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TWO-STAGE TWO-DIMENSIONAL SPATIAL COMPETITION BETWEEN TWO FIRMS

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1. INTRODUCTION

Although urban location problem is well analyzed in two-dimensional space, it is usually examined by one-dimensional space in the literature on spatial competition a la Hotelling (1929). This is presumably due to mathematical tractability. As will be demonstrated later, however, results by one-dimensional models are not always valid in two-dimensional space.

In studying spatial competition of oligopolistic firms, we must be faced with a troublesome obstacle of nonexistence of Nash price equilibrium. That is to say, as demonstrated by d'Aspremont, Gabszewitz and Thisse (1979), there exists no price equilibrium under a linear transportation cost when duopolists locate close. Later, Champsaur and Rochet (1988) and Tabuchi (1989) showed that so as to guarantee the existence of price equilibrium, we have to limit a family of transportation cost function substantially. Without existence of price equilibrium for all locational pairs, payoff functions of the firms are not defined globally, which prevents us from knowing the overall locational behavior of the firms.

A similar argument applies for Nash location equilibrium under a fixed price. It is well known that there exists no location equilibrium in one-dimensional space when the number of firms is three.

These findings imply that we can obtain both price equilibrium and location equilibrium under a very limited set of the number of firms, transportation cost functions, and consumer distribution functions. This forces us to presuppose duopolistic firms, a quadratic transportation cost, and uniform consumer distributions of ellipses and rectangles. These presuppositions will be discussed in later sections.

In this paper, we deal with subgame perfect Nash equilibrium. That is to say, we consider a situation that two firms compete in location in the first stage anticipating the subsequent price competition in the second stage. The locations cannot be altered in the second stage. In spatial competition, such a change in location is seldom done due to irreversible nature of urban buildings. In characteristics competition, such a change in model type is usually very costly because of existence of scale economies in production.

We also consider a case of sequential entry with simultaneous price competition. Specifically, one firm enters the spatial market in the first stage, the other firm enters the market in the second stage, and then they compete in price in the final stage. In other words, the former two stages are a Stackelberg location game while the latter stage is a Nash price game. A comparison is made between this sequential location model with the simultaneous location model.

The paper is organized as follows. The two-stage location-price game is briefly depicted in Section 2, and a link between one-dimensional non-uniform distributions of consumers and two-dimensional uniform distributions of consumers is stated in Section 3.

After solving the second-stage Nash price game, we analyze the first-stage Nash location game when the consumer distribution is given by an ellipse shape in Section 4, and given by an rectangular shape in Section 5. In Section 6, we modify the Nash location game to a Stackelberg location game in the first stage while the second stage of the Nash price game remains the same and examine the case of rectangular distributions of consumers. Section 7 makes a welfare comparison between the Nash location equilibrium in Section 5 and the Stackelberg location equilibrium in Section 6. Section 8 concludes the paper.

2. THE MODEL SETTING

Consumers who purchase a unit of good are uniformly distributed over a compact set $\mathbb{C}$ on $\mathbb{R}^2$, where $\int_{\mathbb{C}} dxdy=1$. In the first stage, anticipating consequences of the second stage, firm 1 locates at $(x_1,y_1)\in\mathbb{C}$ and firm 2 locates at $(x_2,y_2)\in\mathbb{C}$. In the second stage, they choose their own mill price p_1 and p_2 respectively holding the locations fixed. The transportation cost which a consumer has to incur is assumed to be a quadratic function of distance between the consumer and the nearer firm.

Suppose the unit transportation cost is unity without loss of generality, a marginal consumer at (x,y) is indifferent between firms 1 and 2, where

$$p_1 + (x_1-x)^2 + (y_1-y)^2 = p_2 + (x_2-x)^2 + (y_2-y)^2, \qquad (1)$$

which is a straight line.

Assuming zero production cost again without loss of generality, each firm maximizes its profit:

$$\Pi_1 = p_1D_1, \qquad \Pi_2 = p_2(1-D_1) \qquad (2)$$

with respect to location and then price, where $D_1=\int_{\mathbb{C}_1} dxdy$, and $\mathbb{C}_1=\{(x,y)\in\mathbb{C}|$ $p_1+(x_1-x)^2+(y_1-y)^2 \leq p_2+(x_2-x)^2+(y_2-y)^2\}$. The analysis is confined to the case of pure strategies. Note that the measure of the boundary is nil, and so ignored.

In order to get the subgame perfect equilibrium, we first solve the second-stage problem of profit maximization with respect to price given the locations. This is done in the next section.

3. CONVERSION TO ONE-DIMENSION SPACE IN THE SECOND-STAGE PRICE GAME

As the market boundary is a straight line, we can convert a two-dimensional uniform distribution of consumers into a one-dimensional non-uniform distribution of consumers. That will be done in the proof of Corollary 1 below. We therefore investigate the model of one-dimensional non-uniform distribution of consumers first.

Let $f(x)$ denote the density function of consumers in one-dimensional space and $F(x)$ be the cumulative distribution function of consumers, where $F(x) \equiv \int_{\underline{x}}^{x} f(z)dz$ and $F(\overline{x})=1$.

By use of (1) and (2), the first-order conditions for profit maximization with respect to its own price are given by

$$\frac{\partial \Pi_1}{\partial p_1} = F(\hat{x}) - \frac{p_1 f(\hat{x})}{2(x_2-x_1)} = 0, \qquad \frac{\partial \Pi_2}{\partial p_2} = 1-F(\hat{x}) - \frac{p_2 f(\hat{x})}{2(x_2-x_1)} = 0. \quad (3)$$

From these two equations and (1), we have

$$G(\hat{x}) \equiv 2F(\hat{x}) - 1 + \left[\hat{x} - \frac{x_1+x_2}{2}\right] f(\hat{x}) = 0. \quad (4)$$

Equation (4) determines the market boundary $\hat{x}$ in Nash price equilibrium, given x_1 and x_2.

In the one-dimensional non-uniform distribution of consumers, Caplin and Nalebuff (1989) established the following proposition.

Proposition 1

For any given locations of firms and for any concave density function *of consumers, a Nash price equilibrium exists, which is unique.*

Employing Proposition 1, we have the corollary below, which is useful in analyzing spatial competition under two-dimensional consumer distributions.

Corollary 1

A unique Nash price equilibrium exists on a two dimensional plane, for any given locations of firms and for any convex set of uniform consumer distribution.

Proof:

By setting the x axis parallels to the line passing through firms' locations, (x_1,y_1) and (x_2,y_2), the market split line (1) becomes parallel to the y axis. Define a convex set of two-dimensional uniform distribution of consumers as $\mathbb{C}_o=\{(x,y)\in\mathbb{R}^2|\ g(x,y)\leq 0\}$. Then, $D_1=\int_{\underline{x}}^{\hat{x}} h_1(x)-h_2(x)\ dx$, where $\underline{x}=\inf x \in\mathbb{C}_o$, and $y=h_1(x)$ and $y=h_2(x)$ $[h_1(x)\geq h_2(x)]$ are two implicit functions derived from $g(x,y)=0$. Since $\mathbb{C}_o$ is convex, $h_1(x)$ should be concave and $h_2(x)$ should be convex, and hence $h_1(x)-h_2(x)$ $[=f(x)]$ should be concave. Therefore, Proposition 1 applies. ■

Since the existence and uniqueness of Nash price equilibrium are established for any convex set on $\mathbb{R}^2$, we will focus only on this set in the reminder of the paper.

