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AHP analysis of BInary and Ternary Comparisons in incomplete information

by

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1. Introduction

We cleared that AHP applied to binary or ternary comparisons is appropriate evaluation method especially for sports games [1]. In [1] we treated only complete games where every pair of teams has a match. But in such as tournament games, where only survival teams are able to have matches, every pair of teams does not necessarily have matches.

Generally in AHP we often encounter such an incomplete information problem, where some elements of comparison matrix (see [1]) lacks data. For such incomplete information problem we have Harker-Takeda method [2][3], which is a very nice method based on natural and reasonable idea.

But unfortunately applying this to some incomplete information problems, we found a decisive inconsistency. This paper aims to point out this fact and to propose a new method which we call as 2-stage method to solve this inconsistency.

2. Harker and Takeda method (H-T method)

We explain H-T method through a simple example with the following comparison matrix ;

$$(1)\quad A=\begin{bmatrix} 1 & a_{12} & a_{13} & (\ ) \\ a_{12}^{-1} & 1 & (\ ) & (\ ) \\ a_{13}^{-1} & (\ ) & 1 & a_{34} \\ (\ ) & (\ ) & a_{34}^{-1} & 1 \end{bmatrix}$$

where ( ) shows a component without data.

Let $w=[w_1\ w_2\ w_3\ w_4]$ be the principal eigenvector (The eigen vector corresponding to the maximal eigen value) of A, then a_{kl} (for (k, l) without data) must be w_k/w_l. So instead A we introduce

$$(2)\quad \tilde{A}=\begin{bmatrix} 1 & a_{12} & a_{13} & w_1/w_4 \\ a_{12}^{-1} & 1 & w_2/w_3 & w_2/w_4 \\ a_{13}^{-1} & w_3/w_2 & 1 & a_{34} \\ w_4/w_1 & w_4/w_2 & a_{34}^{-1} & 1 \end{bmatrix}$$

and calculated the principal eigen vector of $\tilde{A}$. To do so we solve the following eigen value problem.

$$(3)\quad \begin{bmatrix} 1 & a_{12} & a_{13} & w_1/w_4 \\ a_{12}^{-1} & 1 & w_2/w_3 & w_2/w_4 \\ a_{13}^{-1} & w_3/w_2 & 1 & a_{34} \\ w_4/w_1 & w_4/w_2 & a_{34}^{-1} & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

which leads to

$$(4)\quad \begin{array}{llll} 2w_1+a_{12}w_2+a_{13}w_3 & & &= \lambda w_1 \\ a_{12}^{-1}w_1+3w_1 & & &= \lambda w_2 \\ a_{13}^{-1}w_1 & +2w_3+a_{34}w_4 & &= \lambda w_3 \\ & a_{34}^{-1}w_3+3w_4 & &= \lambda w_4 . \end{array}$$

As a result we are to calculate the principal eigen vector of the following matrix.

$$(5)\quad \overline{A}= \begin{bmatrix} 2 & a_{12} & a_{13} & 0 \\ a_{12}^{-1} & 3 & 0 & 0 \\ a_{13}^{-1} & 0 & 2 & a_{34} \\ 0 & 0 & a_{34}^{-1} & 3 \end{bmatrix}$$

This is the basic idea of H-T method. Generally we can construct $\overline{A}=[\overline{a}_{ij}]$ from $A=[a_{ij}]$ the following simple rule ;

(i) $\overline{a}_{ij}=0 \quad \longleftrightarrow \quad a_{ij}=(\ \ )$

(ii) $\overline{a}_{ii}=1$ plus the number of ( ) i-th row of A

3. Examples

Example 1 Tournament of 4 teams

There are 4 teams 1, 2, 3, 4.

First team 1 and 2 have a match and team 1 wins, and team 3 and 4 have a match and team 3 wins. Next team 1 and 3 have a match and the former wins (Fig 1).

