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PRODUCT AND PRICE COMPETITION WITH NON-UNIFORM CONSUMER DISTRIBUTIONS: A STUDY OF SYMMETRIC AND ASYMMETRIC EQUILIBRIA

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1. INTRODUCTION

Ever since the pioneering contribution of Hotelling (1929), most of the literature on spatial competition has assumed a uniform distribution of consumers (see Gabszewicz and Thisse (1990) for a recent survey and references). Clearly, such a simplifying assumption is due to its mathematical tractability. In the characteristics approach, there are "consumer pockets" corresponding to consumers whose preferences are clustered around some fashionable brands. Similarly, in the urban setting, it is well known that the distribution of households is concentrated around the CBD in a monocentric city.

In a very nice paper, Caplin and Nalebuff (1990) have recently investigated the existence of a price equilibrium in pure strategies for non-uniform distributions of consumers. Essentially, these authors show that log-concavity of consumer densities, together with the linearity of preferences, is a sufficient condition for existence.

In this paper, we study the existence of subgame perfect Nash equilibria in pure strategies for location-price games where the consumer density is non-uniform. Unlike Caplin and Nalebuff, we confine our analysis to a one-dimensional, but unbounded space. The fact that we focus on two-stage location-price games has led us to prove existence of a perfect equilibrium in pure strategies for smaller class of distributions than Caplin and Nalebuff, i.e., we require the distributions to be concave and symmetric. Under some additional, but mild, regularity conditions, we can prove the existence of a symmetric equilibrium. We illustrate this result for a specific class of distributions which includes the uniform and triangular densities as polar cases. Furthermore, when the regularity conditions are not satisfied, we show that no symmetric equilibrium exists for a class of subject distributions. Surprisingly enough, despite the symmetry of a model, asymmetric equilibria may occur. In particular, for triangular distributions, there always exist asymmetric equilibria in pure strategies, even though no symmetric equilibrium exists. Hence, our analysis extends markedly preliminary results by Neven (1986) who studied symmetric local equilibria only.

The secondary purpose of this paper is to revisit Hotelling's duopoly in light of Prescott and Visscher's (1977) approach. These authors observe that many real world location decisions are not made simultaneously, but rather sequentially. Given that most location decisions are irrevocable, the first entrant will strive to anticipate the choices of the subsequent entrants. In this paper, we limit ourselves to the case of two firms. Both firms enter the market sequentially but choose their prices simultaneously in the second stage. Our formulation of sequential entry differs, therefore, from that adopted by Anderson (1987) who considers a two-stage Stackelberg game in location and price. As, in most cases, prices can be revised after the entry of a new firm, a Nash game seems indeed to be better appropriate to model price competition. Moreover, here also, we do not assume that the location space is restricted to the market space. As will be seen, this changes drastically the locational configuration.

The rest of the paper is organized as follows. The model is described in Section 2. In Section 3, we explore the existence of a pure strategy equilibrium in the first-stage location game. In order to glean further insights, we consider a specific family of densities in Section 4. The extreme case of a triangular distribution is investigated in details. In Section 5, the assumption of simultaneous location choices is replaced with that of sequential choices and the corresponding equilibrium is analyzed.

Section 6 concludes the paper.

2. MODEL

There are two firms producing a homogeneous good at constant and equal marginal cost. Without loss of generality, this cost is set equal to zero. There is a continuum of consumers distributed over the unit-line segment [0,1] and their location is denoted by $x \in [0,1]$. Let $f(x)$ be the consumer density function and $F(x)$ be the corresponding cumulative distribution function. Of course, we have $f(x) \geq 0$ for all $x \in [0,1]$, $F(x) \equiv \int_0^x f(y)dy$, and $F(1)=1$ (by normalization of total population).

In the first stage of the game, the two firms choose simultaneously their locations x_1 and x_2 in $\mathbb{R}$, anticipating the outcome of the second stage price subgame. Given x_1 and x_2, in the second stage firms choose simultaneously their (mill) price p_1 and p_2 with $p_1, p_2 \geq 0$.[1]

The transportation cost incurred by consumers is assumed to be a quadratic function of distance (without loss of generality, the transportation rate is normalized to one). Each consumer buys one unit of the good from the firm having the lower full price (i.e., mill price plus transportation cost).

By definition, the marginal consumers are indifferent between purchasing from either firm. Their location $\hat{x}$ is given by

$$\hat{x} = (p_2 - p_1 + x_2^2 - x_1^2)/2(x_2 - x_1). \tag{1}$$

When $x_1 \leq x_2$, the profit functions are respectively

$$\Pi_1 = p_1 F(\hat{x}) \qquad \text{and} \qquad \Pi_2 = p_2[1 - F(\hat{x})]. \tag{2}$$

On the other hand, for $x_1 > x_2$ they are given by

$$\Pi_1 = p_1[1 - F(\hat{x})] \qquad \text{and} \qquad \Pi_2 = p_2 F(\hat{x}). \tag{3}$$

game are well defined and given by:

$$\Pi_1^*(x_1,x_2) = 2(x_2-x_1)F^2(\hat{x})/f(\hat{x}), \quad (5a)$$

$$\Pi_2^*(x_1,x_2) = 2(x_2-x_1)[1-F(\hat{x})]^2/f(\hat{x}). \quad (5b)$$

Furthermore, as the price equilibrium is unique, $\hat{x}$ is univocally determined by the solution to the equation

$$2F(\hat{x})-1+[\hat{x}-x_1/2-x_2/2]f(\hat{x})=0, \quad (6)$$

which is obtained directly by subtracting (4a) from (4b) and by replacing p_2-p_1 in (1).

At a Nash location equilibrium $\underline{x}^N=(x_1^N,x_2^N)$, firm i maximizes $\Pi_i^*(x_i,x_j^N)$ with respect to x_i ($i\neq j$). Differentiating (5a) with respect to x_1, and using (6) yields:

$$\frac{\partial\Pi_1^*}{\partial x_1} = \frac{\partial\Pi_1^*}{\partial \hat{x}}\frac{\partial\hat{x}}{\partial x_1} = \frac{4F(\hat{x})}{f(\hat{x})}\,G(\hat{x})\,\frac{\partial\hat{x}}{\partial x_1}, \quad (7)$$

where

$$G(\hat{x}) \equiv 2(x_2-\hat{x})f(\hat{x})+2-7F(\hat{x})-\{(x_2-\hat{x})f(\hat{x})+2-4F(\hat{x})\}F(\hat{x})f'(\hat{x})/f^2(\hat{x}). \quad (8)$$

Since $\partial\hat{x}/\partial x_1=f(\hat{x})/[6f(\hat{x})+(2\hat{x}-x_1-x_2)f'(\hat{x})]>0$ by application of the implicit function theorem to (5a), $G(\hat{x})=0$ is a <u>necessary</u> condition for location equilibrium.