4. THE FIRST-STAGE LOCATION GAME I: AN ELLIPSE CASE

In the beginning, consider the Nash location equilibrium for a concave distribution of consumers on $\mathbb{R}$. From (2) and (3), each firm's profit already maximized with respect to its own price is expressed as a function of locations:

$$\Pi_1^*(x_1,x_2) = 2(x_2-x_1)F^2(\hat{x})/f(\hat{x}), \qquad (5)$$
$$\Pi_2^*(x_1,x_2) = 2(x_2-x_1)[1-F(\hat{x})]^2/f(\hat{x}).$$

Remember that Π_1^* and Π_2^* exists, and $\hat{x}$ is uniquely determined by (4) for any concave distribution of consumers.

Now, firm i maximizes $\Pi_i^*(x_1,x_2)$ with respect to x_i for i=1,2, knowing the subsequent price competition with perfect foresight. It should be noticed that although each profit function is quasi-concave with respect to its price for concave density (Proposition 1), it is not necessarily concave with respect to its location.[1] This forces us to examine only a limited family of consumer distributions since we cannot investigate the subgame perfect equilibrium without existence of price equilibrium and location equilibrium.

The rest of this section deals with consumer distributions of ellipses including circles, and the next three sections are devoted to consumer distributions of rectangles including squares. Needless to say, these distributions are convex.

Proposition 2

If $\mathbb{C}$ is an ellipse, neither firm locates at the interior region in Nash equilibrium.

Proof:

Immediately from Corollary 1, there exists a unique equilibrium in have

$$H(\hat{x}) \le 4AB-3/2 = 4/\pi-3/2 < 0.$$

We thus showed that $\partial\Pi_1^*/\partial x_1<0$ for all $\hat{x}\le 0$.

Next, when $\hat{x}>0$, $f'(\hat{x})<0$ and $F(\hat{x})>1/2$. So

$$\partial H(\hat{x})/\partial F(\hat{x}) = [8F(\hat{x})-(x_2-\hat{x})f(\hat{x})-2]f'(\hat{x})/f^2(\hat{x})-7 \le 0,$$

as $(x_2-\hat{x})f(\hat{x})<2AB<2/\pi<2$. Therefore, substituting 1/2 for $F(\hat{x})$ in (6), we get

$$H(\hat{x}) \le 2(x_2-\hat{x})f(\hat{x})-3/2-(x_2-\hat{x})f'(\hat{x})/[2f(\hat{x})],$$

$$< 4AB-3/2-Af'(\hat{x})/[2f(\hat{x})] \qquad (7)$$

as $x_2-\hat{x}<A$ and $f(\hat{x})<2B$.

Now, $\sup \hat{x} < A/6$ holds because if $\hat{x}\ge A/6$, then from (4) with $x_1\le 0$ and $x2\le A$,

$$G(\hat{x}) \ge 2F(A/6)-1+(A/6-A/2)f(A/6) = 2[F(A/6)-1/2-f(A/6)A/6].$$

However, since $F(\hat{x})-F(0)>f(\hat{x})\hat{x}$ for all $\hat{x}>0$ due to concavity of an ellipse, the RHS is positive. Thus, (4) does not hold, which means nonexistence of equilibrium for $\hat{x}\ge A/6$.

Since (7) is increasing in $\hat{x}\in(0,A/6]$, it becomes

$$H(\hat{x}) < 4AB-3/2-Af'(A/6)/[2f(A/6)] = 4/\pi-3/2+3/35 < 0.$$

Hence, we showed $\partial\Pi_1^*/\partial x_1<0$ for any $-A\le x_1\le x_2\le A$. Also, as $\partial\Pi_2^*/\partial x_2>0$ is similarly shown owing to symmetry of the model setting, we conclude that no firm locates at an interior point of any ellipse. ■

The intuition of Proposition 2 is that firms locate as far as possible each other to relax price competition. In other words, location competition to acquire a larger market share is dominated by price competition which lessens equilibrium prices and profits when they locate close.

price. Let $\mathbb{C}=\{(x,y)\in\mathbb{R}^2|\ x^2/a^2+y^2/b^2\le 1,\ \pi ab=1\}$. Without losing generality, assume $x_1\le x_2$ and $y_1\le y_2$, where $(x_1,y_1)\in\mathbb{C}$ and $(x_2,y_2)\in\mathbb{C}$.

If we rotate the x-y coordinates by θ, then the boundary of $\mathbb{C}$ becomes $cX^2-dXY+eY^2\le 1$, where X-Y is the new coordinates, and c, d and e are functions only of θ. Two implicit functions derived from this are $h_1(X),h_2(X)=(dX\pm\sqrt{d^2X^2-4ceX^2+4e})/2e$. As $f(X)=h_1(X)-h_2(X)$, we finally arrive at an equation of ellipse:

$$X^2/A^2 + f^2(X)/B^2 = 1,$$

where $A=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$, and $B=2ab/\sqrt{A}$. Thus, rotating the x-y coordinates by $\theta=\arctan((Y_2-Y_1)/(X_2-X_1))$, we can equalize y_1 and y_2 on the X-Y coordinates, and still work with an ellipse, which is symmetric. This means that we can convert the two-dimensional consumer distribution and firm locations into one-dimensional ones.

Now, we first eliminate x_1 in (5) by use of (4) so that Π_1^* becomes a function of $\hat{x}$ and x_2 in one-dimensional space. We then differentiate Π_1^* in the following manner:

$$\frac{\partial\Pi_1^*}{\partial x_1} = \frac{\partial\Pi_1^*}{\partial\hat{x}}\frac{\partial\hat{x}}{\partial x_1} = \frac{4F(\hat{x})}{f(\hat{x})}\frac{\partial\hat{x}}{\partial x_1}H(\hat{x}),$$

where

$$H(\hat{x}) \equiv 2(x_2-\hat{x})f(\hat{x})+2-7F(\hat{x})-\{(x_2-\hat{x})f(\hat{x})+2-4F(\hat{x})\}F(\hat{x})f'(\hat{x})/f^2(\hat{x}). \quad (6)$$

Since $\partial\hat{x}/\partial x_1=f(\hat{x})/[6f(\hat{x})+(2\hat{x}-x_1-x_2)f'(\hat{x})]>0$ by application of the implicit function theorem to (4), $\mathrm{sgn}(\partial\Pi_1^*/\partial x_1)=\mathrm{sgn}(H(\hat{x}))$ always holds.

