

NO.408

Submodular Functions  
and  
Optimization

by

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August 1989



# Submodular Functions and Optimization

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JAPAN

July 1989

(version 1.2)



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## I. INTRODUCTION

(poly-)matroid intersection problem of J. Edmonds [Edm70], the submodular flow problem of J. Edmonds and R. Giles [Edm+Giles77], the independent flow problem of the author [Fuji78a] and the polymatroidal flow problem of R. Hassin [Hassin82] and E. L. Lawler and C. U. Martel [Lawler+Martel82].

Submodular functions are discrete analogues of convex functions. In Chapter 4 we develop a theory of submodular functions from the point of view of convex analysis [Rockafellar70], which we call the submodular analysis. We will make clear the close relationship between the submodular analysis and the results obtained in Chapter 3.

Finally we consider nonlinear optimization problems with submodular constraints in Chapter 5. A decomposition algorithm is shown for a separable convex optimization problem over a base polyhedron and it lays a basis for the algorithms of the other problems such as the lexicographically optimal base problem, the weighted max-min (min-max) problem and the fair resource allocation problem. We also consider a neoflow problem (the submodular flow problem) with a separable convex cost function.

### Basic Definitions and Notations

We denote the set of reals by  $\mathbb{R}$ , the set of rationals by  $\mathbb{Q}$  and the set of integers by  $\mathbb{Z}$ .

For any finite set  $X$  we denote its cardinality by  $|X|$ . When  $X$  is a subset of a set  $Y$ , we write  $X \subseteq Y$ , and when  $X$  is a proper subset of  $Y$  (i.e.,  $X \subseteq Y$  and  $X \neq Y$ ), we write  $X \subset Y$ .

Let  $V$  and  $A$  be finite sets, where  $V$  is called a *vertex set* and  $A$  an *arc set*. We are also given two functions  $\partial^+, \partial^-: A \rightarrow V$ . For each arc  $a \in A$   $\partial^+ a$  is the *initial end-vertex* (or the *tail*) of  $a$  and  $\partial^- a$  is the *terminal end-vertex* (or the *head*) of  $a$ . We call  $G = (V, A; \partial^+, \partial^-)$  a *graph* (or a *directed graph*). When there is no possibility of confusion, we also denote the graph by  $G = (V, A)$ . We often express an arc  $a$  by the ordered pair  $(\partial^+ a, \partial^- a)$  of the end-vertices when such a pair uniquely determines the arc.

When we do not distinguish  $\partial^+ a$  from  $\partial^- a$  for each  $a$  in  $A$  or we are not concerned with the orientations of the arcs, we call the graph  $G = (V, A)$  an *undirected graph* and call each  $a \in A$  an *edge* instead of an arc.

We define for each vertex  $v \in V$

$$\delta^+ v = \{a \mid a \in A, \partial^+ a = v\}, \quad (0.1)$$

$$\delta^- v = \{a \mid a \in A, \partial^- a = v\}. \quad (0.2)$$

Also define for each subset  $U$  of  $V$

$$\Delta^+ U = \{a \mid a \in A, \partial^+ a \in U, \partial^- a \in V - U\}, \quad (0.3)$$

## Chapter I. Introduction

### Introduction

In 1935 H. Whitney [Whit35] introduced the concept of matroid as an abstraction of the linear dependence structure of a set of vectors. Several systems of axioms for defining a matroid are now known, each of which is simple but substantial enough to yield a deep theory in Combinatorial Optimization and to have a lot of applications in practical engineering problems (see [Iri83], [Iri+Fuji81], [Murota87], [Recski]). Matroidal structures are closely related to a class of efficiently solvable combinatorial optimization problems; a careful examination of an efficiently solvable problem often reveals a matroidal structure which underlies the problem.

In 1970 J. Edmonds [Edm70] combined the matroid theory with polyhedral combinatorics and lead us to the concept of polymatroid. A polymatroid polytope, called an independence polytope, is expressed by a system of linear inequalities with  $\{0,1\}$ -coefficients and the right-hand sides given by a submodular function which is the rank function of the polymatroid. The relation between matroids and polymatroids is similar to that between matchings in bipartite graphs and flows in networks.

The rank function of any polymatroid is a monotone nondecreasing submodular function on a Boolean lattice  $2^E$  for a finite set  $E$ . The monotonicity of the rank function does not play any essential rôle in characterizing the combinatorial structure of the polymatroid polytope since the monotonicity is not invariant with respect to translations of the polytope. The concepts of submodular and supermodular systems [Fuji78b,84] naturally come up with this observation. The rank function of a submodular (or supermodular) system is a submodular function on a distributive lattice, a sublattice of a Boolean lattice. The duality is defined between a submodular system and a supermodular system, which dissolves the clumsy definition of polymatroid duality [McDiarmid75]. Submodular systems are not only theoretical generalizations of matroids and polymatroids but also significantly extend the applicability in practical problems.

In Chapter 2 we first introduce the concepts of submodular and supermodular systems and their associated base polyhedra by following the historical generalization sequence of matroids, polymatroids and submodular systems. We then examine algorithmic aspects of submodular systems and basic structures of base polyhedra.

In Chapter 3 we consider a class of network flow problems with submodular boundary constraints, which we call the neoflow problem. It includes the

$$\Delta^-U = \{a \mid a \in A, \partial^-a \in U, \partial^+a \in V - U\}. \quad (0.4)$$

A *directed path* in  $G$  is an alternating sequence  $(v_0, a_1, v_1, a_2, \dots, v_{k-1}, a_k, v_k)$  of vertices  $v_i$  ( $i = 0, 1, \dots, k$ ) and arcs  $a_i$  ( $i = 1, 2, \dots, k$ ) such that  $\partial^+a_i = v_{i-1}$ ,  $\partial^-a_i = v_i$  ( $i = 1, 2, \dots, k$ ).  $v_0$  is the *initial vertex* of the path and  $v_k$  is the *terminal vertex*. We also say that the path is *from*  $v_0$  *to*  $v_k$ . A directed path is called a *directed cycle* if its initial and terminal vertices coincide with each other.

A *bipartite graph*  $G = (V^+, V^-; A)$  is a graph with two disjoint vertex sets  $V^+$  and  $V^-$  and with an arc set  $A$  consisting of arcs  $a$  such that  $\partial^+a \in V^+$  and  $\partial^-a \in V^-$ . A *matching*  $M$  in the bipartite graph  $G$  is a subset of the arc set  $A$  such that for any distinct arcs  $a, a'$  in  $M$  we have  $\partial^+a \neq \partial^+a'$  and  $\partial^-a \neq \partial^-a'$ . Also, a *cover*  $(U^+, U^-)$  of  $G$  is the ordered pair of  $U^+ \subseteq V^+$  and  $U^- \subseteq V^-$  such that for any arc  $a \in A$  we have  $\partial^+a \in U^+$  or  $\partial^-a \in U^-$ .

A graph  $G = (V, A)$  is a *directed tree* if it is connected and for each vertex  $v \in V$  there exists at most one arc  $a \in A$  such that  $\partial^-a = v$ . In a directed tree  $G = (V, A)$  there exists one and only one vertex  $v_0 \in V$  such that  $\delta^-v_0 = \emptyset$ . We call  $v_0$  the *root* of the directed tree  $G$ .



## Chapter II. Submodular Systems and Base Polyhedra

In this chapter we give basic concepts on matroids, polymatroids and submodular systems and show the natural generalization sequences of these concepts from matroids to submodular systems. We also examine the fundamental combinatorial structures of submodular systems and associated polyhedra.

For basic properties of matroids and polymatroids shown without proofs in this chapter, readers should be referred to [Tutte65], [Lawler76], [Welsh76]. The knowledge about matroids and polymatroids is not prerequisite to understanding submodular systems, though it certainly helps.

For general information on submodular and supermodular functions see, e.g., [Edm70], [Faigle87], [Frank + Tardos88], [Fuji84c], and [Lovász83].

### 1. From Matroids to Submodular Systems

#### 1.1. Matroids

The concept of matroid was introduced in 1935 by H. Whitney [Whit35] and independently by B. L. van der Waerden [Waerden37]. The term, matroid, is due to Whitney. As the term indicates, a matroid is an abstraction of linear independence and dependence structure of the set of columns of a matrix.

Let  $E$  be a finite set. Suppose that a family  $\mathcal{I}$  of subsets of  $E$  satisfies the following (I0)–(I2):

$$(I0) \quad \emptyset \in \mathcal{I}.$$

$$(I1) \quad I_1 \subseteq I_2 \in \mathcal{I} \implies I_1 \in \mathcal{I}.$$

$$(I2) \quad I_1, I_2 \in \mathcal{I}, |I_1| < |I_2| \implies \exists e \in I_2 - I_1: I_1 \cup \{e\} \in \mathcal{I}.$$

We call the pair  $(E, \mathcal{I})$  a *matroid*. Each  $I \in \mathcal{I}$  is called an *independent set* of matroid  $(E, \mathcal{I})$  and  $\mathcal{I}$  the *family of independent sets* of matroid  $(E, \mathcal{I})$ .

An independent set which is maximal in  $\mathcal{I}$  with respect to set inclusion is called a *base*.

A subset of  $E$  which is not an independent set is called a *dependent set*. A dependent set which is minimal with respect to set inclusion is called a *circuit*.

We define the *rank function*  $\rho: 2^E \rightarrow \mathbb{Z}$  of matroid  $(E, \mathcal{I})$  by

$$\rho(X) = \max\{|I| \mid I \subseteq X, I \in \mathcal{I}\} \quad (1.1)$$

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for each  $X \subseteq E$ . The rank function  $\rho$  satisfies the following  $(\rho 0)$ – $(\rho 2)$ :

$$(\rho 0) \quad \forall X \subseteq E: 0 \leq \rho(X) \leq |X|.$$

$$(\rho 1) \quad X \subseteq Y \subseteq E \implies \rho(X) \leq \rho(Y).$$

$$(\rho 2) \quad \forall X, Y \subseteq E: \rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y).$$

Any function  $\rho$  satisfying  $(\rho 2)$  is called a *submodular function* on  $2^E$ . From  $(\rho 0)$ – $(\rho 2)$  we see that  $\rho$  has the *unit-increase property*, i.e., for any  $X, Y \subseteq E$  with  $X \subseteq Y$  and  $|X| + 1 = |Y|$  we have  $\rho(Y) = \rho(X)$  or  $\rho(Y) = \rho(X) + 1$ .

Also define the *closure function*  $\text{cl}: 2^E \rightarrow 2^E$  of matroid  $(E, \mathcal{I})$  by

$$\text{cl}(X) = \{e \mid e \in E, \rho(X \cup \{e\}) = \rho(X)\} \quad (1.2)$$

for each  $X \subseteq E$ .

For any independent set  $I \in \mathcal{I}$  and any element  $e \in \text{cl}(I) - I$ , there exists a unique circuit contained in  $I \cup \{e\}$ . Such a circuit is called *the fundamental circuit* with respect to  $I$  and  $e$ , and is denoted by  $C(I|e)$ . For any  $e' \in C(I|e)$ ,  $(I \cup \{e\}) - \{e'\}$  is an independent set.

It is well known (see [Welsh76]) that each of the family  $\mathcal{I}$  of independent sets, the family  $\mathcal{B}$  of bases, the family  $\mathcal{C}$  of circuits, the rank function  $\rho$  and the closure function  $\text{cl}$  uniquely determines the matroid which defines it. Giving a system of axioms for each of the family  $\mathcal{B}$  of bases, the family  $\mathcal{C}$  of circuits, the rank function  $\rho$  and the closure function  $\text{cl}$ , we can define a matroid. We denote such a matroid by  $(E, \mathcal{B})$ ,  $(E, \mathcal{C})$ ,  $(E, \rho)$  and  $(E, \text{cl})$ , respectively. For example,  $(\rho 0)$ – $(\rho 2)$  give the system of axioms for the rank function  $\rho$  of a matroid. Any integer-valued function  $\rho: 2^E \rightarrow \mathbb{Z}$  satisfying  $(\rho 0)$ – $(\rho 2)$  defines a matroid  $(E, \rho)$  with the family  $\mathcal{I}$  of independent sets given by

$$\mathcal{I} = \{I \mid I \subseteq E, \rho(I) = |I|\}. \quad (1.3)$$

For a matroid  $M = (E, \mathcal{B})$  with a family  $\mathcal{B}$  of bases, the family  $\mathcal{B}^*$  of the complements of bases of the matroid is also a family of bases of a matroid. The matroid  $M^* = (E, \mathcal{B}^*)$  is called the *dual matroid* of  $M = (E, \mathcal{B})$ . The dual of  $M^*$  is equal to  $M$ , i.e.,  $(M^*)^* = M$ .

### Examples of a Matroid

(1) For a graph  $G = (V, E)$  with a vertex set  $V$  and an edge set  $E$  let  $G(\mathcal{I})$  be the set of those edge subsets each of which does not contain any cycle of  $G$ . Then  $M(G) = (E, G(\mathcal{I}))$  is a matroid with  $G(\mathcal{I})$  being the family of independent sets. A matroid which can be obtained in this way is called a *graphic matroid*. Given an independence oracle for the membership in the family of independent sets, the graphicness of a matroid is efficiently discerned and if

## 1.2. POLYMATROIDS

graphic, a graph representing it can also efficiently be constructed by combining the algorithms of P. D. Seymour [Seymour80] and the author [Fuji80a] (also see [Bixby+Wagner88]).

(2) For a matrix  $A = [a_1, \dots, a_n]$  over some field with column vectors  $a_1, \dots, a_n$  let  $E = \{1, \dots, n\}$  be the column index set and  $A(\mathcal{I})$  be the set of those subsets of  $E$  each of which forms independent column vectors. Then  $M(A) = (E, \mathcal{I}(A))$  is a matroid with  $\mathcal{I}(A)$  being the family of independent sets. A matroid which can be obtained in this way is called a *matric matroid* or a *linear matroid*. A graphic matroid is a matric matroid represented by the incidence matrix of the associated graph.

(3) For a bipartite graph  $G = (V^+, V^-; A)$  with left and right end-vertex sets  $V^+$  and  $V^-$  and an arc set  $A$ . Define

$$\mathcal{I}^+ = \{\partial^+ M \mid M: \text{a matching of } G\}, \quad (1.4)$$

where  $\partial^+ M$  is the set of left end-vertices of matching  $M$ . Then  $M^+ = (V^+, \mathcal{I}^+)$  is a matroid with  $\mathcal{I}^+$  being the family of independent sets.  $M^+$  is called a *transversal matroid*. Transversal matroids are matric.

(4) Let  $E$  be a finite set with cardinality  $|E| = n > 0$ . For any nonnegative integer  $k \leq n$  define

$$\mathcal{I}_k = \{I \mid I \subseteq E, |I| \leq k\}. \quad (1.5)$$

Then  $(E, \mathcal{I}_k)$  is a matroid. It is called a *uniform matroid* of rank  $k$  and is usually denoted by  $U_{k,n}$ . In particular,  $U_{n,n} = (E, 2^E)$  is called a *free matroid* and  $U_{0,n} = (E, \{\emptyset\})$  a *trivial matroid*.

## 1.2. Polymatroids

Let  $E$  be a finite set and  $\rho$  be a function from  $2^E$  to  $\mathbf{R}$ . Here,  $\mathbf{R}$  is the set of reals but throughout this monograph  $\mathbf{R}$  can be any totally ordered additive group such as the set  $\mathbf{Z}$  of integers, the set  $\mathbf{Q}$  of rationals etc. unless otherwise stated. Suppose that the function  $\rho: 2^E \rightarrow \mathbf{R}$  satisfies

$$(\rho 0) \quad \rho(\emptyset) = 0.$$

$$(\rho 1) \quad X \subseteq Y \subseteq E \implies \rho(X) \leq \rho(Y).$$

$$(\rho 2) \quad \forall X, Y \subseteq E: \rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y).$$

The pair  $(E, \rho)$  is called a *polymatroid* and  $\rho$  the *rank function* of the polymatroid ([Edm70]). The rank function  $\rho$  is a monotone nondecreasing submodular function on  $2^E$  with  $\rho(\emptyset) = 0$  and does not necessarily have the unit-increase property as the rank function of a matroid. When  $\rho$  is the rank function of a matroid, polymatroid  $(E, \rho)$  is called *matroidal*.

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Define

$$P_{(+)}(\rho) = \{x \mid x \in \mathbb{R}^E, x \geq 0, \forall X \subseteq E: x(X) \leq \rho(X)\}, \quad (1.6)$$

where for each  $X \subseteq E$  and  $x \in \mathbb{R}^E$

$$x(X) = \sum_{e \in X} x(e). \quad (1.7)$$

$P_{(+)}(\rho)$  is called the *independence polyhedron* associated with polymatroid  $(E, \rho)$ . Also define

$$B(\rho) = \{x \mid x \in P_{(+)}(\rho), x(E) = \rho(E)\}. \quad (1.8)$$

We call  $B(\rho)$  the *base polyhedron* associated with polymatroid  $(E, \rho)$ . The base polyhedron  $B(\rho)$  is always nonempty. (See Fig. 1.1.)

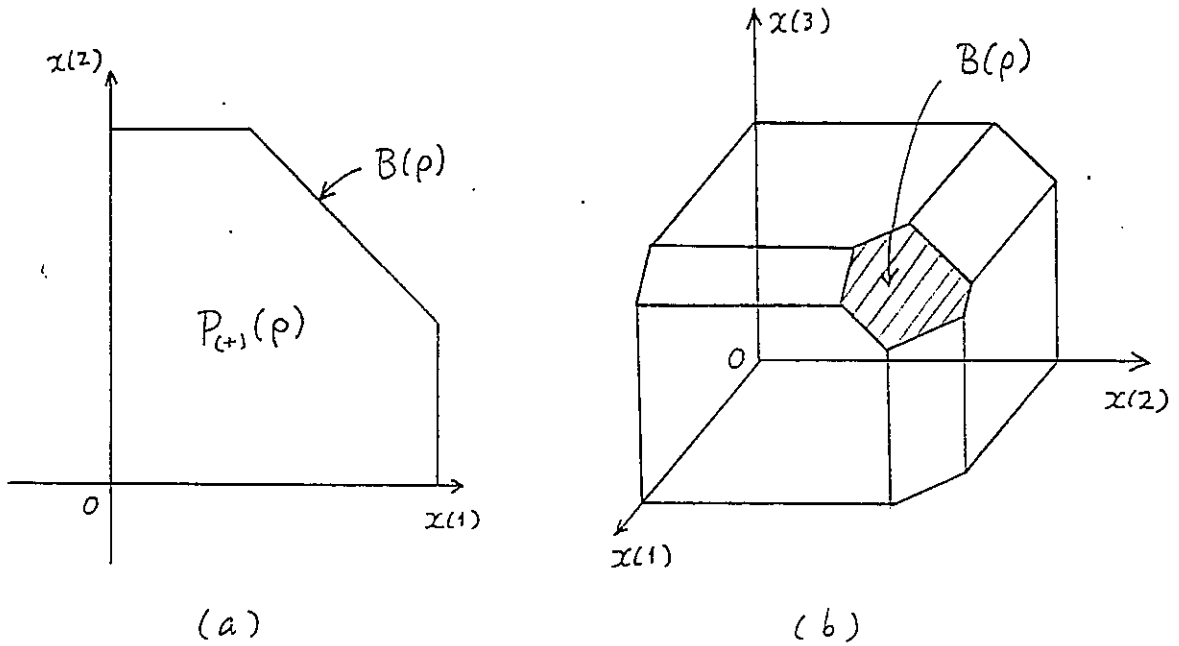


Figure 1.1.

It can be shown (see [Edm70]) that the convex hull in  $\mathbb{R}^E$  of the characteristic vectors of the independent sets (or bases) of a matroid on  $E$  with the rank function  $\rho$ , where  $\mathbb{R}$  is the set of reals, is the independence polyhedron (or the base polyhedron) associated with the matroidal polymatroid  $(E, \rho)$ .

Each  $x \in P_{(+)}(\rho)$  is called an *independent vector* and each  $x \in B(\rho)$  a *base* of polymatroid  $(E, \rho)$ .



## 1.2. POLYMATROIDS

For an independent vector  $x \in P_{(+)}(\rho)$  define

$$\mathcal{D}(x) = \{X \mid X \subseteq E, x(X) = \rho(X)\}. \quad (1.9)$$

$\mathcal{D}(x)$  is closed with respect to set union and intersection, i.e.,  $X, Y \in \mathcal{D}(x) \implies X \cup Y, X \cap Y \in \mathcal{D}(x)$ . For, if  $X, Y \in \mathcal{D}(x)$ ,

$$0 = \rho(X) - x(X) + \rho(Y) - x(Y) \geq \rho(X \cup Y) - x(X \cup Y) + \rho(X \cap Y) - x(X \cap Y) \geq 0, \quad (1.10)$$

where  $\rho(X \cup Y) - x(X \cup Y) \geq 0$  and  $\rho(X \cap Y) - x(X \cap Y) \geq 0$  since  $x \in P_{(+)}(\rho)$ . It follows that  $X \cup Y, X \cap Y \in \mathcal{D}(x)$ . So,  $\mathcal{D}(x)$  is a distributive lattice with set union and intersection as the lattice operations, join and meet. Denote the unique maximal element of  $\mathcal{D}(x)$  by  $\text{sat}(x)$ , i.e.,

$$\text{sat}(x) = \bigcup \{X \mid X \subseteq E, x(X) = \rho(X)\}. \quad (1.11)$$

The function,  $\text{sat}: P_{(+)}(\rho) \rightarrow 2^E$ , is called the *saturation function* (see [Fuji78a]). Informally,  $\text{sat}(x)$  is the set of the saturated components of  $x$ . More precisely,

$$\text{sat}(x) = \{e \mid e \in E, \forall \alpha > 0: x + \alpha \chi_e \notin P_{(+)}(\rho)\}, \quad (1.12)$$

where  $\chi_e$  is the unit vector with  $\chi_e(e) = 1$  and  $\chi_e(e') = 0$  ( $e' \in E - \{e\}$ ). The saturation function is a generalization of the closure function of a matroid.

For an independent vector  $x \in P_{(+)}(\rho)$  and an element  $e \in \text{sat}(x)$  define

$$\mathcal{D}(x, e) = \{X \mid e \in X \subseteq E, x(X) = \rho(X)\}. \quad (1.13)$$

We have  $\mathcal{D}(x, e) \subseteq \mathcal{D}(x)$  and  $\mathcal{D}(x, e)$  is a sublattice of  $\mathcal{D}(x)$ . Denote the unique minimal element of  $\mathcal{D}(x, e)$  by  $\text{dep}(x, e)$ , i.e.,

$$\text{dep}(x, e) = \bigcap \{X \mid e \in X \subseteq E, x(X) = \rho(X)\}. \quad (1.14)$$

For each  $e \in E - \text{sat}(x)$  we define  $\text{dep}(x, e) = \emptyset$ . The function,  $\text{dep}: P_{(+)}(\rho) \times E \rightarrow 2^E$ , is called the *dependence function* (see [Fuji78a]). For  $x \in P_{(+)}(\rho)$  and  $e \in \text{sat}(x)$ ,

$$\text{dep}(x, e) = \{e' \mid e' \in E, \exists \alpha > 0: x + \alpha(\chi_e - \chi_{e'}) \in P_{(+)}(\rho)\}. \quad (1.15)$$

The dependence function is a generalization of a fundamental circuit of a matroid.

For  $x \in P_{(+)}(\rho)$  and  $e \in E - \text{sat}(x)$  define

$$\hat{c}(x, e) = \max\{\alpha \mid \alpha \in \mathbb{R}, x + \alpha \chi_e \in P_{(+)}(\rho)\}, \quad (1.16)$$

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which is called the *saturation capacity* associated with  $x$  and  $e$ . The saturation capacity is also expressed as

$$\hat{c}(x, e) = \min\{\rho(X) - x(X) \mid e \in X \subseteq E\}. \quad (1.17)$$

For any  $\alpha$  such that  $0 \leq \alpha \leq \hat{c}(x, e)$  we have  $x + \alpha\chi_e \in P_{(+)}(\rho)$ .

For  $x \in P_{(+)}(\rho)$ ,  $e \in \text{sat}(x)$  and  $e' \in \text{dep}(x, e) - \{e\}$ , define

$$\tilde{c}(x, e, e') = \max\{\alpha \mid \alpha \in \mathbb{R}, x + \alpha(\chi_e - \chi_{e'}) \in P_{(+)}(\rho)\}, \quad (1.18)$$

which is called the *exchange capacity* associated with  $x$ ,  $e$  and  $e'$ . The exchange capacity is also expressed as

$$\tilde{c}(x, e, e') = \min\{\rho(X) - x(X) \mid e \in X \subseteq E, e' \notin X\}. \quad (1.19)$$

(See Fig. 1.2.) For any  $\alpha$  such that  $0 \leq \alpha \leq \tilde{c}(x, e, e')$  we have  $x + \alpha(\chi_e - \chi_{e'}) \in P_{(+)}(\rho)$ .

