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NICE DEMAND AND CONCAVIFIABLE SMOOTH PREFERENCES - A Further Implication

by
Sho-Ichiro Kusumoto

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THE UNIVERSITY OF TSUKUBA

Institute of Socio-Economic Planning
Sakura, Ibaraki 305, Japan
Telephone : 0298-53-5182

Cable. UNTUKU TSUCHIURA
Telex : 3652580 UNTUKU J

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ABSTRACT

Given strongly convex preferences, the differentiability of demand and the concavifiability of smooth preferences will allow us to select a utility function which is determinate, generating inferior demand as well as normal one. Taking the new approach we shall establish three fundamental theorems justifying the hypothetical argument which Pareto had made and since was controversial from an ordinalist's view point.(Samuelson[1974]).

1. Introduction

As a main result, this paper contains that, at points where the income derivative of the marginal utility of money vanishes, the Parato-Edgeworth-Kannai*1 complements of a normal (resp. inferior) commodity are normal (resp. inferior), and, conversely for the substitutes.

2. A Framework for Local Demand Analysis

We shall begin with the postulates and their local properties that our analysis will be based upon.. In so doing we shall follow Debreu[1972], Kannai[1986] and Hurwicz, Jordan and Kannai[1987].

2.0 Smooth Preferences

A basic problem in the consumer theory is the existence of a twice continuously differentiable utility function u from the consumption space Y to the set of real numbers R, such that its derivative Du is everywhere in Y a strictly positive multiple of a vector field g, that is, for a function λ from Y to the strictly positive number, $Du=\lambda g$. Assume that Y be the interior of the positive cone of R^n. Debreu[1972] studied three ways of approaching the questions of smooth preferences and proved the following three postulates to be equivalent. Assume them.

(i) A monotone, continuous, complete and transitive binary relation R^* on Y of class C^2, (ii) a vector field $g(x)>0$ from Y to the unit sphere of R^n of C^1, satisfying the (local) intergrability conditions on Y, and (iii) a real valued function $u(x)$ on Y of C^2 satisfying $Du(x)>0$ for every x in Y.

Assume in adition R^* is strongly convex on an open convex subset X of Y.

2.1 A Continuous Demand Function

Let $\bar{p}$ be an n dimensional price vector and $\bar{m}$ be a positive real number, such that, with respect to R*, $\bar{x}$ is a unique, maximal point in the budget set $\{y \in X; \bar{p}y \leq \bar{m}\}$. Then, there are two open neighborhoods, N of $\bar{p}$ and M of $\bar{m}$, such that, for each (p,m) in NxM, single-valued function $d_i(p,m), i=1,..,n$, are well-defined and continuous in NxM. Then, $\bar{x}=d(\bar{p},\bar{m})$, $\bar{p}\bar{x}=\bar{m}$. Likewise if $x=d(p,m)$, then $px=m$.

2.2 Local Properties

In a neighborhood of a certain point $\bar{x}$, let us take a new coordinates $\{\xi_1,..,\xi_{n-1}\}$, which are paralell to the hyperplane $H(\bar{x})=\{y \in X; \bar{p}y=\bar{p}\bar{x}\}$ and tangent at $\bar{x}$ to the indifference hyperface $I(\bar{x})=\{y \in X; yR^*\bar{x}, \bar{x}R^*y\}$. Now read $\bar{x}$ the new origin, i.e. $\bar{\xi}=0$. Then, X may read Ξ, $H(\bar{x})$*2 reads $\xi_n=0$ and $g_i(\bar{x}), i=1,..,n$ may read, respectively, $\rho_i(\xi), i=1,..,n-1$, and $\rho_n(\xi)=1$. Hessian $DDu(\bar{x})$ reads $DD\phi(\bar{\xi})$ and, for an orthogonal transformation T,*3 $DD\phi(\bar{\xi})=T^t DDu(\bar{x})T$. We denote $DD\phi(\bar{\xi})$ by $A_n(\bar{\xi})$. Note that $\bar{p}\bar{\xi}=\bar{p}\bar{x}=\bar{m}$ and the scalar λ is invariant.
The n-1 by n-1 principal submatrix $A_{n-1}(.)$ is restricted to the subspace orthogonal to $(p_1,..,p_n)$. Let $\alpha_i(\bar{\xi}), i=1,..,n-1$, denote the eigen values of $A_{n-1}(\bar{\xi})$. Then, by a wellknown fact, A_{n-1} can be diagonalized by $\alpha_i, i=1,..,n-1$ so that (by letting two bars indicate a matrix)