For any ellipse distribution, however, $f'(\hat{x})\ge 0$ and $F(\hat{x})\le 1/2$ hold for $\hat{x}\le 0$ due to symmetry, and so

$$H(\hat{x}) \le 2(x_2-\hat{x})f(\hat{x})+2-7F(\hat{x}) \le (6\hat{x}-7x_1+x_2)f(\hat{x})/4-3/2,$$

by substitution of $F(\hat{x})$ in (4). As $\hat{x}\le 0$, $x_1\ge -A$, $x_2\le A$, $f(\hat{x})\le 2B$ and $\pi AB=1$, we

5. THE FIRST STAGE LOCATION GAME II: A RECTANGULAR CASE

We will prove in this section that firms never locate at the interior region in any rectangular distribution. Although the proof becomes more complicated, we can fully characterize Nash location equilibria for any rectangular distribution of consumers. Furthermore, we can also obtain Stackelberg location equilibrium in the next section, and examine social welfare of these equilibria in Section 7.

Now, since rectangles are convex, there always exists a unique price equilibrium for any location pair from Corollary 1. We will thus focus our analysis only on location equilibrium hereafter as in the case of ellipses. Let us start from several lemmas.

Lemma 1

If $\mathbb{C}$ is a rectangle, then one of the firms locates either at a corner or at a center of one side.

Proof:

Consider a uniform distribution of consumers on a rectangle whose lengths of sides are c by 1/c. Without loss of generality, assume $0 \le x_1 \le x_2 \le c$ and $0 \le y_1 \le y_2 \le 1/c$. Define x_a such that $(x_a, 0)$ is on the market split line of (1); x_b such that $(x_b, 1/c)$ is on (1); y_a such that $(0, y_a)$ is on (1); and y_b such that (c, y_b) is on (1). Notice that the line of (1) is at right angles to the line passing through (x_1, y_1) and (x_2, y_2). Denote the slope of the latter line be $\alpha \equiv (y_2 - y_1)/(x_2 - x_1)$.

Depending upon the values of x_a, x_b, y_a and y_b, divisions of the rectangular market are classified into four cases as drawn in Figure 1(i), 1(ii), 1(iii) and 1(iv).

[Figure 1 about here]

Case (i) [$c \le x_a$, $0 \le x_b \le c$, $1/c \le y_a$, $0 \le y_b \le 1/c$]

The profits are respectively given by

$$\Pi_1 = p_1[1-(c-x_b)^2/2\alpha] \quad \text{and} \quad \Pi_2 = p_2(c-x_b)^2/2\alpha.$$

Calculating the first-order conditions of $\partial\Pi_1/\partial p_1=0$ and $\partial\Pi_2/\partial p_2=0$, we have

$$\Pi_2^* = \frac{x_2-x_1}{2\alpha}(c-x_b)^3.$$

Differentiating this expression with respect to x_2,

$$\frac{\partial\Pi_2^*}{\partial x_2} = \frac{(c-x_b)^2}{2\alpha}\left[2(c-x_b) - \frac{3(x_2+c-2x_b)}{2+\alpha/(c-x_b)^2}\right] > \frac{(c-x_b)[\alpha-(c-x_b)^2]}{\alpha[2+\alpha/(c-x_b)^2]}$$

$$\ge \frac{(c-x_b)^2 x_b}{\alpha[2+\alpha/(c-x_b)^2]} \ge 0 \qquad \text{for all } x_2<c.$$

The first inequality is implied by $x_2<c$, and the second is followed from $\alpha=(c-x_b)/(1/c-y_b)\ge c(c-x_b)$. Note that if $\alpha\to+\infty$, then $x_1\to x_2$, which leads to $x_b<0$, i.e., $\alpha\to\infty$ does not fall in case (i).

Similarly, we can show $\partial\Pi_2^*/\partial y_2>0$ for all $y_2<1/c$. Hence, firm 2 does not locate inside the rectangle, but rather locates at the corner $(c,1/c)$ in this market division case.

Case (ii) [$0 \le x_b \le x_a \le c$, $1/c \le y_a$, $y_b \le 0$]

Each profit is given by

$$\Pi_1 = p_1(x_a+x_b)/2c \quad \text{and} \quad \Pi_2 = p_2[1-(x_a+x_b)/2c].$$

Calculating $\partial\Pi_1/\partial p_1=0$ and $\partial\Pi_2/\partial p_2=0$, we get

$$\Pi_2^* = \frac{x_2-x_1}{2c}(2c-x_a-x_b)^2.$$

Differentiating this with respect to x_2, we have

$$\frac{\partial\Pi_2^*}{\partial x_2} = \frac{2c-x_a-x_b}{2c}[4c/3+x_1/3-x_2+\alpha(y_1+y_2-1/c)/3]$$

$$> \frac{2c-x_a-x_b}{6c}[c-\alpha/c] \geq 0 \qquad \text{for all } x_2<c.$$

The first inequality is followed from $x_2<c$ and $x_1,y_1,y_2 \geq 0$; and the second is due to the fact that the absolute value of the slope of equation (1) is greater than or equal to that of the diagonal of the rectangle, i.e., $|-1/\alpha|\geq 1/c^2$.

Next, differentiating Π_2^* with respect to y_2, we get

$$\frac{\partial\Pi_2^*}{\partial y_2} = \frac{2c-x_a-x_b}{3c}(1/c-2y_2) = 0.$$

Therefore, the optimal location of firm 2 is the center of the side (c,1/2c) in case (ii).

Case (iii) $[c\leq x_a,\ x_b\leq 0,\ 0\leq y_b\leq y_a\leq 1/c]$

Similar computation as in case (ii), we obtain that

$$\frac{\partial\Pi_2^*}{\partial x_2} = c(2/c-y_a-y_b)(c-2x_2)/3 = 0 \quad \text{and} \quad \frac{\partial\Pi_2^*}{\partial y_2} > 0 \qquad \text{for all } y_2<1/c.$$

That is, the optimal location of firm 2 is the center of the side (c/2,1/c).

Case (iv) $[0\leq x_a\leq c,\ x_b\leq 0,\ 0\leq y_b\leq 1/c,\ y_b\leq 0]$

Similar calculations as in case (i) yield $\partial\Pi_1^*/\partial x_1< 0$ for all $x_1>0$, and $\partial\Pi_1^*/\partial y_1<0$ for all $y_1>0$. Thus, the optimal location of firm 1 is the corner (0,0). ■

Since there are four corners and four sides in a rectangle, there exist eight possible locations for one of the duopolists in this game. Due to symmetry of rectangles, however, it suffices to analyze only two possible locations of firm 1: (0,0) and (0,1/2c), where $c\in(0,\infty)$. The best locational reply of firm 2 against the former location is analyzed in Lemma 2, and that against the latter location is examined in Lemma 3.

Lemma 2

If $\mathbb{C}$ is a rectangle and if one firm locates at a corner, then the other firm locates at a center of one side which is farthest from the corner.