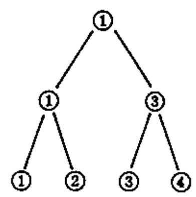


Fig 1

Then we have the following incomplete comparison matrix (→[1]),

$$(6)\quad A= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & \theta & \theta & (\ \ ) \\ \theta^{-1} & 1 & (\ \ ) & (\ \ ) \\ \theta^{-1} & (\ \ ) & 1 & \theta \\ (\ \ ) & (\ \ ) & \theta^{-1} & 1 \end{bmatrix} \end{array} \quad (\theta>1)$$

where θ is a parameter greater then unity. Applying H-T method we have

$$(7)\quad \overline{A}= \begin{bmatrix} 2 & \theta & \theta & 0 \\ \theta^{-1} & 3 & 0 & 0 \\ \theta^{-1} & 0 & 2 & \theta \\ 0 & 0 & \theta^{-1} & 3 \end{bmatrix}$$

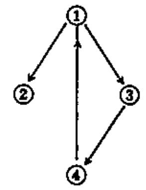
, whose principal eigen vector is

$$(8)\quad w = \begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ [& 1 & \theta^{-1} & \theta^{-1} & \theta^{-2} \;] \end{array} \qquad (\lambda_{max} = 4).$$

The result is rather strange ; By our intuition team 3 is the second winter, so it is to be stronger than team 2. But our calculation shons that team 3 is equivalent to team 2.

Example 2 Another incomplete game. Assume that we have the result shown in Fig 2. Using H-T method we have

$$(9)\quad \overline{A} = \begin{bmatrix} 1 & \theta & \theta & \theta^{-1} \\ \theta^{-1} & 3 & 0 & 0 \\ \theta^{-1} & 0 & 2 & \theta \\ \theta & 0 & \theta^{-1} & 2 \end{bmatrix}$$



(ⓘ→ⓙ means that ⓘ defeats ⓙ)

Fig 2

For $\theta = 1.5$, 2, 5 and 10 we have,

$$\begin{array}{lccccl} & 1 & 2 & 3 & 4 & \\ w = [& 1 & 0.567 & 1.06 & 1.05\ ] & (\lambda_{max} = 4.13,\ \theta = 1.5) \\ w = [& 1 & 0.355 & 1.08 & 1.06\ ] & (\lambda_{max} = 4.41,\ \theta = 2\ ) \\ w = [& 1 & 0.051 & 1.10 & 1.05\ ] & (\lambda_{max} = 6.96,\ \theta = 5\ ) \\ w = [& 1 & 0.010 & 1.01 & 1.17\ ] & (\lambda_{max} = 11.3,\ \theta = 10\ ). \end{array}$$

For any values of θ team 3 and 4 have higher evaluation than team 1, but this clearly contradicts with the fact that team 1 wins two matches and loses one, and team 3 and 4 win one match and lose one.

This example reveals a decisive defect of H-T method.

4. 2-stage method

We propose another method for incomplete information problems, which let us call 2-stage method. This is also based on natural idea and covers the defect of H-T method.

The stages of 2-stage method are as follows ;

stage 1 For the first approximation of w_i (the i-th component of principal eigen vector w) we take the geometric mean of the i-th row of the given incomplete comparison matrix (like A in (1) or (6)).

And we estimate a_{kl} for blank component (k, l) by $\overline{a}_{kl} = w_k/w_l$. Then we have an estimated complete comparsion matrix $\overline{A}$.

stage 2 Calculate pricipal eigen vector $\overline{w} = [\overline{w}_1, \cdots \overline{w}_n]$ of $\overline{A}$. Then $\overline{w}_i$ is the evaluation of the i-th object $(i = 1, 2, \cdots, n)$.