(1) $$A_{n-1}(\bar{\xi})=\|\alpha_i(\bar{\xi})\delta_{ij}\|_{i,j=1}^{n-1}$$

Denote by $S(\bar{\xi})$ the n-1 by n-1 principal submatrix of Jacobian $D\rho(\bar{\xi})$. Then, $S(\bar{\xi})=A_{n-1}(\bar{\xi})$ and

(2) $$A_n(\bar{\xi})= \lambda(\bar{\xi}) \left\|\begin{matrix} S(\bar{\xi}) & \beta(\bar{\xi})^t \\ \beta(\bar{\xi}) & \Lambda(\xi) \end{matrix}\right\|$$

where $\beta=\{\beta_1,..,\beta_{n-1}\}$, $\beta_i(\bar{\xi})=\beta_i=\partial\rho_i(\bar{\xi})/\partial\xi_n=\partial\phi_n(\bar{\xi})/\partial\xi_i=\partial\lambda(\bar{\xi})/\partial\xi_i, i=1,,$

,n-1, and $\Lambda(\bar{\xi})=\Lambda=\partial\lambda(\bar{\xi})/\partial\xi_n/\lambda$. Then, the determinant of Hessian $A_n(\bar{\xi})$ is given if each α_i, i=1,.,n-1 does not vanish at $\bar{\xi}$, as

(3) $$|A_n(\bar{\xi})|=\lambda(\bar{\xi})^n\Pi_{i=1}^{n-1}\alpha_i(\bar{\xi})\{\Lambda(\bar{\xi})-\Sigma_{i=1}^{n-1}\beta_i^2(\bar{\xi})/\alpha_i(\bar{\xi})\}$$

The non-vanishing Gaussian curvature*4, formulated in Debreu[1972 612] is

(4) $$|A_{n-1}(\bar{\xi})|=|S(\bar{\xi})|=\Pi_{i=1}^{n-1}\alpha_i(\bar{\xi})$$

and the one-point conditions, examined by Kannai[1977]), is given simply as, for ξ near $\bar{\xi}$,

(5) $$\Lambda(\xi)-\Sigma_{i=1}^{n-1}\beta_i^2(\xi)/\alpha_i(\xi)\leqq 0$$

Wherever $\alpha_i(\bar{\xi})=0$ for some i, we can take and use instead limsup $\beta_i^2(\xi^\nu)/\alpha_i(\xi^\nu)$ by taking $\xi^\nu\to\bar{\xi}$, for which $\alpha_i(\xi^\nu)\neq 0$, i=1,.,n-1, along a nearby indifference hypersurface. (See Smale[1974] for this.)

We shall say;

<u>$\bar{\xi}$ is a differentiability point if (and only if)</u> $\alpha_i(\bar{\xi})\neq 0$ i=1,.,n-1,

Irrespective of whether or not $\bar{\xi}$ is a differentiability point, we shall say;

<u>A strictly quasi-concave utility function</u> ϕ <u>on</u> Ξ, <u>representing</u> R*, <u>is concavifiable near</u> $\bar{\xi}$, <u>if (and only if) the Fenchel-Kannai condition</u> (5) <u>holds</u>.

Thus, if $\alpha_i(\xi)<0$ everywhere in Ξ, R* is representable by means of a twice continuously differentiable, concave function, in a compact set of Ξ. Furthermore, if ξ is a differentiability point, condition (5) is equivalent to that $\lambda(\xi(q,m))$ is non-increasing in m, i.e.

(6) $$\partial\lambda(\xi)/\partial m/\lambda(\xi)=\Lambda(\xi)-\Sigma_{i=1}^{n-1}\beta_i^2(\xi)/\alpha_i(\xi)\leqq 0.$$

The income Lipschitz condition,*5 introduced by Uzawa[1959] for obtaining both continuity of demand function in m and the uniqueness of R*, plays here another role. In the system $\{\xi_1,..,\xi_n\}$, it is equal to saying that, for some $\gamma>0$, for all ξ near $\bar{\xi}$, and for all i's for which $\alpha_i\neq 0$, (by letting one bar show an absolute value)