Proof:

Let $(x_1,y_1)=(0,0)$ and $c\geq1$ without loss of generality. We will compute the equilibrium profits of firm 2 corresponding the four cases appeared in Lemma 1.

Case (i)

From Lemma 1(i), we have $(x_2^*,y_2^*)=(c,1/c)$ iff $c=1$. (If $c\neq1$, case(i) does not occur.) The corresponding profit is $\Pi_2^{*i}=1/2$.

Case (ii)

From Lemma 1(ii), we have $(x_2^*,y_2^*) = (c,1/2c)$. The corresponding profit is $\Pi_2^{*ii}=(12c+1/c^3)^2/288\geq169/288$ since $c\geq1$.

Case (iii)

From Lemma 1(iii), we have $(x_2^*,y_2^*)=(c/2,1/c)$ and $\Pi_2^{*iii}=(12/c+c^3)^2/288\leq$ 169/288 since $c\geq1$.

Case (iv)

Calculating $\partial\Pi_2^*/\partial x_2=0$ and $\partial\Pi_2^*/\partial y_2=0$, and substituting $x_1=y_1=0$, we get

$$2(x_a-x_2)(2+\alpha/x_a^2) + x_2 - 2x_a = 0, \tag{8}$$

$$(x_a/\alpha-2\alpha x_2)(2+\alpha/x_a^2) + 1/x_a + \alpha x_2 = 0. \tag{9}$$

Subtracting (9) from (8) multiplied by α, we have

$$(\alpha^2-1)(1/\alpha+1/x_a^2) = 0.$$

Thus, α should be unity in equilibrium of case (iv), and so $x_2^*=y_2^*$.

Moreover, using $\partial\Pi_1/\partial p_1=0$, $\partial\Pi_2/\partial p_2=0$ and the definition of x_a, we obtain

$$x_2^* = 2x_a^* - 1/x_a^*. \tag{10}$$

From (8) and (10), we finally get $x_2^*=y_2^*=(\sqrt{33}-3)/\sqrt{2\sqrt{33}+2}$ and $\Pi_2^{*iV}=$ $(207-33\sqrt{33})/32$, for $c\le\sqrt{2\sqrt{33}+2}/(\sqrt{33}-3)$. For $c>\sqrt{2\sqrt{33}+2}/(\sqrt{33}-3)$, we have $x_2^*=1/c$, which does not satisfy the first-order condition. This implies that Π_2^{*iv} for $c>\sqrt{2\sqrt{33}+2}/(\sqrt{33}-3)$ is smaller than Π_2^{*iv} for $c\le\sqrt{2\sqrt{33}+2}/(\sqrt{33}-3)$.

Comparing the above four values of Π_2^*, we conclude that Π_2^{*ii} is the largest. ■

Lemma 3

If $\mathbb{C}$ *is a rectangle with side lengths of* c *by* 1/c, *and if one firm locates at a center of one side, then the other firm locates at a center of one of three other sides. More precisely, given the firm 1's location of* (0,1/2c), *firm 2 locates at* (c/2,0) *or* (c/2,1/c) *if* $c \le c_o$, *and locates at* (c,1/2c) *if* $c\ge c_o$, *where* $c_o=\sqrt{3\sqrt{2}-\sqrt{13}}\simeq 0.798$.

Proof:

Let $(x_1,y_1)=(0,1/2c)$, where $c\in(0,\infty)$ without loss of generality. Similar to the previous lemma, we compute the equilibrium profits of firm 2 for the four cases.

Case (i)

If this is the case, firm 2 locates at (c,1/c) from Lemma 1(i). The condition of $y_b\ge 0$ is satisfied if $c\le(7/12)^{1/4}$, and the condition of $x_b\ge 0$ is satisfied if $c\ge(5/12)^{1/4}$. That is, $\Pi_2^{*i}=[c^2+1/4c^2+\sqrt{(c^2+1/4c^2)^2+16}]^3/512$ iff $c\in((5/12)^{1/4},(7/12)^{1/4})$. Otherwise, case (i) is not applied. This result is valid too when firm 2 locates at (c,0).

Case (ii)

From Lemma 1(ii), firm 2 locates at (c,1/2c), and earns the profit of $\Pi_2^{*ii}=c^2/2$. Of course, the conditions of $0\leq x_a, x_b\leq c$ are satisfied for all c>0 since $x_a=x_b=c/2$.

Case (iii)

From Lemma 1(iii), firm 2 locates at (c/2,1/c). Its profit is given by $\Pi_2^{*iii}=(c^3+5/c)^2/144$ iff $c\leq 1$. This result also applies when firm 2 locates at (c/2,0) too.

Case (iv)

This case does not occur because of the following reason. $x_b\leq c$ holds for $c\geq(7/12)^{1/4}$, and $y_b\leq 1/c$ holds for $c\leq(5/12)^{1/4}$. This means that both $x_b\leq c$ and $y_b\leq 1/c$ are not simultaneously satisfied when locations of the two firms are (0,1/2c) and (c,1/c).

Consider first the comparison between Π_2^{*i} and Π_2^{*ii} for all $c\in((5/12)^{1/4},(7/12)^{1/4})$ since Π_2^{*i} is defined only within this interval. As Π_2^{*i} and Π_2^{*ii} are increasing for all $c\in((5/12)^{1/4},(7/12)^{1/4})$, and as Π_2^{*i} at $c=(7/12)^{1/4}$ is strictly less than Π_2^{*ii} at $c=(5/12)^{1/4}$, we conclude $\Pi_2^{*ii}>\Pi_2^{*i}$ for all $c\in((5/12)^{1/4},(7/12)^{1/4})$. That is, we can drop case (i) as a candidate for firm 2's best location reply, which gives a proof of the former part of Lemma 3.

The latter part of Lemma 3 can be shown by comparison Π_2^{*ii} with Π_2^{*iii} for $c\leq 1$. Taking the root and subtracting, we have $\sqrt{\Pi_2^{*ii}}-\sqrt{\Pi_2^{*iii}}=(c^4-6\sqrt{2}c^2+5)/(12c)$, which is zero at $c_o\simeq 0.798(<1)$. Consequently, we obtain $\Pi_2^{*ii}\lesseqgtr\Pi_2^{*iii}$ for $c\lesseqgtr c_o$, which gives a proof of the latter part. ■

Consider the case that c is within the interval of $((1/3)^{1/4},c_o)$. Given the firm 1's location of (0,1/2c), firm 2 chooses to locate at (c/2,0) or (c/2,1/c) rather than (c,1/2c) from Lemma 3. The distance between the two firms in the former two cases ($\sqrt{1+c^4}/2c$) is smaller than that in the latter case (c/2), and the firm 2's share in the former two cases ($(5+c^4)/12$) is smaller than that in the latter case (1/2). Since such a location would intensify price competition due to closeness to the other firm, and would be disadvantageous because of the smaller equilibrium share, it seems irrational in any situation under one-dimensional uniform distributions of consumers.