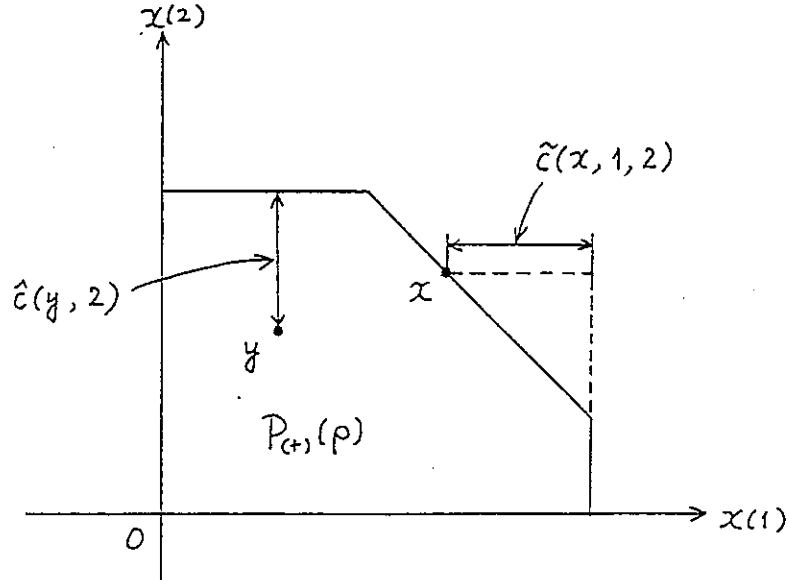


Figure 1.2.

A vector  $v \in \mathbb{R}^E$  such that  $x \leq v$  for any  $x \in P_{(+)}(\rho)$  is called a *dominating vector* of  $(E, \rho)$ . For a dominating vector  $v$  of the polymatroid  $P = (E, \rho)$  define  $\rho_{(v)}^*: 2^E \rightarrow \mathbb{R}$  by

$$\rho_{(v)}^*(X) = v(X) + \rho(E - X) - \rho(E) \quad (X \subseteq E). \quad (1.20)$$

## 1.2. POLYMATROIDS

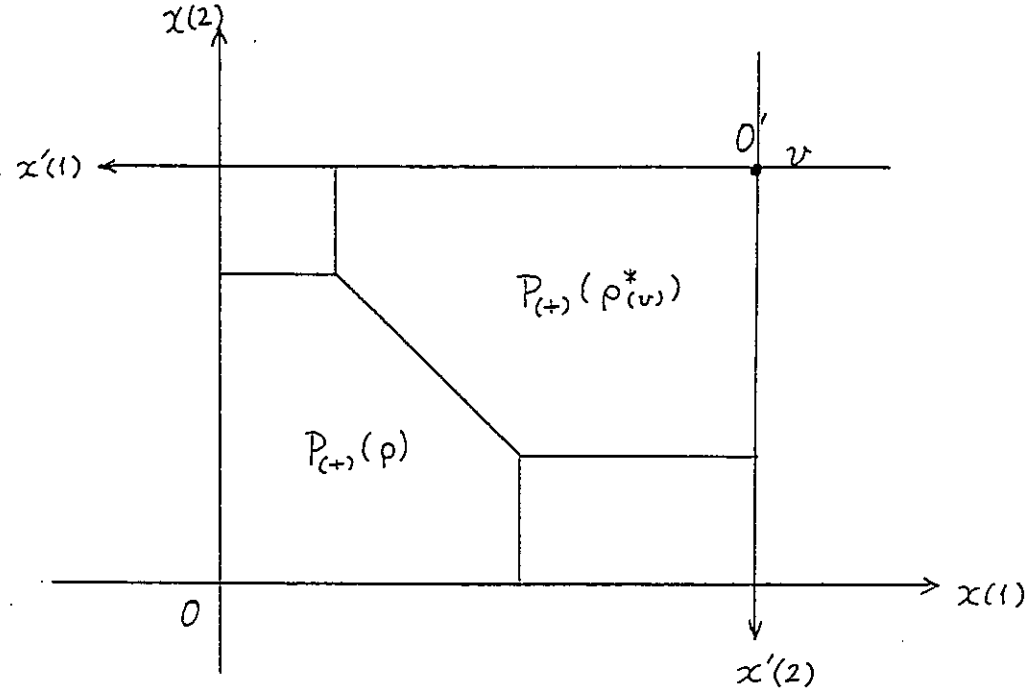


Figure 1.3.

Then  $P_{(+)}^* = (E, \rho_{(+)}^*)$  is a polymatroid and is called the *dual polymatroid* of  $P = (E, \rho)$  with respect to  $v$  ([McDiarmid75]). (See Fig. 1.3.)

For most polymatroids there is no reasonable and physically meaningful way of choosing a dominating vector  $v$  and the arbitrariness of dual polymatroids remains, though the choice of  $v = (\rho(\{e\}): e \in E)$  may be reasonable. For matroidal polymatroids, the matroid duality corresponds to the polymatroid duality with respect to the vector  $\mathbf{1}$  of all the components being equal to 1.

### Examples of a Polymatroid

(1) **Multiterminal flows:** For a capacitated flow network  $\mathcal{N} = (G = (V, A), c, S^+, S^-)$  with the underlying graph  $G = (V, A)$ , a nonnegative capacity function  $c: A \rightarrow \mathbb{R}_+$  and  $S^+$  and  $S^-$  are, respectively, the set of entrances and the set of exits, where  $S^+, S^- \subseteq V$  and  $S^+ \cap S^- = \emptyset$ . We assume without loss of generality that there is no arc entering  $S^+$  or leaving  $S^-$ . A function  $\varphi: A \rightarrow \mathbb{R}$  is a *feasible flow* in  $\mathcal{N}$  if

$$\forall a \in A: 0 \leq \varphi(a) \leq c(a), \quad (1.21)$$

$$\forall v \in V - (S^+ \cup S^-): \partial\varphi(v) = 0, \quad (1.22)$$

## II. SUBMODULAR SYSTEMS AND BASE POLYHEDRA

where  $\partial\varphi: V \rightarrow \mathbf{R}$  is the *boundary* of  $\varphi$  defined by

$$\partial\varphi(v) = \sum_{a \in \delta^+ v} \varphi(a) - \sum_{a \in \delta^- v} \varphi(a) \quad (v \in V). \quad (1.23)$$

Here,  $\delta^+ v$  (or  $\delta^- v$ ) is the set of the arcs whose initial (or terminal) end-vertices are  $v$ . It is shown (see [Megiddo74]) that the set  $\{(\partial\varphi)^{S^+} \mid \varphi: \text{a feasible flow in } \mathcal{N}\}$  is the independence polyhedron of a polymatroid on the set  $S^+$  of entrances, where  $(\partial\varphi)^{S^+}$  is the restriction of  $\partial\varphi$  on  $S^+$ . Similarly, we have a polymatroid on the set  $S^-$  of exits.

(2) **Linear polymatroids:** Let  $A$  be a matrix with the column index set  $E$ . Suppose that  $E$  is partitioned into nonempty disjoint subsets  $F_1, F_2, \dots, F_n$ , and define  $\tilde{E} = \{1, 2, \dots, n\}$ . Also define for each  $X \subseteq \tilde{E}$

$$\rho(X) = \text{rank } A^X, \quad (1.24)$$

where  $A^X$  is the submatrix of  $A$  formed by the columns of  $A$  with the indices in  $\bigcup\{F_i \mid i \in X\}$ . We see that  $(\tilde{E}, \rho)$  is a polymatroid, which is called a *linear polymatroid*.

(3) **Entropy functions:** Let  $x_1, x_2, \dots, x_n$  be random variables taking on values in a finite set  $\{1, 2, \dots, N\}$ . For the set  $E = \{x_1, \dots, x_n\}$  of the random variables, define for each nonempty subset  $X$  of  $E$

$$h(X) = \begin{cases} \text{the entropy of } X \text{ in the Shannon sense} & (X \neq \emptyset) \\ 0 & (X = \emptyset). \end{cases} \quad (1.25)$$

For example, if  $X = \{x_1, x_2, \dots, x_k\}$  ( $1 \leq k \leq n$ ),

$$h(X) = - \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N p(x_1 = i_1, \dots, x_k = i_k) \log_2 p(x_1 = i_1, \dots, x_k = i_k), \quad (1.26)$$

where  $p(x_1 = i_1, \dots, x_k = i_k)$  is the probability of the event that  $x_1 = i_1, \dots, x_k = i_k$ . The function  $h: 2^E \rightarrow \mathbf{R}_+$  is called an *entropy function*. Entropy function  $h$  is a monotone nondecreasing submodular function with  $h(\emptyset) = 0$ , i.e.,  $(E, h)$  is a polymatroid. See [Fuji78c] for polymatroidal problems in the Shannon information theory.

(4) **Convex games:** Consider a characteristic-function game (see, e.g., [Shubik82]). Let  $E = \{1, 2, \dots, n\}$  be a set of  $n$  persons, called *players*. A characteristic function  $v$  is a nonnegative function defined on the set of coalitions which are subsets of  $E$ .

### 1.3. SUBMODULAR SYSTEMS

A characteristic-function game  $(E, v)$  is called a *convex game* ([Shapley]) if the characteristic function  $v$  is supermodular, i.e.,

$$\forall X, Y \subseteq E: v(X) + v(Y) \leq v(X \cup Y) + v(X \cap Y). \quad (1.27)$$

The *core* of the game  $(E, v)$  is the set of payoff vectors defined by

$$\{x \mid x \in \mathbb{R}^E, \forall X \subseteq E: x(X) \geq v(X), x(E) = v(E)\}. \quad (1.28)$$

Define the function  $v^\# : 2^E \rightarrow \mathbb{R}$  by

$$v^\#(E - X) = v(E) - v(X) \quad (X \subseteq E). \quad (1.29)$$

Then we can show that  $v^\#$  is a polymatroid rank function and that the core given by (1.28) coincides with the base polyhedron  $B(v^\#)$  of the polymatroid.

### 1.3. Submodular Systems

Let  $E$  be a finite set and  $\mathcal{D}$  be a collection of subsets of  $E$  which forms a distributive lattice with set union and intersection as the lattice operations, join and meet, i.e., for each  $X, Y \in \mathcal{D}$  we have  $X \cup Y, X \cap Y \in \mathcal{D}$ . We assume that  $\emptyset, E \in \mathcal{D}$ , i.e.,  $\emptyset$  and  $E$  are, respectively, the unique minimal and maximal elements of  $\mathcal{D}$ . Let  $f: \mathcal{D} \rightarrow \mathbb{R}$  be a submodular function on the distributive lattice  $\mathcal{D}$ , i.e.,

$$\forall X, Y \in \mathcal{D}: f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \quad (1.30)$$

We have the following fundamental lemma concerning submodular functions.

**Lemma 1.1:** Let  $f: \mathcal{D} \rightarrow \mathbb{R}$  be an arbitrary submodular function on a distributive lattice  $\mathcal{D} \subseteq 2^E$ . Then the set of all the minimizers of  $f$  given by

$$\mathcal{D}_0 = \{X \mid X \in \mathcal{D}, f(X) = \min\{f(Y) \mid Y \in \mathcal{D}\}\} \quad (1.31)$$

forms a sublattice of  $\mathcal{D}$ , i.e., for any  $X, Y \in \mathcal{D}_0$  we have  $X \cup Y, X \cap Y \in \mathcal{D}_0$ .

(Proof) For any  $X, Y \in \mathcal{D}_0$ ,

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \quad (1.32)$$

where  $\min\{f(X \cup Y), f(X \cap Y)\} \geq f(X) = f(Y)$ . Hence we must have  $f(X \cup Y) = f(X \cap Y) = f(X) (= f(Y))$ , i.e.,  $X \cup Y, X \cap Y \in \mathcal{D}_0$ . Q.E.D.

## II. SUBMODULAR SYSTEMS AND BASE POLYHEDRA

For a submodular function  $f$  on a distributive lattice  $\mathcal{D} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = 0$ , we call the pair  $(\mathcal{D}, f)$  a *submodular system* on  $E$ , and  $f$  the *rank function* of the submodular system (see [Fuji78b, 84]). We call  $f(E)$  the *rank* of  $(\mathcal{D}, f)$ .

Define a polyhedron in  $\mathbb{R}^E$  by

$$P(f) = \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{D}: x(X) \leq f(X)\}, \quad (1.33)$$

We call  $P(f)$  the *submodular polyhedron* associated with submodular system  $(\mathcal{D}, f)$ . Also define

$$B(f) = \{x \mid x \in P(f), x(E) = f(E)\}. \quad (1.34)$$

We call  $B(f)$  the *base polyhedron* associated with submodular system  $(\mathcal{D}, f)$ . (See Fig. 1.4.)

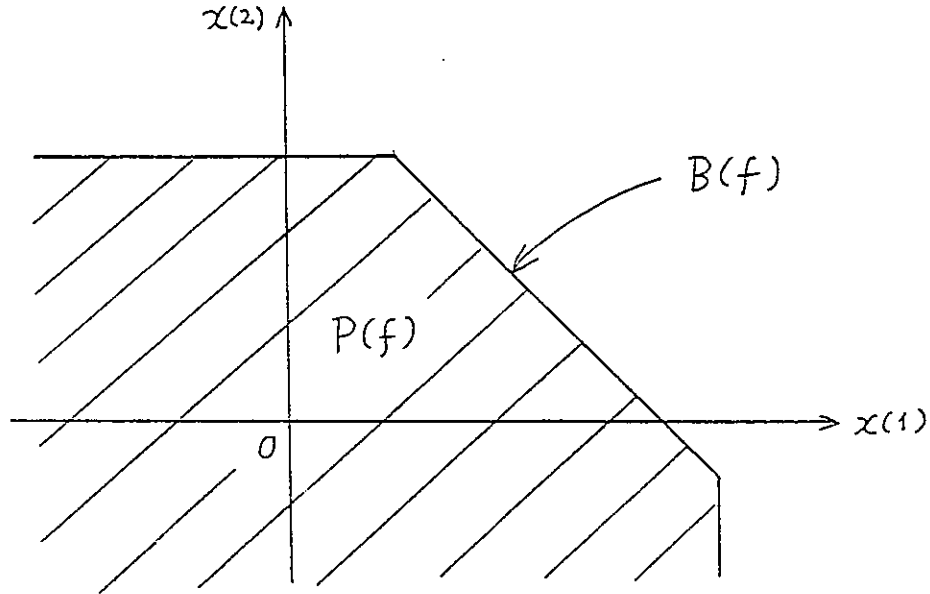


Figure 1.4.

**Proposition 1.2:** The base polyhedron  $B(f)$  is the set of all the maximal vectors in the submodular polyhedron  $P(f)$ . In particular,  $B(f) \neq \emptyset$ . Here, the order  $\leq$  among vectors in  $\mathbb{R}^E$  is defined as follows: for  $x, y \in \mathbb{R}^E$  we have  $x \leq y$  if and only if  $x(e) \leq y(e)$  for all  $e \in E$ .

### 1.3. SUBMODULAR SYSTEMS

A vector in the base polyhedron  $B(f)$  is called a *base* of  $(\mathcal{D}, f)$  and a vector in the submodular polyhedron  $P(f)$  is called a *subbase* of  $(\mathcal{D}, f)$ . From Proposition 1.2 we see that for any subbase  $x$  of  $(\mathcal{D}, f)$  there exists a base  $y$  of  $(\mathcal{D}, f)$  such that  $x \leq y$ .

The saturation function, the dependence function, the saturation capacity and the exchange capacity introduced for polymatroids can easily be extended for submodular systems. For any subbase  $x \in P(f)$  define

$$\mathcal{D}(x) = \{X \mid x \in \mathcal{D}, x(X) = f(X)\}. \quad (1.35)$$

Then  $\mathcal{D}(x)$  is a sublattice of  $\mathcal{D}$ . (This follows from Lemma 1.1 since  $\mathcal{D}(x)$  is the set of minimizers of the nonnegative submodular function  $f - x: \mathcal{D} \rightarrow \mathbb{R}$ .) The unique maximal element of  $\mathcal{D}(x)$  is denoted by  $\text{sat}(x)$ .  $\text{sat}: P(f) \rightarrow 2^E$  is the *saturation function*.

For any subbase  $x \in P(f)$  and  $e \in \text{sat}(x)$ , define

$$\mathcal{D}(x, e) = \{X \mid e \in X \in \mathcal{D}, x(X) = f(X)\}. \quad (1.36)$$

Then  $\mathcal{D}(x, e)$  is a sublattice of  $\mathcal{D}$ . (Note that  $\mathcal{D}(x, e)$  is the set of minimizers of the nonnegative submodular function  $f - x$  on the distributive lattice  $\mathcal{D}(e) \equiv \{X \mid e \in X \in \mathcal{D}\}$ .) The unique minimal element of  $\mathcal{D}(x, e)$  is denoted by  $\text{dep}(x, e)$ . For any  $e \in E - \text{sat}(x)$  we define  $\text{dep}(x, e) = \emptyset$ .  $\text{dep}: P(f) \times E \rightarrow 2^E$  is the *dependence function*.

For any  $x \in P(f)$  and  $e \in E - \text{sat}(x)$  the *saturation capacity*  $\hat{c}(x, e)$  is defined by

$$\hat{c}(x, e) = \min\{f(X) - x(X) \mid e \in X \in \mathcal{D}\}. \quad (1.37)$$

For a nonnegative  $\alpha$ , we have  $x + \alpha\chi_e \in P(f)$  if and only if  $0 \leq \alpha \leq \hat{c}(x, e)$ .

Moreover, for any  $x \in P(f)$ ,  $e \in \text{sat}(x)$  and  $e' \in \text{dap}(x, e) - \{e\}$  the *exchange capacity*  $\tilde{c}(x, e, e')$  is defined by

$$\tilde{c}(x, e, e') = \min\{f(X) - x(X) \mid e \in X \in \mathcal{D}, e' \notin X\}. \quad (1.38)$$

For a nonnegative  $\alpha$ , we have  $x + \alpha(\chi_e - \chi_{e'}) \in P(f)$  if and only if  $0 \leq \alpha \leq \tilde{c}(x, e, e')$ . (Note that if  $x \in B(f)$ , then  $x + \alpha(\chi_e - \chi_{e'}) \in P(f)$  implies  $x + \alpha(\chi_e - \chi_{e'}) \in B(f)$ .)

A function  $g: \mathcal{D} \rightarrow \mathbb{R}$  on the distributive lattice  $\mathcal{D}$  is called a *supermodular function* if  $-g$  is a submodular function, i.e.,

$$\forall X, Y \in \mathcal{D}: g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y). \quad (1.39)$$

The pair  $(\mathcal{D}, g)$  is called a *supermodular system*. Define

$$P(g) = \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{D}: x(X) \geq g(X)\}, \quad (1.40)$$

## II. SUBMODULAR SYSTEMS AND BASE POLYHEDRA

$$B(g) = \{x \mid x \in P(g), x(E) = g(E)\}. \quad (1.41)$$

$P(g)$  is called the *supermodular polyhedron* and  $B(g)$  the *base polyhedron* associated with the supermodular system  $(\mathcal{D}, g)$ . A vector in  $P(g)$  is called a *superbase* and a vector in  $B(g)$  a *base* of  $(\mathcal{D}, g)$ .

A function which is submodular and at the same time supermodular is called a *modular function*. For a modular function  $x: \mathcal{D} \rightarrow \mathbb{R}$ ,  $P(x)$  should be considered as either a submodular polyhedron or a supermodular polyhedron according as we consider  $x$  as a submodular function or a supermodular function. There will be no confusion by the use of this notation.

If we consider the dual order  $\leq^*$  of the ordinary order  $\leq$  among  $\mathbb{R}$  (where the dual order  $\leq^*$  is the one such that  $\alpha \leq^* \beta$  if and only if  $\beta \leq \alpha$ , for each  $\alpha, \beta \in \mathbb{R}$ ), then a supermodular function  $g: \mathcal{D} \rightarrow \mathbb{R}$  with respect to  $(\mathbb{R}, \leq)$  is a submodular function with respect to  $(\mathbb{R}, \leq^*)$ . In the same way the supermodular polyhedron  $P(g)$  and a superbase  $x \in P(g)$  with respect to  $(\mathbb{R}, \leq)$  is a submodular polyhedron and a subbase with respect to  $(\mathbb{R}, \leq^*)$ .

For a submodular system  $(\mathcal{D}, f)$  on  $E$  define a function  $f^\#: \overline{\mathcal{D}} \rightarrow \mathbb{R}$  as follows.

$$\overline{\mathcal{D}} = \{E - X \mid X \in \mathcal{D}\}, \quad (1.42)$$

$$f^\#(E - X) = f(E) - f(X) \quad (X \in \mathcal{D}). \quad (1.43)$$

$f^\#$  is a supermodular function on the dual lattice  $\overline{\mathcal{D}}$  of  $\mathcal{D}$ . We call the pair  $(\overline{\mathcal{D}}, f^\#)$  the *dual supermodular system* of  $(\mathcal{D}, f)$ . (See Fig. 1.5.) Similarly, we define the *dual submodular system*  $(\overline{\mathcal{D}}, g^\#)$  of a supermodular system  $(\mathcal{D}, g)$ , where  $\mathcal{D}$  is defined by (1.42) and  $g^\#$  by (1.43) with  $f$  replaced by  $g$ .

**Lemma 1.3:** We have  $B(f) = B(f^\#)$  and  $(f^\#)^\# = f$ .

The proof is immediate and is omitted. It should be noted that Lemma 1.3 holds for *any* function  $f$  on  $\mathcal{D}$  without submodularity.

The submodular/supermodular polyhedron and the base polyhedron of a submodular system also arise from more general functions. A family  $\mathcal{F} \subseteq 2^E$  is called an *intersecting family* if for each intersecting  $X, Y \in \mathcal{F}$  (i.e.,  $X, Y \in \mathcal{F}$  and  $X \cap Y \neq \emptyset$ ) we have  $X \cup Y, X \cap Y \in \mathcal{F}$ . A function  $f: \mathcal{F} \rightarrow \mathbb{R}$  on the intersecting family  $\mathcal{F}$  is called *intersecting-submodular* if for each intersecting  $X, Y \in \mathcal{F}$  we have the submodularity inequality

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \quad (1.44)$$

Moreover, a family  $\mathcal{F} \subseteq 2^E$  is called a *crossing family* if for each crossing  $X, Y \in \mathcal{F}$  (i.e.,  $X, Y \in \mathcal{F}$ ,  $X \cap Y \neq \emptyset$ ,  $X - Y \neq \emptyset$ ,  $Y - X \neq \emptyset$ , and  $X \cup Y \neq E$ ) we have  $X \cup Y, X \cap Y \in \mathcal{F}$ . A function  $f: \mathcal{F} \rightarrow \mathbb{R}$  on the crossing



### 1.3. SUBMODULAR SYSTEMS

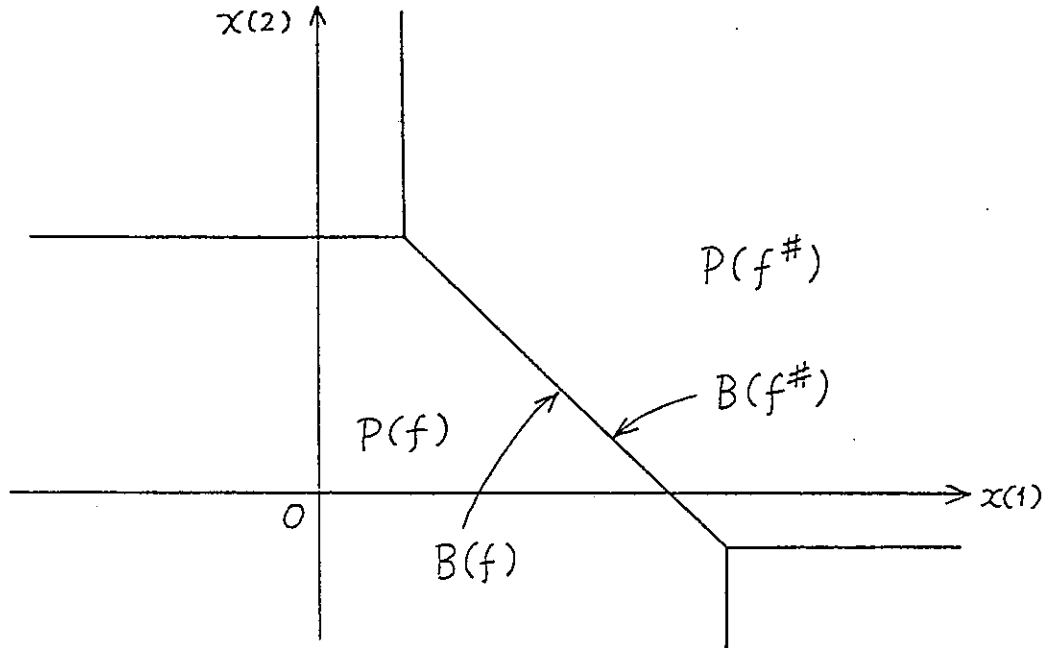


Figure 1.5.

family  $\mathcal{F}$  is called *crossing-submodular* if for each crossing  $X, Y \in \mathcal{F}$  we have the submodularity inequality (1.44) (see [Edm+Giles77]). Note that the term, *submodular*, without any modifier is used for such a function  $f: \mathcal{D} \rightarrow \mathbb{R}$  on a distributive lattice  $\mathcal{D}$  that satisfies (1.44) for all  $X, Y \in \mathcal{D}$ .

The following theorem, due to the author [Fuji84b], concerning intersecting- and crossing-submodular functions on intersecting and crossing families plays a very important rôle in the combinatorial optimization problems described by intersecting- or crossing-submodular functions on intersecting or crossing families, and reveals the essential combinatorial structures of the problems (which will also be seen in Chapter III).

**Theorem 1.4** [Fuji84b]:

(i) Let  $f$  be an intersecting-submodular function on an intersecting family  $\mathcal{F} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{F}$  and  $f(\emptyset) = 0$ . Define

$$P(f) = \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{F}: x(X) \leq f(X)\}. \quad (1.45)$$

Then there exists a unique submodular system  $(\mathcal{D}_1, f_1)$  on  $E$  such that

$$P(f) = P(f_1). \quad (1.46)$$

Moreover, if  $f$  is integer-valued, then  $f_1$  is also integer-valued.