(7) $|\beta_i(\xi)/\alpha_i(\xi)|\leqq\gamma$

As has shown by Kannai[1986], (7) implies (5), Thus,

<u>A strictly quasi-concave utility function</u> ϕ <u>is concavifiable near</u> $\bar{\xi}$, <u>if demand functions</u> f_i, $i=1,..,n$, <u>satisfy the Uzawa-Kannai Lipschitz condition</u> (7),

where $\xi_i=f_i(q,m)$, $i=1,..,n$.*6

2.3 Least Concave Utility Function

Denote by U the set of concave utility functions on Ξ, each representing R* equivalently. U is weakly ordered by the relation that $\Phi\varepsilon U$ is more concave than $\phi\varepsilon U$, if there is a strictly increasing, concave function μ, for which $\Phi(\xi)=\mu\{\phi(\xi)\}$. Debreu[1976] has shown that if $U\neq\emptyset$, U has a least element. Thus, the least concave utility function is unique up to a positive linear transformation, hence <u>determinate</u> as a <u>cardinal</u> utility; see Lange[1936] for this definition.

Denote by $\Gamma(\xi)$ the <u>L.H.S.</u> of (5). Then, ϕ is concavifiable near $\bar{\xi}$ if and only if $\Gamma(\xi)\leqq 0$ near $\bar{\xi}$. Take a point ξ^0 in an open convex subset containing $\bar{\xi}$ in such a way that $\Gamma(\xi^0)=\sup\Gamma(\xi^\nu)$ on a sequence, $\{\xi^\nu\}_{\nu=1}^{\infty}$, $\xi^\nu\varepsilon I(\xi^0)$, $\xi^\nu\to\xi^0$. Then, by Kannai[1977][1986], we shall say;

<u>A utility function</u> ϕ <u>is chosen to be least concave, if (and only if), for some</u> ξ^0 <u>near</u> $\bar{\xi}$, $\Gamma(\xi^0)=0$.

2.4 Price and Income Derivatives

In this and following sections, we shall take an advantage of treating the derivatives rather classically in the p,x coordinates than in the q,ξ coordinates. Recall $x_i=\sum_{j=1}^{n}t_{ij}\xi_j$ and $q_j=\sum_{i=1}^{n}t_{ij}p_i$. Without ambiguity, let

$$\bar{A}_n=\bar{A}_n(x)=\left\|\begin{matrix} u_{ij} & p_i \\ p_j & 0 \end{matrix}\right\| \quad i,j=1,..,n;\ u_{ij}=u_{ij}(x),\ x=d(p,m),\ px=m$$

Let $\bar{A}_n(i_1,.,i_k;j_1,.,j_k)$ denotes a submatrix obtained by deleting rows $i_1,.,i_k$ and columns $j_1,.,j_k$ from $\bar{A}_n$, where $1\leqq i_1\leqq,.,\leqq i_k\leqq n+1$; $1\leqq j_1\leqq,.,\leqq j_k\leqq n+1$. Then, the own-price derivative $\partial d_n(\bar{p},\bar{m})/\partial p_n$ is decomposed into two terms

(8) $\partial d_n(\bar{p},\bar{m})/\partial p_n=\lambda(\bar{x})|\bar{A}_{n-1}|/|\bar{A}_n|+d_n(\bar{p},\bar{m})|\bar{A}_n(n+1;n)|/|\bar{A}_n|$

provided that $|\bar{A}_n|\neq 0$ at $\bar{x}=d(\bar{p},\bar{m})$. We say the first term of the R.H.S. the substitution (Slutsky)*7 term and the second the income term. It is easy to check that $\bar{x}$ is a differentiability point if and only if $|\bar{A}_n(\bar{x})|=|A_{n-1}(\xi)|\neq 0$.*8 At $\bar{x}$, where $|\bar{A}_n(\bar{x})|=0$, we take the limsup of own price derivatives over a convergent sequence of the own price derivatives not vanishing for nearby regular prices where $u(x)=u(\bar{x})$ and $|\bar{A}_n(x)|\neq 0$. Hurwicz et al[1987] found out that $\{-|\bar{A}_n(n+1;n)|/|\bar{A}_n|\}^2$ is bounded above by a positive scalar multiple of $\{-|\bar{A}_{n-1}|/|\bar{A}_n|\}$, if utility is concave. This implies that the negative substitution term could not be dominated by an (infinite) asymptotic income term.