However, such behavior takes place in certain cases in two-demensional (or one-dimensional non-uniform) distributions. The intuitive reason is understood if we compare the number of marginal consumers in the above example. A simple calculation yields that the number of marginal consumers in the former two cases is less than that in the latter case, i.e., the ratio of the former to the latter is $c\sqrt{1 + c^4}$, which is smaller than unity for all $c \in ((1/3)^{1/4},c_o)$. Apparently, when the number of marginal consumers becomes smaller, firms do not lower prices to acquire additional marginal consumers. They would rather raise prices to increase the revenue from non-marginal consumers. We thus find the possibility that price competition can be relaxed by locating closer if the number of marginal consumers is reduced.

With regard to the exclusion of interior location, we establish the following proposition as in the case of ellipse.

Proposition 3

If $\mathbb{C}$ is a rectangle, neither firm locates at the interior region in Nash equilibrium.

Proof:

From Lemma 1, one of the two firms, say firm 1, locates either at a corner or a center of one side. If it locates at a corner, then firm 2 locates at a center of one side from Lemma 2. If firm 1 locates at a center of one side, then firm 2 locates at a center of another side from the former part of Lemma 3. Thus, neither firm locates inside the rectangle. ■

It should be noted from Propositions 2 and 3 that the exclusion of interior location holds under both rectangular and ellipse distributions of consumers. This may imply that the non-interior equilibrium location is a necessary consequence of dominance of price competition over location one.

The next proposition fully characterizes the Nash location equilibrium in the rectangular case.

Proposition 4

If $\mathbb{C}$ is rectangle with side lengths of c *by* 1/c, *then the two-stage Nash equilibrium locations are given by*

$$
\begin{aligned}
(x_1^*, y_1^*, x_2^*, y_2^*) &= (c/2, 0, c/2, 1/c) && \text{for } c<c_o, \\
&= (c/2, 0, c/2, 1/c) \text{ or } (0, 1/2c, c, 1/2c) && \text{for } c_o \le c \le 1/c_o, \\
&= (0, 1/2c, c, 1/2c), && \text{for } c>1/c_o.
\end{aligned}
$$

Proof:

From Lemmas 1 and 2, we know that one firm should locate at a center of one side.

(a) $c<c_o$

Suppose firm 1 locates at (0,1/2c). Then, firm 2 will locate at (c/2,1/c) from Lemma 3. Conversely, however, (0,1/2c) of firm 1's location is not the best reply against (c/2,1/c) of firm 2's location due to the following reason. If we rotate the rectangle by $\pi/2$, the side lengths become 1/c by c, and c_o is replaced with $1/c_o$, which is of course greater than c_o. Hence, the best reply of firm 1 is not (0,1/2c), but (c/2,0) from Lemma 3. Conversely, if firm 1 locates at (c/2,0), then firm 2 locates at (c/2,1/c) due to symmetry.

(b) $c_o \le c \le 1$

If firm 1 locates at (0,1/2c), then firm 2 locates at (c,1/2c) from Lemma 3. By symmetry, the reverse is also true. Similarly, if firm 1 locates at (c/2,0), then firm 2 locates at (c/2,1/c), and the reverse is true too.

(c) $c>1$

Because of the symmetric structure of the model, the symmetric results should be obtained for $c > 1$. ∎

In brief, when the rectangle is close or equal to a square, there exist two location equilibria; and when the rectangle is long and slender, there exists a unique location equilibrium. Firms succeed to relax price competition by locating apart each other in the latter case, but not necessarily in the former case. Nonetheless, we will show in the next section that if firms enter the market sequentially rather than simultaneously, they always succeed to locate apart.

6. SEQUENTIAL ENTRY IN THE RECTANGULAR CASE

This section continues to assume rectangular distributions of consumers, but the model is modified to <u>sequential choice of location</u>, i.e., the first stage is a Stackelberg leader-follower location game while the second stage is a Nash price subgame.

Mathematically, firm 1 (the leader) maximizes its profit of $\Pi_1^*(x_1,y_1,x_2,y_2)$ with respect to x_1 and y_1, replacing x_2 and y_2 with firm 2's (the follower's) reaction functions $x_2=R_x(x_1,y_1)$ and $y_2=R_y(x_1,y_1)$, which are derived from the maximization of $\Pi_2^*(x_1,y_1,x_2,y_2)$ with respect to x_2 and y_2. So as to obtain the Stackelberg-location and Nash-price equilibrium, let us begin with some lemmas as usual.

Lemma 4

If $\mathbb{C}$ *is a rectangle with side lengths of* c *by* 1/c, *and if* $x_1 \le x_2$ *and* $y_1 \le y_2$, *then*

$$\frac{\partial \Pi_1^{*ii}}{\partial x_2} > 0, \qquad \frac{\partial \Pi_1^{*ii}}{\partial y_2} \gtreqless 0 \quad \text{for } y_2 \gtreqless \frac{1}{2c}, \tag{11}$$

$$\frac{\partial \Pi_1^{*iv}}{\partial x_2} > 0, \qquad \frac{\partial \Pi_1^{*iv}}{\partial y_2} > 0, \tag{12}$$

where the Roman numerals at the superscripts correspond to those in the proof of Lemma 1.

Proof:

Without losing generality, assume $c \ge 1$.

Case (ii)

Differentiating Π_1^{*ii} with respect to x_2, we have

$$\frac{\partial \Pi_1^{*ii}}{\partial x_2} = \frac{x_a+x_b}{6c}\left[3x_2 - x_1 + \frac{y_2-y_1}{c(x_2-x_1)} + 2c - \frac{y_2-y_1}{x_2-x_1}(y_1+y_2)\right]$$

$$\geq \frac{x_a+x_b}{6c}\left[3x_2 - x_1 + \frac{y_2-y_1}{c(x_2-x_1)} + c(2-cy_1-cy_2)\right]$$

since $(y_2-y_1)/(x_2-x_1)\leq c^2$. As $y_1,y_2\leq 1/c$, the last term in the brackets is nonnegative, and hence $\partial\Pi_1^{*ii}/\partial x_2\geq 0$. If the equality were to hold, then $x_1=x_2=0$, which does not occur in case (ii). Thus, we conclude $\partial\Pi_1^{*ii}/\partial x_2>0$.

Next, differentiating Π_1^{*ii} with respect to y_2, we obtain

$$\frac{\partial \Pi_1^{*ii}}{\partial y_2} = \frac{x_a+x_b}{3c}(2y_2-1/c).$$

This means that $\partial\Pi_1^{*ii}/\partial y_2 \gtreqless 0$ for $y_2 \gtreqless 1/2c$.

Case (iii)

The same operation as above yields $\partial\Pi_1^{*iii}/\partial x_2 \gtreqless 0$ for $x_2 \gtreqless c/2$, and $\partial\Pi_1^{*iii}/\partial y_2>0$.