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(ii) Let  $f$  be a crossing-submodular function on a crossing family  $\mathcal{F} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{F}$  and  $f(\emptyset) = 0$ . Define

$$B(f) = \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{F}: x(X) \leq f(X), x(E) = f(E)\} \quad (1.47)$$

and suppose  $B(f) \neq \emptyset$ . Then there exists a unique submodular system  $(\mathcal{D}_2, f_2)$  on  $E$  such that

$$B(f) = B(f_2). \quad (1.48)$$

Moreover, if  $f$  is integer-valued, so is  $f_2$ .

**Theorem 1.5** [Fuji84b]:

(i) Let  $f$  be an intersecting-submodular function on an intersecting family  $\mathcal{F} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{F}$  and  $f(\emptyset) = 0$ . Then the rank function  $f_1$  of the submodular system  $(\mathcal{D}_1, f_1)$  in (i) of Theorem 1.4 is given by

$$f_1(Y) = \min \left\{ \sum_{i \in I} f(X_i) \mid \{X_i \mid i \in I\}: \text{a partition of } Y, \right. \\ \left. X_i \in \mathcal{F} (i \in I) \right\} \quad (1.49)$$

for each  $Y \subseteq E$ , where  $f_1(Y) = +\infty$  (i.e., the minimum is taken over the empty set) if and only if  $Y \notin \mathcal{D}_1$ .

(ii) Let  $f$  be a crossing-submodular function on a crossing family  $\mathcal{F} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{F}$  and  $f(\emptyset) = 0$ . Define

$$f_p(E) = \min \left\{ \sum_{i \in I} f(X_i) \mid \{X_i \mid i \in I\}: \text{a partition of } E, \right. \\ \left. X_i \in \mathcal{F} (i \in I) \right\}, \quad (1.50)$$

$$(f^\#)_p(E) = \max \left\{ \sum_{i \in I} f^\#(X_i) \mid \{X_i \mid i \in I\}: \text{a partition of } E, \right. \\ \left. E - X_i \in \mathcal{F} (i \in I) \right\}, \quad (1.51)$$

where  $f^\#(X) = f(E) - f(E - X)$  ( $E - X \in \mathcal{F}$ ). Then  $B(f)$  defined by (1.47) is nonempty if and only if

$$f(E) = f_p(E) = (f^\#)_p(E) \quad (1.52)$$

or

$$\sum_{j \in J} f^\#(Y_j) \leq f(E) \leq \sum_{i \in I} f(X_i) \quad (1.53)$$

### 1.3. SUBMODULAR SYSTEMS

for any partitions  $\{X_i \mid i \in I\}$  and  $\{Y_j \mid j \in J\}$  of  $E$  with  $X_i \in \mathcal{F}$  ( $i \in I$ ) and  $E - Y_j \in \mathcal{F}$  ( $j \in J$ ).

Moreover, if  $B(f) \neq \emptyset$ , the rank function  $f_2$  of  $(\mathcal{D}_2, f_2)$  in (ii) of Theorem 1.4 is given, in terms of its dual, by

$$\begin{aligned} f_2^\#(Y) &= f(E) - f_2(E - Y) \\ &= \max \left\{ \sum_{i \in I} (f_p)^\#(X_i) \mid \{X_i \mid i \in I\}: \text{a partition of } Y, \right. \\ &\quad \left. E - X_i \in \mathcal{F}_p (i \in I) \right\}, \end{aligned} \quad (1.54)$$

where

$$\begin{aligned} f_p(Y) &= \min \left\{ \sum_{i \in I} f(X_i) \mid \{X_i \mid i \in I\}: \text{a partition of } Y, \right. \\ &\quad \left. X_i \in \mathcal{F} (i \in I) \right\} \quad (Y \subseteq E), \end{aligned} \quad (1.55)$$

$$\mathcal{F}_p = \{X \mid X \subseteq E, f_p(X) < +\infty\}, \quad (1.56)$$

$$(f_p)^\#(X) = f(E) - f_p(E - X) \quad (E - X \in \mathcal{F}_p), \quad (1.57)$$

$$\mathcal{D}_2 = \{X \mid X \subseteq E, f_2(X) < +\infty\} \quad (1.58)$$

and the minimum (or maximum) taken over the empty set is equal to  $+\infty$  (or  $-\infty$ ).

Since  $B(f) = B(f^\#)$ , beginning with  $f^\#$  instead of  $f$ , we can also obtain a dual formula for  $f_2$ .

It should be noted that for a crossing-submodular function  $f$  on a crossing family  $\mathcal{F}$  the polyhedron  $P(f)$  defined by (1.45) may not be a submodular polyhedron but that any intersection of  $P(f)$  with a hyperplane  $x(E) = k$  (const.), if not empty, is a base polyhedron (see Fig. 1.6).

#### Examples of a Submodular System

Matroids and polymatroids are examples of a submodular system. We show some non-polymatroidal submodular systems.

(1) **Cut functions:** A typical non-polymatroidal submodular system arises from network flows.

Let  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$  be a capacitated network with the underlying graph  $G = (V, A)$  and the lower and upper capacity functions  $\underline{c}: A \rightarrow \mathbf{R} \cup \{-\infty\}$  and  $\bar{c}: A \rightarrow \mathbf{R} \cup \{+\infty\}$  such that  $\underline{c}(a) \leq \bar{c}(a)$  for each arc  $a \in A$ . Define a function  $\kappa_{\underline{c}, \bar{c}}: 2^E \rightarrow \mathbf{R} \cup \{+\infty\}$  by

$$\kappa_{\underline{c}, \bar{c}}(U) = \sum_{a \in \Delta^+ U} \bar{c}(a) - \sum_{a \in \Delta^- U} \underline{c}(a) \quad (U \subseteq V), \quad (1.59)$$

## II. SUBMODULAR SYSTEMS AND BASE POLYHEDRA

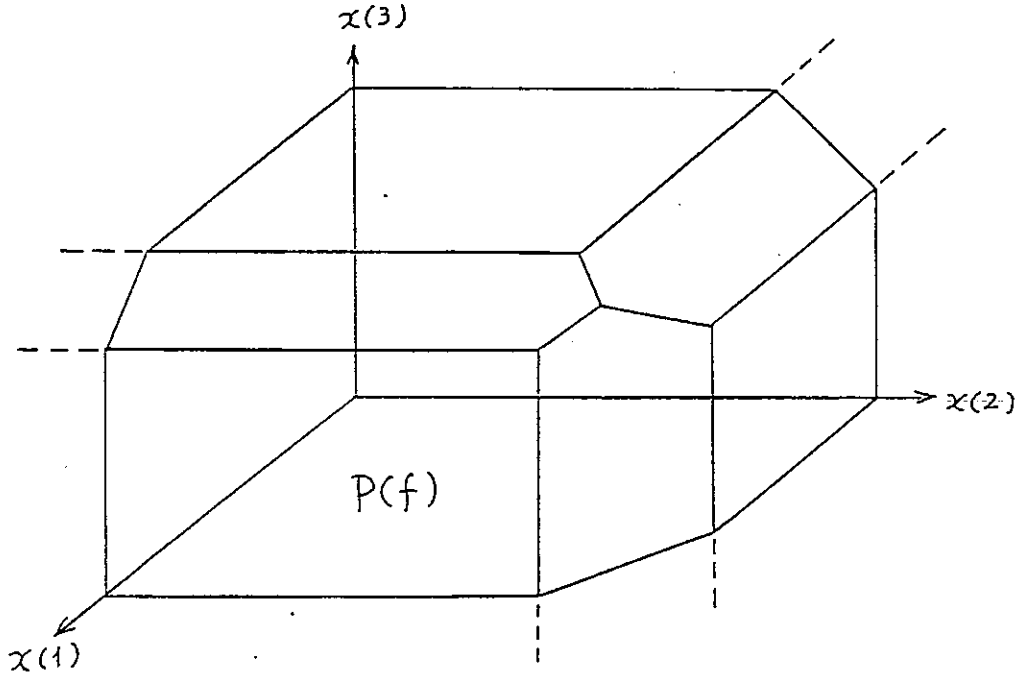


Figure 1.6.

where  $\Delta^+U$  is the set of arcs leaving  $U$  in  $G$  and  $\Delta^-U$  is the set of arcs entering  $U$  in  $G$ . For each  $U, W \in 2^V$  such that  $\kappa_{\underline{c}, \bar{c}}(U) < +\infty$  and  $\kappa_{\underline{c}, \bar{c}}(W) < +\infty$ , we have  $\kappa_{\underline{c}, \bar{c}}(U \cup W) < +\infty$ ,  $\kappa_{\underline{c}, \bar{c}}(U \cap W) < +\infty$  and

$$\begin{aligned} & \kappa_{\underline{c}, \bar{c}}(U) + \kappa_{\underline{c}, \bar{c}}(W) - \kappa_{\underline{c}, \bar{c}}(U \cup W) - \kappa_{\underline{c}, \bar{c}}(U \cap W) \\ &= \sum \{ \bar{c}(a) - \underline{c}(a) \mid a \in A, \partial^+ a \in U \cap W, \partial^- a \in V - (U \cup W) \} \\ & \quad + \sum \{ \bar{c}(a) - \underline{c}(a) \mid a \in A, \partial^- a \in U \cap W, \partial^+ a \in V - (U \cup W) \} \\ & \geq 0. \end{aligned} \tag{1.60}$$

Therefore,  $\mathcal{D}(\underline{c}, \bar{c}) \subseteq 2^E$  defined by

$$\mathcal{D}(\underline{c}, \bar{c}) = \{U \mid U \subseteq V, \kappa_{\underline{c}, \bar{c}}(U) < +\infty\} \tag{1.61}$$

is a distributive lattice with  $\emptyset, V \in \mathcal{D}(\underline{c}, \bar{c})$ . Denoting the restriction of  $\kappa_{\underline{c}, \bar{c}}$  to  $\mathcal{D}(\underline{c}, \bar{c})$  by  $\kappa_{\underline{c}, \bar{c}}$  again, we have a submodular function  $\kappa_{\underline{c}, \bar{c}}$  on the distributive lattice  $\mathcal{D}(\underline{c}, \bar{c})$ , where  $\kappa_{\underline{c}, \bar{c}}(\emptyset) = \kappa_{\underline{c}, \bar{c}}(V) = 0$ . We call  $\kappa_{\underline{c}, \bar{c}}$  the *cut function* associated with network  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$ . The cut function  $\kappa_{\underline{c}, \bar{c}}$  is not monotone nondecreasing for nontrivial networks. A *feasible flow*  $\varphi$  in  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$  is a function  $\varphi: A \rightarrow \mathbb{R}$  such that  $\underline{c}(a) \leq \varphi(a) \leq \bar{c}(a)$  for any  $a \in A$ . The set of the boundaries  $\partial\varphi$  of feasible flows  $\varphi$  in  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$  is

## 2.1. FUNDAMENTAL OPERATIONS ON SUBMODULAR SYSTEMS

given by

$$\begin{aligned}\partial\Phi &\equiv \{\partial\varphi \mid \varphi: A \rightarrow \mathbb{R}, \forall a \in A: \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a)\} \\ &= B(\kappa_{\underline{c}, \bar{c}}),\end{aligned}\tag{1.62}$$

which is the base polyhedron associated with the submodular system  $(\mathcal{D}(\underline{c}, \bar{c}), \kappa_{\underline{c}, \bar{c}})$ . (See (1.23) for the definition of the boundary  $\partial\varphi$ .)

The fact that each base in  $B(\kappa_{\underline{c}, \bar{c}})$  is expressed as the boundary  $\partial\varphi$  of a feasible flow  $\varphi$  in  $\mathcal{N}$  can be shown by the use of the feasible circulation theorem of A. Hoffman [Hoffman60] as follows.

For any  $x \in B(\kappa_{\underline{c}, \bar{c}})$ , consider a new vertex  $s \notin V$  and new arcs  $(s, v)$  ( $v \in V$ ), and define  $\underline{c}(s, v) = \bar{c}(s, v) = x(v)$  ( $v \in V$ ). Denote the augmented network by  $\mathcal{N}' = (G' = (V \cup \{s\}, A \cup \{(s, v) \mid v \in V\}), \underline{c}, \bar{c})$ . There exists a feasible circulation (a feasible flow with the zero boundary) in  $\mathcal{N}'$  if (and only if) for every  $U \subseteq V \cup \{s\}$  we have

$$\sum_{a \in \Delta^+ U} \bar{c}(a) \geq \sum_{a \in \Delta^- U} \underline{c}(a),\tag{1.63}$$

where  $\Delta^+$  and  $\Delta^-$  are with respect to  $G'$ . (1.63) is equivalent to  $x \in B(\kappa_{\underline{c}, \bar{c}})$ . Therefore, there exists a feasible circulation  $\varphi$  in  $\mathcal{N}'$ . Restricting  $\varphi$  to  $A$ , we obtain a required feasible flow in  $\mathcal{N}$  whose boundary is equal to  $x$ .

The converse,  $\partial\Phi \subseteq B(\kappa_{\underline{c}, \bar{c}})$ , is immediate.

**(2) Cross-free families:** For a finite set  $E$  let  $\mathcal{F} \subseteq 2^E$  be a cross-free family, i.e., for each  $X, Y \in \mathcal{F}$   $X$  and  $Y$  do not cross, where we assume  $\emptyset, E \in \mathcal{F}$ . Then for any function  $f: \mathcal{F} \rightarrow \mathbb{R}$  with  $f(\emptyset) = 0$ , if  $B(f) \equiv \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{F}: x(X) \leq f(X), x(E) = f(E)\} \neq \emptyset$ ,  $B(f)$  is a base polyhedron due to Theorem 1.4, since  $\mathcal{F}$  is a crossing family and  $f$  is a crossing-submodular function.

Moreover, if  $\mathcal{F}$  is *laminar*, i.e., for each  $X, Y \in \mathcal{F}$   $X \cap Y \neq \emptyset$  implies  $X \subseteq Y$  or  $X \supseteq Y$ , then for any function  $f$  on  $\mathcal{F}$   $P(f) \equiv \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{F}: x(X) \leq f(X)\}$  is a submodular polyhedron due to Theorem 1.4. Note that laminar  $\mathcal{F}$  is an intersecting family and that  $f$  is an intersecting-submodular function on  $\mathcal{F}$ .

## 2. Submodular Systems

In this section we give basic properties of submodular systems.

### 2.1. Fundamental Operations on Submodular Systems

We show several fundamental operations on a submodular system  $(\mathcal{D}, f)$  on  $E$ .

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### (a) Reductions and contractions by sets

For any  $A \in \mathcal{D}$  define

$$\mathcal{D}^A = \{X \mid A \supseteq X \in \mathcal{D}\}, \quad (2.1)$$

$$f^A(X) = f(X) \quad (X \in \mathcal{D}^A). \quad (2.2)$$

Then  $(\mathcal{D}^A, f^A)$  is a submodular system on  $A$  and is called the *reduction* or the *restriction* of  $(\mathcal{D}, f)$  to  $A$ . We denote  $(\mathcal{D}^A, f^A)$  by  $(\mathcal{D}, f) \cdot A$  or  $(\mathcal{D}, f) - (E - A)$ .

Also define for  $A \in \mathcal{D}$

$$\mathcal{D}_A = \{X - A \mid A \subseteq X \subseteq \mathcal{D}\}, \quad (2.3)$$

$$f_A(X) = f(X \cup A) - f(A) \quad (X \in \mathcal{D}_A). \quad (2.4)$$

Then we have a submodular system  $(\mathcal{D}_A, f_A)$  on  $E - A$ , which is called the *contraction* of  $(\mathcal{D}, f)$  by  $A$  and is denoted by  $(\mathcal{D}, f)/A$  or  $(\mathcal{D}, f) \times (E - A)$ .

We call a submodular system obtained by repeated reductions and/or contractions of  $(\mathcal{D}, f)$  by sets a *set minor* of  $(\mathcal{D}, f)$ .

**Lemma 2.1:** For any  $A \in \mathcal{D}$  let  $x^A$  be a base of the reduction  $(\mathcal{D}, f) \cdot A$  of submodular system  $(\mathcal{D}, f)$  to  $A$  and  $x_A$  be a base of the contraction  $(\mathcal{D}, f)/A$  of  $(\mathcal{D}, f)$  by  $A$ . Then the direct sum  $\hat{x} = x^A \oplus x_A$  of  $x^A$  and  $x_A$  defined by

$$(x^A \oplus x_A)(e) = \begin{cases} x^A(e) & (e \in A) \\ x_A(e) & (e \in E - A) \end{cases} \quad (2.5)$$

is a base of  $(\mathcal{D}, f)$ . Conversely, for any base  $\hat{x}$  of  $(\mathcal{D}, f)$  satisfying  $\hat{x}(A) = f(A)$ , restricting  $\hat{x}$  on  $A$  (or on  $E - A$ ) yields a base of  $(\mathcal{D}, f) \cdot A$  (or  $(\mathcal{D}, f)/A$ ).

(Proof) Easy.

Q.E.D.

**Lemma 2.2:** For any  $X \in \mathcal{D}$  there exists a subbase  $x \in P(f)$  such that  $x(X) = f(X)$ . Furthermore, for any  $X \in 2^E - \mathcal{D}$  and any  $K > 0$  there exists a subbase  $X \in P(f)$  such that  $x(X) \geq K$ .

(Proof) The first half follows from Lemma 2.1. The second half is shown as follows. For any  $X \in 2^E - \mathcal{D}$  there exist  $e \in X$  and  $e' \in E - X$  such that for each  $Y \in \mathcal{D}$  with  $e \in Y$  we have  $e' \in Y$ . Hence for any  $y \in P(f)$ ,  $y_d \equiv y + d(\chi_e - \chi_{e'})$  belongs to  $P(f)$  for any  $d > 0$ . Therefore, the value,  $y_d(X) = y(X) + d$ , can be made arbitrarily large.

Q.E.D.

### (b) Reductions and contractions by vectors

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For any vector  $x \in \mathbb{R}^E$  define a function  $f^x: 2^E \rightarrow \mathbb{R}$  by

$$f^x(X) = \min\{f(Z) + x(X - Z) \mid X \supseteq Z \in \mathcal{D}\} \quad (2.6)$$

for each  $X \subseteq E$ . Then the function  $f^x: 2^E \rightarrow \mathbb{R}$  is a submodular function on the Boolean lattice  $2^E$ . For, denoting by  $Z_X$  a minimizer  $Z$  of the right-hand side of (2.6), we have for each  $X, Y \subseteq E$

$$\begin{aligned} f^x(X) + f^x(Y) &= f(Z_X) + x(X - Z_X) + f(Z_Y) + x(Y - Z_Y) \\ &\geq f(Z_X \cup Z_Y) + x((X \cup Y) - (Z_X \cup Z_Y)) \\ &\quad + f(Z_X \cap Z_Y) + x((X \cap Y) - (Z_X \cap Z_Y)) \\ &\geq f^x(X \cup Y) + f^x(X \cap Y). \end{aligned} \quad (2.7)$$

We call the submodular system  $(2^E, f^x)$  the *reduction of  $(\mathcal{D}, f)$  by vector  $x$* .

Define

$$P(f)^x = \{y \mid y \in P(f), y \leq x\}, \quad (2.8)$$

which is the set of subbases of  $(\mathcal{D}, f)$  smaller than or equal to  $x$  (see Fig. 2.1).

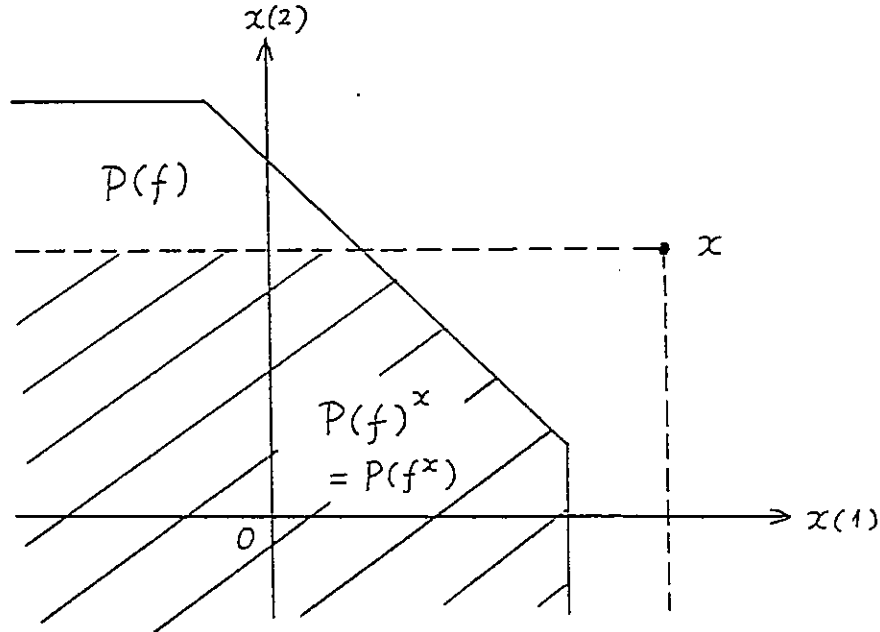


Figure 2.1.

**Theorem 2.3:** The submodular polyhedron associated with the reduction  $(2^E, f^x)$  of  $(\mathcal{D}, f)$  by vector  $x$  is given by

$$P(f^x) = P(f)^x. \quad (2.9)$$

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When  $x$  is a superbase of  $(\mathcal{D}, f)$ , i.e.  $x \in P(f^\#)$ , then

$$B(f^x) = B(f)^x, \quad (2.10)$$

where  $B(f)^x = \{y \mid y \in B(f), y \leq x\}$ .

(Proof) For any  $y \in P(f^x)$ , we see from (2.6)

$$\forall X \in \mathcal{D}: y(X) \leq f(X), \quad (2.11)$$

$$\forall e \in E: y(e) \leq x(e). \quad (2.12)$$

Hence  $y \in P(f)^x$ . Conversely, for any  $y \in P(f)^x$  we have (2.11) and (2.12). For any  $Z, X \subseteq E$  with  $X \supseteq Z \in \mathcal{D}$ ,

$$y(X) = y(Z) + y(X - Z) \leq f(Z) + x(X - Z). \quad (2.13)$$

Hence  $y \in P(f^x)$ . Moreover, (2.10) follows from (2.9) since from Proposition 1.2 there exists a base  $y \in B(f)$  such that  $y \leq x$ . Q.E.D.

For a supermodular system  $(\mathcal{D}, g)$  on  $E$  we define the reduction of  $(\mathcal{D}, g)$  by a vector  $x \in \mathbb{R}^E$  in a dual manner. Define  $g_x: 2^E \rightarrow \mathbb{R}$  by

$$g_x(X) = \max\{g(Z) + x(X - Z) \mid X \supseteq Z \in \mathcal{D}\}, \quad (2.14)$$

$$P(g)_x = \{y \mid y \in P(g), y \geq x\}. \quad (2.15)$$

Then  $(2^E, g_x)$  is the *reduction* of  $(\mathcal{D}, g)$  by  $x$  and its associated supermodular polyhedron is given by

$$P(g_x) = P(g)_x \quad (2.16)$$

due to Theorem 2.1.

From Lemma 2.2, Theorem 2.3 and (2.6) we have the following min-max relation.

$$\min\{f(Z) + x(E - Z) \mid Z \in \mathcal{D}\} = \max\{y(E) \mid y \in P(f), y \leq x\}. \quad (2.17)$$

In Particular, for  $x = \mathbf{0}$  (2.17) becomes

$$\min\{f(Z) \mid Z \in \mathcal{D}\} = \max\{y(E) \mid y \in P(f), y \leq \mathbf{0}\}. \quad (2.18)$$

Next, for any subbase  $x \in P(f)$  define a function  $f_x: 2^E \rightarrow \mathbb{R}$  by

$$f_x(X) = \min\{f(Z) - x(Z - X) \mid X \subseteq Z \in \mathcal{D}\} \quad (2.19)$$



## 2.1. FUNDAMENTAL OPERATIONS ON SUBMODULAR SYSTEMS

for each  $X \subseteq E$ . Similarly as in (2.7), we see that  $f_x: 2^E \rightarrow \mathbb{R}$  is a submodular function on the Boolean lattice  $2^E$ . We call the submodular system  $(2^E, f_x)$  the *contraction of  $(\mathcal{D}, f)$  by vector  $x \in P(f)$* .

**Theorem 2.4:** For any subbase  $x \in P(f)$  we have

$$(f_x)^\# = (f^\#)_x, \quad (2.20)$$

$$B(f_x) = B(f)_x. \quad (2.21)$$

(Proof) Since  $x \in P(f)$ , we have  $f_x(E) = f(E)$ . (2.20) easily follows from (2.19) and the definition of the dual function. Furthermore, from (2.20), Theorem 2.3 and the duality shown in Lemma 1.3 together with the remarks given after it, we have

$$B(f_x) = B((f_x)^\#) = B((f^\#)_x) = B(f^\#)_x = B(f)_x \quad (2.22)$$

Q.E.D.

The contraction of  $(\mathcal{D}, f)$  by  $x \in P(f)$  corresponds to the reduction of its dual supermodular system  $(\overline{\mathcal{D}}, f^\#)$  by  $x$  (see Fig 2.2).