Slutsky decomposition for $\lambda(\bar{x})$ is given likewise;

(9) $\quad -\partial\lambda(\bar{x})/\partial p_n=\lambda(\bar{x})|\bar{A}_n(n;n+1)|/|\bar{A}_n|+d_n(\bar{p},\bar{m})|\bar{A}_n(n+1;n+1)|/|\bar{A}_n|$

where $\bar{A}_n(n+1;n+1)=A_n$ by definition and the second term is a scalar multiple of income derivative of λ; $\partial\lambda/\partial m$, which is equal to (6) when $\xi=\bar{\xi}$.

To a Marshallian consumer; a maximizer of $\{u(x)-\lambda px\}$ with respect to x, given (p,λ), demand function d_i, $i=1,.,n$, is a function of (p,λ), characterized by $u_i(d(p,\lambda))-\lambda p_i=0$. Then, the price derivative of $d_n(\bar{p},\bar{\lambda})$, where $\bar{m}=\Sigma_{i=1}^{n}\bar{p}_i d_i(\bar{p},\bar{\lambda})$; see Hicks[1956 12-15], can be decomposed into two terms;

(10) $\quad \partial d_n(\bar{p},\bar{\lambda})/\partial p_n=\bar{\lambda}|\bar{A}_{n-1}|/|\bar{A}_n|+d_n(\bar{p},\bar{\lambda})|\bar{A}_n(n+1;n)|/|\bar{A}_n|$

Thus, the same results may apply to (10) as well.

3. Theorems

In the framework above described, we shall consider the price-income behaviour of a consumer in a neighborhood of a certain demand point $\bar{x}=d(\bar{p},\bar{m})$, at which the (asymptotic) income derivative of marginal utility of money vanishes, hence the utility function can be chosen to be least concave. See (5) and (6) above.

3.1 Theorems 1 and 2

Saying that $\bar{x}$ is a differentiability point is equivalent to saying that $|Dd^{-1}(\bar{x})|\neq 0$ for the inverse demand function $d_i^{-1}, i=1,..n$, as remarked in Debreu[1972 612]. Thus, whenever restricted to an open, convex subset of $d^{-1}(X)$, here equated to N below, the single-valued continuous demand functions defined in subsection 2.1 $d_i(p,\bar{m}), i=1,..,n$, are equal to the uniquely determined functions $h_i, i=1,..,n$, of C^1, such that, for an open convex subset C of X, $\bar{x}\varepsilon C$, $\bar{x}=h(\bar{p})$, $h^{-1}(\bar{x})=\bar{p}\varepsilon N$, $x=h(p)$ for $x\varepsilon C, p=h^{-1}(x)\varepsilon N$, h^{-1} is one to one on N and $h^{-1}(C)=N$. From homogeneity of, $d_i(p,m), i=1,..,n$ are differentiable also in m.

Theorem 1: Suppose, at a differentiability point $\bar{x}=d(\bar{p},\bar{m})$, the cross derivatives; $u_{ij}(\bar{x}), i,j=1,..,n$, possess the positive signs, i.e. $u_{ij}(\bar{x})>0$, for $i,j=1,..,s$ and $i,j=s+1,..,n$, and $u_{ij}(\bar{x}), i,j=1,..,n$, possess the negative signs, i.e. $u_{ij}(\bar{x})<0$ for $i=1,..,s$ and $j=s+1,..,n$, where $1\leq s\leq n-1$. Then,

$$\{\partial d_i(\bar{p},\bar{m})/\partial m\}\{\partial d_n(\bar{p},\bar{m})/\partial m\}<0,\ i=1,..,s,$$

and

$$\{\partial d_j(\bar{p},\bar{m})/\partial m\}\{\partial d_n(\bar{p},\bar{m})/\partial m\}>0,\ j=s+1,..,n-1,$$

provided that

$$\partial\lambda(\bar{p},\bar{m})/\partial m=0.$$

Remark 1: The conclusion of Theorem 1 will be invariant even with independents involved in the two groups of commodities, if the Hessian matrix is indecomposable.

Remark 2: As a corollary to Theorem 1, all goods are complements and normal, if s=0. *9

Moreover, in the proof below, we shall see Remark 2 hold even with a concave function with a nonvanishing Hessian matrix. For the definition of complements for this case, see Kannai[1980 116].

