $k \in N$. We are to show that for k = 1,..., n-1, and all $a,b \in A$ with $a <^* b$,

$$f(\omega_k(ba)) = \alpha_k f(b) + (\alpha_n - \alpha_k) f(a) + \mu.$$

Hence, a similar induction analysis in the proof of Theorem 1 provides the desired result in the general case.

We have two cases to examine: for an essential $j \in N$ and some $c \in A$, either $\omega_j(bb) \lesssim^* \omega_j(ac)$ or $\omega_j(ac) <^* \omega_j(bb)$. Suppose first that $\omega_j(bb) \lesssim^* \omega_j(ac)$ for some $c \in A$ and an essential j. Since j is essential and $a <^* b$, A5 implies that $\omega_j(ab) <^* \omega_j(bb)$. Thus by A1, $\omega_j(ab) <^* \omega_j(ac)$, so by A5*, $b <^* c$. Therefore, by A6*, $\omega_k(ba) \lesssim^* \omega_k(bc)$. Then it follows from Theorem 1(2) that

$$\begin{aligned}
&f(\omega_j(\omega_k(ba)\omega_k(bc))) \\
&\qquad = \alpha_j f(\omega_k(ba)) + (\alpha_n - \alpha_j) f(\omega_k(bc)) + \mu \\
&\qquad = \alpha_j f(\omega_k(ba)) + (\alpha_n - \alpha_j)\{\alpha_k f(b) + (\alpha_n - \alpha_k) f(c) + \mu\} + \mu, \\
&f(\omega_k(\omega_j(bb)\omega_j(ac))) \\
&\qquad = \alpha_k f(\omega_j(bb)) + (\alpha_n - \alpha_k) f(\omega_j(ac)) + \mu \\
&\qquad = \alpha_k(\alpha_n f(b) + \mu) + (\alpha_n - \alpha_k)\{\alpha_j f(a) + (\alpha_n - \alpha_j) f(c) + \mu\} + \mu,
\end{aligned}$$

A7* and Theorem 1(1) imply that the above two equations must be equal, so that rearrangement gives

$$\alpha_j f(\omega_k(ba)) = \alpha_j \{\alpha_k f(b) + (\alpha_n - \alpha_k) f(a) + \mu\}.$$

Since j is essential, $\alpha_j \neq 0$. Therefore, we obtain the desired result.

Suppose next that $\omega_j(ac) <^* \omega_j(bb)$ for some $c \in A$ and an essential j. By A5, $\omega_j(aa) <^* \omega_j(ab)$. By A7, $\omega_j(\omega_j(ab)\omega_j(ab)) \sim^* \omega_j(\omega_j(aa)\omega_j(bb))$. Therefore, the hypothesis in the preceding paragraph holds, so that

$$\begin{aligned} f(\omega_k(\omega_j(ab)\omega_j(aa))) &= \alpha_k f(\omega_j(ab)) + (\alpha_n - \alpha_k) f(\omega_j(aa)) + \mu \\ &= \alpha_k\{\alpha_j f(a) + (\alpha_n - \alpha_j) f(b) + \mu\} \\ &\qquad + (\alpha_n - \alpha_k)(\alpha_n f(a) + \mu) + \mu. \end{aligned}$$

Since $\omega_k(aa) \lesssim^* \omega_k(ba)$ by A6*, Theorem 1(2) implies

$$\begin{aligned} f(\omega_j(\omega_k(aa)\omega_k(ba))) &= \alpha_j f(\omega_k(aa)) + (\alpha_n - \alpha_j) f(\omega_k(ba)) + \mu \\ &= \alpha_j(\alpha_n f(a) + \mu) + (\alpha_n - \alpha_j) f(\omega_k(ba)) + \mu. \end{aligned}$$

By A7* and Theorem 1(2), $f(\omega_j(\omega_k(aa)\omega_k(ba))) = f(\omega_k(\omega_j(ab)\omega_j(aa)))$. Thus it follows from the preceding paragraph that

$$\begin{aligned} &\alpha_j(\alpha_n f(a) + \mu) + (\alpha_n - \alpha_j) f(\omega_k(ba)) \\ &\qquad = \alpha_k\{\alpha_j f(a) + (\alpha_n - \alpha_j) f(b) + \mu\} + (\alpha_n - \alpha_k)(\alpha_n f(a) + \mu). \end{aligned}$$

This is rearranged to give

$$(\alpha_n - \alpha_j) f(\omega_k(ba)) = (\alpha_n - \alpha_j)\{\alpha_k f(b) + (\alpha_n - \alpha_k) f(a) + \gamma\}.$$

Since j is essential, $\alpha_j \neq \alpha_n$. Thus the desired result is otained.

[Q.E.D.]

6. CONCLUSION

The main purpose in this paper has been to introduce the weak and strong forms of the multi-symmetric structures, and then develop their numerical representations. The multi-symmetric structures postulate an n-ary operation to generalize a binary operation in the bisymmetric structures. We also developed the numerical representations for multiple operations, and then applied them to derive a common axiomatic charecterization for Choquet expected utility and Quiggin's anticipated utility.

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