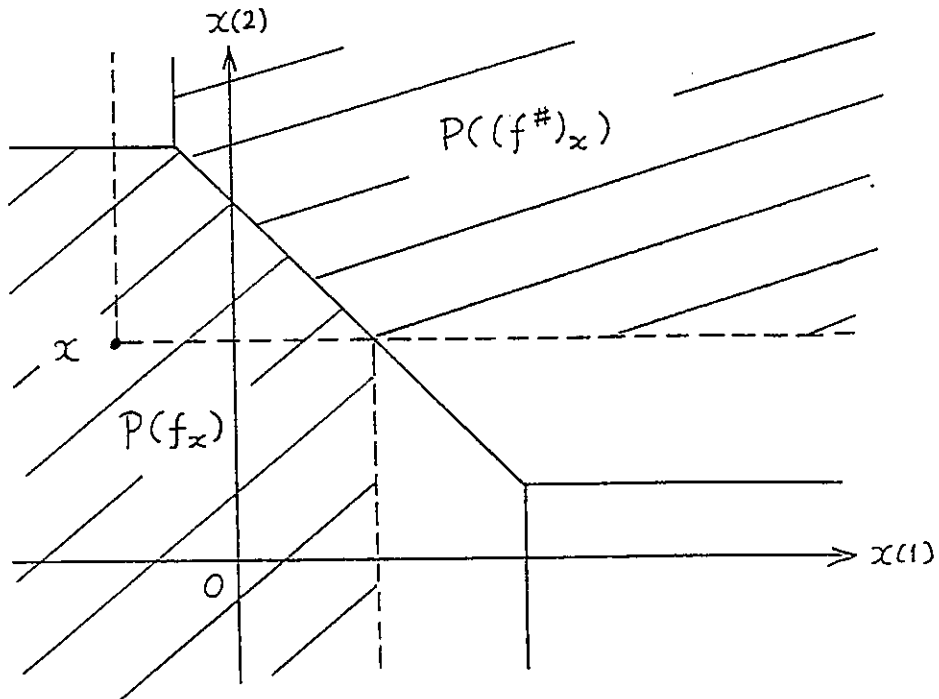


Figure 2.2.

## II. SUBMODULAR SYSTEMS AND BASE POLYHEDRA

For any subbase  $x \in P(f)$  define

$$[P(f)]_x = \{y \mid y \in \mathbf{R}^E, x \vee y \in P(f)\}, \quad (2.23)$$

where  $x \vee y$  is the vector in  $\mathbf{R}^E$  defined by  $(x \vee y)(e) = \max\{x(e), y(e)\}$  ( $e \in E$ ), i.e., the join of  $x$  and  $y$  in the vector lattice  $\mathbf{R}^E$ . We can show

$$P(f_x) = [P(f)]_x. \quad (2.24)$$

In a dual manner we define the contraction  $(2^E, g^x)$  of a supermodular system  $(\mathcal{D}, g)$  by a vector  $x \in P(g)$ , where  $g^x = ((g^\#)^x)^\#$ .

A submodular system obtained by repeated reductions and/or contractions of submodular system  $(\mathcal{D}, f)$  by vectors in  $\mathbf{R}^E$  is called a *vector minor* of  $(\mathcal{D}, f)$ .

**Theorem 2.5:** If vectors  $x, y \in \mathbf{R}^E$  satisfy (i)  $x \leq y$ , (ii)  $B(f)_x \neq \emptyset$  and (iii)  $B(f)_y \neq \emptyset$ , then we have  $B(f)_x^y (= (B(f)_x)^y = (B(f)_y)_x) \neq \emptyset$ .

(Proof) Since  $x \in P(f)$ ,  $y \in P(f^\#)$  and  $x \leq y$ , we have  $y \in P(f^\#)_x = P((f_x)^\#)$ . Hence,  $B(f)_x^y = B(f_x)_y = B((f_x)^\#)_y \neq \emptyset$ . Q.E.D.

For any  $x \in \mathbf{R}^E$  the rank of the reduction of  $(\mathcal{D}, f)$  by  $x$  is denoted by  $r_f(x) (= f^x(E))$ . The function  $r_f: \mathbf{R}^E \rightarrow \mathbf{R}$  is called the *vector rank function* of  $(\mathcal{D}, f)$ . From the definition,  $r_f$  is a concave function (see (2.6)).

### (c) Translations and sums

For any vector  $x \in \mathbf{R}^E$  the *translation* of a submodular system  $(\mathcal{D}, f)$  by  $x$  is the submodular system whose rank function is given by  $f + x: \mathcal{D} \rightarrow \mathbf{R}$ , where  $x$  should be considered as a set function (a modular function) on  $2^E$  by  $x(X) = \sum_{e \in X} x(e)$  ( $X \subseteq E$ ). For the translation  $(\mathcal{D}, f + x)$ ,

$$P(f + x) = P(f) + \{x\}, \quad (2.25)$$

$$B(f + x) = B(f) + \{x\}, \quad (2.26)$$

where  $x$  in the right-hand sides is in  $\mathbf{R}^E$  and the sums in the right-hand sides denote the vector sum (see Fig. 2.3).

It should be noted that a contraction of  $(\mathcal{D}, f)$  by  $x \in P(f)$  followed by a translation by  $-x$  corresponds to an ordinary contraction of a (poly-)matroid (see [Fuji80b]). The combinatorial structure of the submodular polyhedron and the base polyhedron is invariant with respect to translations and the rank function can be made monotone nondecreasing by an appropriate translation (see Lemma 2.19 in Section 2.3.a). Therefore, the monotonicity of the rank function plays no essential rôle in the theory of submodular systems but sometimes makes it easier for us to find an initial feasible solution in algorithms. Also,

## 2.1. FUNDAMENTAL OPERATIONS ON SUBMODULAR SYSTEMS

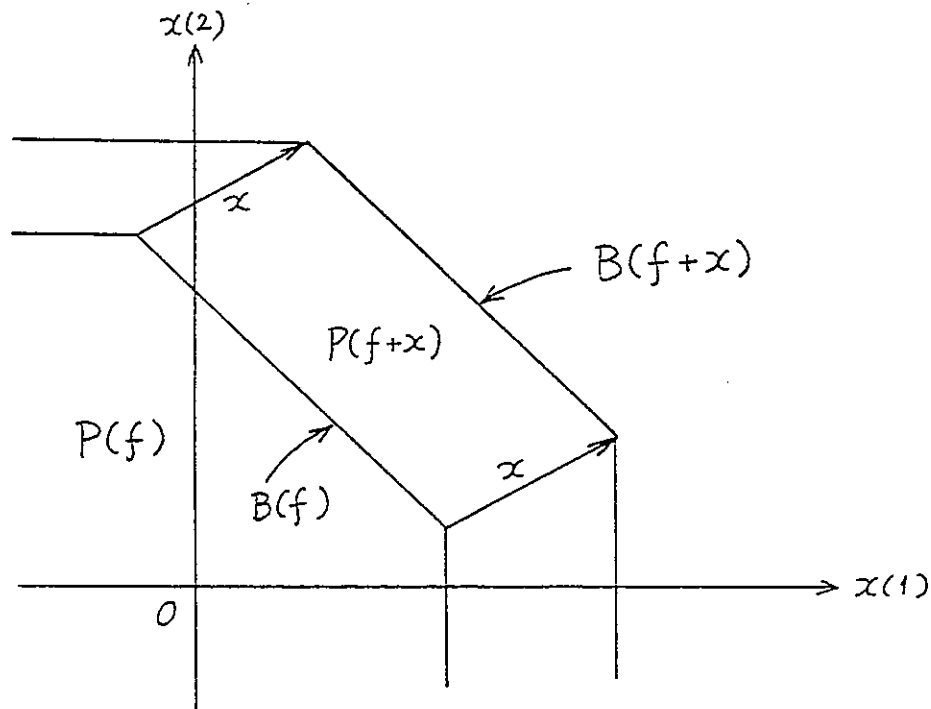


Figure 2.3.

any results obtained in (poly-)matroid theory which are invariant with respect to translations can easily be extended to submodular systems. For example, we will generalize the polymatroid intersection theorem of Edmonds [Edm70] to submodular systems in Section 3.1.

For two submodular systems  $(\mathcal{D}_1, f_1)$  and  $(\mathcal{D}_2, f_2)$  on  $E$  the *sum* of the two submodular systems is defined as the submodular system  $(\mathcal{D}_1 \cap \mathcal{D}_2, f_1 + f_2)$  on  $E$ . We have

$$P(f_1 + f_2) = P(f_1) + P(f_2), \quad (2.27)$$

$$B(f_1 + f_2) = B(f_1) + B(f_2), \quad (2.28)$$

where the sums in the right-hand sides denote the vector sum. (Relations (2.27) and (2.28) will be shown in Section 3.2.) Note that a translation is a special case of a sum.

### (d) Other operations

For a submodular system  $(\mathcal{D}, f)$  on  $E$  let  $r$  be an arbitrary nonnegative element in  $\mathbb{R}$ . Define

$$f_{-r}(X) = f(X) \quad (X \in \mathcal{D} - \{E\}), \quad (2.29)$$

$$f_{-r}(E) = f(E) - r. \quad (2.30)$$

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Then  $(\mathcal{D}, f_{-r})$  is also a submodular system on  $E$  and is called the  $r$ -truncation of  $(\mathcal{D}, f)$ . Similarly, we define the  $r$ -truncation,  $(\mathcal{D}, g_{+r})$ , of a supermodular system  $(\mathcal{D}, g)$  by

$$g_{+r}(X) = g(X) \quad (X \in \mathcal{D} - \{E\}), \quad (2.31)$$

$$g_{+r}(E) = g(E) + r. \quad (2.32)$$

The dual of the  $r$ -truncation of the dual supermodular system of the submodular system  $(\mathcal{D}, f)$  is called the  $r$ -elongation of  $(\mathcal{D}, f)$  ([Tomi81a]) (see Fig. 2.4).

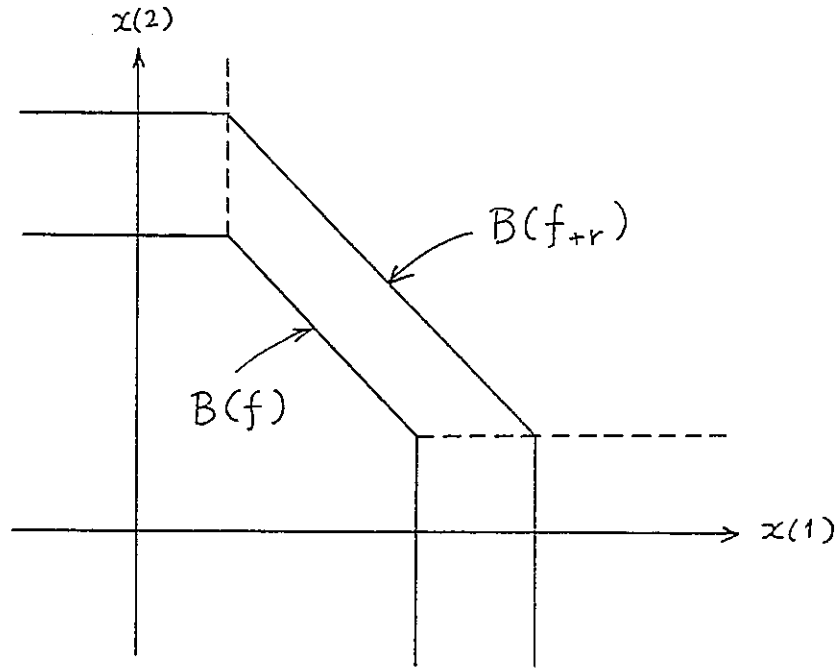


Figure 2.4.

Denoting the  $r$ -elongation of  $(\mathcal{D}, f)$  by  $(\mathcal{D}, f_{+r})$ , we have

$$f_{+r} = ((f^\#)_{+r})^\#. \quad (2.33)$$

Similarly, in a dual manner we define the  $r$ -elongation of a supermodular system.

For any partition  $\Pi = \{A_1, A_2, \dots, A_l\}$  of  $E$  a subset  $X$  of  $E$  is said to be compatible with  $\Pi$  if, for each  $A_i \in \Pi$ ,  $A_i \cap X \neq \emptyset$  implies  $A_i \subseteq X$ . Define

$$\mathcal{D}(\Pi) = \{X \mid X \in \mathcal{D}, X \text{ is compatible with } \Pi\} \quad (2.34)$$

and denote the restriction of  $f$  to  $\mathcal{D}(\Pi)$  by  $f_\Pi$ . The pair  $(\mathcal{D}(\Pi), f_\Pi)$  is a submodular system on  $E$  and is called the aggregation of  $(\mathcal{D}, f)$  by  $\Pi$ . Aggregations play

## 2.2. GREEDY ALGORITHM

a fundamental rôle in a decomposition theory for submodular systems [Fuji83] which generalizes the decomposition theory of graphs by Tutte [Tutte66].

### 2.2. Greedy Algorithm

In this section we consider a linear optimization problem over the base polyhedron and give an algorithm, called a *greedy algorithm*, for solving the problem. Before getting into the problem, we first examine the structure of the distributive lattice  $\mathcal{D} \subseteq 2^E$  and show the one-to-one correspondence between the set of distributive lattices  $\mathcal{D} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{D}$  and the set of partially ordered sets (posets) on partitions of  $E$ .

#### (a) Distributive lattices and posets

For a distributive lattice  $\mathcal{D} \subseteq 2^E$  the cardinality  $|\mathcal{D}|$  of  $\mathcal{D}$  can be as large as  $2^{|E|}$  and listing all the elements of  $\mathcal{D}$  to represent it is not practical even for medium-sized  $E$ . We shall show how to efficiently express a distributive lattice as a structured system, a poset, on  $E$ .

Let  $\mathcal{D} \subseteq 2^E$  be a distributive lattice with  $\emptyset, E \in \mathcal{D}$ . A sequence of monotone increasing elements of  $\mathcal{D}$

$$\mathcal{D}: S_0 \subset S_1 \subset \cdots \subset S_k \quad (2.35)$$

is called a *chain* of  $\mathcal{D}$  and  $k$  is the *length* of the chain  $\mathcal{C}$ . If there exists no chain which contains chain  $\mathcal{C}$  as a proper subsequence,  $\mathcal{C}$  is called a *maximal chain* of  $\mathcal{D}$ . If  $\mathcal{C}$  given by (2.35) is a maximal chain, we have  $S_0 = \emptyset$  and  $S_k = E$ .

For each  $e \in E$  define

$$D(e) = \bigcap \{X \mid e \in X \in \mathcal{D}\}. \quad (2.36)$$

$D(e)$  is the unique minimal element of  $\mathcal{D}$  containing  $e$ . Note that for any  $e \in E$  and  $e' \in D(e)$  we have

$$D(e') \subseteq D(e). \quad (2.37)$$

Also let  $G(\mathcal{D}) = (E, A(\mathcal{D}))$  be a (directed) graph with a vertex set  $E$  and an arc set  $A(\mathcal{D})$  given by

$$A(\mathcal{D}) = \{(e, e') \mid e \in E, e' \in D(e)\}. \quad (2.38)$$

Suppose the graph  $G(\mathcal{D})$  is decomposed into strongly connected components  $G_i = (F_i, A_i)$  ( $i \in I$ ). Let  $\preceq_{\mathcal{D}}$  be the partial order on the set of the strongly connected components  $\{G_i \mid i \in I\}$  naturally determined by the decomposition. That is,  $G_{i_1} \preceq_{\mathcal{D}} G_{i_2}$  for  $i_1, i_2 \in I$  if and only if there exists a directed path from

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a vertex of  $G_{i_2}$  to a vertex of  $G_{i_1}$ . Note that from (2.37)  $G(\mathcal{D})$  is transitively closed (i.e., if there is a directed path from a vertex  $v_1$  to a vertex  $v_2$ , then there is an arc  $(v_1, v_2)$  in  $G(\mathcal{D})$ ). Therefore, if  $G_{i_1} \preceq_{\mathcal{D}} G_{i_2}$ , there exists an arc from any vertex of  $G_{i_2}$  to any vertex of  $G_{i_1}$ .

Denote the set of the vertex sets  $F_i$  ( $i \in I$ ) of the strongly connected components  $G_i$  ( $i \in I$ ) by

$$\Pi(\mathcal{D}) = \{F_i \mid i \in I\}. \quad (2.39)$$

$\Pi(\mathcal{D})$  is a partition of  $E$ . In the following we regard  $\preceq_{\mathcal{D}}$  as a partial order on  $\Pi(\mathcal{D})$  by identifying  $G_i$  with  $F_i$  for each  $i \in I$ .

Now, we have obtained a poset  $\mathcal{P}(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$ , which is called the *poset derived from distributive lattice  $\mathcal{D}$* . An example of a distributive lattice  $\mathcal{D}$  and the poset  $\mathcal{P}(\mathcal{D})$  derived from it is shown in Fig. 2.5.

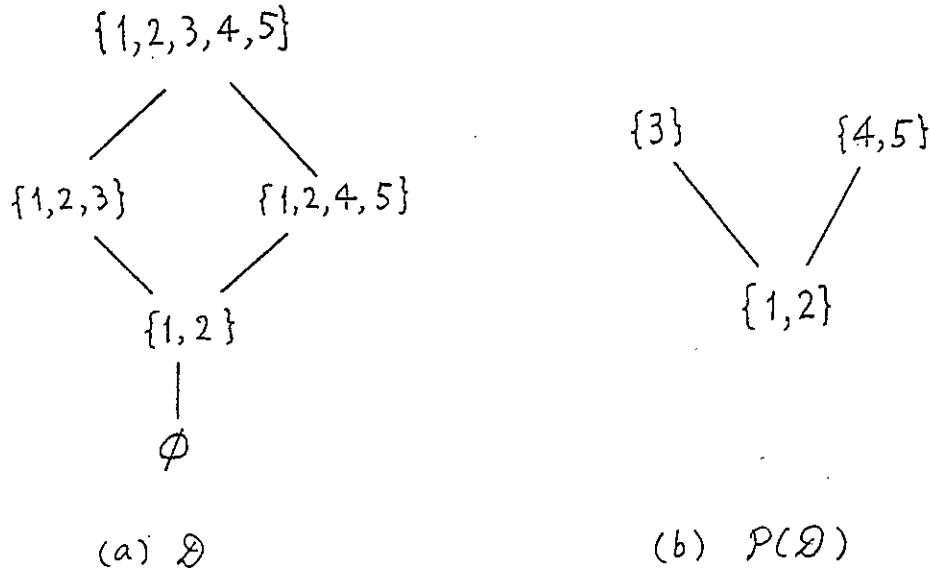


Figure 2.5.

For a (general) poset  $\mathcal{P} = (P, \preceq)$ , a set  $J \subseteq P$  is called a (*lower*) *ideal* of  $\mathcal{P}$  if for each  $e, e' \in P$  we have

$$e \preceq e' \in J \implies e \in J. \quad (2.40)$$

**Theorem 2.6** [Birkhoff37]: Let  $\mathcal{D} \subseteq 2^E$  be a distributive lattice with  $\emptyset, E \in \mathcal{D}$ . Then, for the poset  $\mathcal{P}(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq_{\mathcal{D}})$  derived from  $\mathcal{D}$  the following (i) and (ii) hold.

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(i) For each ideal  $J$  of  $\mathcal{P}(\mathcal{D})$ ,

$$\bigcup \{F \mid F \in J\} \in \mathcal{D}. \quad (2.41)$$

(ii) For each  $X \in \mathcal{D}$ , there exists an ideal  $J$  of  $\mathcal{P}(\mathcal{D})$  such that

$$X = \bigcup \{F \mid F \in J\}. \quad (2.42)$$

(Proof) (i): Put  $X = \bigcup \{F \mid F \in J\}$ . Since  $J$  is an ideal of  $\mathcal{P}(\mathcal{D})$ , it follows from the definition of  $\mathcal{P}(\mathcal{D})$  that  $D(e) \subseteq X$  for each  $e \in X$ . Then,  $X = \bigcup \{D(e) \mid e \in X\}$  and we have  $X \in \mathcal{D}$  since  $D(e) \in \mathcal{D}$  from (2.36).

(ii): For a given  $X \in \mathcal{D}$ ,  $X$  and any  $F \in \Pi(\mathcal{D})$  do not cross, i.e., either  $F \subseteq X$  or  $F \subseteq E - X$ , due to the definition of  $\mathcal{P}(\mathcal{D})$ . Therefore,  $J = \{F \mid F \in \Pi(\mathcal{D}), F \subseteq X\}$  is a partition of  $X$ . Moreover, because of the definition of  $\mathcal{P}(\mathcal{D})$  " $F_1 \leq_{\mathcal{D}} F_2 \subseteq X$ " implies " $F_1 \subseteq X$ ". Consequently,  $J = \{F \mid F \in \Pi(\mathcal{D}), F \subseteq X\}$  is a desired ideal of  $\mathcal{P}(\mathcal{D})$ . Q.E.D.

From Theorem 2.6, (2.41) (or (2.42)) determines a one-to-one correspondence between  $\mathcal{D}$  and the set of all the ideals of  $\mathcal{P}(\mathcal{D})$ . It should be noted that for any poset  $\mathcal{P} = (P, \leq)$  on a partition  $P$  of  $E$  the set  $\mathcal{D}(\mathcal{P})$  defined by

$$\mathcal{D}(\mathcal{P}) = \{\tilde{J} \mid J: \text{an ideal of } \mathcal{P}\}, \quad (2.43)$$

$$\tilde{J} = \bigcup \{F \mid F \in J\} \quad (2.44)$$

forms a distributive lattice with  $\emptyset, E \in \mathcal{D}(\mathcal{P})$ . We can also see that the mapping which assigns each  $\mathcal{D}$  to its derived posets  $\mathcal{P}(\mathcal{D})$  is a one-to-one correspondence between the set of distributive lattices  $\mathcal{D} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{D}$  and that of posets  $\mathcal{P} = (P, \leq)$  on partitions  $P$  of  $E$ .

From Theorem 2.5 we can easily show

**Corollary 2.7:** Given a distributive lattice  $\mathcal{D} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{D}$ , let

$$\mathcal{C}: \emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = E \quad (2.45)$$

be an arbitrary maximal chain of  $\mathcal{D}$ . Then we have

$$\Pi(\mathcal{D}) = \{S_i - S_{i-1} \mid i = 1, 2, \dots, k\}. \quad (2.46)$$

In particular, the length of any maximal chain of  $\mathcal{D}$  is independent of the choice of a maximal chain and is equal to  $|\Pi(\mathcal{D})|$ .

We call  $\mathcal{D}$  *simple* if the partition  $\Pi(\mathcal{D})$  is composed of singletons of  $E$  only, i.e.,  $\Pi(\mathcal{D}) = \{\{e\} \mid e \in E\}$ . For a simple distributive lattice  $\mathcal{D}$  we regard  $\mathcal{P}(\mathcal{D})$

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as a poset on  $E$  and write  $\mathcal{P}(\mathcal{D}) = (E, \leq_{\mathcal{D}})$ . Conversely, the set of all the (lower) ideals of a poset  $\mathcal{P} = (E, \leq)$  on  $E$  forms a simple distributive lattice  $\mathcal{D} \subseteq 2^E$  and we denote such a simple  $\mathcal{D}$  by  $2^{\mathcal{P}}$ . A submodular system  $(\mathcal{D}, f)$  with simple  $\mathcal{D}$  is called *simple*.

For a non-simple submodular system  $(\mathcal{D}, f)$  on  $E$ , define

$$\hat{X} = \{F \mid F \in \Pi(\mathcal{D}), F \subseteq X\} \quad (X \in \mathcal{D}), \quad (2.47)$$

$$\hat{\mathcal{D}} = \{\hat{X} \mid X \in \mathcal{D}\}, \quad (2.48)$$

$$\hat{f}(\hat{X}) = f(X) \quad (X \in \mathcal{D}). \quad (2.49)$$

Then we have a simple submodular system  $(\hat{\mathcal{D}}, \hat{f})$  on  $\Pi(\mathcal{D})$ , which we call the *simplification* of  $(\mathcal{D}, f)$ .

### (b) Greedy algorithm

For a submodular system  $(\mathcal{D}, f)$  on  $E$  we consider a linear optimization problem described as follows.

$$P_w: \text{Minimize } \sum_{e \in E} w(e)x(e) \quad (2.50a)$$

$$\text{subject to } x \in B(f), \quad (2.50b)$$

where  $w: E \rightarrow \mathbb{R}$  is a given weight function. An optimal solution of  $P_w$  is called a *minimum-weight base* of  $(\mathcal{D}, f)$  with respect to the weight function  $w$ . Similarly, a *maximum-weight base* of  $(\mathcal{D}, f)$  with respect to the weight function  $w$  is an optimal solution of Problem  $P_{-w}$  with the weight function  $-w$ .

Fundamental structural properties of the base polyhedron  $B(f)$  is given by the following theorems.

**Theorem 2.8:** The base polyhedron  $B(f)$  is pointed (or has extreme points) if and only if  $\mathcal{D}$  is simple, i.e.,  $\mathcal{D} = 2^{\mathcal{P}}$  for some poset  $\mathcal{P} = (E, \leq)$ .

(Proof) A polyhedron is pointed if and only if its characteristic cone (or recession cone) does not contain any line ([Stoer+Witzgall], [Rockafellar]). The characteristic cone of  $B(f)$  is the solution set of the following system of inequalities and an equation:

$$x(X) \leq 0 \quad (X \in \mathcal{D}), \quad (2.51)$$

$$x(E) = 0. \quad (2.52)$$

Therefore,  $B(f)$  is pointed if and only if the system of equations

$$x(X) = 0 \quad (X \in \mathcal{D}) \quad (2.53)$$



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has the unique solution  $x = \mathbf{0}$ , where note that  $E \in \mathcal{D}$ . We see from Corollary 2.7 that (2.53) is equivalent to

$$x(F) = 0 \quad (F \in \Pi(\mathcal{D})). \quad (2.54)$$

System (2.54) has the unique solution  $x = \mathbf{0}$  if and only if  $|F| = 1$  for each  $F \in \Pi(\mathcal{D})$ , i.e.,  $\mathcal{D}$  is simple. Q.E.D.

It should be noted that the rank of the coefficient matrix of the left-hand side of (2.53) is equal to the height of  $\mathcal{D}$ , the length of maximal chains of  $\mathcal{D}$ .