Theorem 2: For a concave utility function u, suppose $u_{ij}(\bar{x})\geqq 0, i,j=1,.,n,\ i\neq j$, at a differentiability point $\bar{x}$, and suppose that the Hessian $DDu(\bar{x})$ is indecomposable there. Then, each demand function $d_i, i=1,.,n$, will possess a positive income derivative, that is,

$$\partial d_i(\bar{p},\bar{m})/\partial m > 0,\ i=1,.,n. \quad {}^{*10}$$

3.2 Proofs of Theorems 1, 2, Remarks 1 and 2

We take care of the proofs synthetically.

Correspond the notations to those used in Section 2. By the concavifiability near a differentiability point $\bar{x}$, $A_{n-1}(\bar{\xi})$ has all eigen values, $\alpha_i(\bar{\xi}), i=1,.,n-1$, negative, when they are restricted to the coordinates $\{\xi_1,.,\xi_{n-1}\}$ with $\xi_n=0$. Assume that $\alpha_1\leqq\alpha_2\leqq\alpha_3\leqq,.,\leqq\alpha_{n-1}$, without loss of generality. Let $\alpha^*_i, i=1,.,n$, designate the eigen values of $A_n(\bar{x})$ in the coordinates $\{x_1,.,x_n\}$ in such a way that $\alpha^*_1\leqq\alpha^*_2\leqq\alpha^*_3\leqq,.,\leqq\alpha^*_{n-1}\leqq\alpha^*_n$. Then, by a Sturmian separation theorem; see Bellman[1960 119], $\alpha^*_{h-1}\leqq\alpha_{h-1}\leqq\alpha^*_h, h=2,.,n$. Thus, $A_{n-1}(\bar{x})$ is nagative definite and has all diagonals $u_{ii}(\bar{x}), i=1,.,n-1$, negative. By assumption, $u_{ij}(\bar{x})>0, i,j=1,.,s$ and $i,j=s+1,.,n-1$, and, $u_{ij}(\bar{x})<0, i=1,.,s; j=s+1,.,n-1$ for $s\leqq n-1$. By a theorem of Morishima [1952] (and see also Basset et al[1967 227]), the inverse of A_{n-1}

exists and is of known sign pattern in a way that

(11) $(-1)^{i+j}|A_{n-1}(i;j)|/|A_{n-1}|>0, i=1,..,s; j=s+1,..,n-1,$

$(-1)^{i+j}|A_{n-1}(i;j)|/|A_{n-1}|<0, i,j=1,..,s; i,j=s+1,..,n-1.$

This may be slightly generalized by weakening the strict inequality to the weak one about u_{ij}, if A_{n-1} is indecomposable; see for this Basset et al[1968 Corollary, Theorem 7 454-5].

Let $\bar{A}=\bar{A}_n$ in case of no ambiguity and $\bar{B}=\bar{A}^{-1}$. Then, it is well known that, if $\bar{A}(n,n+1;n,n+1)$ is invertible,

(12) $\bar{B}(1,..,n-1;1,..,n-1)$=the inverse of $\bar{A}(1,..,n-1;1,..,n-1)$

$-\ \bar{A}(1,..,n-1;n,n-1)[\bar{A}(n;n+1;n,n+1)]^{-1}\bar{A}(n,n+1;1,..,n-1)$

where $\bar{A}(n,n+1,n,n+1)=A_{n-1}$ by definition. In view of (4) and by a well known fact, $|A_{n-1}|\neq 0$ if and only if $|\bar{A}|\neq 0$. By Jacobi's theorem of reciprocal determinant, we can have;

(13) $$\bar{B}(1,..,n-1;1,..,n-1)=\left|\left|\begin{matrix}|\bar{A}(n;n)| & -|\bar{A}(n;n+1)| \\ -|\bar{A}(n+1;n)| & |\bar{A}(n+1;n+1)|\end{matrix}\right|\right||\bar{A}|^{-1}$$

and

(14) $|\bar{B}(1,..,n-1;1,..,n-1)|=|A_{n-1}|/|\bar{A}|<0,$

implying that if $|\bar{A}(n+1;n+1)|=|A_n|=0$, which is assumed,

(15) $|\bar{A}(n+1;n)|=|\bar{A}(n;n+1)|\neq 0$

and also

(16) $\partial d_n(p,m)/\partial m=-|\bar{A}(n+1;n)|/|\bar{A}|$

$$= -\frac{\bar{p}_n-\Sigma_{h\neq n}\Sigma_{k\neq n}(-1)^{h+k}|A_{n-1}(h;k)|/|A_{n-1}|\ u_{nh}\bar{p}_k}{|\bar{B}(1,..,n-1;1,..,n-1)^{-1}|},$$

where the denominator is negative by (14).