Suppose further that f(x) is <u>symmetric</u> about x=1/2. If f(x) is differentiable, then because f(x) is concave, we have $f'(\hat{x})\geq 0$ for all $\hat{x}\in[0,1/2)$, $f'(\hat{x})\leq 0$ for all $\hat{x}\in(1/2,1]$, and $f'(1/2)=0$.[2] Therefore, as $F(1/2)=1/2$, for a symmetric location equilibrium - if any -, it must be that

$$G(1/2) = (2x_2^N-1)f(1/2) - 3/2 = 0. \quad (9)$$

Differentiating (7), and using (9) to obtain the symmetric locations, we have another <u>necessary</u> condition for such an equilibrium:

We seek subgame perfect Nash equilibria. Hence, we solve the game by backward induction, i.e., we first deal with price competition for any given pair of locations. The following result, due to Caplin and Nalebuff (1990), establishes the existence of a price equilibrium in pure strategies for a wide class of consumer density functions.

Proposition 1

If the transportation cost is quadratic, then for any given locations of firms and for any log-concave consumer density function, there exists a Nash price equilibrium. Furthermore this equilibrium is unique.

Assuming $x_1 \le x_2$, the first-order conditions for equilibrium price are given by

$$\frac{\partial \Pi_1}{\partial p_1} = F(\hat{x}) - \frac{p_1 f(\hat{x})}{2(x_2-x_1)} = 0, \tag{4a}$$

$$\frac{\partial \Pi_2}{\partial p_2} = 1-F(\hat{x}) - \frac{p_2 f(\hat{x})}{2(x_2-x_1)} = 0. \tag{4b}$$

We know from Proposition 1 that (4a) and (4b) give a unique price equilibrium if f(x) is log-concave, and hence we confine f(x) to be log-concave hereafter. (Notice that any concave function is log-concave because the logarithm is an increasing and concave function.) Expressions similar to (4a) and (4b) can be obtained when $x_1 > x_2$.

3. THE FIRST-STAGE LOCATION GAME

Consider the location game for log-concave consumer density functions. Throughout this section and the next one, we assume $x_1 \le x_2$ unless explicitly mentioned. From (2) and (4a)-(4b), the payoff functions of the location

$$\left.\frac{\partial^2 \Pi_1^*}{\partial x_1^{\,2}}\right|_{x_1^N+x_2^N=1} = -\frac{8f^2(1/2)\{f(1/2)+2\} + f''(1/2)}{48f^3(1/2)} \leq 0. \tag{10}$$

Consider now the case of a location pair which is not necessarily symmetric. Eliminating x_1 and x_2 from $\partial\Pi_1^*/\partial x_1=0$ and $\partial\Pi_2^*/\partial x_2=0$, we have

$$H(\hat{x}) \equiv [1-2F(\hat{x})]f^2(\hat{x}) - [1-F(\hat{x})]F(\hat{x})f'(\hat{x}) = 0, \tag{11}$$

which must be satisfied for any location equilibrium. By using (11), we will prove existence and uniqueness of a symmetric equilibrium in Proposition 2, where a further restriction on $f(x)$ is imposed. It should be noticed that whereas existence and uniqueness of a price equilibrium are guaranteed for any log-concave consumer density, these properties do not necessarily hold for location equilibria.

<u>Lemma 1</u>

<u>If</u> $f(x)$ <u>is symmetric about</u> $x=1/2$ <u>and concave</u>, <u>then</u> $H(\hat{x})$ <u>is positive for all</u> $\hat{x}\in(0,\hat{x}_a)$, <u>and negative for all</u> $\hat{x}\in(1-\hat{x}_a,1)$, <u>where</u> $F(\hat{x}_a)=1/3$.

<u>Proof</u>:

Consider the function of $I(\hat{x})\equiv f^2-2Ff'$. Since $I(0)\geq 0$ and $I'(\hat{x})=-2Ff''\geq 0$, $I(\hat{x})$ is nonnegative. Using this fact, (11) becomes

$$H(\hat{x}) \geq (1-3F)Ff' \geq 0, \qquad \forall\ \hat{x}\in(0,\hat{x}_a),$$

where the first (the second) inequality holds if $f''=0$ ($f'=0$) for all $\hat{x}\in(0,\hat{x}_a)$. In other words, $H(\hat{x})=0$ iff $f(\hat{x})=1$ for all $\hat{x}\in(0,\hat{x}_a)$. If so, however, we have $H(\hat{x})=1-2F>0$ for any $\hat{x}\in(0,\hat{x}_a)$. Thus, the former part of Lemma 1 is proven.

As $f(x)$ is symmetric, we have $f'(\hat{x})=-f'(1-\hat{x})$ and $F(1-\hat{x})=1-F(\hat{x})$. Consequently, $H(\hat{x})=-H(1-\hat{x})$ holds for all $\hat{x}\in[0,1]$. Hence, the latter part of

Lemma 1 is also proven. ∎

Lemma 2

If $f(x)$ *is symmetric about* $x=1/2$ *and concave, and* $f'''(x)$ *is nonnegative for all* $x\in(\hat{x}_b,1/2)$, *then* $H''(\hat{x})$ *is nonpositive for all* $\hat{x}\in(\hat{x}_b,1/2]$, *and nonnegative for all* $\hat{x}\in(1/2,1-\hat{x}_b)$, *where* $F(\hat{x}_b)=1/18$.

Proof:

Differentiating (11) twice, we obtain

$$H''(\hat{x}) = (1-2F)f'^2-8f'f^2-(1-F)Ff''' \leq (1-18F)f'^2-(1-F)Ff''' \leq 0,\ \forall\hat{x}\in(\hat{x}_b,1/2].$$

The first inequality follows from $I(\hat{x})\geq 0$ and $f'(\hat{x})\geq 0$ for all $\hat{x}\in(\hat{x}_b,1/2]$, and the second is implied by $1/2\geq F(\hat{x})>1/18$ and $f'''(\hat{x})\geq 0$ for all $\hat{x}\in(\hat{x}_b,1/2]$.