**Theorem 2.9:** The base polyhedron  $B(f)$  is bounded if and only if  $\mathcal{D}$  is the Boolean lattice  $2^E$ , i.e.,  $\mathcal{D}$  is simple and complemented.

(Proof) If  $\mathcal{D} = 2^E$ , then  $B(f)$  is included in the following bounded solution set of

$$x(e) \leq f(\{e\}) \quad (e \in E), \quad x(E) = f(E). \quad (2.55)$$

On the other hand, if  $\mathcal{D} \neq 2^E$ , there exist distinct elements  $e, e' \in E$  such that for each  $X \in \mathcal{D}$   $e \in X$  implies  $e' \in X$ . Then for any base  $x \in B(f)$  the ray or half-line

$$x + \alpha(\chi_e - \chi_{e'}) \quad (\alpha \geq 0) \quad (2.56)$$

is contained in  $B(f)$ . So,  $B(f)$  is not bounded. Q.E.D.

When  $(\mathcal{D}, f)$  is not simple, Problem  $P_w$  is unbounded if  $w(e) \neq w(e')$  for any  $e, e' \in F \in \Pi(\mathcal{D})$ . Therefore, if  $P_w$  has an optimal solution,  $w: E \rightarrow \mathbf{R}$  is constant on each  $F \in \Pi(\mathcal{D})$ , and hence it suffices to consider the simplification of  $(\mathcal{D}, f)$ .

We suppose without loss of generality that in the minimum-weight base problem  $P_w$  described by (2.50)  $B(f)$  is pointed, i.e.,  $\mathcal{D} = 2^{\mathcal{P}}$  with  $\mathcal{P} = (E, \preceq)$ .

**Theorem 2.10** [Fuji+Tomi83]: Problem  $P_w$  in (2.50) has a finite optimal solution if and only if  $w: E \rightarrow \mathbf{R}$  is a monotone nondecreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbf{R}, \leq)$ , i.e.,  $\forall e, e' \in E: e \preceq e' \implies w(e) \leq w(e')$ .

(Proof) *The "if" part:* Suppose that  $w$  is a monotone nondecreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbf{R}, \leq)$  and that the distinct values of weights  $w(e)$  ( $e \in E$ ) are given by

$$w_1 < w_2 < \cdots < w_p. \quad (2.57)$$

Define

$$A_i = \{e \mid e \in E, w(e) \leq w_i\} \quad (i = 1, 2, \dots, p), \quad (2.58)$$

where note that  $A_p = E$ . The sets  $A_i$  ( $i = 1, 2, \dots, p$ ) form a chain  $A_1 \subset A_2 \subset \cdots \subset A_p$  of  $\mathcal{D}$ . From Lemma 2.1 there exists a base  $x \in B(f)$  such that

$$x(A_i) = f(A_i) \quad (i = 1, 2, \dots, p). \quad (2.59)$$

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Then for any base  $y \in B(f)$  we have from (2.57)–(2.59)

$$\begin{aligned}
 & \sum_{e \in E} w(e)y(e) - \sum_{e \in E} w(e)x(e) \\
 &= \sum_{i=1}^p \sum_{e \in A_i - A_{i-1}} w_i(y(e) - x(e)) \\
 &= \sum_{i=1}^p \{w_i(y(A_i) - x(A_i)) - w_i(y(A_{i-1}) - x(A_{i-1}))\} \\
 &= \sum_{i=1}^{p-1} (w_i - w_{i+1})(y(A_i) - x(A_i)) + w_p(y(A_p) - x(A_p)) \\
 &= \sum_{i=1}^{p-1} (w_{i+1} - w_i)(f(A_i) - y(A_i)) \\
 &\geq 0,
 \end{aligned} \tag{2.60}$$

where we define  $A_0 = \emptyset$ , and recall  $A_p = E$ . This shows the optimality of  $x$ .

*The “only if” part:* Suppose that  $w$  is not a monotone nondecreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbb{R}, \leq)$ , i.e., for some  $k$  ( $1 \leq k < p$ )  $A_k$  defined by (2.58) does not belong to  $\mathcal{D}$ . Then there exist elements  $e \in A_k$  and  $e' \in E - A_k$  such that for any  $X \in \mathcal{D}$  with  $e \in X$  we have  $e' \in X$ . Hence, for any base  $x \in B(f)$  we have  $\tilde{c}(x, e, e') = +\infty$  and  $x + \alpha(\chi_e - \chi_{e'}) \in B(f)$  for any  $\alpha > 0$ . Since  $w(e) < w(e')$ , Problem  $P_w$  is unbounded. Q.E.D.

**Corollary 2.11:** Let  $P_w'$  be the problem given by  $P_w$  where  $B(f)$  is replaced by  $P(f)$ . Problem  $P_w'$  has a finite optimal solution if and only if  $w: E \rightarrow \mathbb{R}$  is a nonpositive monotone nondecreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbb{R}, \leq)$ .

Furthermore, we have the following

**Theorem 2.12:** Suppose that  $w$  is a monotone nondecreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbb{R}, \leq)$ , i.e., the sets  $A_i$  ( $i = 1, 2, \dots, p$ ) defined by (2.58) form a chain of  $\mathcal{D}$ . For each  $i = 1, 2, \dots, p$  let  $f_i$  be the rank function of the set minor  $(\mathcal{D}, f) \cdot A_i / A_{i-1}$ , where  $A_0 = \emptyset$ . Then the set of all the optimal solutions of Problem  $P_w$  is given by

$$\begin{aligned}
 & B(f_1) \oplus B(f_2) \oplus \dots \oplus B(f_p) \\
 &= \{x_1 \oplus x_2 \oplus \dots \oplus x_p \mid x_i \in B(f_i) \ (i = 1, 2, \dots, p)\},
 \end{aligned} \tag{2.61}$$

where the direct sum  $\oplus$  is defined by (2.5). That is,  $x \in B(f)$  is an optimal solution of  $P_w$  if and only if  $x$  restricted on  $A_i - A_{i-1}$  is a base of  $(\mathcal{D}, f) \cdot A_i / A_{i-1}$  for each  $i = 1, 2, \dots, p$ .

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(Proof) It follows from the proof of the "if" part of Theorem 2.9 that  $x \in B(f_1) \oplus \cdots \oplus B(f_p)$  is an optimal solution of Problem  $P_w$ .

On the other hand, if  $x$  is an optimal solution, then we must have  $\text{dep}(x, e) \subseteq A_i$  for each  $i = 1, 2, \dots, p$  and  $e \in A_i$ . This implies  $x(A_i) = f(A_i)$  ( $i = 1, 2, \dots, p$ ) since  $A_i = \cup\{\text{dep}(x, e) \mid e \in A_i\}$ . Therefore, we have  $x \in B(f_1) \oplus \cdots \oplus B(f_p)$  due to Lemma 2.1. Q.E.D.

**Theorem 2.13:** A base  $x \in B(f)$  is an optimal solution of Problem  $P_w$  if and only if for each  $e, e' \in E$  such that  $e' \in \text{dep}(x, e)$  we have

$$w(e) \geq w(e'). \quad (2.62)$$

(Proof) The "only if" part is trivial. The "if" part follows from Theorem 2.12. For, if (2.62) holds for each  $e, e' \in E$  such that  $e' \in \text{dep}(x, e)$ , then we have (2.59) for  $A_i$  ( $i = 1, 2, \dots, p$ ) defined by (2.58). Note that  $A_i = \cup\{\text{dep}(x, e) \mid e \in A_i\}$  for  $i = 1, 2, \dots, p$ . Q.E.D.

Theorem 2.13 says that the local optimality with respect to elementary transformations implies the global optimality.

The proof of Theorem 2.10 provides us with an algorithm, called a *greedy algorithm*, for solving Problem  $P_w$ .

A sequence  $(e_1, e_2, \dots, e_n)$  of all the elements of  $E$  (a linear or total ordering of  $E$ ) is called a *linear extension* of  $\mathcal{P} = (E, \preceq)$  if  $i \leq j$  whenever  $e_i \preceq e_j$  ( $i, j = 1, 2, \dots, n$ ). Furthermore, a linear extension  $(e_1, e_2, \dots, e_n)$  of  $\mathcal{P} = (E, \preceq)$  is called *monotone nondecreasing* with respect to  $w: E \rightarrow \mathbb{R}$  if  $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_n)$ . Such a monotone nondecreasing linear extension of  $\mathcal{P} = (E, \preceq)$  exists if and only if  $w$  is a monotone nondecreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbb{R}, \leq)$ .

### A greedy algorithm

- 1° Find a monotone nondecreasing linear extension  $(e_1, e_2, \dots, e_n)$  of  $\mathcal{P} = (E, \preceq)$  with respect to  $w$ .
- 2° Define a vector  $x \in \mathbb{R}^E$  by

$$x(e_i) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \dots, n), \quad (2.63)$$

where for each  $i = 1, 2, \dots, n$   $S_i$  is the set of the first  $i$  elements of  $(e_1, e_2, \dots, e_n)$  and  $S_0 = \emptyset$ . Then  $x$  is a minimum-weight base of  $(\mathcal{D}, f)$  with respect to weight  $w$ .

(End)

It should be noted that  $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$  is a maximal chain of  $\mathcal{D}$  containing  $A_i$  ( $i = 1, 2, \dots, p$ ) defined by (2.58) and that conversely

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such a maximal chain of  $\mathcal{D}$  gives a monotone nondecreasing linear extension  $(e_1, e_2, \dots, e_n)$  of  $\mathcal{P}$  by  $\{e_i\} = S_i - S_{i-1}$  ( $i = 1, 2, \dots, n$ ). Also note that due to Theorem 2.12 every extreme minimum-weight base can be obtained by the greedy algorithm by appropriately choosing a monotone nondecreasing linear extension  $(e_1, e_2, \dots, e_n)$  in Step 1<sup>o</sup>.

**Corollary 2.14:** Problem  $P_w$  has a unique optimal solution if and only if  $w: E \rightarrow \mathbf{R}$  is a one-to-one monotone increasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbf{R}, \leq)$ .

**Theorem 2.15:** Let  $f: \mathcal{D} \rightarrow \mathbf{R}$  be a function on a simple distributive lattice  $\mathcal{D} \subseteq 2^E$  such that  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = 0$ . Define

$$B(f) = \{x \mid x \in \mathbf{R}^E, \forall X \in \mathcal{D}: x(X) \leq f(X), x(E) = f(E)\}. \quad (2.64)$$

Then, the greedy algorithm described above works for  $B(f)$  defined by (2.64) if and only if  $f$  is a submodular function on  $\mathcal{D}$ .

(Proof) It suffices to show the "only if" part. Suppose that the greedy algorithm works for  $B(f)$ . Then, for each maximal chain

$$\mathcal{C}: \emptyset = S_0 \subset S_1 \subset \dots \subset S_n = E \quad (2.65)$$

of  $\mathcal{D}$  the vector  $x \in \mathbf{R}^E$  defined by (2.63) belongs to  $B(f)$ . For any incomparable  $X, Y \in \mathcal{D}$  choose a maximal chain  $\mathcal{C}$  of (2.65) containing  $X \cap Y$  and  $X \cup Y$  and define  $x$  by (2.63). Since  $x \in B(f)$ , by the definition of  $x$  we have

$$x(X) \leq f(X), x(Y) \leq f(Y), x(X \cup Y) = f(X \cup Y), x(X \cap Y) = f(X \cap Y). \quad (2.66)$$

Hence we have

$$\begin{aligned} f(X) + f(Y) &\geq x(X) + x(Y) \\ &= x(X \cup Y) + x(X \cap Y) = f(X \cup Y) + f(X \cap Y). \end{aligned} \quad (2.67)$$

It follows that  $f$  is a submodular function on  $\mathcal{D}$ .

Q.E.D.

The linear programming dual of Problem  $P_w$  in (2.50) is given by

$$P_w^*: \text{Maximize } \sum_{X \in \mathcal{D}} \lambda(X) f(X) \quad (2.68a)$$

$$\text{subject to } \sum \{\lambda(X) \mid X \in \mathcal{D}, e \in X\} \geq w(e) \quad (e \in E), \quad (2.68b)$$

$$\lambda(X) \leq 0 \quad (X \in \mathcal{D} - \{E\}). \quad (2.68c)$$

### 2.3. STRUCTURES OF BASE POLYHEDRA

For an optimal solution  $x$  obtained by the greedy algorithm we have

$$\begin{aligned}
 \sum_{e \in E} w(e)x(e) &= \sum_{i=1}^p w_i(f(A_i) - f(A_{i-1})) \\
 &= \sum_{i=1}^{p-1} (w_i - w_{i+1})f(A_i) + w_p f(E), \tag{2.69}
 \end{aligned}$$

where  $w_i$ ,  $A_i$  ( $i = 1, 2, \dots, p$ ) are defined by (2.57) and (2.58) and  $A_0 = \emptyset$ . Define

$$\lambda(A_i) = w_i - w_{i+1} \quad (i = 1, 2, \dots, p-1), \tag{2.70}$$

$$\lambda(E) = w_p \tag{2.71}$$

and  $\lambda(X) = 0$  for other  $X \in \mathcal{D}$ . Then  $\lambda$  is a dual feasible solution and it follows from (2.69) that the values of the objective functions of the dual problems coincide with each other. Hence  $\lambda$  given above is an optimal solution of the dual problem  $P_w^*$ . From (2.70) and (2.71), for any *integral*  $w$  such that the primal problem  $P_w$  has an optimal solution there exists an *integral* optimal solution of the dual problem  $P_w^*$ . A system of linear inequalities and equations with this property is called *totally dual integral* ([Hoffman74], [Edm+Giles77]).

**Theorem 2.16:** The system of inequalities and an equation given by

$$x(X) \leq f(X) \quad (X \in \mathcal{D} - \{E\}), \tag{2.72}$$

$$x(E) = f(E) \tag{2.73}$$

is totally dual integral.

Since the system of (2.72) and (2.73) is totally dual integral, if the rank function  $f$  is integer-valued, each face of  $B(f)$  contains an integral vector and in particular, each vertex is integral ([Hoffman74], [Edm+Giles77]), which can also be seen from the greedy algorithm (cf. Theorem 2.18 below).

**Corollary 2.17:** The system of inequalities

$$x(X) \leq f(X) \quad (X \in \mathcal{D}) \tag{2.74}$$

is totally dual integral.

(Proof) The proof is similar to that of Theorem 2.16. Use Corollary 2.11.

Q.E.D.

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### 2.3. Structures of Base Polyhedra

Suppose that  $(\mathcal{D}, f)$  is a simple submodular system on  $E$ .

#### (a) Extreme points and rays

From the greedy algorithm and Corollary 2.14 we have

**Theorem 2.18** (The extreme point theorem) [Fuji+Tomi83]: A base  $x \in B(f)$  is an extreme point of the base polyhedron  $B(f)$  if and only if for a maximal chain

$$C: \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \quad (2.75)$$

of  $\mathcal{D}$  we have

$$x(e_i) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \dots, n), \quad (2.76)$$

where  $\{e_i\} = S_i - S_{i-1}$ .

Theorem 2.18, when  $\mathcal{D} = 2^E$ , has been shown in [Edm70], [Shapley71] and [Lovász83].

Define a vector  $\bar{\alpha} \in \mathbb{R}^E$  by

$$D(e) = \bigcap \{X \mid e \in X \in \mathcal{D}\}, \quad (2.77)$$

$$\bar{\alpha}(e) = f(D(e)) - f(D(e) - \{e\}) \quad (e \in E). \quad (2.78)$$

Then, by the submodularity of  $f$  and Theorem 2.18, vector  $\bar{\alpha}$  is the least upper bound (or the join) of all the extreme points of  $B(f)$  in the vector lattice  $\mathbb{R}^E$ . In particular, for each  $X \in \mathcal{D}$ ,

$$f(X) \leq \bar{\alpha}(X), \quad (2.79)$$

since there is an extreme base  $x \in B(f)$  such that  $x(X) = f(X)$ . Because of (2.79),  $\bar{\alpha}$  can be used for estimating an upper bound of  $f$ , i.e.,

$$\begin{aligned} \max\{f(X) \mid X \in \mathcal{D}\} &\leq \max\{\bar{\alpha}(X) \mid X \in \mathcal{D}\} \\ &\leq \sum \{\bar{\alpha}(e) \mid e \in E, \bar{\alpha}(e) > 0\}. \end{aligned} \quad (2.80)$$

Also note that given any base  $x \in B(f)$  a lower bound of  $f$  is given by

$$\min\{f(X) \mid X \in \mathcal{D}\} \geq \min\{x(X) \mid X \in \mathcal{D}\} \geq \sum \{x(e) \mid e \in E, x(e) < 0\}. \quad (2.81)$$

### 2.3. STRUCTURES OF BASE POLYHEDRA

Moreover, denote by  $\underline{\alpha}$  the greatest lower bound (or the meet) of all the extreme points of  $B(f)$  in the vector lattice  $\mathbb{R}^E$ .  $\underline{\alpha}$  is given similarly as (2.77) and (2.78) in a dual form:

$$D^*(e) = \bigcup \{X \mid e \notin X \in \mathcal{D}\} \quad (e \in E), \quad (2.82)$$

$$\underline{\alpha}(e) = f(D^*(e) \cup \{e\}) - f(D^*(e)) \quad (e \in E). \quad (2.83)$$

**Lemma 2.19:** The rank function  $f$  of a simple submodular system  $(\mathcal{D}, f)$  is monotone nondecreasing if and only if  $\underline{\alpha} \geq \mathbf{0}$ . In other words,  $f$  is monotone nondecreasing if and only if every extreme point of the base polyhedron  $B(f)$  belongs to the nonnegative orthant  $\mathbb{R}_+^E$ .

(Proof) This easily follows from Theorem 2.18 and the definition of  $\underline{\alpha}$ . Q.E.D.

It follows from Theorem 2.18 that if the rank function  $f$  of  $(\mathcal{D}, f)$  is integer-valued, every extreme point of  $B(f)$  is integral. In particular, we have the following polyhedral characterization of matroids due to Edmonds [Edm70].

**Corollary 2.20** [Edm70]: For a matroidal submodular system  $(2^E, \rho)$ , where  $\rho$  is the rank function of a matroid  $M$  on  $E$ , the base polyhedron  $B(f)$  is the convex hull of the characteristic vectors of all the bases of the matroid  $M$ .

From Theorem 1.4 we can easily see that if  $f$  is a nonnegative integer-valued crossing-submodular function on a crossing family  $\mathcal{F}$  and if the polyhedron given by

$$B(f_0^1) = \{x \mid x \in B(f), \forall e \in E: 0 \leq x(e) \leq 1\} \quad (2.84)$$

is nonempty, then  $B(f_0^1)$  is the integral base polyhedron obtained by the reduction of  $B(f)$  by vector  $\mathbf{1} = (1(e) = 1: e \in E)$  and the contraction by zero vector  $\mathbf{0}$  and is a base polyhedron of a matroid. Hence, from Corollary 2.20 we see that

$$\{X \mid X \subseteq E, \forall Y \in \mathcal{F}: |X \cap Y| \leq f(Y), |X| = f(E)\} \quad (2.85)$$

is a family of bases of a matroid. This is a result by A. Frank and É. Tardos [Frank+Tardos81].

If the rank function  $f$  of a simple submodular system  $(\mathcal{D}, f)$  is integer-valued and has the unit-increase property (i.e., for any  $X, Y \in \mathcal{D}$  with  $X \subseteq Y$  and  $|X| + 1 = |Y|$  we have  $f(Y) = f(X)$  or  $f(Y) = f(X) + 1$ ), then all the extreme points of  $B(f)$  are  $\{0, 1\}$ -vectors. Therefore, we can develop a matroid-like theory for such a submodular system  $(\mathcal{D}, f)$ , which corresponds to

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Faigle's geometry on the poset  $\mathcal{P} = (E, \preceq)$  with  $\mathcal{D} = 2^{\mathcal{P}}$  [Faigle79], [Faigle80]. In fact, Theorem 2.18 gives a polyhedral characterization of the family of bases of Faigle's geometry.

Next, denote the characteristic cone of the base polyhedron  $B(f)$  by  $C(f)$ , which is given by

$$C(f) = \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{D}: x(X) \leq 0, x(E) = 0\}. \quad (2.86)$$

Let  $G = (E, A)$  be the graph with vertex set  $E$  and arc set  $A$  which represents the Hasse diagram of the poset  $\mathcal{P} = (E, \preceq)$ , i.e.,  $(e, e') \in A$  if and only if  $e$  covers  $e'$  (or  $e' \prec e$  and there is no element  $e'' \in E$  such that  $e' \prec e'' \prec e$ ). Also define a capacity function  $c$  on  $A$  by

$$c(a) = +\infty \quad (a \in A). \quad (2.87)$$

Then we easily see that  $C(f)$  in (2.86) coincides with the set of the boundaries  $\partial\varphi$  of nonnegative flows  $\varphi$  in the network  $\mathcal{N} = (G = (E, A), c)$ . Consequently, we have

**Theorem 2.21** (The extreme ray theorem) [Tomi83]: The extreme rays of the characteristic cone  $C(f)$  of the base polyhedron  $B(f)$  are exactly those represented by the vectors

$$\chi_e - \chi_{e'} \quad (2.88)$$

for all  $e, e' \in E$  such that  $e$  covers  $e'$  in  $\mathcal{P} = (E, \preceq)$ .

Theorem 2.10 characterizes the dual cone  $C^*(f)$  of  $C(f)$ , where

$$C^*(f) = \{y \mid y \in \mathbb{R}^E, \forall x \in C(f): \sum_{e \in E} x(e)y(e) \leq 0\}. \quad (2.89)$$

Theorem 2.10 says that  $-C^*(f)$  consists of monotone nondecreasing functions from  $\mathcal{P} = (E, \prec)$  to  $(\mathbb{R}, \leq)$  or that  $C^*(f)$  consists of monotone nonincreasing functions from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbb{R}, \leq)$ .  $C^*(f)$  is generated by the characteristic vectors of the (lower) ideals of  $\mathcal{P}$ .

### (b) Elementary transformations of bases

For any base  $x$  of submodular system  $(\mathcal{D}, f)$  on  $E$  and for  $e, e' \in E$  such that  $e' \in \text{dep}(x, e) - \{e\}$  we have

$$x + \alpha(\chi_e - \chi_{e'}) \in B(f) \quad (0 \leq \alpha \leq \tilde{c}(x, e, e')). \quad (2.90)$$



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The transformation of base  $x \in B(f)$  into such a base  $x + \alpha(\chi_e - \chi_{e'}) \in B(f)$  is called an *elementary transformation* of base  $x \in B(f)$ .

The following theorem is important from an algorithmic point of view.

**Theorem 2.22:** For any two bases  $x, y \in B(f)$  base  $x$  can be transformed into base  $y$  by at most  $\lfloor |E|^2/4 \rfloor$  repeated elementary transformations such that each component  $x(e)$  with  $x(e) < y(e)$  monotonically increases and each component  $x(e)$  with  $x(e) > y(e)$  monotonically decreases.

(Proof) Consider the following algorithm.

- 1° If  $x = y$ , then stop.
- 2° Choose any element  $e \in E$  such that  $x(e) < y(e)$ .
- 3° Choose any element  $e' \in \text{dep}(x, e)$  such that  $x(e') > y(e')$ . Put  $\alpha \leftarrow \min\{y(e) - x(e), x(e') - y(e'), \tilde{c}(x, e, e')\}$ .
- 4° If  $\alpha < y(e) - x(e)$ , then put  $x \leftarrow x + \alpha(\chi_e - \chi_{e'})$  and go to Step 3°. Otherwise ( $\alpha = y(e) - x(e)$ ) put  $x \leftarrow x + \alpha(\chi_e - \chi_{e'})$  and go to Step 1°.

Note that if  $x \neq y$ , there is an element  $e$  such that  $x(e) < y(e)$  and that if  $x(e) < y(e)$ , there is an element  $e' \in \text{dep}(x, e)$  such that  $x(e') > y(e')$ , since otherwise, putting  $X = \text{dep}(x, e)$ , we would have  $f(X) = x(X) < y(X)$ , a contradiction. Also note that if  $\alpha < y(e) - x(e)$  in Step 4°, we have  $\alpha = \min\{x(e') - y(e'), \tilde{c}(x, e, e')\}$  and the number of elements in  $\{e'' \mid e'' \in \text{dep}(x, e), x(e'') > y(e'')\}$  decreases by at least one after the elementary transformation  $x \leftarrow x + \alpha(\chi_e - \chi_{e'})$ . Therefore, the case where  $\alpha < y(e) - x(e)$  is repeated at most  $|S^-|$  times, where  $S^- = \{e \mid e \in E, x(e) > y(e)\}$  for the initial base  $x$ . Defining  $S^+ = \{e \mid e \in E, x(e) < y(e)\}$  for the initial  $x$ , the total number of the elementary transformations is at most  $|S^+| \times |S^-|$ , which is bounded by  $\lfloor |E|^2/4 \rfloor$ . Q.E.D.

Consider a capacitated network  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$  with an underlying graph  $G$  and lower and upper capacity functions  $\underline{c}, \bar{c}: A \rightarrow \mathbb{R}$  with  $\underline{c} \leq \bar{c}$ . Let  $\kappa_{\underline{c}, \bar{c}}: 2^V \rightarrow \mathbb{R}$  be the cut function associated with the network  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$  (see (1.59) in Section 1.3). The set of the boundaries of feasible flows in  $\mathcal{N}$  is the base polyhedron associated with the submodular system  $(2^V, \kappa_{\underline{c}, \bar{c}})$ . Note that there exists a feasible circulation (a feasible flow  $\varphi$  such that  $\partial\varphi = 0$ ) in  $\mathcal{N}$  if and only if  $0 \in B(\kappa_{\underline{c}, \bar{c}})$ . Since  $\partial\underline{c} \in B(\kappa_{\underline{c}, \bar{c}})$ , there exists a feasible circulation in  $\mathcal{N}$  if and only if the base  $\partial\underline{c} \in B(\kappa_{\underline{c}, \bar{c}})$  is transformed into  $0$  by repeated elementary transformations as in Theorem 2.22. A standard algorithm for finding a feasible circulation by the use of a max-flow algorithm consists of such repeated elementary transformations (cf. [Hoffman60], [Ford+Fulkerson62]).