Observe first, if s=0, then, $\partial d_n(\bar{p},\bar{m})/\partial m>0$. This will be used for Theorem 2. For the other income derivatives, $\partial d_j(\bar{p},\bar{m})/\partial m, j=1,..,n-1$, we have a 2 by n-1 matrix

(17) $\bar{B}(1,..,n-1;n,n+1)=\bar{B}(1,..,n-1;1,..n-1)\bar{A}(1,..,n-1;n,n+1)A_{n-1}^{-1}$

whose second row corresponds to them. Namely,

(18) $\partial d_j(\bar{p},\bar{m})/\partial m=(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|, j=1,..,n-1$

This satisfies the identity; $\Sigma_{h=1}^{n}(-1)^{n+1+h}|A(n+1;h)|\bar{p}_h/|\bar{A}|=1$ (one of them must be positive!). (16)(17)&(18) give;

(19) $\{\partial d_j(\bar{p},\bar{m})/\partial m\}\{\partial d_n(\bar{p},\bar{m})/\partial m\}^{-1}$

$$=-\Sigma_{h\neq n}(-1)^{h+j}\{|A_{n-1}(h;j)|/|A_{n-1}|\}u_{nh}$$

$$+\{|\bar{A}(n+1;n+1)|/|\bar{A}(n+1;n)|\}\Sigma_{h\neq n}(-1)^{h+j}\{|A_{n-1}(h;j)|/|A_{n-1}|\}\bar{p}_h$$

Again, if s=0, then, by (11) for s=0, the R.H.S. is positive, even if $|\bar{A}(n+1;n+1)|=|A_n|$ does not vanish at $\bar{x}$. By (16), all income derivatives of demand are positive at $\bar{x}$, where $\partial\lambda/\partial m\leqq 0$; see (6) in 2.2. This completes the proof of Theorem 2. However this is not true any more, if $s\geqq 1$. But note that, in the 1st term, for each h,

$$u_{nh}\ (-1)^{h+j}|A_{n-1}(h;j)|/|A_{n-1}|>0$$

*11

if j=1,..,s (because if h=1,..,s, $u_{nh}<0$ and by (11) $(-1)^{h+j}|A_{n-1}(h;j)|/|A_{n-1}|<0$, and if h=s+1,..,n-1, $u_{nh}>0$ and $(-1)^{h+j}|A_{n-1}(h;j)|/|A_{n-1}|>0$), hence their sum over h becomes positive. Similarly for j=s+1,..,n-1, but the sign reversed;

$$u_{nh}(\bar{x})(-1)^{h+j}|A_{n-1}(h;j)|/|A_{n-1}|<0,\ j=s+1,..,n-1.$$

Thus, Theorem 1 holds, provided that $\bar{A}(n+1;n+1)=A_n$ vanishes at $\bar{x}$.

3.3 Independents

Two extreme but very suggestive cases, in which independents are involved in such a way that the Hessian of a least concave utility is (completely) decomposable, are pointed out.

(i) In case $u_{nh}(\bar{x})=0, h=1,.,n-1$, note, in the proof above, both the 2nd term in the numerator of (16) and the 1st term in (19) are 0. Hence, $\partial d_n(\bar{p},\bar{m})/\partial m=-\bar{p}_n|A_{n-1}|/|\bar{A}|>0$ while $\partial d_j(\bar{p},\bar{m})/\partial m=0$ if $A_n(\bar{x})$ vanishes. This holds whatever cross derivatives $u_{ij}(\bar{x}), i,j\neq n$ may take. In fact, when $u_{ij}(\bar{x})=0, i,j=1,.,n-1$, and $u_{ii}(\bar{x})<0, i=1,.,n-1$, and $u_{nn}(\bar{x})=0$, $|\bar{A}(n+1;n)|/|\bar{A}|=-\bar{p}_n^{-1}<0$ and $|A(n+1;j)|/|\bar{A}|=0$.
(ii) However, if $u_{nn}(\bar{x})<0$, the signs of $\partial d_j(p,m)/\partial m, j=1,.,n-1$ are indeterminate. Suppose further s=0 so that $u_{ij}\geqq 0, i,j=1,.,n, i\neq j$. in (11). Then they are positive. In fact, $u_{ij}=0, i,j=1,.,n$, will imply $\partial d_n(\bar{p},\bar{m})/\partial m=-\bar{p}_n/u_{nn}(-\Sigma_{i=1}^{n}p_i^2/u_{ii})>0$ and $\partial d_j(\bar{p},\bar{m})/\partial m=-\bar{p}_j/u_{jj}(-\Sigma_{i=1}^{n}\bar{p}_i^2/u_{ii})>0$.*12

Including (i) and (ii), the following (iii) through (v) are more than a corollary to Theorem 2, because of the decomposability.