As $f(x)$ is symmetric, $H''(\hat{x})=-H''(1-\hat{x})$ holds for all $\hat{x}\in[0,1]$. Thus, $H''(\hat{x})\geq 0$ for all $\hat{x}\in(1/2,1-\hat{x}_b)$. ∎

We are now ready to prove the existence and uniqueness of an equilibrium as follows.

Proposition 2

Assume that $f(x)$ *is concave, symmetric about* $x=1/2$ *and such that* $f'''(x)$ *is nonnegative for all* $x\leq 1/2$. *Then, there is a unique Nash location equilibrium. Furthermore, this equilibrium is symmetric and given by* $x_1^N=1/2-3/[4f(1/2)]$ *and* $x_2^N=1/2+3/[4f(1/2)]$.

Proof:

We know that $H(\hat{x})>0$ for all $\hat{x}\in(0,\hat{x}_a)$ by Lemma 1, and $H''(\hat{x})\leq 0$ for all $\hat{x}\in(\hat{x}_b,1/2)$ by Lemma 2. Recall that $\hat{x}_a>\hat{x}_b$, $H(0)\geq 0$ and $H(1/2)=0$. Therefore, as $H''(\hat{x})\leq 0$ for $\hat{x}\in[\hat{x}_a,1/2)$, we have $H(\hat{x})>0$ for $\hat{x}\in[\hat{x}_a,1/2)$. Hence, $H(\hat{x})=0$ does not have a solution belonging to $(0,1/2)$.

A similar argument shows that $H(\hat{x})=0$ does not have a solution in $(1/2,1)$. Thus, (11) has at most three solutions: $\hat{x}=0,1/2,1$. However, it is obvious that $\hat{x}=0$ and $\hat{x}=1$ do not correspond to a location equilibrium because the profit of either firm is zero. Hence, $\hat{x}=1/2$ is the only value that satisfies the pair of the first-order conditions for a location equilibrium.

Next, let us check the local maximum condition (10) for $\hat{x}=1/2$. Since $f(1/2)\geq 1$ for any symmetric concave function, (10) holds if $f''(1/2)\geq -24$. However, applying the mean value theorem, we obtain

$$f''(1/2) \geq [f'(1/2)-f'(1/4)]/(1/2-1/4) = -4f'(1/4)$$
$$\geq -4[f(1/4)-f(0)]/(1/4-0) \geq -16f(1/4).$$

The first inequality follows from $f'(\hat{x})$ decreasing and convex for $\hat{x}\leq 1/2$ and $f'(1/2)=0$; the second inequality is implied by the concavity of $f(\hat{x})$ whereas the last is due to $f(0)\geq 0$. Therefore, (10) is satisfied if $f(1/4)\leq 24/16$.

Suppose, on the contrary, that $f(1/4)>24/16$. Because of concavity, we have $F(1/4)\geq f(1/4)/8$ and $F(1/2)-F(1/4)\geq f(1/4)/4$. Consequently, $F(1/2)\geq 3f(1/4)/8>9/16$, which is a contradiction, and hence, $f(1/4)\leq 24/16$.

Thus, $\hat{x}=1/2$ is the only value for which both $\partial\Pi_1^*/\partial x_1=0$ and $\partial\Pi_2^*/\partial x_2=0$ are satisfied; furthermore, it corresponds to a local maximum for Π_i^* with respect to x_i (i=1,2). Consequently, $\hat{x}=1/2$ corresponds to a global maximum of Π_i^* with respect to x_i (i=1,2) when $x_1\leq x_2$. As the function $H(\hat{x})$, as defined by (11), can be shown to be invariant with respect to a permutation of indices, Lemmas 1 and 2 hold regardless of the relative position of x_1 and x_2. Hence, by permuting indices, we can repeat the above argument for the case where $x_1\geq x_2$ since no restriction was imposed on $\hat{x}$. This implies that $\hat{x}=1/2$ corresponds to a global maximizer with respect to x_i (i=1,2) for all possible localities.

Finally, the equilibrium locations are derived from (9) and from the symmetry of the equilibrium. ∎

Thus, under some regularity conditions on the consumer density function (f is symmetric and twice continuously differentiable, f' is decreasing and convex), there always exists a unique symmetric location equilibrium. Geometrically, the regularity conditions mean that the changes in the slope of f(x) can be large in the vicinity of the market end points but small near the center. Stated differently, the distribution can not be "too" different from the uniform one. When the distribution is uniform, the equilibrium locations are situated at -1/4 and 5/4, that is, <u>outside</u> the market. This surprising result reflects the fact that price competition under quadratic transportation costs is very fierce indeed. As the distribution becomes more concentrated, f(1/2) increases and firms locate closer to the market center. Eventually, they will lie inside the market. In these cases, the price competition effect is outweighing by the demand effect generated by the high concentration of consumers around the center. According to Neven (1986, Proposition 3), the distance to the center from any equilibrium location is greater than 3/8.

It is worth noting here that when f"' does not exist or is negative, no symmetric equilibrium may exist. In general, we obtain the following:

<u>Proposition 3</u>

<u>For any log-concave distribution of consumers which is symmetric about</u> x=1/2, <u>there exits no symmetric equilibrium if the derivative of the consumer distribution is discontinuous at</u> x=1/2.

Proof:

Although $f'(\hat{x})$ is not defined at $\hat{x}=1/2$, we have $\lim_{\hat{x}\to 1/2-0} f'(\hat{x})>0$ and $\lim_{\hat{x}\to 1/2+0} f'(\hat{x})<0$. Thus, as $F(1/2)=1/2$, we obtain $\lim_{\hat{x}\to 1/2-0} H'(\hat{x})<0$ and $\lim_{\hat{x}\to 1/2+0} H'(\hat{x})>0$. This means that (11) does not hold at $\hat{x}=1/2$. In other words, $\hat{x}=1/2$ does not satisfy the first-order conditions for location equilibrium. ■

Notice that as the derivative of any convex but log-concave functions is discontinuous at $\hat{x}=1/2$, no symmetric equilibrium exists although the model setting is symmetric.[3] We thus observe a wide class of consumer distribution functions which do not generate a symmetric outcome. However, in such cases, asymmetric equilibria may exist as will be seen in Proposition 6. In the next result, we provide upper bounds on equilibrium market shares and profits corresponding these equilibria.