For a (directed) graph  $G = (V, A)$  and a  $\{0, 1\}$ -valued function  $\varphi: A \rightarrow \{0, 1\}$  define the graph  $G_\varphi$  as the one obtained from  $G$  by reorienting arcs  $a \in A$  such that  $\varphi(a) = 1$ . We say  $\varphi$  defines the reorientation  $G_\varphi$ . A graph

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$G = (V, A)$  is *strongly  $k$ -connected* ( $k$ : a positive integer) if for each nonempty proper subset  $U$  of vertex set  $V$  there exist at least  $k$  arcs from  $U$  to  $V - U$ . We call  $\varphi: A \rightarrow \{0, 1\}$  a *strongly  $k$ -connected reorientation* if  $G_\varphi$  is strongly  $k$ -connected. Define a capacity function  $c: A \rightarrow \mathbf{R}$  by

$$c(a) = 1 \quad (a \in A), \quad (2.91)$$

and let  $\kappa: 2^E \rightarrow \mathbf{R}$  be the associated cut function, i.e.,  $\kappa(U) = c(\Delta^+ U) = |\Delta^+ U|$  for  $U \subseteq V$ . Also define

$$\kappa^{(k)}(U) = \begin{cases} \kappa(U) - k & (U \in 2^V - \{\emptyset, V\}) \\ 0 & (U \in \{\emptyset, V\}). \end{cases} \quad (2.92)$$

Then  $\kappa^{(k)}: 2^V \rightarrow \mathbf{R}$  is a crossing-submodular function on  $2^V$  and defines a base polyhedron  $B(\kappa^{(k)})$ , if  $B(\kappa^{(k)}) \neq \emptyset$ , due to Theorem 1.4.  $B(\kappa^{(k)})$  is called the  *$k$ -abridgment* of  $B(\kappa)$  in [Tomi81a, 81b]. It can easily be seen that

$$B(\kappa^{(k)}) = \{\partial\varphi \mid \varphi \text{ defines a strongly } k\text{-connected reorientation of } G\}. \quad (2.93)$$

Here, we assume that the underlying totally ordered additive group  $\mathbf{R}$  is the set  $\mathbf{Z}$  of integers. Also note that the strong  $k$ -connectedness of a reorientation of  $G$  depends only on the boundary  $\partial\varphi$  of  $\varphi$  which defines the reorientation. Because of this fact and Theorem 2.22 we obtain a theorem of Frank [Frank82a]:

"Let  $G'$  and  $G''$  be strongly  $k$ -connected reorientations of  $G$ . Then there exists a sequence of strongly  $k$ -connected reorientations  $G' = G_0, G_1, \dots, G_m = G''$  of  $G$  such that for each  $i = 1, 2, \dots, m$   $G_i$  is obtained by reorienting arcs in a directed path or a directed cycle in  $G_{i-1}$ ."

Note that an elementary transformation of a base (or a boundary  $\partial\varphi$ ) in  $B(\kappa^{(k)})$  corresponds to a transformation of the reorientation of  $G$  (defined by  $\varphi$ ) by reversing the arcs in a directed path and possibly directed cycles and that reversing arcs in a directed cycle does not change the boundary  $\partial\varphi$ .

### (c) Tangent cones

For any base  $x \in B(f)$  associated with submodular system  $(\mathcal{D}, f)$  on  $E$  the *tangent cone* of  $B(f)$  at  $x$ , denoted by  $TC(B(f), x)$ , is defined by

$$TC(B(f), x) = \{\lambda y \mid \lambda \geq 0, y \in \mathbf{R}^E, x + y \in B(f)\}. \quad (2.94)$$

Here, the underlying totally ordered additive group is assumed to be the set  $\mathbf{R}$  of reals.

Given a base  $x \in B(f)$ , we call an ordered pair  $(e, e')$  of elements of  $E$  an *exchangeable pair* associated with  $x$  if  $e' \in \text{dep}(x, e) - \{e\}$ .

## 2.4. RELATED POLYHEDRA

**Theorem 2.23:** The tangent cone  $\text{TC}(\mathbf{B}(f), x)$  of  $\mathbf{B}(f)$  at a base  $x$  is generated by the set of the following vectors:

$$\chi_e - \chi_{e'} \quad (e \in E, e' \in \text{dep}(x, e) - \{e\}). \quad (2.95)$$

In other words, for any vector  $y \in \text{TC}(\mathbf{B}(f), x)$  there exist some nonnegative coefficients  $\lambda(e, e')$  for exchangeable pairs  $(e, e')$  such that

$$y = \sum \{\lambda(e, e')(\chi_e - \chi_{e'}) \mid (e, e') : \text{an exchangeable pairs associated with } x\}. \quad (2.96)$$

(Proof) Let  $C$  be the cone generated by the vectors in (2.95). It follows from the definition of dependence function that

$$C \subseteq \text{TC}(\mathbf{B}(f), x). \quad (2.97)$$

Suppose that there exists a vector  $y \in \text{TC}(\mathbf{B}(f), x) - C$ . Then there exists a vector  $w \in \mathbb{R}^E$  such that

$$\forall z \in C: \sum_{e \in E} w(e)z(e) \geq 0, \quad (2.98)$$

$$\sum_{e \in E} w(e)y(e) < 0. \quad (2.99)$$

From Theorem 2.13 and (2.98) the base  $x$  is a minimum-weight base with respect to the weight function  $w$  but (2.99) implies that for a sufficiently small  $\alpha > 0$   $x + \alpha y$  is a base and the weight of the base  $x + \alpha y$  is smaller than that of  $x$ . This is a contradiction. So, we must have  $C = \text{TC}(\mathbf{B}(f), x)$ . Q.E.D.

A constructive proof of this theorem for polymatroids was given in [Fuji 78a, Lemma 9]. It should also be noted that Theorems 2.21 and 2.23 are closely related. We can show Theorem 2.23 by using Theorem 2.21 and vice versa.

Suppose that  $(\mathcal{D}, f)$  is a simple submodular system on  $E$  and that  $x$  is an extreme point of  $\mathbf{B}(f)$ . Then,

$$\mathcal{D}(x) = \{X \mid X \in \mathcal{D}, x(X) = f(X)\} \quad (2.100)$$

is also a simple distributive lattice and let  $\mathcal{P}(x) = (E, \preceq_x)$  be the poset corresponding to  $\mathcal{D}(x)$  (i.e.,  $\mathcal{D}(x)$  is the set of all the ideals of  $\mathcal{P}(x)$ ). Let  $H(x) = (E, A(x))$  be the Hasse diagram representing the poset  $\mathcal{P}(x)$ . We can show that

$$\chi_e - \chi_{e'} \quad ((e, e') \in A(x)) \quad (2.101)$$

constitute the unique minimal system of generators of tangent cone  $\text{TC}(\mathbf{B}(f), x)$  at  $x$ . Let  $y$  be another extreme point of  $\mathbf{B}(f)$  and  $H(y) = (E, A(y))$  be the

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Hasse diagram representing the poset  $\mathcal{P}(y)$ . We can also show that the two extreme points  $x$  and  $y$  of  $B(f)$  are adjacent if and only if there exist an arc  $(e, e') \in A(x)$  and its reorientation  $(e', e) \in A(y)$  such that contracting (or shortcircuiting) the arc  $(e, e')$  in  $H(x)$  and contracting the arc  $(e', e)$  in  $H(y)$  yield the same graph. The proof is left as an exercise (see [Fuji84d] for more detail).

### 2.4. Related Polyhedra

We show some polyhedra which are closely related to base polyhedra and submodular/supermodular polyhedra.

#### (a) Generalized polymatroids

A. Frank [Frank81] introduced the concept of *generalized polymatroid*. Suppose that a submodular system  $(\mathcal{D}_1, f')$  and a supermodular system  $(\mathcal{D}_2, g')$  on  $E'$  satisfy

$$\begin{aligned} \forall X \in \mathcal{D}_1, \forall Y \in \mathcal{D}_2: X - Y \in \mathcal{D}_1, Y - X \in \mathcal{D}_2, \\ f'(X) - g'(Y) \geq f'(X - Y) - g'(Y - X). \end{aligned} \quad (2.102)$$

Then the polyhedron  $P(f', g')$  defined by

$$P(f', g') = \{x \mid x \in \mathbb{R}^{E'}, \forall X \in \mathcal{D}_1: x(X) \leq f'(X), \\ \forall Y \in \mathcal{D}_2: x(Y) \geq g'(Y)\} \quad (2.103)$$

is called a *generalized polymatroid* or *g-polymatroid*. (The same polyhedron is also considered by R. Hassin [Hassin82] for the case where  $\mathcal{D}_1 = \mathcal{D}_2 = 2^{E'}$ .)

Generalized polymatroids are characterized by the following theorem, which is implicit in [Frank81] (also see [Schrijver84]) (see Fig. 2.6).

**Theorem 2.24** [Fuji84a]: For the base polyhedron  $B(f)$  associated with a submodular system  $(\mathcal{D}, f)$  on  $E$ , the projection of  $B(f)$  along an axis  $e \in E$  on the hyperplane  $x(e) = 0$  is a generalized polymatroid  $P(f', g')$  in  $\mathbb{R}^{E'}$  with  $E' = E - \{e\}$ , where

$$\mathcal{D}_1 = \{X \mid e \notin X \in \mathcal{D}\}, \quad (2.104)$$

$$\mathcal{D}_2 = \{E - X \mid e \in X \in \mathcal{D}\}, \quad (2.105)$$

$f'$  is the restriction of  $f$  to  $\mathcal{D}_1$  and  $g'$  is the restriction of  $f^\#$  to  $\mathcal{D}_2$ .

Conversely, every generalized polymatroid in  $\mathbb{R}^{E'}$  is obtained in this way. For each generalized polymatroid  $P(f', g')$  with  $\mathcal{D}_i \subseteq 2^{E'}$  ( $i = 1, 2$ ) such a base

## 2.4. RELATED POLYHEDRA

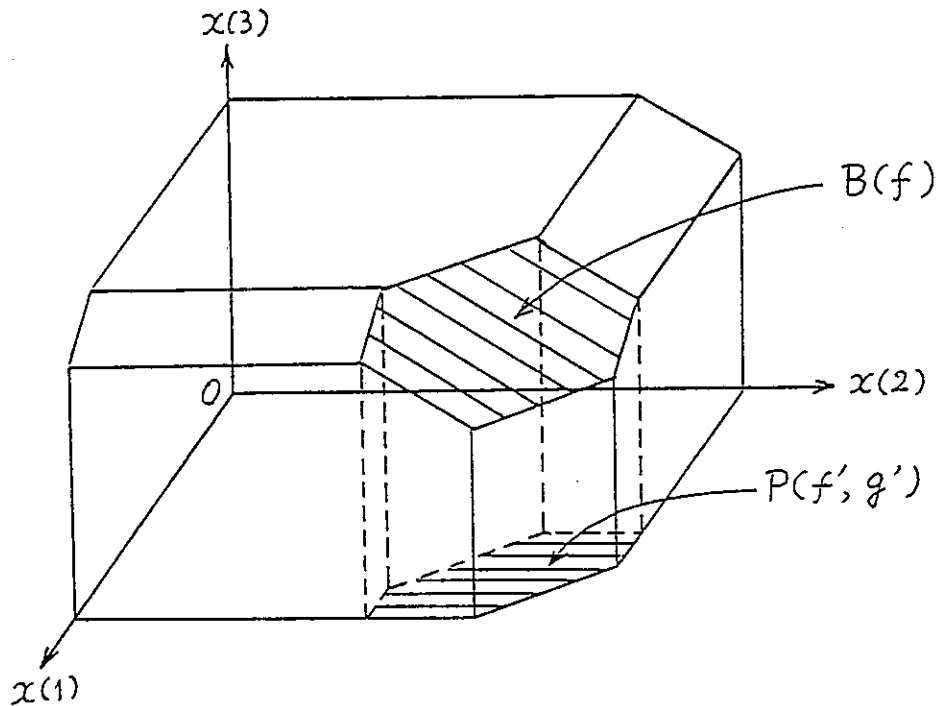


Figure 2.6.

polyhedron  $B(f)$  in  $\mathbb{R}^E$  with  $E = E' \cup \{e\}$  is unique up to translation along the new axis  $e$ , and the two polyhedra  $P(f', g')$  and  $B(f)$  are isomorphic with each other under the projection of the hyperplane  $x(E) = f(E)$  onto the hyperplane  $x(e) = 0$  along the axis  $e$ .

The proof is left as an exercise. It follows from Theorem 2.24 that extreme points, extreme rays, faces etc. of  $P(f', g')$  are characterized by the corresponding results for  $B(f)$  (cf. [Fuji84d]). Moreover, since the greedy algorithm works for  $B(f)$ , it also works for  $P(f', g')$  (cf. [Hassin82]).

Frank [Frank81] originally defined generalized polymatroids in terms of intersecting families. This corresponds to the fact (Theorem 1.4) that a crossing-submodular function on a crossing family determines the base polyhedron associated with a submodular system. Note that if  $\mathcal{D} \subseteq 2^E$  is a crossing family, then  $\mathcal{D}_i$  ( $i = 1, 2$ ) defined by (2.104) and (2.105) are intersecting families. (For more details on generalized polymatroids, see [Frank+Tardos88].)

### (b) Pseudomatroids

The concept of pseudomatroid was introduced by R. Chandrasekaran and S. N. Kabadi [Chandrasekaran+Kabadi88]. The same or similar concepts were independently considered by A. Bouchet [Bouchet87] as  $\Delta$ -matroid, A. Dress

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[Dress + Havel86] as metroid, M. Nakamura [Nakamura88] as universal polymatroid and L. Qi [Qi88] as ditroid. We shall consider pseudomatroids of Chandrasekaran and Kabadi from the point of view of submodular systems.

Denote by  $3^E$  the set of all the ordered pairs  $(X, Y)$  of disjoint subsets of  $E$ . Let  $f: 3^E \rightarrow \mathbb{R}$  be a function with  $f(\emptyset, \emptyset) = 0$  such that for each  $(X_i, Y_i) \in 3^E$  ( $i = 1, 2$ )

$$\begin{aligned} & f(X_1, Y_1) + f(X_2, Y_2) \\ & \geq f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) + f(X_1 \cap X_2, Y_1 \cap Y_2). \end{aligned} \quad (2.106)$$

Define a polyhedron

$$P_*(f) = \{x \mid x \in \mathbb{R}^E, \forall (X, Y) \in 3^E: x(X) - x(Y) \leq f(X, Y)\}. \quad (2.107)$$

The polyhedron  $P_*(f)$  is called a *pseudomatroid* [Chandrasekaran+Kabadi88] and  $f$  its *rank function* (see Fig. 2.7).

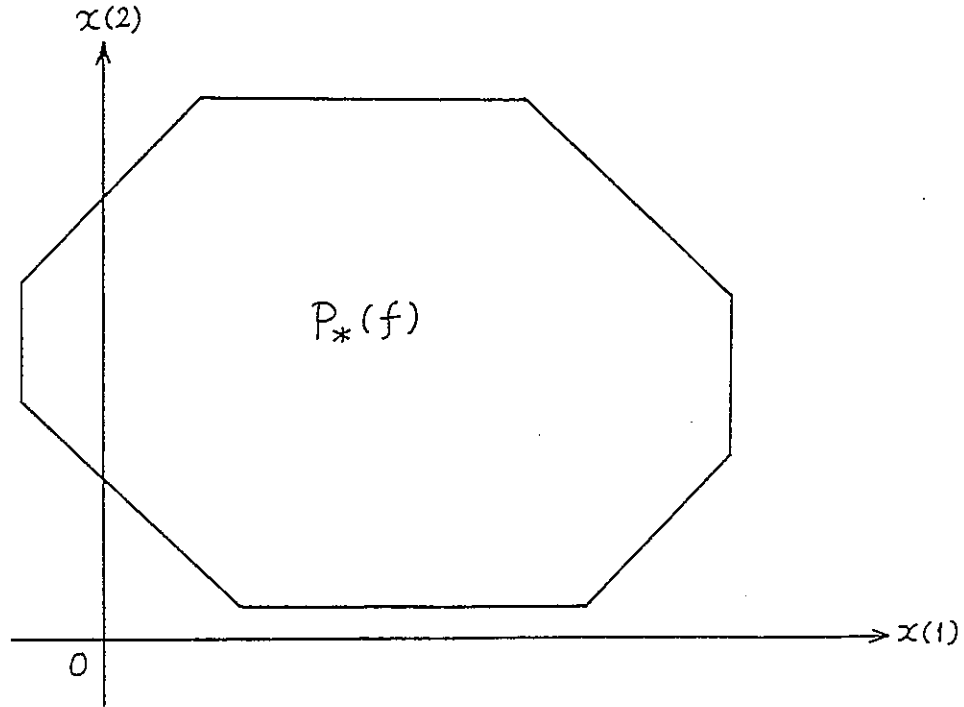


Figure 2.7.

We call a pair  $(S, T) \in 3^E$  such that  $S \cup T = E$  an *orthant* of  $\mathbb{R}^E$ . For each orthant  $(S, T)$  denote by  $2^{(S, T)}$  the set of all the pairs  $(X, Y)$  such that  $X \subseteq S$  and  $Y \subseteq T$ .

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Now, choose an orthant  $(S, T)$ . Then for each  $(X_i, Y_i) \in 2^{(S, T)}$  ( $i = 1, 2$ ) we have from (2.106)

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f(X_1 \cup X_2, Y_1 \cup Y_2) + f(X_1 \cap X_2, Y_1 \cap Y_2). \quad (2.108)$$

This means that for the orthant  $(S, T)$  the function  $f': 2^E \rightarrow \mathbb{R}$  defined by

$$f'(X) = f(S \cap X, T \cap X) \quad (X \subseteq E) \quad (2.109)$$

is a submodular function on  $2^E$ . Define

$$P_{(S, T)}(f) = \{x \mid x \in \mathbb{R}^E, \forall (X, Y) \in 2^{(S, T)}: x(X) - x(Y) \leq f(X, Y)\}. \quad (2.110)$$

The polyhedron  $P_{(S, T)}(f)$  is expressed by the submodular polyhedron  $P(f')$  associated with the submodular system  $(2^E, f')$  as follows.

$$P_{(S, T)}(f) = \{x \mid y \in P(f'), \forall e \in S: x(e) = y(e), \forall e \in T: x(e) = -y(e)\}. \quad (2.111)$$

Therefore, the combinatorial properties of  $P_{(S, T)}(f)$  are the same as  $P(f')$  and the greedy algorithm described in Section 2.2.b works for  $P_{(S, T)}(f)$  *mutatis mutandis* as for  $P(f')$ . Define

$$B_{(S, T)}(f) = \{x \mid x \in P_{(S, T)}(f), x(S) - x(T) = f(S, T)\}. \quad (2.112)$$

We call  $B_{(S, T)}(f)$  the *base polyhedron in the orthant*  $(S, T)$  of the pseudomatroid  $P_*(f)$ .

From Theorem 2.18 we have

**Corollary 2.25:** A vector  $x \in \mathbb{R}^E$  is an extreme point of  $B_{(S, T)}(f)$  in (2.112) if and only if for a maximal chain

$$C: \emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = E \quad (2.113)$$

of  $2^E$  we have

$$\begin{aligned} & f(S \cap A_i, T \cap A_i) - f(S \cap A_{i-1}, T \cap A_{i-1}) \\ &= \begin{cases} x(A_i - A_{i-1}) & \text{if } A_i - A_{i-1} \subseteq S, \\ -x(A_i - A_{i-1}) & \text{if } A_i - A_{i-1} \subseteq T. \end{cases} \end{aligned} \quad (2.114)$$

for each  $i = 1, 2, \dots, n$ .

We also have

## II. SUBMODULAR SYSTEMS AND BASE POLYHEDRA

**Lemma 2.26:** For each orthant  $(S, T)$  of  $\mathbb{R}^E$  we have

$$B_{(S,T)}(f) \subseteq P_*(f). \quad (2.115)$$

(Proof) Suppose  $x \in B_{(S,T)}(f)$ . Then, for any  $(X, Y) \in 3^E$  we have from (2.106)

$$\begin{aligned} & x(X) - x(Y) - f(X, Y) \\ &= x(X) - x(Y) - f(X, Y) + x(S) - x(T) - f(S, T) \\ &\leq x(S - Y) - x(T - X) - f(S - Y, T - X) \\ &\quad + x(S \cap X) - x(T \cap Y) - f(S \cap X, T \cap Y) \\ &\leq 0. \end{aligned} \quad (2.116)$$

Hence  $x \in P_*(f)$ . Q.E.D.

We see from Lemma 2.26 that  $B_{(S,T)}(f)$  for each orthant  $(S, T)$  is a face of  $P_*(f)$ . Since

$$P_*(f) = \bigcap \{P_{(S,T)}(f) \mid (S, T): \text{an orthant of } \mathbb{R}^E\}, \quad (2.117)$$

for any  $w: E \rightarrow \mathbb{R}$  we have from Lemma 2.26

$$\begin{aligned} & \max \left\{ \sum_{e \in E} w(e)x(e) \mid x \in P_{(S,T)}(f) \right\} \\ & \geq \max \left\{ \sum_{e \in E} w(e)x(e) \mid x \in P_*(f) \right\} \\ & \geq \max \left\{ \sum_{e \in E} w(e)x(e) \mid x \in B_{(S,T)}(f) \right\}. \end{aligned} \quad (2.118)$$

For an orthant  $(S, T)$  such that  $w(e) \geq 0$  ( $e \in S$ ) and  $w(e) \leq 0$  ( $e \in T$ ), (2.118) holds with equality. Therefore, the problem of maximizing the linear function  $\sum_{e \in E} w(e)x(e)$  over the pseudomatroid  $P_*(f)$  is solved by maximizing the same linear function over  $P_{(S,T)}(f)$  or  $B_{(S,T)}(f)$ . Hence the greedy algorithm as in Corollary 2.25 works for the pseudomatroid  $P_*(f)$  and the union of all the extreme points of  $B_{(S,T)}(f)$  for all the orthants  $(S, T)$  is exactly the set of all the extreme points of  $P_*(f)$ . Also, we see from Lemma 2.26 that for each  $(X, Y) \in 3^E$

$$f(X, Y) = \max \{x(X) - x(Y) \mid x \in P_*(f)\}, \quad (2.119)$$

so that  $f$  is uniquely determined by  $P_*(f)$ .

**Theorem 2.27** [Chandrasekaran+Kabadi88], [Nakamura88]: For any function  $f: 3^E \rightarrow \mathbb{R}$  define

$$P_*(f) = \{x \mid x \in \mathbb{R}^E, \forall (X, Y) \in 3^E: x(X) - x(Y) \leq f(X, Y)\}. \quad (2.120)$$



## 2.4. RELATED POLYHEDRA

The greedy algorithm of the type described in Corollary 2.25 works for  $P_*(f)$  if and only if  $f$  is the rank function of a pseudomatroid.

(Proof) It suffices to show the "only if" part. Suppose that the greedy algorithm as in Corollary 2.25 works for  $P_*(f)$ . Then, for any  $(X_i, Y_i) \in 3^E$  ( $i = 1, 2$ ) choose a maximal chain  $C$  of (2.113) containing  $(X_1 \cap X_2) \cup (Y_1 \cap Y_2)$  and  $((X_1 \cup X_2) - (Y_1 \cup Y_2)) \cup ((Y_1 \cup Y_2) - (X_1 \cup X_2))$  in it and define the vector  $x \in \mathbb{R}^E$  by (2.114), where  $(S, T)$  is an orthant such that  $(X_1 \cup X_2) - (Y_1 \cup Y_2) \subseteq S$  and  $(Y_1 \cup Y_2) - (X_1 \cup X_2) \subseteq T$ . Since from the assumption we have  $x \in P_*(f)$ , by the definition of  $x$  we have

$$x(X_i) - x(Y_i) \leq f(X_i, Y_i) \quad (i = 1, 2), \quad (2.121)$$

$$\begin{aligned} & x((X_1 \cup X_2) - (Y_1 \cup Y_2)) - x((Y_1 \cup Y_2) - (X_1 \cup X_2)) \\ & = f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \end{aligned} \quad (2.122)$$

$$x(X_1 \cap X_2) - x(Y_1 \cap Y_2) = f(X_1 \cap X_2, Y_1 \cap Y_2). \quad (2.123)$$

From (2.121)–(2.123),

$$\begin{aligned} & f(X_1, Y_1) + f(X_2, Y_2) \\ & \geq x(X_1) - x(Y_1) + x(X_2) - x(Y_2) \\ & = x((X_1 \cup X_2) - (Y_1 \cup Y_2)) - x((Y_1 \cup Y_2) - (X_1 \cup X_2)) \\ & \quad + x(X_1 \cap X_2) - x(Y_1 \cap Y_2) \\ & = f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) \\ & \quad + f(X_1 \cap X_2, Y_1 \cap Y_2). \end{aligned} \quad (2.124)$$

It follows that  $f$  is the rank function of a pseudomatroid. Q.E.D.

**Theorem 2.28** [Chandrasekaran+Kabadi88]: The system of inequalities

$$x(X) - x(Y) \leq f(X, Y) \quad ((X, Y) \in 3^E) \quad (2.125)$$

is totally dual integral.