Example 3 We analyse the problem shown in Example 1 by the 2-stage method. First we calculate geometric mean of each row. The results are shown in the last column in (10).

$$(10)\quad A=\left[\begin{array}{cccc} 1 & \theta & \theta & (\ ) \\ \theta^{-1} & 1 & (\ ) & (\ ) \\ \theta^{-1} & (\ ) & 1 & \theta \\ (\ ) & (\ ) & \theta^{-1} & 1 \end{array}\right]\begin{array}{l} \theta^{2/3} \\ \theta^{-1/2} \\ 1 \\ \theta^{-1/2} \end{array}$$

So we have first approximation

$$w_1=\theta^{2/3},\ w_2=\theta^{-1/2},\ w_3=1,\ w_4=\theta^{-1/2}$$

than

$$\overline{a}_{14}=w_1/w_4=\theta^{7/6},\ \overline{a}_{23}=w_2/w_3=\theta^{-1/2},\ \overline{a}_{24}=w_2/w_4=1$$

So we have

$$(11)\quad \overline{A}=\left[\begin{array}{cccc} 1 & \theta & \theta & \theta^{7/6} \\ \theta^{-1} & 1 & \theta^{-1/2} & 1 \\ \theta^{-1} & \theta^{1/2} & 1 & \theta \\ \theta^{-5/6} & 1 & \theta^{-1} & 1 \end{array}\right]$$

For $\theta=2$ we have

$$(12)\quad \overline{A}=\left[\begin{array}{cccc} 1 & 2 & 2 & 2.245 \\ 0.5 & 1 & 0.707 & 1 \\ 0.5 & 1.414 & 1 & 2 \\ 0.4454 & 1 & 0.5 & 1 \end{array}\right],$$

whose principal eigen vector is

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ w=[& 1 & 0.442 & 0.631 & 0.396 &]. \end{array}\quad (\lambda_{max}=4.04)$$

The result fits our intuition. We have almost same results for other valuse of θ.

Example 4 We analyse the problem shown in Example 2 by 2-stage method. We follow the same procedure as Example 3 ;

$$(13)\quad A=\left[\begin{array}{cccc} 1 & \theta & \theta & \theta^{-1} \\ \theta^{-1} & 1 & (\ ) & (\ ) \\ \theta^{-1} & (\ ) & 1 & \theta \\ \theta & (\ ) & \theta^{-1} & 1 \end{array}\right]\quad \begin{array}{ll} \theta^{1/4} & =w_1 \\ \theta^{-1/2} & =w_2 \\ 1 & =w_3 \\ 1 & =w_4 \end{array}$$

$$(14)\quad \overline{A}=\left[\begin{array}{cccc} 1 & \theta & \theta & \theta^{-1} \\ \theta^{-1} & 1 & \theta^{-1/2} & \theta^{-1/2} \\ \theta^{-1} & \theta^{1/2} & 1 & \theta \\ \theta & \theta^{1/2} & \theta^{-1} & 1 \end{array}\right]$$

$$
= \begin{bmatrix} 1 & 2 & 2 & 0.5 \\ 0.5 & 1 & 0.707 & 0.707 \\ 0.5 & 1.414 & 1 & 2 \\ 2 & 1.414 & 0.5 & 1 \end{bmatrix} \qquad (\theta = 2)
$$

$$
w = [\; 1 \quad 0.537 \quad 0.928 \quad 0.946 \;] \qquad (\lambda_{max} = 4.403)
$$

Thus we have a reasonable result.

5. Conclusion and further discussion

We point out the depect of H-T method and propose 2-stage method which cover the depect through some examples. But we cannot show theoretically or analytically why H-T method has the defect and 2-stage method covers it.

For example in the field of linear statistical inference we can state that the least square method is best linear estimating method (Gauss-Markoff Theorem). But we cannot state that 2-stage method is universally better than H-T method in AHP problem.

This depends on the structure of the eigen nector method itself. Although this is very excellent, this has no model and is based on no principle, unlike the least square method is besed on the principle requiring to minimize the sum of squres of errors.

We must construct a model or a principle of eigen nector method in AHP, without which we cannot proceed further.

[references]

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[3] P.T.Harker, "Alternative Modes of Questioning in the Analytic Hierarchy Plocess", submitted to Mathematical Modelling on the AHP