Proposition 4

For any concave distribution of consumers which is symmetric about $x=1/2$, the equilibrium market share of one firm is at most twice as large as that of the other, and the equilibrium profit of one firm is at most four times as large as that of the other.

Proof:

From Lemma 1, there exists no location equilibrium for $\hat{x}\in(0,x_a)$, which corresponds to $F\in(0,1/3)$. That is, the market share of each firm should exceed 1/3, and so the market share of one firm is at most twice as large as that of the other.

Next, the ratio of the profits is simply given by $F^2/(1-F)^2$ from (5a) and (5b). Since F lies within the interval of [1/3,2/3] in location equilibrium, we immediately get $F^2/(1-F)^2 \in [1/4,4]$. ■

We will see in Proposition 6 that the above upper bounds are the best possible for concave consumer distributions.

4. FUNCTIONAL SPECIFICATION OF THE CONSUMER DENSITY

Since functions Π_i^* (i=1,2) are not necessarily quasi-concave, we are not able to obtain further results without using specific, but meaningful functional forms for the consumer density.

Consider the following family of consumer densities:

$$f(x) = \frac{\alpha+1}{\alpha}(1 - |2x-1|^{\alpha}), \quad \text{for} \quad x \in [0,1], \tag{12}$$

where $\alpha \geq 1$. Density (12) is concave and symmetric about 1/2 for any $\alpha \geq 1$. Moreover, (12) is general enough to cover distributions from the triangular distribution ($\alpha=1$) to the uniform one ($\alpha \to \infty$). These distributions are the two polar cases of the concave function family (12), which is analyzed below.

In the first place, computing the necessary condition (10) for a symmetric location equilibrium, we can easily establish that for $1 \leq \alpha < 2$ in (12), there exists no symmetric Nash location equilibrium. When $\alpha=1$, (12) becomes a triangular distribution, and hence Proposition 3 can directly apply to show the nonexistence of symmetric location equilibrium.[4] When $1 < \alpha < 2$, the necessary condition (10) is violated because $\lim_{x \to 1/2 \pm 0} f''(x) = -\infty$.

This result and Proposition 3 imply that although two identical firms compete under the identical conditions, they do not earn identical profits when the second derivative of the consumer density function at the market

4-1. UNIFORM DISTRIBUTION ($\alpha \to \infty$)

To start with, let us examine the case of the uniform distribution. As seen in Proposition 2, there is a symmetric Nash location equilibrium given by $(x_1^N, x_2^N) = (-1/4, 5/4)$ whereas $\Pi_1^*(\underline{x}^N) = \Pi_2^*(\underline{x}^N) = 3/4$.[5] Notice that the equilibrium location can be obtained directly by using (5)-(8) when $f(x)=1$ for all $x \in [0,1]$.

4-2. TRIANGULAR DISTRIBUTION ($\alpha=1$)

By contrast, we consider the other polar case of a triangular distribution, i.e., $f(x)=2-2|2x-1|$, for all $x \in [0,1]$. We show in Proposition 6 that although no symmetric location equilibrium exists, asymmetric location equilibria do exist with such a distribution.

<u>Proposition 6</u>

<u>When the distribution of consumers is symmetric and triangular, there exist two asymmetric Nash location equilibria, which are given by</u>

$$(x_1^N, x_2^N) = \begin{cases} (-\sqrt{6}/9, 5\sqrt{6}/18) \\ (1-5\sqrt{6}/18, 1+\sqrt{6}/9). \end{cases}$$

<u>Proof</u>:

By computing (11) for the triangular distribution, we get the following solutions for $\hat{x}$ (i.e., the location of the marginal consumers): $0, 1/\sqrt{6}, 1/2, 1-1/\sqrt{6}, 1$. However, $\hat{x}=0$ and $\hat{x}=1$ do not correspond to Nash equilibria because one firm earns zero profit, and $\hat{x}=1/2$ does not correspond to an equilibrium by Proposition 3. Hence, only two solutions are left: $\hat{x}=1/\sqrt{6}$ and $\hat{x}==1-1/\sqrt{6}$. The corresponding candidate-equilibrium locations are respectively $(x_1^N, x_2^N)=(-\sqrt{6}/9, 5\sqrt{6}/18)$ and $(1-5\sqrt{6}/18, 1+\sqrt{6}/9)$, which are center is large enough in absolute value, or when the first derivative is discontinuous at the center. Surprisingly, the violation of the local condition is here sufficient to destroy any symmetric equilibrium.

On the other hand, for $\alpha \geq 2$ we have a unique symmetric equilibrium.

<u>Proposition 5</u>

<u>If</u> $\alpha \geq 2$ <u>in (12), then there exists the unique symmetric Nash location equilibrium given by</u> $(x_1^N, x_2^N) = ((1/2-\alpha/4)/(\alpha+1), (5\alpha/4+1/2)/(\alpha+1))$.

<u>Proof</u>:

Using Proposition 2, the existence and uniqueness can be assured by checking that the third derivative of (12) is nonnegative for $\hat{x} \in [0,1/2]$. The values of x_i^N are directly obtained by (9). ■

Since α is interpreted as a degree of flatness in the concave family of consumer distributions (12), we may state from the above results that the flatter the distribution of consumers is, the more likely the existence of a unique symmetric equilibrium is, which is in accord with our discussion following Proposition 2. Besides, we observe that the more concentrated the distribution of consumers is (i.e., α smaller), the closer the firms locate.

The equilibrium locations are plotted in Figure 1, where x_i^N for $\alpha=1$ are given by Proposition 6 below, and x_i^N for $\alpha \geq 2$ are obtained from Proposition 5. For $\alpha \in (1,2)$, $\underline{x}^N$ has been computed numerically. Two asymmetric equilibria are obtained; they are represented in solid and dotted lines respectively.

[Insert Figure 1 about here]

obtained from (6) and $G(\hat{x})=0$. We will show below that the first location pair is indeed a Nash equilibrium. Since the setting is symmetric about 1/2, if the first pair is a Nash equilibrium, then the second pair is also a Nash equilibrium.

[i] Let us show that $x_1=-\sqrt{6}/9$ is the best reply against $x_2=5\sqrt{6}/18$.