(Proof) The present theorem follows from Lemma 2.26 and Theorem 2.16. Q.E.D.

The concept of pseudomatroid explained above can easily be generalized as follows. Let  $\mathcal{F}$  be a subset of  $3^E$  such that for each  $(X_i, Y_i) \in \mathcal{F}$  ( $i = 1, 2$ ) the two pairs  $((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2))$  and  $(X_1 \cap X_2, Y_1 \cap Y_2)$  belong to  $\mathcal{F}$ . Also let  $f: \mathcal{F} \rightarrow \mathbb{R}$  be a function such that for each  $(X_i, Y_i) \in \mathcal{F}$  ( $i = 1, 2$ )  $f$  satisfies (2.106). The class of pseudomatroids in this generalized

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sense includes as special cases submodular and supermodular polyhedra, base polyhedra and generalized polymatroids.

The author [Fuji84e] considered a distributive lattice  $\hat{\mathcal{F}} \subseteq 3^E$  such that for each  $(X_i, Y_i) \in \hat{\mathcal{F}}$  ( $i = 1, 2$ ) we have  $(X_1 \cup X_2, Y_1 \cap Y_2), (X_1 \cap X_2, Y_1 \cup Y_2) \in \hat{\mathcal{F}}$  and a submodular function  $\hat{f}: \hat{\mathcal{F}} \rightarrow \mathbb{R}$  such that for each  $(X_i, Y_i) \in \hat{\mathcal{F}}$  ( $i = 1, 2$ )

$$\hat{f}(X_1, Y_1) + \hat{f}(X_2, Y_2) \geq \hat{f}(X_1 \cup X_2, Y_1 \cap Y_2) + \hat{f}(X_1 \cap X_2, Y_1 \cup Y_2). \quad (2.126)$$

The total dual integrality of the system of inequalities

$$x(X) - x(Y) \leq \hat{f}(X, Y) \quad ((X, Y) \in \hat{\mathcal{F}}) \quad (2.127)$$

was shown in [Fuji84e] under a mild additional condition given as follows:

If for  $(X_i, Y_i) \in \hat{\mathcal{F}}$  ( $i = 1, 2$ )

$$X_1 \supseteq X_2, Y_1 \subseteq Y_2, X_1 \cap Y_2 \neq \emptyset \quad (2.128)$$

and

$$X_2 \neq \emptyset \text{ or } Y_1 \neq \emptyset \quad (2.129)$$

Then we have

$$(X_1 - Y_2, Y_1), (X_2, Y_2 - X_1) \in \hat{\mathcal{F}}, \quad (2.130)$$

$$\hat{f}(X_1, Y_1) + \hat{f}(X_2, Y_2) \geq \hat{f}(X_1 - Y_2, Y_1) + \hat{f}(X_2, Y_2 - X_1). \quad (2.131)$$

The class of polyhedra defined by

$$\hat{P}(\hat{f}) = \{x \mid x \in \mathbb{R}^E, \forall (X, Y) \in \hat{\mathcal{F}}: x(X) - x(Y) \leq \hat{f}(X, Y)\} \quad (2.132)$$

includes as a special case the intersection of two base polyhedra (see [Fuji84e]) and is different from that of pseudomatroids (also see [Qi89]).

### 2.5. Submodular Systems of Network Type [Tomi+Fuji81]

Let  $\mathcal{N} = (G = (V, A), c)$  be a capacitated network with an underlying graph  $G = (V, A)$  and a nonnegative upper capacity function  $c: A \rightarrow \mathbb{R}_+$ , where the lower capacity function is regarded as the zero function. The cut function  $\kappa_c: 2^E \rightarrow \mathbb{R}$  associated with the network  $\mathcal{N}$  is given by

$$\kappa_c(U) = \sum_{a \in \Delta^+ U} c(a) \quad (U \in V), \quad (2.133)$$

where  $\Delta^+ U$  is the set of arcs leaving  $U$  (see Section 1.3). Without loss of generality we assume that  $G$  is a simple graph and that each arc  $a \in A$  is

## 2.5. SUBMODULAR SYSTEMS OF NETWORK TYPE

identified with the ordered pair  $(\partial^+ a, \partial^- a)$  of its end-vertices. (2.133) can be rewritten as

$$\kappa_c(U) = \sum_{u \in U} \sum_{v \in U - \{u\}} c(u, v) - \sum_{\{u, v\} \in \binom{U}{2}} (c(u, v) + c(v, u)) \quad (U \in V), \quad (2.134)$$

where for each arc  $(u, v) \in A$  we write  $c((u, v))$  as  $c(u, v)$ ,  $\binom{U}{2}$  is the set of all the two-element subsets of  $U$ , and we define  $c(u, v) = 0$  for  $(u, v) \notin A$ . For any finite set  $X$  and any integer  $i$  with  $0 \leq i \leq |X|$  we denote by  $\binom{X}{i}$  the set of all the  $i$ -element subsets of  $X$ .

For any non-zero set function  $f: 2^V \rightarrow \mathbb{R}$  there exist functions  $f^{(i)}: \binom{V}{i} \rightarrow \mathbb{R}$  ( $0 \leq i \leq |V|$ ) such that

$$f(X) = \sum_{i=0}^{|X|} \sum_{Y \in \binom{X}{i}} f^{(i)}(Y) \quad (X \subseteq V). \quad (2.135)$$

By the Möbius inversion formula  $f^{(i)}$  ( $0 \leq i \leq |V|$ ) are uniquely determined from  $f$  as

$$f^{(i)}(X) = \sum_{Y \subseteq X} (-1)^{|X-Y|} f(Y). \quad (2.136)$$

Let  $k$  be an integer such that  $0 \leq k \leq |V|$ ,  $f^{(k)} \neq 0$  and  $f^{(i)} = 0$  ( $k+1 \leq i \leq |V|$ ). The function  $f$  is called a set function of order  $k$  (see [Tomi80b]). We see from (2.134) that any cut function is of order 2 if  $c \neq 0$  (also see (2.140) and (2.141) below).

A submodular system  $(2^V, f)$  is said to be of network type if  $f$  is equal to the cut function  $\kappa_c: 2^V \rightarrow \mathbb{R}$  associated with a network having a nonnegative capacity function  $c$ .

**Theorem 2.29** [Tomi+Fuji81]: Suppose that  $(2^V, f)$  is a submodular system on  $V$ .  $(2^V, f)$  is of network type if and only if the following (i)–(iii) hold:

- (i) The order of  $f$  is less than or equal to 2.
- (ii) For the functions  $f^{(i)}$  ( $0 \leq i \leq |V|$ ) in (2.135),

$$f^{(0)} = 0, \quad f^{(1)} \geq 0, \quad (2.137)$$

$$\sum_{u \in V} f^{(1)}(\{u\}) = - \sum_{U \in \binom{V}{2}} f^{(2)}(U). \quad (2.138)$$

- (iii) For each  $X \subseteq V$ ,

$$\sum_{u \in X} f^{(1)}(\{u\}) \leq - \sum_{\substack{U \in \binom{V}{2} \\ U \cap X \neq \emptyset}} f^{(2)}(U). \quad (2.139)$$

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(Proof) *The "only if" part:* If  $(2^V, f)$  is of network type and  $f = \kappa_c$  for a network with a nonnegative capacity function  $c$ , then from (2.134) we have

$$f^{(0)} = 0, \quad f^{(1)}(\{u\}) = \sum_{v \in U - \{u\}} c(u, v), \quad (2.140)$$

$$f^{(2)}(\{u, v\}) = -(c(u, v) + c(v, u)). \quad (2.141)$$

Since the capacity function  $c$  is nonnegative, (i)–(iii) easily follow from (2.140) and (2.141).

*The "if" part:* Suppose (i)–(iii) hold. Consider the bipartite graph  $\hat{G} = (W^+, W^-; \hat{A})$  with the left and right vertex sets  $W^+$  and  $W^-$  and with the arc set  $\hat{A}$  defined by

$$W^+ = V, \quad W^- = \binom{V}{2}, \quad (2.142)$$

$$\hat{A} = \{(u, U) \mid u \in U \in \binom{V}{2}\}. \quad (2.143)$$

Let  $\hat{\mathcal{N}} = (\hat{G}, \hat{c})$  be a network with the underlying graph  $\hat{G}$  and a nonnegative capacity function  $\hat{c}$  such that for each  $a \in \hat{A}$   $\hat{c}(a)$  is sufficiently large, where  $W^+$  is the entrance vertex set and  $W^-$  the exit vertex set of  $\hat{\mathcal{N}}$ . Then, it follows from the assumption that there exists a flow  $\varphi: \hat{A} \rightarrow \mathbb{R}$  in  $\hat{\mathcal{N}}$  such that

$$\partial\varphi(u) = f^{(1)}(\{u\}) \quad (u \in W^+ (= V)), \quad (2.144)$$

$$-\partial\varphi(U) = f^{(2)}(U) \quad (U \in W^- (= \binom{V}{2})), \quad (2.145)$$

due to the supply-demand theorem for bipartite networks ([Gale57], [Ford+Fulkerson62]). Choose one such flow  $\varphi: \hat{A} \rightarrow \mathbb{R}_+$ . Define

$$A = \{(u, v) \mid u, v \in V, u \neq v\}, \quad (2.146)$$

$$c(u, v) = \varphi((u, \{u, v\})) \quad ((u, v) \in A). \quad (2.147)$$

Let  $\mathcal{N}$  be the network with the underlying graph  $G = (V, A)$  and the nonnegative capacity function  $c$  defined by (2.146) and (2.147). From (2.144)–(2.147) we have (2.140) and (2.141). Since the order of  $f$  is less than or equal to 2,  $f$  coincides with the cut function  $\kappa_c$  associated with the network  $\mathcal{N}$ . Q.E.D.

We see from this theorem that for a submodular system  $(2^V, f)$  on  $V$  with the rank function of order at most 2 the problem of discerning whether the submodular system  $(2^V, f)$  is of network type or not can be solved by a max-flow computation for the network  $\hat{\mathcal{N}}$  defined in the above proof, since (iii)

## 2.5. SUBMODULAR SYSTEMS OF NETWORK TYPE

in Theorem 2.29 can directly be checked and (iii) together with (2.138) is a necessary and sufficient condition for the existence of a feasible flow in the bipartite network  $\hat{\mathcal{N}}$ .

Moreover, when  $(2^V, f)$  is of network type, consider the minimum-cost network-realization problem defined as follows: Find a network  $\mathcal{N} = (G = (V, A), c)$  with  $\kappa_c = f$  such that the cost

$$\sum_{a \in A} \gamma(a)c(a) \tag{2.148}$$

is as small as possible, where  $\gamma(a)$  is the realization cost per unit capacity for each arc  $a \in A$ . As is seen from the above proof, this problem is reduced to a Hitchcock transportation problem for the bipartite network  $\hat{\mathcal{N}}$  defined in the above proof (see [Tomi+Fuji81]).



## Chapter III. Neoflows

In this chapter we consider generalizations of classical flow problems of Ford and Fulkerson to flow problems with boundary constraints described by submodular functions. The new flow problems to be treated are the submodular flow problem, the independent flow problem and the polymatroidal flow problem, and they are, in a sense, equivalent. Because of this we call the class of these flow problems and other possible equivalent ones *the neoflow problem* and each of them *a neoflow problem*. We give a theory and algorithms for the neoflow problem.

### 3. The Intersection Problem

In this section we consider the problem of finding a maximum common subbase of two submodular systems and some related problems.

#### 3.1. The Intersection Theorem

Let  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) be two submodular systems on  $E$  and consider the following problem.

$$P_1: \text{Maximize } x(E) \quad (3.1a)$$

$$\text{subject to } x \in P(f_1) \cap P(f_2). \quad (3.1b)$$

Equivalently, we express this problem as follows. Let  $E'$  be a copy of  $E$  and we regard  $(\mathcal{D}_2, f_2)$  as a submodular system on  $E'$ . Also let  $G = (E, E'; A)$  be the bipartite graph with the left and right vertex sets  $E$  and  $E'$  and with the arc set  $A = \{(e, e') \mid e \in E\}$ , where  $e' \in E'$  is a copy of  $e \in E$ , i.e.,  $A$  gives a natural bijection between  $E$  and its copy  $E'$  (see Fig 3.1). Furthermore, we consider a capacity function  $c: A \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $c(a) = +\infty$  ( $a \in A$ ). Then Problem  $P_1$  in (3.1) is equivalent to the following problem

$$P_1': \text{Maximize } \partial\varphi(E) \quad (3.2a)$$

$$\text{subject to } (\partial\varphi)^E \in P(f_1), \quad (3.2b)$$

$$-(\partial\varphi)^{E'} \in P(f_2), \quad (3.2c)$$

where  $\varphi: A \rightarrow \mathbb{R}$  is a flow,  $\partial\varphi$  is the boundary of  $\varphi$  in  $\mathcal{N}$  and  $(\partial\varphi)^E$  and  $(\partial\varphi)^{E'}$  are, respectively, the restrictions of  $\partial\varphi$  to  $E$  and  $E'$ . A flow  $\varphi: A \rightarrow \mathbb{R}$  satisfying (3.2b) and (3.2c) is called a *feasible flow* in  $\mathcal{N} = (G = (E, E'; A), c, S_1, S_2)$ ,

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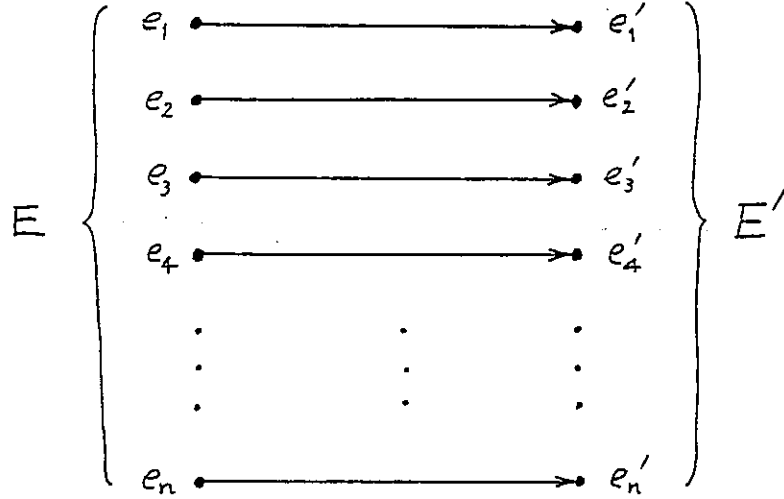


Figure 3.1.

where  $S_i = (\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ). As we will see later, Problem  $P_1'$  is a special case of a neoflow problem.

#### (a) Preliminaries

We need some preliminaries to furnish an algorithm for solving the intersection problem described by (3.1) or (3.2).

Consider a submodular system  $(\mathcal{D}, f)$  on  $E$ . In the following we give several lemmas which are obtained by a direct adaptation of the results shown in [Fuji78a] for polymatroids.

**Lemma 3.1:** Suppose  $x \in P(f)$ ,  $u \in \text{sat}(x)$  and  $v \in \text{dep}(x, u) - \{u\}$ . For any  $\alpha$  such that  $0 < \alpha \leq \tilde{c}(x, u, v)$  define  $y = x + \alpha(\chi_u - \chi_v)$ . Then,  $y \in P(f)$  and

$$\text{sat}(y) = \text{sat}(x). \quad (3.3)$$

(Proof) From the definition of the exchange capacity we have  $y \in P(f)$ . Also, since for any  $X \in \mathcal{D}$  such that  $X \supseteq \text{sat}(x)$  we have  $y(X) = x(X)$  and  $\text{sat}(x)$  is the unique maximal tight set  $X$  such that  $x(X) = f(X)$ , we have  $\text{sat}(y) = \text{sat}(x)$ . Q.E.D.



### 3.1. THE INTERSECTION THEOREM

**Lemma 3.2:** Under the same assumption as in Lemma 3.1,

$$\hat{c}(y, w) = \hat{c}(x, w) \quad (w \in E - \text{sat}(x)). \quad (3.4)$$

(Proof) Put  $X_0 = \text{sat}(x)(= \text{sat}(y))$ . Since  $x(X_0) = f(X_0)$  and  $y(X_0) = f(X_0)$ , we have for any  $X \in \mathcal{D}$

$$\begin{aligned} f(X) - y(X) &= f(X) - y(X) + f(X_0) - y(X_0) \\ &\geq f(X \cup X_0) - y(X \cup X_0) + f(X \cap X_0) - y(X \cap X_0) \\ &\geq f(X \cup X_0) - y(X \cup X_0) \\ &= f(X \cup X_0) - x(X \cup X_0), \end{aligned} \quad (3.5)$$

and similarly,

$$f(X) - x(X) \geq f(X \cup X_0) - x(X \cup X_0). \quad (3.6)$$

Hence the lemma follows from (1.37). Q.E.D.

**Lemma 3.3:** For any  $x \in P(f)$  let  $u_1, u_2$  and  $v_2$  be three distinct elements of  $E$  such that

$$u_i \in \text{sat}(x) \quad (i = 1, 2), \quad (3.7)$$

$$v_2 \in \text{dep}(x, u_2), \quad v_2 \notin \text{dep}(x, u_1). \quad (3.8)$$

For any  $\alpha$  such that  $0 < \alpha \leq \tilde{c}(x, u_2, v_2)$  define

$$y = x + \alpha(\chi_{u_2} - \chi_{v_2}). \quad (3.9)$$

Then we have  $u_1 \in \text{sat}(y)$  and

$$\text{dep}(y, u_1) = \text{dep}(x, u_1). \quad (3.10)$$

(Proof) From Lemma 3.1 we have  $u_1 \in \text{sat}(y)$ . Also we have  $u_2 \notin \text{dep}(x, u_1)$ , since otherwise we would have  $\text{dep}(x, u_2) \subseteq \text{dep}(x, u_1)$  by the minimality of  $\text{dep}(x, u_2)$  and hence  $v_2 \in \text{dep}(x, u_1)$ . Therefore, putting  $X_0 = \text{dep}(x, u_1)$ , we have  $y(X_0) = x(X_0) = f(X_0)$  and  $y(X) = x(X)$  for any  $X \in \mathcal{D}$  with  $X \subseteq X_0$ . (3.10) follows from the definition of dependence function. Q.E.D.

**Lemma 3.4:** For any  $x \in P(f)$  let  $u_1, u_2, v_1$  and  $v_2$  be four distinct elements of  $E$  satisfying (3.7), (3.8) and

$$v_1 \in \text{dep}(x, u_1). \quad (3.11)$$

Then for the vector  $y$  defined by (3.9) for any  $\alpha$  with  $0 < \alpha \leq \tilde{c}(x, u_2, v_2)$ , we have

$$\tilde{c}(y, u_1, v_1) = \tilde{c}(x, u_1, v_1). \quad (3.12)$$

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(Proof) For any  $z \in P(f)$  and  $X_0 \in \mathcal{D}$  such that  $z(X_0) = f(X_0)$  we have

$$f(X) - z(X) \geq f(X \cap X_0) - z(X \cap X_0) \quad (X \in \mathcal{D}). \quad (3.13)$$

For  $X_0 = \text{dep}(x, u_1)$  we have from Lemma 3.3

$$y(X_0) = x(X_0) = f(X_0) \quad (3.14)$$

and since  $u_2, v_2 \notin \text{dep}(x, u_1)$ , we have

$$y(X) = x(X) \quad (X \subseteq X_0, X \in \mathcal{D}). \quad (3.15)$$

Since (3.13) holds for  $z = x, y$ , (3.12) follows from (3.14) and (3.15). Q.E.D.

**Lemma 3.5:** For any  $x \in P(f)$  let  $u_i, v_i$  ( $i = 1, 2, \dots, q$ ) be  $2q$  distinct elements of  $E$  such that

$$u_i \in \text{sat}(x), \quad v_i \in \text{dep}(x, u_i) \quad (i = 1, 2, \dots, q), \quad (3.16)$$

$$v_i \notin \text{dep}(x, u_j) \quad (1 \leq i < j \leq q). \quad (3.17)$$

For any  $\alpha_i$  ( $i = 1, 2, \dots, q$ ) satisfying  $0 < \alpha_i \leq \tilde{c}(x, u_i, v_i)$  ( $i = 1, 2, \dots, q$ ) define a vector  $y \in \mathbb{R}^E$  by

$$y = x + \sum_{i=1}^q \alpha_i (\chi_{u_i} - \chi_{v_i}). \quad (3.18)$$

Then,

$$y \in P(f), \quad (3.19)$$

$$\text{sat}(y) = \text{sat}(x), \quad (3.20)$$

$$\tilde{c}(y, w) = \tilde{c}(x, w) \quad (w \in E - \text{sat}(x)). \quad (3.21)$$

(Proof) Considering the elementary transformations in the order of the pairs  $(u_1, v_1), (u_2, v_2), \dots, (u_q, v_q)$ , the present lemma can be shown by repeatedly applying Lemmas 3.1–3.4. Q.E.D.

It should be noted that we have (3.19)–(3.21) if (3.16) and (3.17) hold for an appropriate numbering of  $u_i$ 's and  $v_i$ 's.

Lemma 3.5 will play a very fundamental rôle in developing algorithms for solving the intersection problem and other problems.

#### (b) An algorithm and the intersection theorem

### 3.1. THE INTERSECTION THEOREM

We now consider Problem  $P_1'$  described by (3.2). Given a feasible flow  $\varphi$  in network  $\mathcal{N} = (G = (E, E'; A), c, S_1, S_2)$ , define the *auxiliary network*  $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$  associated with  $\varphi$  as follows.  $G_\varphi = (V, A_\varphi)$  is a directed graph, called the *auxiliary graph* associated with  $\varphi$ , with vertex set  $V$  and arc set  $A_\varphi$  given by

$$V = E \cup E' \cup \{s^+, s^-\}, \quad (3.22)$$

$$A_\varphi = S_\varphi^+ \cup A_\varphi^+ \cup A^* \cup B^* \cup A_\varphi^- \cup S_\varphi^-, \quad (3.23)$$

where

$$S_\varphi^+ = \{(s^+, v) \mid v \in S^+ - \text{sat}^+(\partial^+ \varphi)\}, \quad (3.24)$$

$$A_\varphi^+ = \{(u, v) \mid v \in \text{sat}^+(\partial^+ \varphi), u \in \text{dep}^+(\partial^+ \varphi, v) - \{v\}\}, \quad (3.25)$$

$$A^* = A, \quad (3.26)$$

$$B^* = \{(e', e) \mid e \in E\}, \quad (3.27)$$

$$A_\varphi^- = \{(u, v) \mid u \in \text{sat}^-(\partial^- \varphi), v \in \text{dep}^-(\partial^- \varphi, u) - \{u\}\}, \quad (3.28)$$

$$S_\varphi^- = \{(v, s^-) \mid v \in S^- - \text{sat}^-(\partial^- \varphi)\}. \quad (3.29)$$

Here,  $\partial^+ \varphi = (\partial \varphi)^E$ ,  $\partial^- \varphi = -(\partial \varphi)^{E'}$ , and  $\text{sat}^+$  and  $\text{dep}^+$  ( $\text{sat}^-$  and  $\text{dep}^-$ ) are, respectively, the saturation function and the dependence function defined with respect to submodular system  $(\mathcal{D}_1, f_1)$  on  $E$  ( $(\mathcal{D}_2, f_2)$  on  $E'$ ). Note that  $B^*$  is the set of the reorientations of arcs of  $A$ . We also define the capacity function  $c_\varphi: A_\varphi \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$c_\varphi(a) = \begin{cases} \hat{c}^+(\partial^+ \varphi, v) & (a = (s^+, v) \in S_\varphi^+), \\ \tilde{c}^+(\partial^+ \varphi, v, u) & (a = (u, v) \in A_\varphi^+), \\ +\infty & (a \in A^* \cup B^*), \\ \tilde{c}^-(\partial^- \varphi, u, v) & (a = (u, v) \in A_\varphi^-), \\ \hat{c}^-(\partial^- \varphi, v) & (a = (v, s^-) \in S_\varphi^-), \end{cases} \quad (3.30)$$

where  $\hat{c}^+$  and  $\tilde{c}^+$  ( $\hat{c}^-$  and  $\tilde{c}^-$ ) are, respectively, the saturation capacity and the exchange capacity defined with respect to submodular system  $(\mathcal{D}_1, f_1)$  on  $E$  ( $(\mathcal{D}_2, f_2)$  on  $E'$ ). Figure 3.2 shows the auxiliary network  $\mathcal{N}_\varphi$ .

Let us consider an algorithm described as follows. We call a directed path from  $s^+$  to  $s^-$  of the minimum number of arcs a *shortest path* from  $s^+$  to  $s^-$ . We assume an oracle for exchange capacities for  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ).

#### Algorithm for the intersection problem

**Input:** a feasible flow  $\varphi$  in  $\mathcal{N} = (G = (E, E'; A), c, S_1, S_2)$ .

**Output:** a maximum flow  $\varphi$  in  $\mathcal{N}$ .

### III. NEOFLOWS

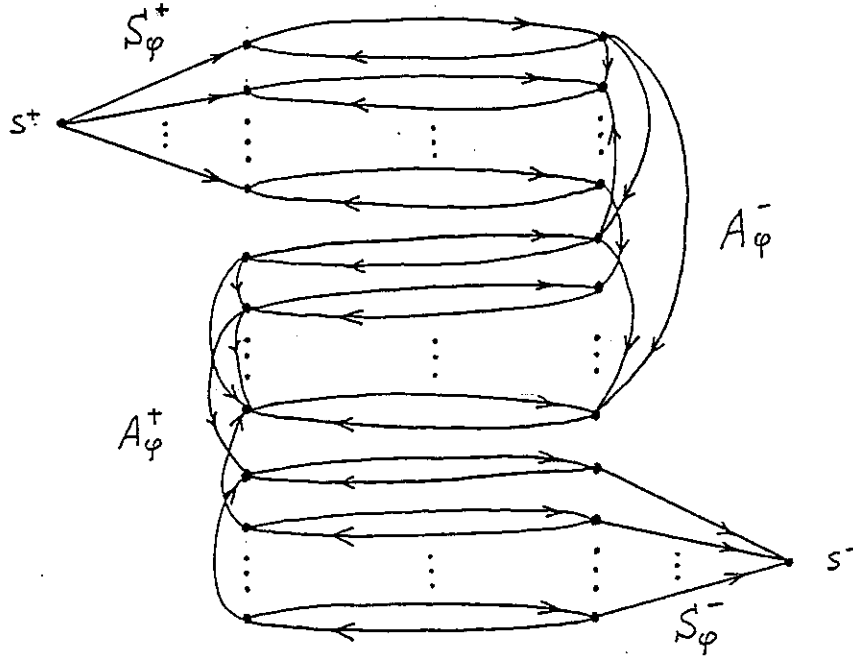


Figure 3.2.