Case (iv)

Differentiating Π_1^{*iv} with respect to x_2 and manipulating, we have

$$\frac{\partial \Pi_1^{*iv}}{\partial x_2} = \frac{(x_2-x_1)^2x_a^2[3x_2+2(1-x_ay_a)/y_a]}{2(y_2-y_1)[2(x_2-x_1)+(y_2-y_1)/x_a^2]} > 0$$

since $x_a,y_a\leq 1$ and $x_2>x_1$. [$x_1=x_2$ does not occur in case(iv)].

$\partial\Pi_1^{*iv}/\partial y_2>0$ can be shown by similar calculation. ■

Lemma 5

If $\mathbb{C}$ is a rectangle, then the first entrant never chooses a corner location.

Proof:

Assume $c\geq 1$ without losing generality. If firm 1 chooses to locate at

(0,0), then firm 2 chooses (c,1/2c) from Lemma 2, and $\Pi_1^*=(3c-1/4c^3)^2/18$. On the other hand, if firm 1 locates at (0,1/2c), then firm 2 chooses (c,1/2c) form Lemma 3, and so $\Pi_1^*=c^2/2$. Since the latter profit is greater than the former, Lemma 5 is shown. ■

Lemma 6

If $\mathbb{C}$ is a rectangle, then the second entrant chooses to locate at a center of a side.

Proof:

As firm 1 (the leader) does not locate at a corner from Lemma 5, an interior location of firm 2 [case (iv)] is excluded. We will prove here that a corner location of firm 2 [case (i)] is also excluded.

In case (i), firm 2 (the follower) necessarily locates at a corner O(0,0).[2] Then, the maximum profit that firm 1 could obtain is $(207-33\sqrt{33})/32$ from Lemma 2(iv) (with permutation of firm indices). On the other hand, if firm 1 located at a center of a shorter side, then firm 2 would locate at a center of another shorter side, and hence firm 1 earns the profit that is $\max\{c^2/2, 1/2c^2\}$ from Lemma 3(ii). Comparing these profits, it might be possible for firm 2 to locate at a corner when $c\in(c_1^{-1}, c_1)$, where $c_1\equiv\sqrt{(207-33\sqrt{33})/16}\simeq1.05$.

Now, if firm 2 locates at O in Figure 2, then

$$y_a = \frac{1}{8x_1}[x_1^2 + y_1^2 + \sqrt{(x_1^2+y_1^2)^2+32x_1y_1}] \le \frac{1}{c}$$

is necessary to hold so that case (iv) applies. That is, the location of firm 1 is in $\mathbb{C}_2$, where

$$\mathbb{C}_2 = \{(x_1,y_1)|\ (x_1+c)^2+(y_1-2/c)^2\le c^2+4/c^2,\ c/2\le x_1\le c,\ 1/2c\le y_1\le 1/c,\ y_1\le x_1\}.$$

The last inequality does not lose generality due to the symmetric nature of rectangles. In other words, the case of $y_1>x_1$ can be similarly shown by interchanging c with c^{-1} in the proof below. Thus, confining firm 1's location to $\mathbb{C}_2$ (which is the shaded area in Figure 2), and limiting the range of one side to (c_1^{-1},c_1), we will prove that for any firm 1's location within the shaded area, firm 2 is sure to locate at A(0,1/2c), but not at O(0,0).

[Figure 2 about here]

(1) For $c\geq 1$

As $x_1>1$ does not satisfy the first inequality in $\mathbb{C}_2$ for $c\geq 1$, we can limit the range of x_1 to [c/2,1]. From (11) in Lemma 4, firm 2's profit at A is smaller if firm 1 is at $R(x_1,1/2c)$ rather than at $P(x_1,y_1)$, i.e.,

$$\Pi_2^{*ii}(x_1,y_1,0,1/2c) \geq \Pi_2^{*ii}(x_1,1/2c,0,1/2c) = \frac{x_1}{18c}(x_1+2c)^2. \qquad (13)$$

On the other hand, from (12) in Lemma 4, firm 2's profit at O is larger if firm 1 is at $Q(x_1,x_1)$ rather than at $P(x_1,y_1)$, i.e.,

$$\Pi_2^{*iv}(x_1,y_1,0,0) \leq \Pi_2^{*iv}(x_1,x_1 0,0) = \frac{x_1}{128}(x_1+\sqrt{x_1^2+8})^3. \qquad (14)$$

By comparing (13) with (14), we can say that A is preferred to O by firm 2, [i.e., $\Pi_2^{*ii}(x_1,y_1,0,1/2c)> \Pi_2^{*iv}(x_1,y_1,0,0)$] if $\Psi(x_1)>0$ for all $x_1\in\mathbb{C}_2$ and $c\in[1,c_1)$, where

$$\Psi(x_1) \equiv x_1 + 2c - 3\sqrt{c}(x_1+3)^{3/2}/8.$$

Since a simple calculation yields that $\Psi''(x_1)<0$, $\Psi'(1/2)<0$ and $\Psi(1)>0$, we have $\Psi(x_1)>0$, which means that A is preferred to O by firm 2.

(2) For $c<1$

We divide the range of x_1 into the following two intervals.

[2a] $1/2c \le x_1 \le 1/2c+1/4$

As $\Psi''(x_1)<0$, $\Psi'(1/2)<0$ and $\Psi(1/2c+1/4)>0$ $\forall c \in (c_1^{-1},1)$, we have $\Psi(x_1)>0$, implying that A is preferred to O by firm 2 for $x_1 \in [1/2c, 1/2c+1/4]$.

[2b] $1/2c+1/4 \le x_1 \le c$

Consider the line $y=x-1/4$, which is shown to be outside $\mathbb{C}_2$ for $x \ge 1/2c+1/4$. Therefore, again from (11) in Lemma 4, firm 2's profit at A is smaller if firm 1 is at $S(x_1,x_1-1/4)$ rather than at $P(x_1,y_1)$, i.e.,

$$\Pi_2^{*ii}(x_1,y_1,0,1/2c) \ge \Pi_2^{*ii}(x_1,x_1-1/4,0,1/2c)$$
$$= \frac{1}{18cx_1}[2x_1^2+(2c-1/2-1/c)x_1+(1/16+1/4c+1/4c^2)]^2. \quad (15)$$

By comparing (15) with (14), we can say that A is preferred to O by firm 2, [i.e., $\Pi_2^{*ii}(x_1,y_1,0,1/2c) > \Pi_2^{*iv}(x_1,y_1,0,0)$] if $\phi(x_1)>0$ for all $x_1 \in \mathbb{C}_2$ and $c \in (c_1^{-1},1)$, where

$$\phi(x_1) \equiv 2x_1^2+(2c-1/2-1/c)x_1+1/16+1/4c+1/4c^2-3\sqrt{c}x_1(x_1+3)^{3/2}/8.$$

After some computations, we get $\phi'''(x_1)<0$, $\phi''(1)>0$, $\phi'(1)<0$ and $\phi(1)>0$ for all $c \in (c_1^{-1},1)$. Hence, $\phi(x_1)>0$, which means that A is preferred to O by firm 2 for $x_1 \in [1/2c+1/4,c]$. ■

Proposition 5

If $\mathbb{C}$ is a rectangle, then each firm locates at a center of a shorter side in Stackelberg equilibrium.