When $\hat{x}\in(0,1/2)$, the sign of $\partial\Pi_1^*/\partial x_1$ is given by the sign of $G(\hat{x})=-16\hat{x}^2+10\hat{x}/\sqrt{6}+1$. Clearly, $G(\hat{x})\gtreqless 0$ for $\hat{x}\lesseqgtr 1/\sqrt{6}$, which corresponds to $x_1\lesseqgtr -\sqrt{6}/9$ (given $x_2=5\sqrt{6}/18$). Therefore, in order to prove that $x_1=-\sqrt{6}/9$ yields a global maximum, it suffices to show that $G(\hat{x})\le 0$ for all $\hat{x}\in(1/2,1)$ because Π_1^* is continuous with respect to $\hat{x}$.

When $\hat{x}\in(1/2,1)$, $1/2<F(\hat{x})<1$ and $f'(\hat{x})=-4$ so that

$$G(\hat{x}) < 2(x_2-\hat{x})f(\hat{x}) - 3/2 + 2(x_2-\hat{x})/f(\hat{x}). \tag{13}$$

If $\hat{x}\ge x_2(=5\sqrt{6}/18)$, then the RHS of (13) is negative. If $\hat{x}<x_2$, then as $f(\hat{x})=4(1-\hat{x})\in(4-4x_2,2)$, the RHS of (13) is less than $4(x_2-1/2)-3/2+(x_2-1/2)/(4-4x_2)\simeq-0.50<0$. Thus, $G(\hat{x})<0$ for all $\hat{x}\in(1/2,1)$, and hence $x_1=-\sqrt{6}/9$ is firm 1's best reply against $x_2=5\sqrt{6}/18$.

[ii] We next prove that $x_2=5\sqrt{6}/18$ is the best reply against $x_1=-\sqrt{6}/9$.

When $\hat{x}\in(0,1/2)$, we have

$$\frac{\partial\Pi_2^*}{\partial x_2} = \frac{\partial\hat{x}/\partial x_2}{2\hat{x}^3}(\hat{x}-1/\sqrt{6})[-64\hat{x}^3(17/48-\hat{x}^2)-40\sqrt{6}\hat{x}^2(7/20-\hat{x}^2)/3-14\hat{x}/3-\sqrt{6}]. \tag{14}$$

Since $\partial\hat{x}/\partial x_2>0$, it is clear that $\partial\Pi_2^*/\partial x_2\gtreqless 0$ for $\hat{x}\lesseqgtr 1/\sqrt{6}$, which corresponds to $x_2\lesseqgtr 5\sqrt{6}/18$ (given $x_1=\sqrt{6}/9$). Therefore, in order to prove that $x_2=5\sqrt{6}/18$ yields a global maximum, it suffices to show that $\partial\Pi_2^*/\partial x_2\le 0$ for all $\hat{x}\in(1/2,1)$ because Π_2^* is also continuous for all $\hat{x}\in(0,1)$.

When $\hat{x}\in(1/2,1)$, we have

$$\frac{\partial\Pi_2^*}{\partial x_2} = \frac{2[1-F(\hat{x})]\partial\hat{x}/\partial x_2}{f(\hat{x})}[-(2\hat{x}-1)(9-8\hat{x})-4(1-\hat{x})/\sqrt{6}]. \tag{15}$$

Since $\partial\hat{x}/\partial x_2>0$, we get $\partial\Pi_2^*/\partial x_2<0$ for all $\hat{x}\in(1/2,1)$.

Hence, $(-\sqrt{6}/9, 5\sqrt{6}/18)$ is a Nash location equilibrium. Since two firms are identical and the triangle is symmetric about $x=1/2$, $(1-5\sqrt{6}/18, 1+\sqrt{6}/9)$ is also a Nash location equilibrium. ■

The market share of firm 1 is exactly 1/3 when $\hat{x}=1/\sqrt{6}$, and 2/3 when $\hat{x}=1-1/\sqrt{6}$. The corresponding profits are $(\Pi_1^*(\underline{x}^N), \Pi_2^*(\underline{x}^N)) = (7/54, 14/27)$ and $(14/27, 7/54)$ respectively. Notice that these values correspond to the upper bounds identified in Proposition 4. Furthermore, we observe that these profit values are less than the profit earned by each firm in the uniform case obtained in the previous subsection. The concentration of consumers around the center attracts the two firms, which lessens the distance between them and intensifies price competition. The result is a decrease in the equilibrium profits of both firms, despite their asymmetry in location.[6]

It should be noticed that the nonexistence of a symmetric equilibrium follows from the discontinuity of the reaction functions near $x_1+x_2=1$ (see Figure 2). The discontinuities occur because the profit function Π_1^* (Π_2^*) has two local maxima and one local minimum when $4/5<x_2<1$ ($0<x_1<1/5$) - this corresponds to the case (b) in the footnote 4.

[Insert Figure 2 about here]

Unlike the uniform distribution, the triangular distribution, though symmetric, leads to <u>asymmetric</u> locations, prices and profits. In other words, <u>competition by two identical firms under identical conditions results in an asymmetric Nash location equilibrium</u> if consumers are concentrated around the center.[7] This result shows the lack of robustness of the symmetric equilibrium which often appears in the literature on spatial competition.

5. SEQUENTIAL ENTRY

Until now, we have been focusing only upon the simultaneous game in locations. However, as discussed in the introduction, it may be more realistic to assume that firms enter the market <u>sequentially</u> while price competition remains simultaneous. More precisely, the first stage is a Stackelberg leader-follower game in location, but the second stage is a Nash subgame in price.

Formally, the model of Section 2 has to be modified in the following manner. Firm 1 (the leader) maximizes its profit of $\Pi_1^*(x_1,x_2)$ with respect to x_1, replacing x_2 by firm 2 (the follower) 's reaction function $x_2=R(x_1)$, which is itself derived from the maximization of $\Pi_2^*(x_1,x_2)$ with respect to x_2. The resulting Stackelberg location equilibrium is denoted by $\underline{x}^S=(x_1^S,x_2^S)$.

The first-order conditions for such an equilibrium can be written as

$$\frac{d\Pi_1^*}{dx_1} = \frac{\partial\Pi_1^*}{\partial x_1} + \frac{\partial\Pi_1^*}{\partial x_2}\frac{dR}{dx_1}, \tag{16a}$$

$$\frac{\partial\Pi_2^*}{\partial x_2} = 0, \tag{16b}$$

where dR/dx_1 is equal to $-(\partial^2\Pi_2^*/\partial x_2\partial x_1)/(\partial^2\Pi_2^*/\partial x_2^2)$.