Suppose the Hessian $A_n(\bar{x})$ is (completely) decomposable so that, for some number s, $1\leqq s\leqq n-1, u_{ij}(\bar{x})=0, i,j=1,.,s; j=s+1,.,n$. Then, $|A_{n-1}(\bar{x})|\neq 0$ implies (iii) $|\bar{A}(n+1;n)|/|\bar{A}|=\bar{p}_n|A_{n-1}|/|\bar{A}| -\Sigma_{h=s+1}^{n-1}\Sigma_{k=s+1}^{n-1}(-1)^{h+k}\{|A_{n-1}(h;k)|/|A_{n-1}|\}u_{nh}\bar{p}_k/\{|A_{n-1}|/|\bar{A}|\}<0$, if $u_{ij}(\bar{x})\geqq 0, i,j=1,.,s; i,j=s+1,.,n, (i\neq j)$. (iv) $|\bar{A}(n+1;j)|/|\bar{A}|=0, j=1,.,s$, if $|A_n(\bar{x})|=0$ (v) Suppose further $u_{ij}(\bar{x})\geqq 0$ $(i\neq j), i,j=1,.,s$ or $i,j=s+1,.,n$, Then, $(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|=\{-|\bar{A}(n+1;n+1)|/|\bar{A}|\}\{\Sigma_{h=1}^{s}(-1)^{h+j}|A_{n-1}(h;j)|\bar{p}_h/|A_{n-1}|\geqq 0$ with equality only when $|A_n(\bar{x})|=0$ for each $j=1,.,s$. For each $j=s+1,.,n-1$, it is equal to; $|\bar{A}(n+1;n)|/|\bar{A}|\}\{\Sigma_{h=s+1}^{n-1}(-1)^{h+j}|A_{n-1}(h;j)|u_{nh}/|A_{n-1}|\}-\{|\bar{A}(n+1;n+1)|/|\bar{A}|\}\{\Sigma_{h=s+1}^{n-1}(-1)^{h+j}|A_{n-1}(h;j)|\bar{p}_n/|A_{n-1}|\}>0$

To see (iii)-(v), let $A_{n-1}=A, B=A^{-1}$ and read the formulas (12) and (17) etc., A,B and s for $\bar{A},\bar{B}$ and n-1, respectively. Then, by assumption, $A(1,.,s;s+1,.,n-1)=A(s+1,.,n-1;1,.,s)^t=0$. This implies

$$B(1,..,s;1,..,s)=[A(1,..,s;1,..,s)]^{-1}.$$

Likewise we have

$$B(s+1,..,n-1;s+1,..,n-1)=[A(s+1,..,n-1;s+1,..,n-1)]^{-1}$$

and $B(s+1,..,n-1;1,..,s)=B(1,..,s;s+1,..,n-1)^t=0$. Hence, $|A(h;j)|=0$, $h=1,..,s;j=s+1,..,n-1$. Also note $|A(h;j)|u_{nh}(\bar{x})=0$ since $|A(h;j)|=0$ when $u_{nh}(\bar{x})\neq 0$ and $u_{nh}(\bar{x})=0$ when $|A(h;j)|\neq 0$, for each $j=1,..,s$. This establishes (iv). By carefully adapting the above result to (16) and (19), we are able to obtain (iii) and (v) easily. Thus, we establish a theorem for independents.

Theorem 3: Suppose that independents are involved so that the Hessian $A_n(\bar{x})$ is (completely) decomposable at a differentiability point $\bar{x}$, in such a way that $u_{ij}(\bar{x})=0, i,j=1,..,s;j=s+1,..,n$, for a concave utility u,[*13]

(i) $\partial d_i(\bar{p},\bar{m})/\partial m=0$, $i=1,..,s$, if $|A_n(\bar{x})|=0$.