Proof:

According to Lemma 6, firm 2 (the follower) necessarily chooses to locate at a center of a side. Knowing this, it should be optimal for firm 1 (the leader) to locate at a center of a shorter side to earn the profit of $c^2/2$, where $c \ge 1$. ■

Note that if firm 1 locates at a center of a longer side, its profit is $1/2c^2$, which is equal to or smaller than $c^2/2$. Thus, with the exception that $\mathbb{C}$ is a square, we observe that the Stackelberg location equilibrium is unique whereas the Nash location equilibrium is not when $c \in [c_o, 1/c_o]$. We also observe that the firm's profit in Stackelberg location equilibrium is greater than or equal to that in Nash location equilibrium. Consequently, we conclude that <u>firms may be worse off if they choose to locate simultaneously rather than sequentially</u>.

7. WELFARE COMPARISON IN THE RECTANGULAR CASE

Let us finally conduct a welfare comparison of the above subgame perfect equilibrium locations and the social optimal locations. In the absence of production costs, we evaluate the welfare loss solely by the sum of the total transportation costs incurred by consumers who are uniformly distributed on a rectangle of c by 1/c.

The welfare loss, defined by the sum of quadratic distance costs between consumers and their nearest firms,[3] is given by

$$\ell \equiv \sum_{i=1}^{2} \iint_{\mathbb{C}_i} (x_i-x)^2+(y_i-y)^2 dxdy,$$

where $\mathbb{C}_i=\{(x,y)\in[0,c]\times[0,1/c] \mid (x_i-x)^2+(y_i-y)^2\leq(x_j-x)^2+(y_j-y)^2,\ i\neq j\}$. As obtained earlier, Nash and Stackelberg equilibrium locations are given by $[(x_1,y_1),(x_2,y_2)]=[(0,1/2c),(c,1/2c)]$ or $[(c/2,0),(c/2,c)]$. The welfare loss in either pair of locations is the same and computed as $\ell=(c^2+1/c^2)/12$.

On the other hand, the social optimum locations are calculated by differentiating ℓ with respect to x_i, and y_i respectively, and are given by $[(x_1,y_1),(x_2,y_2)]=[(c/4,1/2c),(3c/4,1/2c)]$ for $c\geq1$, and $[(c/2,1/4c),(c/2,3/4c)]$ for $c\leq1$. The welfare in either pair is identical and given by $\ell=(c^2+1/c^2)/96$. Comparing the two values of ℓ, we establish the following proposition.

Proposition 6

The welfare loss of Nash or Stackelberg equilibrium locations is eight times as large as that of the social optimum locations.

It may be worth noting that the former value (1/12) for two-dimensional case is four times as large as the latter value (1/48) for one-dimensional case where consumers are uniformly distributed over [0,1], and firms locate in [0,1]. This result implies that one-dimensional modeling leads to substantial underestimation of the welfare loss if real urban space is two-dimensional.

Finally, in either dimensional case, we confirm that price competition in duopoly is so fierce that firms have to locate far apart in equilibrium, which results in greater loss of welfare than the loss in social optimum.

8. CONCLUSIONS

Throughout this paper, we have assumed that there are two firms competing in two-dimensional location first and then in mill price, and that the transportation cost is a quadratic function of distance. We showed first that a unique Nash price equilibrium exists on a two-dimensional space for any pair of firm locations if the consumer distribution is uniform and is a convex set (Corollary 1).

Second, we proved that if the convex set is given by any ellipse or rectangle, then neither firm locates at the interior region in Nash two-stage (location then price) equilibrium (Propositions 2 and 3). This means that as the price competition keeps their locations apart whereas the location competition brings them near, the former competition dominates the latter. This may explicate recent tendency in suburbanization of retail firms.

Third, we showed in the rectangular case that each firm locates at a center of one side opposite to each other in Nash location equilibrium, and that multiple location equilibria exist when the rectangle is close to a square while a unique location equilibrium exists when the rectangle is long and slender (Proposition 4). It should be mentioned that although the price competition is so fierce that firms do not locate in the interior region, they do not locate to maximize the distance between the two. A similar result is obtained by Neven and Thisse (1990) although they consider horizontal and vertical differentiation instead of two-horizontal differentiation which is done here.

Fourth, we identified three factors in location choice: (a) farther location to relax (Bertrand) price competition; (b) closer location to acquire customers; and (c) a certain location which reduces the number of marginal customers. Factors (a) and (b) are too obvious to explain. Factor (c), on the other hand, should also be taken into account in two-dimensional models. This is because the number of marginal consumers, which is related to intensity of the price competition, varies according to their locations on a two-dimensional case. To put it plainly, firms can raise prices and hence profits when there are few marginal customers that firms want to acquire. Notice that factor (c) never emerges under one-dimensional uniform distribution of consumers.

Fifth, we modified the game of Nash location and Nash price to that of Stackelberg location and Nash price in Section 6. We then obtained a unique

Stackelberg location equilibrium for any rectangular (except square) distribution of consumers (Proposition 5). Comparing it with the Nash one, we showed that the sequential choice of location is desirable for the duopolistic firms than the simultaneous choice of location.

Sixth, we conducted a comparison of welfare loss defined by the sum of the transportation costs. We showed that the welfare loss of Nash equilibrium locations is equal to that of Stackelberg equilibrium locations, and that they are eight times as large as that of the social optimum locations (Proposition 6). Furthermore, we also demonstrated that the difference in the two-dimensional case is as large again as that in the one-dimensional case. These results would stress the need of a certain government intervention.

Finally, it should be noted that the above conclusions may not hold if firms are allowed to locate outside the market As demonstrated by Tabuchi and Thisse (1989) a completely different outcome may emerge in such a situation. Identical firms may locate asymmetrically in equilibrium.

FOOTNOTES

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[1] However, Tabuchi and Thisse (1989) showed that if f(x) is symmetric and sgn(f'(x))=sgn(f"'(x)), then there exists a unique Nash location

equilibrium.

[2] To simplify mathematical computations, firm 2's location is set to (0,0) instead of (c,1/c).

[3] Since both Nash and Stackelberg equilibrium locations are found to be symmetric, consumers necessarily go to their nearest firm.