Computing (16a), we get

$$\frac{d\Pi_1^*}{dx_1} = \frac{4F(\hat{x})K(\hat{x})}{[2f^2(\hat{x})+\{1-F(\hat{x})\}f'(\hat{x})]f(\hat{x})},$$

where $K(\hat{x}) \equiv \{1-2F(\hat{x})+g'(x_1)\}f^2(\hat{x})-\{1-F(\hat{x})\}F(\hat{x})f'(\hat{x})$. (17)

Note that since the denominator is positive for all $\hat{x}\in(0,1)$, $\text{sgn}(d\Pi_1^*/dx_1)=$

$\mathrm{sgn}(K(\hat{x}))$. Unfortunately, (17) is too complicated to derive analytical results for the distributions given by (12). Nevertheless, if we set $\alpha=1$ (triangular distribution) or $\alpha\to\infty$ (uniform distribution), we can obtain the following proposition.

Proposition 7

If the distribution of consumers is either uniform or symmetric triangular, then the first entrant necessarily locates at the market center.

Proof:

[i] Uniform distribution

Some standard manipulations show that firm 2's reaction function is $x_2=x_1/3+4/3$. Using this and (6), (17) is such that $\mathrm{sgn}(d\Pi_1^*/dx_1)=\mathrm{sgn}(1-2x_1)$. Therefore, the optimum location of the first entrant is $x_1^S=1/2$.

[ii] Triangular distribution

(a) Assume that $\hat{x}\le 1/2$.

By using (6) and (14), (5a) depends only upon $X\equiv\hat{x}^2 \in[0,1/4]$ as follows:

$$\Pi_1^*(X) = \frac{X(-32X^3+28X^2-4X-1)}{12X^2-4X-1} .$$

Differentiating this expression with respect to X, we get

$$\Pi_1^{*\prime}(X) = \frac{L(X)}{(12X^2-4X-1)^2} ,$$

where $L(X) \equiv -768X^5+720X^4-96X^3-56X^2+8X+1$. Differentiating L(X) yields $L'(X)=8(1-2X)M(X)$, where $M(X)\equiv 1-12X-60X^2+240X^3$. Differentiating M(X) gives $M'(X)=12(-1-10X+60X^2)$. Since $M'(0)<0$ and $M'(1/4)>0$, $M'(X)$ changes its sign only once at $\tilde{X}\in[0,1/4]$. Moreover, as $M(0)>0$ and $M(1/4)<0$, M(X) and hence $L'(X)$ changes its sign only once at $\bar{X}\in[0,1/4]$. Hence, $\Pi_1^{*\prime}(X)>0$ is first increasing and then decreasing on [0,1/4] because $\mathrm{sgn}(\Pi_1^{*\prime}(X))=\mathrm{sgn}(L(X))$.

However, as $\Pi_1^{*\prime}(0)>0$ and $\Pi_1^{*\prime}(1/4)>0$, we can conclude that $\text{sgn}(d\Pi_1^*(x_1,x_2)/dx_1)=\text{sgn}(d\Pi_1^*(\hat{x})/d\hat{x})$ is positive for all $\hat{x}\in[0,1/2]$.

(b) Assume that $\hat{x}>1/2$.

Calculating $K(\hat{x})$ as given by (17), we get

$$\text{sgn}(d\Pi_1^*/dx_1) = \text{sgn}(6(1-\hat{x})^2-1+2g'(x_1)), \quad \text{for} \quad \hat{x}\in(1/2,1]. \tag{18}$$

We now show that (18) is positive so that it is optimal for firm 1 to choose the central location.

Setting (15) equal to zero and using (6), we get firm 2's reaction function $R(x_1)$. Differentiating $R(x_1)$ with respect to x_1, we obtain

$$R'(x_1) = -\frac{1}{4} + \frac{9(1-x_1)}{4A} + \frac{24[1+3(1-x_1)/A]}{(3-3x_1+A)^2}$$

$$> -\frac{1}{4} + \frac{9(1-x_1)}{20} + \frac{24}{5(8-3x_1)} \equiv N(x_1),$$

where $A \equiv \sqrt{9(1-x_1)^2+16} < 5$ for all $x_1\in(0,1/2]$. (The reason for $x_1>0$ is due to the fact that $x_2=R(x_1)$ lies above $x_1+x_2=1$ since $\hat{x}>1/2$, and the case of $x_1>1/2$ can be dealt with similarly by symmetry of the problem.)

However, as $N'(x_1) = 9(-9x_1^2+48x_1-32)/[20(3x_1-8)^2]<0$, $\forall x_1\in[0,1/2]$, we have $R'(x_1)>N(0)=4/5$. From this and (18), we can show that $\text{sgn}(d\Pi_1^*/dx_1)=\text{sgn}(6(1-\hat{x})^2+3/5)$ is positive for all $\hat{x}\in(1/2,1]$.

Hence, $x_1^S=1/2$ is the optimum location of the first entrant. ■

The Stackelberg equilibrium locations can be computed as follows:

(i) for the uniform distribution,

$$(x_1^S,x_2^S) = \begin{cases} (1/2,\ 3/2) \\ (1/2,-1/2) \end{cases}$$

and

$$(\Pi_1^*(\underline{x}^S),\Pi_2^*(\underline{x}^S)) = (8/9,2/9),$$

(ii) for the triangular distribution,

$$(x_1^S, x_2^S) = \begin{cases} (1/2,\ 1.443) \\ (1/2, -0.443) \end{cases}$$

and

$$(\Pi_1^*(\underline{x}^S), \Pi_2^*(\underline{x}^S)) = (0.715, 0.089).$$

In either case, <u>the first mover advantage is substantial</u>. The profit of the firm 1 is four times as large as that of firm 2 in the uniform case, and approximately eight times in the triangular case. The latter exhibits more unequal profit differential due to the high concentration of consumers around the center.

Furthermore, under the uniform distribution of consumers, we have $(x_1^S, x_2^S) = (0,1)$ when the location space is restricted to [0,1]. When this constraint - the justification of which is far from being clear - is relaxed, we obtain a totally different pattern as one firm (the leader) locates at the market center and the other (the follower) outside the market space.

6. CONCLUSIONS

The following remarks are in order.

(i) A lot of work remains to be done to derive general conditions on the consumer densities guaranteeing the existence of a perfect location-price equilibrium in pure strategies. In this respect, it should be clear that Proposition 2 is far from being as tight as the characterization obtained by Caplin and Nalebuff for the existence of a price equilibrium. However, in view of Figure 1, the set of location equilibria seems to be very sensitive to the specification of the consumer distributions. Consequently, it is probably a very hard task to get general results.