(ii) Suppose further $u_{ij}(\bar{x})\geqq 0$ $(i\neq j), i,j=1,..,s;i,j=s+1,..,n$, then,

$$\partial d_i(\bar{p},\bar{m})/\partial m\geqq 0, i=1,..,s;\ \partial d_j(\bar{p},\bar{m})/\partial m>0, j=s+1,..,n,$$

with equality only when $|A_n(\bar{x})|=0$, for $i=1,..,s$.

3.4 Theorems at a Non-Differentiability Point

Retain the hypothesis in Theorem 1 at a non-differentiability point $\bar{x}$, and define the aymptotic income derivative of marginal utility of money, $\partial\lambda(\bar{x})/\partial m$, to be, for $\partial\lambda(x)/\partial m=|A_n(x)|/|\bar{A}_n(x)|$,

$$\partial\lambda(\bar{x})/\partial m=\limsup\{\partial\lambda(x)/\partial m\}=0,\ \text{as } x \text{ approaches } \bar{x},$$

where x is a differentiability point near $\bar{x}$ and is indifferent with $\bar{x}$. Since $u_{ij}(\bar{x}), i,j=1,..,n$, are continuous, equation (19) approaches

$$\Sigma_{h\neq n}(-1)^{n+1+j}\lim |A_{n-1}(h;j)|u_{nh}(x)/|A_{n-1}(x)|$$

where $A_{n-1}(h;j)$ is a submatrix of $A_{n-1}(x)$, as x approaches $\bar{x}$. Thus, local behavior of demand $d_i(p,m), i=1,..,n$, at $(\bar{p},\bar{m})$ where they are not differentiable, can be approximated by their derivatives at nearby regular prices, because the differentiability points are dense. This will stand for Theorems 2 and 3.

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FOOTNOTES

* I have much benefited from conversations with Professors Satoru Fujishige, Kazuo Murota and Yoshihiko Otani and also from criticisms(and encouragement as well) provided by Professors Hiroshi Atsumi, Lionel McKenzie and Akira Yamazaki on earlier versions. This is a shortened, revised version of the most recent one (Kusumoto[1987]), which takes into account comments made by the referee of this journal.

1 See Kannai[1980] for the definition. Cf-Pareto[1929,1971]

2 That is, $\bar{p}x=\bar{m}$.

3 $T^t=T^{-1}$ See Hurwicz et al[1987 p.180].

4 Denote by $c_n(\xi)$ the curvature, then,

$$c_n(\bar{\xi})=\begin{vmatrix} S(\bar{\xi}) & \beta^t & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

5 demands $d_i(p,m)$, $i=1,.,n$, each, satisfy a uniformLipschitz condition in income m at $(\bar{p},\bar{m})$, if, for each p near $\bar{p}$, there exists a finite number $\gamma>0$, such that, for any m and m' near $\bar{m}$,

$$|d_i(p,m')-d_i(p,m)|\leqq\gamma|m'-m|.$$

6 $x=T\xi$ and $q=pT$.

7 See Slutsky[1952]

8 $|\bar{A}_n(\bar{x})|=\Sigma_{i=1}^{n-1}\alpha_i(\xi)$. See Fenchel[1956pp.500-501] for this. See also the proofs below and footnote 4 above. For the non-vanishing bordered Hessian condition, $|\bar{A}_n(\bar{x})|\neq 0$, see earlier works by Dhrymes [1967], Katzner[1968] and Barten, Kloek and Lemper[1969].

9 A fact that at least one commodity is normal implies this.

10 See Dhrymes[1968] and Chipman[1977] for the case in which the Hessian does not vanish.

11 If the n th commodity were complementary with each of the first s commodities, that is, $u_{nh}>0$, $h=1,.s$, whereas it were substitutive for each of the last n-1-s commodities; $u_{nh}<0, h=s+1,.,n-1$, then, the signs would be reversed in Theorem 1. Without loss of generality, we assumed that $u_{nh}<0, h=1,.,s$ and $u_{nh}>0, h=s+1,.,n-1$.

12 Stigler[1950] pointed out this without a formal proof.

13 Only for a least concave utility u, that $u_{ij}=0$ is meaningful. Note however that for any concave utility function U there is an increasing real concave function ϕ, for which $U=\phi(u)$, hence $U_{ij}>0$ implies $u_{ij}<0$.