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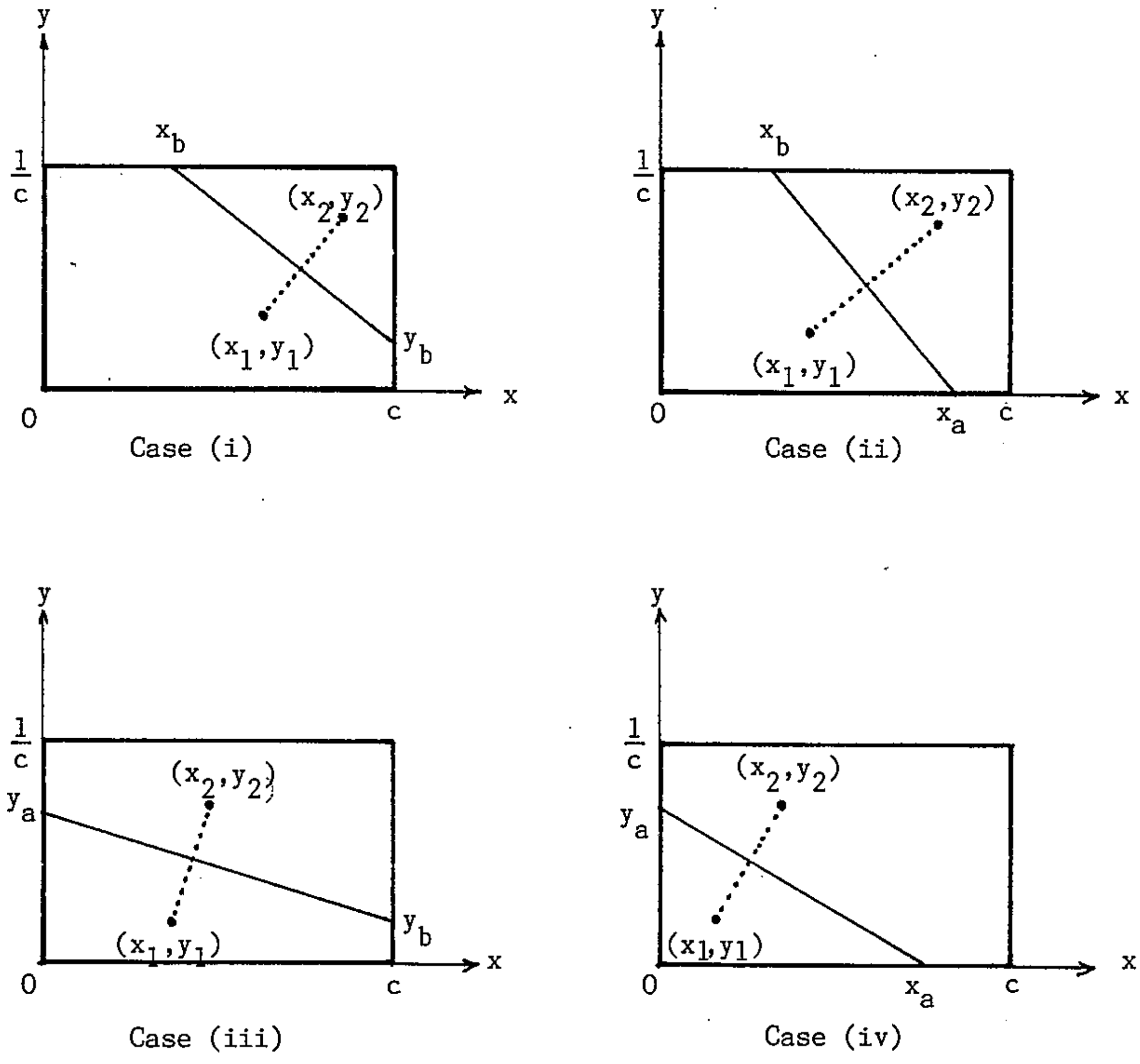
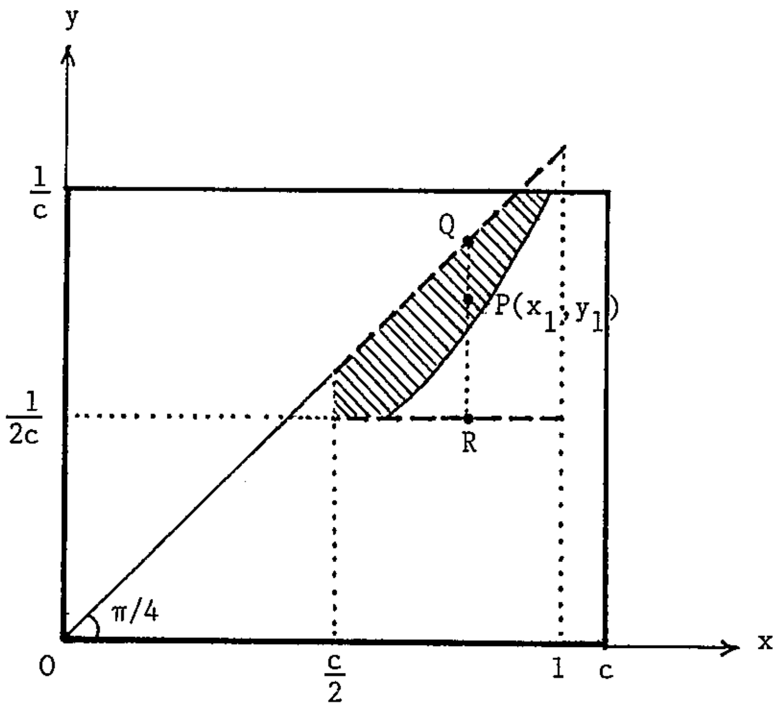
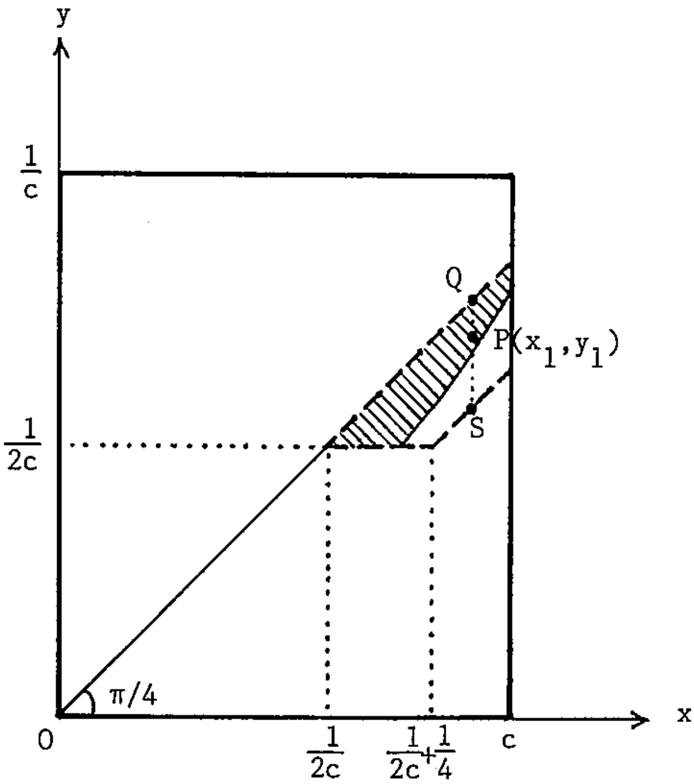


Figure 1 Four cases of market division under a rectangular distribution of consumers



(1) c≧1



(2) c<1

Figure 2 Possible location of firm 1 (the leader) if firm 2 (the follower) were to locate at the corner 0