(ii) We have shown that moving (even slightly) from the uniform distribution may destroy the existence of a symmetric equilibrium, even though the model remains symmetric. This casts some doubts on the robustness of the results derived under the (standard) assumption of a uniform distribution and invites us to pay much more attention to the existence of asymmetric equilibria.

(iii) Besides the non-uniform consumer distribution, another important extension of the spatial competition model is to deal with multi-dimensional spaces. Neven and Thisse (1990) have studied a simple horizontal-vertical product differentiation model with a uniform distribution of consumers. In this setting, an asymmetric equilibrium may arise too. Combined with a non-uniform distribution, it seems reasonable to expect the occurrence of asymmetric equilibria to be reinforced. This would suggest that systematic emphasis put on symmetric equilibria (in one-dimensional uniform models) is not well founded.

(iv) As a final remark, let us say that the above analysis has also shed some light on the role of the assumption that firms must locate inside the market space. Indeed, relaxing the apparently innocuous assumption may leads to quite different equilibrium outcomes. For example, in the sequential location game, this yields a completely different locational configuration and uncovers a first-mover advantage which cannot appear in the standard setting. Here also, these results invite us to revisit some results in horizontal product differentiation in the context of a model with a bounded market space but with an unbounded location space.

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FOOTNOTES

1 The analysis is confined to the case of pure strategies.

2 If f(x) is a.e. differentiable but continuous, the argument must be modified. Subsection 4-2 deals with such a case.

3 Examples of these functions are:

$$f(x)=\mu_1 x^{\mu_2} \text{ for } x\in[0,1/2], \quad f(x)=\mu_1(1-x)^{\mu_2} \text{ for } x\in(1/2,1],$$

$$f(x)=\mu_3 e^{\mu_4 x} \text{ for } x\in[0,1/2], \qquad f(x)=\mu_3 e^{\mu_4(1-x)} \text{ for } x\in(1/2,1],$$

where μ's are positive constants.

[4] When the distribution is triangular, we can compute directly Π_1^* as defined by (5a) when x_1 and x_2 are almost symmetric about the market center. After some tedious calculations, we obtain

$$\Pi_1^*(1-x_2-\epsilon,x_2) - \Pi_1^*(1-x_2,x_2) = (1-x_2)\epsilon/2 + O(\epsilon^2),$$

and

$$\Pi_1^*(1-x_2+\epsilon,x_2) - \Pi_1^*(1-x_2,x_2) = 5(x_2-4/5)\epsilon/12 + O(\epsilon^2),$$

where $\epsilon>0$ is small enough. That is: (a) for $x_2\leq 4/5$, we have $\Pi_1^*(1-x_2-\epsilon,x_2)> \Pi_1^*(1-x_2,x_2)$; (b) for $4/5<x_2<1$, $\Pi_1^*(1-x_2-\epsilon,x_2)>\Pi_1^*(1-x_2,x_2)$ and $\Pi_1^*(1-x_2+\epsilon,x_2)>\Pi_1^*(1-x_2,x_2)$; (c) for $x_2\geq 1$, $\Pi_1^*(1-x_2+\epsilon,x_2)>\Pi_1^*(1-x_2,x_2)$. Therefore, when x_1 and x_2 are symmetric about the center, x_1 is not a best reply for firm 1 against x_2 for firm 2 (in case (b), x_1 is even a local maximizer of Π_1^*). The same holds for firm 2.

[5] It may be worth noting that $(x_1^N,x_2^N)=(x_\ell,x_u)$ and $\Pi_1^*(\underline{x}^N)=\Pi_2^*(\underline{x}^N)=(x_u-x_\ell)/2$ when the strategy space is restricted to a compact interval $[x_\ell,x_u]$ such that $1/2\leq x_u\leq 5/4$ and $x_\ell+x_u=1$.

[6] When the strategy space of firm location is restricted to $x\in[0,1]$, Proposition 6 still holds, but the equilibrium values are $(x_1^N,x_2^N)=$ $(0,\ (\sqrt{33}-3)/\sqrt{2\sqrt{33}+2})$, $(1-(\sqrt{33}-3)/\sqrt{2\sqrt{33}+2},\ 1)$, and $(\Pi_1^*(\underline{x}^N),\Pi_2^*(\underline{x}^N))=$ $((15-\sqrt{33})/64,\ (207-33\sqrt{33})/64)$, $((207-33\sqrt{33})/64),\ (15-\sqrt{33})/64)$ respectively. As in the nonrestricted case, the profits in the restricted triangular case are smaller than the profit in the restricted uniform case, whose value is

1/2.

[7] Interestingly, the same result holds when the triangular distribution has positive and equal values at the market endpoints, however close it is to the uniform distribution. The result also remains valid for log-concave but convex distributions, such as exponential distributions, which are very popular in urban population distribution models.

FIGURE CAPTIONS

Figure 1 Equilibrium locations

Figure 2 Reaction functions in the triangular consumer distribution

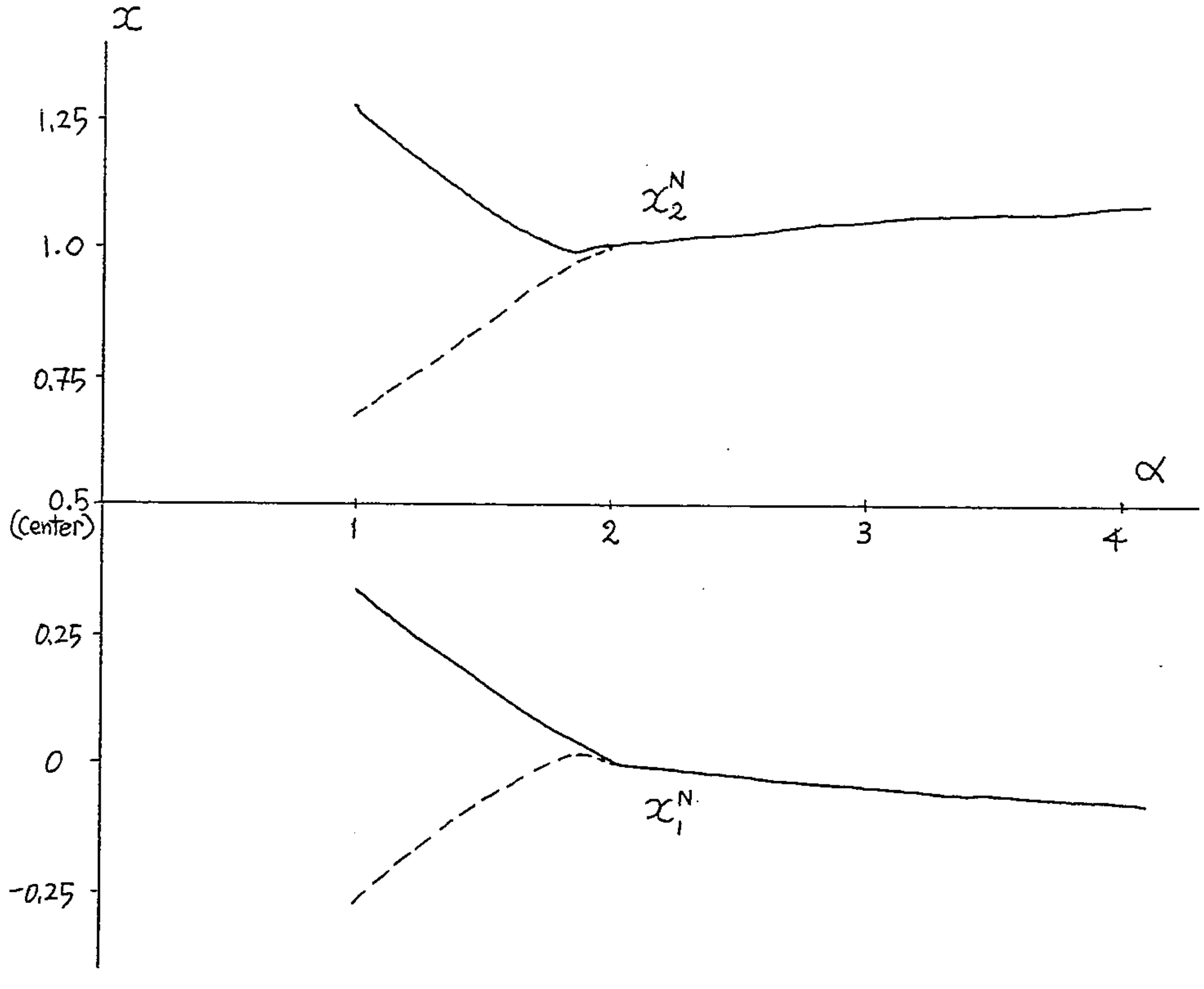

Figure 1 Equilibrium locations

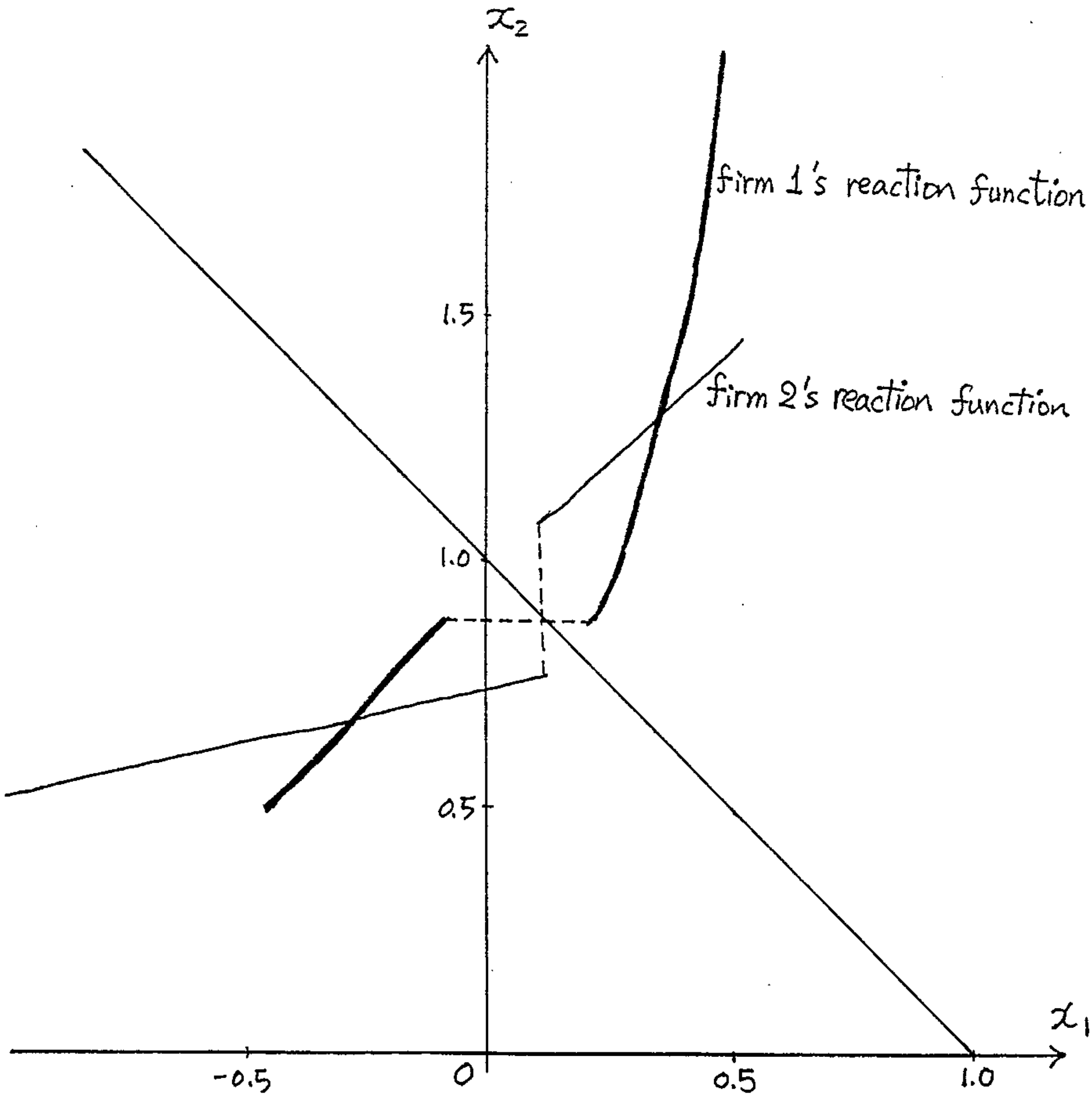

Figure 2 Reaction functions in the triangular consumer distribution