**Step 1:** Construct the auxiliary network  $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$ .

**Step 2:** If there exists no directed path from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ , then the algorithm terminates and the current  $\varphi$  is a maximum flow in  $\mathcal{N}$ .

Otherwise, find a shortest path  $P$  from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  and put

$$\alpha \leftarrow \min\{c_\varphi(a) \mid a \text{ is an arc in } P\}, \quad (3.31)$$

$$\varphi(a) \leftarrow \begin{cases} \varphi(a) + \alpha & (a = (e, e') \in A \text{ and } a \text{ is in } P), \\ \varphi(a) - \alpha & (a = (e, e') \in A \text{ and } (e', e) \text{ is in } P). \end{cases} \quad (3.32)$$

Go to Step 1.

(End)

**Theorem 3.6:** The flows  $\varphi$  obtained in Step 2 of the algorithm are feasible flows in  $\mathcal{N}$ . Moreover, each time we carry out (3.31) and (3.32), the flow value of  $\varphi$  increases by  $\alpha > 0$  given by (3.31).

(Proof) In Step 2 we find a shortest path  $P$  from  $s^+$  to  $s^-$  in the auxiliary network  $\mathcal{N}_\varphi$ . Let  $(u_1^+, v_1^+), \dots, (u_p^+, v_p^+)$  be the arcs in  $A_\varphi^+$  lying on  $P$  in this order and  $(u_1^-, v_1^-), \dots, (u_q^-, v_q^-)$  be the arcs in  $A_\varphi^-$  lying on  $P$  in this order. By the way of choosing path  $P$ , for each  $i, j$  such that  $1 \leq i < j \leq p$  ( $q$ ) we have

$$u_i^+ \notin \text{dep}^+(\partial^+ \varphi, v_j^+) \quad (v_i^- \notin \text{dep}^-(\partial^- \varphi, u_j^-)). \quad (3.33)$$

### 3.1. THE INTERSECTION THEOREM

Therefore, the first half of the present theorem follows from Lemma 3.5. The second half is easy. Q.E.D.

**Theorem 3.7:** If there exists no directed path from  $s^+$  to  $s^-$  in the auxiliary network  $\mathcal{N}_\varphi$  in Step 2 of the algorithm, the current flow  $\varphi$  is a maximum flow in  $\mathcal{N}$ .

(Proof) Let  $U^*$  be the set of the vertices in  $V$  which are reachable from  $s^+$  along directed paths in  $\mathcal{N}_\varphi$  (see Fig. 3.3).

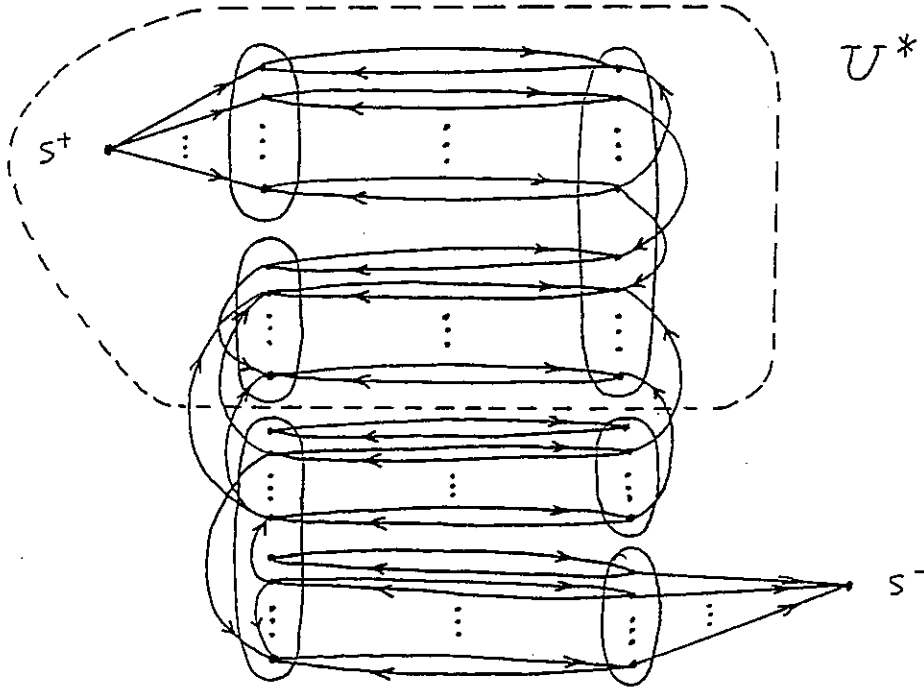


Figure 3.3.

Then for each  $v \in E - U^*$  and  $u \in E' \cap U^*$  we have

$$\text{dep}^+(\partial^+ \varphi, v) \subseteq E - U^*, \quad (3.34)$$

$$\text{dep}^-(\partial^- \varphi, u) \subseteq E' \cap U^*. \quad (3.35)$$

Therefore,

$$\partial^+ \varphi(E - U^*) = f_1(E - U^*), \quad (3.36)$$

$$\partial^- \varphi(E' \cap U^*) = f_2(E' \cap U^*). \quad (3.37)$$

Moreover, from the definition of  $U^*$  and the auxiliary network  $\mathcal{N}_\varphi$  we see

$$E \cap U^* = \{e \mid e' \in E' \cap U^*\} (\subseteq E). \quad (3.38)$$

### III. NEOFLOWS

From (3.36)–(3.38),

$$\begin{aligned}\partial^+ \varphi(E) &= \partial^+ \varphi(E - U^*) + \partial^- \varphi(E' \cap U^+) \\ &= f_1(E - U^*) + f_2(E' \cap U^*). \end{aligned} \quad (3.39)$$

On the other hand, for any feasible flow  $\hat{\varphi}$  in  $\mathcal{N}$  and for any  $U \subseteq E \cup E'$  such that  $E - U \in \mathcal{D}_1$  and  $E' \cap U \in \mathcal{D}_2$  we have

$$\begin{aligned}\partial^+ \hat{\varphi}(E) &= \partial^+ \hat{\varphi}(E - U) + \partial^- \hat{\varphi}(E' \cap U) \\ &\leq f_1(E - U) + f_2(E' \cap U). \end{aligned} \quad (3.40)$$

It follows from (3.39) and (3.40) that the current flow  $\varphi$  is a maximum flow in  $\mathcal{N}$ . Q.E.D.

Since there exists a maximum flow in  $\mathcal{N}$  (this fact will also algorithmically be proven later), Theorems 3.6 and 3.7 together with the proof of Theorem 3.7 show the following theorem. Note that when  $f_1$  and  $f_2$  are integer-valued, the algorithm given above terminates after a finite number of steps, starting with an integral feasible flow in  $\mathcal{N}$ , and then gives an integral maximum flow.

**Theorem 3.8:** For Problem  $P_1'$  described by (3.2) we have

$$\begin{aligned}\max\{\partial^+ \varphi(E) \mid \varphi: \text{a feasible flow in } \mathcal{N}\} \\ = \min\{f_1(X) + f_2(E' - X') \mid X \in \mathcal{D}_1, E' - X' \in \mathcal{D}_2\}, \end{aligned} \quad (3.41)$$

where  $X' = \{x' \mid x \in X\} \subseteq E'$ .

Moreover, if  $f_1$  and  $f_2$  are integer-valued, there exists an integral maximum flow in  $\mathcal{N}$ .

Rewriting Theorem 3.8 for Problem  $P_1$ , we also have

**Theorem 3.9 (The Intersection Theorem):** For Problem  $P_1$  described by (3.1),

$$\begin{aligned}\max\{x(E) \mid x \in P(f_1) \cap P(f_2)\} \\ = \min\{f_1(X) + f_2(E - X) \mid X \in \mathcal{D}_1, E - X \in \mathcal{D}_2\}. \end{aligned} \quad (3.42)$$

Moreover, if  $f_1$  and  $f_2$  are integer valued, the maximum on the left-hand side of (3.42) is attained by an integral vector  $x \in P(f_1) \cap P(f_2)$ .

Theorem 3.9 is a generalization of the intersection theorem for polymatroids due to Edmonds [Edm70]. The algorithm given in this section is a direct adaptation of the one given in [Fuji78a].

### 3.1. THE INTERSECTION THEOREM

#### (c) A refinement of the algorithm

When  $f_1$  and  $f_2$  are not integer-valued, the algorithm shown in Section 3.1.b may not terminate after finitely many steps. We shall show an algorithm due to Schönsleben [Schönsleben80] and Lawler and Martel [Lawler+Martel82] which modifies Step 2 of the algorithm and terminates after repeating Step 2  $O(|V|^3)$  times.

Let  $\pi: V(= E \cup E') \rightarrow \{1, 2, \dots, 2n\}$  be one-to-one mapping which denotes a numbering of the vertices in  $V$ , where  $n = |E|$ . We also define  $\pi(s^+) = 0$  and  $\pi(s^-) = 2n + 1$ . We represent any directed path  $P$  from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  by the sequence  $(s^+, v_1, v_2, \dots, v_p, s^-)$  of vertices in  $P$  and denote by  $\pi(P)$  the sequence  $(\pi(v_1), \pi(v_2), \dots, \pi(v_p))$  of the numbering indices of the vertices. For any directed paths  $P_1$  and  $P_2$  from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  we say  $P_1$  is *lexicographically smaller than*  $P_2$  if  $\pi(P_1)$  is lexicographically smaller than  $\pi(P_2)$ . We call the lexicographically smallest one in the set of all the shortest paths from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  the *lexicographically shortest path* from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ .

Now, we modify the part of finding a shortest path  $P$  in Step 2 of the algorithm as follows.

- (\*) If there exists a directed path from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ , find the lexicographically shortest path  $P$  from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ .

**Theorem 3.10** ([Schönsleben80], [Lawler+Martel82] for polymatroids): If we modify Step 2 of the algorithm given in Section 3.1.b for Problem  $P_1'$  as above, the algorithm finds a maximum flow in  $\mathcal{N}$  after repeating Step 2  $O(|E|^3)$  times.

(Proof) Denote by  $W_k \subseteq V \cup \{s^+, s^-\}$  be the set of the vertices which are reachable by a directed path from  $s^+$  having  $k$  arcs but not less than  $k$  arcs. Suppose  $W_0 = \{s^+\}$  and  $s^- \in W_p$ . Let  $P$  be the lexicographically shortest path in  $\mathcal{N}_\varphi$ . Also suppose that the arcs of  $A_\varphi^+$  lying on  $P$  appear in order as

$$(u_1^+, v_1^+), (u_2^+, v_2^+), \dots, (u_l^+, v_l^+) \quad (3.43)$$

from  $s^+$  to  $s^-$ . Put  $x = \partial^+ \varphi$  and define

$$y = x + \alpha(\chi_{v_1^+} - \chi_{u_1^+}), \quad (3.44)$$

where  $\alpha$  is defined by (3.31) for the current  $\varphi$ . Consider vertices  $w, z \in E$  such that

$$w \notin \text{dep}^+(x, z), \quad w \in \text{dep}^+(y, z). \quad (3.45)$$

Then we must have

$$u_1^+ \in \text{dep}^+(x, z), \quad v_1^+ \notin \text{dep}^+(x, z), \quad (3.46)$$

### III. NEOFLOWS

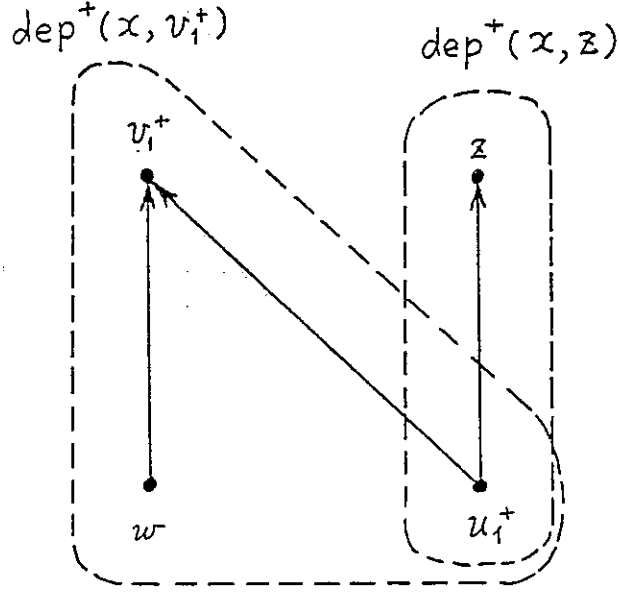


Figure 3.4.

$$w \in \text{dep}^+(x, v_1^+). \quad (3.47)$$

(See Fig. 3.4.) Here we may have  $v_1^+ = w$  or  $z = u_1^+$ .

Now, suppose

$$u_1^+ \in W_k, \quad v_1^+ \in W_{k+1} \quad (3.48)$$

for some  $k$  ( $1 \leq k \leq 2n$ ). Then we have  $z \in W_{k_1}$  for some positive integer  $k_1$  with  $k_1 \leq k+1$ .

- (1) If vertex  $w$  is not reachable from  $s^+$  in  $\mathcal{N}_\varphi$ , then adding arc  $(w, z)$  to  $\mathcal{N}_\varphi$  does not change the set of shortest paths from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ .
- (2) Suppose  $w \in W_{k_2}$  for some  $k_2$  ( $1 \leq k_2 \leq 2n$ ). (Note that  $k_2 \geq k$ .)
  - (2-1) If  $k_2 > k$  or  $k_1 < k+1$ , then adding arc  $(w, z)$  to  $\mathcal{N}_\varphi$  does not change the set of shortest paths from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ .
  - (2-2) If  $k_2 = k$ ,  $k_1 = k+1$  and there exists no shortest paths from  $s^+$  to  $s^-$  which include vertex  $z$ , then adding arc  $(w, z)$  to  $\mathcal{N}_\varphi$  does not change the set of shortest paths from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$ .
  - (2-3) If  $k_2 = k$ ,  $k_1 = k+1$  and there exists a shortest path from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  which includes vertex  $z$ , then we have

$$\pi(v_1^+) < \pi(z) \quad (3.49)$$

since  $P$  is the lexicographically shortest path.



### 3.2. DISCRETE SEPARATION THEOREM

Because of Lemma 3.5 we can repeatedly apply the above argument to arcs  $(u_i^+, v_i^+)$  ( $i = 2, \dots, l$ ) in (3.43).

We can also apply the argument for  $A_\varphi^+$  to  $A_\varphi^-$  *mutatis mutandis*.

For each vertex  $w \in V \cup \{s^+\}$  define

$$\begin{aligned} p_\varphi(w) \\ = \min\{\pi(v) \mid v \in V, (w, v) \text{ lies on a shortest path from } s^+ \text{ to } s^- \text{ in } \mathcal{N}_\varphi\}. \end{aligned} \quad (3.50)$$

Also denote by  $\varphi^*$  the feasible flow obtained from  $\varphi$  by transformation (3.32), and let  $P^*$  be the lexicographically shortest path in  $\mathcal{N}_{\varphi^*}$ . Since at least one arc lying on  $P$  is missing in  $\mathcal{N}_{\varphi^*}$ , it follows from the above argument that

- (i) (the length of  $P^*$ )  $\geq$  (the length of  $P$ ) + 1 or
- (ii) (the length of  $P^*$ ) = (the length of  $P$ ) and  
 $p_\varphi \leq p_{\varphi^*}$  with  $p_\varphi(v) < p_{\varphi^*}(v)$  for some  $v \in V \cup \{s^+\}$ .

Case (ii) occurs consecutively  $O(|V|^2)$  times and hence Step 2 is repeated  $O(|V|^3)$  ( $=O(|E|^3)$ ) times. Q.E.D.

The above proof technique is due to R. E. Bixby (cf. [Cunningham84]). It should be noted that Theorem 3.10 together with Theorem 3.7 shows the existence of a maximum flow in  $\mathcal{N}$  for any totally ordered additive group  $\mathbb{R}$  and that the modified algorithm finds a maximum flow in  $\mathcal{N}$  by changing feasible flows  $O(|V|^3)$  times.

The above algorithm corresponds to the Edmonds-Karp algorithm for classical maximum flows [Edm + Karp72]. A complexity improvement over the above algorithm is shown in [Tardos + Tovey + Trick86] by generalizing the idea of layered networks due to E. A. Dinits [Dinits70].

### 3.2. Discrete Separation Theorem

A. Frank [Frank82b] showed the following.

**Theorem 3.11** (Discrete Separation Theorem): Let  $(\mathcal{D}_1, f)$  be a submodular system on  $E$  and  $(\mathcal{D}_2, g)$  be a supermodular system on  $E$ . Then,

$$g \leq f \implies \exists x \in \mathbb{R}^E: g \leq x \leq f. \quad (3.51)$$

Moreover, if  $f$  and  $g$  are integer-valued, there exists an integral  $x \in \mathbb{R}^E$  such that  $g \leq x \leq f$ . Here,  $g \leq f$  means  $\forall X \in \mathcal{D}_1 \cap \mathcal{D}_2: g(X) \leq f(X)$  and  $g \leq x$  ( $x \leq f$ ) means  $\forall X \in \mathcal{D}_2: g(X) \leq x(X)$  ( $\forall X \in \mathcal{D}_1: x(X) \leq f(X)$ ).

### III. NEOFLOWS

We shall prove this theorem by the use of the intersection theorem (Theorem 3.9).

(Proof of Theorem 3.11) Suppose  $g \leq f$ . Then from the intersection theorem,

$$\begin{aligned} & \max\{x(E) \mid x \in P(f) \cap P(g^\#)\} \\ &= \min\{f(X) + g^\#(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \min\{f(X) + g(E) - g(X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= g(E). \end{aligned} \tag{3.52}$$

Therefore, there exists a vector  $x$  in  $P(f) \cap B(g)$  ( $= P(f) \cap B(g^\#)$ ). This vector  $x$  satisfies  $g \leq x \leq f$ . Moreover, if  $f$  and  $g$  are integer-valued, then from (3.52) and the intersection theorem there exists integral  $x \in P(f) \cap B(g)$ , which satisfies  $g \leq x \leq f$ . Q.E.D.

We can also show the intersection theorem by using the discrete separation theorem (Theorem 3.11) as follows.

Let  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) be submodular systems on  $E$ . Define

$$k = \min\{f_1(X) + f_2(E - X) \mid X \in \mathcal{D}_1, E - X \in \mathcal{D}_2\}. \tag{3.53}$$

Also define  $\hat{f}_2: \mathcal{D}_2 \rightarrow \mathbb{R}$  by

$$\hat{f}_2(X) = \begin{cases} f_2(X) & (X \in \mathcal{D}_2 - \{E\}), \\ k & (X = E). \end{cases} \tag{3.54}$$

Note that  $\hat{f}_2: \mathcal{D}_2 \rightarrow \mathbb{R}$  is a submodular function since  $k \leq f_2(E)$ ; in fact,  $(\mathcal{D}_2, \hat{f}_2)$  is the  $(f_2(E) - k)$ -truncation of  $(\mathcal{D}_2, f_2)$ . Then we have  $\hat{f}_2^\#(\emptyset) = 0 = f_1(\emptyset)$  and for each  $X \in \mathcal{D}_1 \cap \overline{\mathcal{D}_2} - \{\emptyset\}$

$$\begin{aligned} \hat{f}_2^\#(X) &= \hat{f}_2(E) - \hat{f}_2(E - X) \\ &= k - f_2(E - X) \\ &\leq f_1(X). \end{aligned} \tag{3.55}$$

It thus follows from the discrete separation theorem that there exists a vector  $x \in \mathbb{R}^E$  such that  $\hat{f}_2^\# \leq x \leq f_1$ . Since  $P(f_1) \cap P(\hat{f}_2^\#) \neq \emptyset$ , we have  $P(f_1) \cap B(\hat{f}_2) (= P(f_1) \cap B(\hat{f}_2^\#)) \neq \emptyset$ . Consequently,

$$\begin{aligned} & \max\{x(E) \mid x \in P(f_1) \cap P(f_2)\} \\ & \geq \max\{x(E) \mid x \in P(f_1) \cap P(\hat{f}_2)\} \\ & = k \\ & = \min\{f_1(X) + f_2(E - X) \mid X \in \mathcal{D}_1, E - X \in \mathcal{D}_2\} \\ & \geq \max\{x(E) \mid x \in P(f_1) \cap P(f_2)\}, \end{aligned} \tag{3.56}$$

### 3.3. THE COMMON BASE PROBLEM

where the last inequality follows from the fact (the weak duality) that  $x(E) \leq f_1(X) + f_2(E - X)$  for any  $x \in (f_1) \cap P(f_2)$  and any  $X \in \mathcal{D}_1$  with  $E - X \in \mathcal{D}_2$ . This proves (3.42).

Moreover, the integrality part of the intersection theorem follows from the integrality part of the discrete separation theorem and the above argument.

We have thus shown the equivalence of the intersection theorem and the discrete separation theorem.

Using the discrete separation theorem, we show (2.27) and (2.28) in Section 2.1.c of Chapter 2. For two submodular systems  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) on  $E$  let  $f$  be the submodular function on  $\mathcal{D}_1 \cap \mathcal{D}_2$  defined by

$$f(X) = f_1(X) + f_2(X) \quad (X \in \mathcal{D}_1 \cap \mathcal{D}_2). \quad (3.57)$$

We write  $f$  as  $f_1 + f_2$ . We can easily see that the relation,  $P(f_1 + f_2) \supseteq P(f_1) + P(f_2)$ , holds.

Conversely, for any  $x \in P(f_1 + f_2)$  we have

$$\forall X \in \mathcal{D}_1 \cap \mathcal{D}_2: x(X) \leq f_1(X) + f_2(X), \quad (3.58)$$

which is rewritten as

$$\forall X \in \mathcal{D}_1 \cap \mathcal{D}_2: x(X) - f_2(X) \leq f_1(X). \quad (3.59)$$

It follows from (3.59) and the discrete separation theorem that there exists a vector  $y \in \mathbb{R}^E$  such that

$$\forall X \in \mathcal{D}_2: x(X) - f_2(X) \leq y(X), \quad (3.60)$$

$$\forall X \in \mathcal{D}_1: y(X) \leq f_1(X). \quad (3.61)$$

Defining  $z = x - y$ , we have from (3.60)  $z \in P(f_2)$  and from (3.61)  $y \in P(f_1)$ . We thus have  $x = y + z \in P(f_1) + P(f_2)$ . Therefore,  $P(f_1 + f_2) \subseteq P(f_1) + P(f_2)$ . This completes the proof of the fact that  $P(f_1 + f_2) = P(f_1) + P(f_2)$ .

Since  $(f_1 + f_2)(E) = f_1(E) + f_2(E)$ , we also have  $B(f_1 + f_2) = B(f_1) + B(f_2)$ .

### 3.3. The Common Base Problem

Let  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) be two submodular systems on  $E$ . The *common base problem* for submodular systems  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) is to discern whether there is a common base  $x \in B(f_1) \cap B(f_2)$  and, if any, to find one such common base. Clearly, that  $f_1(E) = f_2(E)$  is necessary for the existence of a common base.

### III. NEOFLOWS

**Theorem 3.12:** Let  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) be submodular systems on  $E$  with  $f_1(E) = f_2(E)$ . Then there exists a common base  $x \in B(f_1) \cap B(f_2)$  if and only if  $f_2^\# \leq f_1$ , i.e.,

$$\forall X \in \mathcal{D}_1 \cap \overline{\mathcal{D}}_2: f_2^\#(X) \leq f_1(X). \quad (3.62)$$

Moreover, if  $f_i$  ( $i = 1, 2$ ) are integer-valued and a common base exists, then there exists an integral common base.

(Proof) If there exists a common base  $x \in B(f_1) \cap B(f_2)$ , then since  $B(f_1) \cap B(f_2) = P(f_1) \cap P(f_2^\#)$ , we have  $f_2^\# \leq x \leq f_1$ . Conversely, if  $f_2^\# \leq f_1$ , then there exists a vector  $x \in \mathbb{R}^E$  such that  $f_2^\# \leq x \leq f_1$ , due to the discrete separation theorem (Theorem 3.11). Hence  $x \in P(f_1) \cap P(f_2^\#) = B(f_1) \cap B(f_2^\#) = B(f_1) \cap B(f_2)$ .

Moreover, the integrality part also follows from the integrality part of the discrete separation theorem. Q.E.D.

Note that, in Theorem 3.12,  $f_2^\# \leq f_1$  if and only if  $f_1^\# \leq f_2$ , since  $f_1(E) = f_2(E)$ . Also note that " $f_2^\# \leq f_1$ " is equivalent to:

$$\forall X \in \mathcal{D}_1 \cap \overline{\mathcal{D}}_2: f_1(X) + f_2(E - X) \geq f_2(E) (= f_1(E)). \quad (3.63)$$

From the intersection theorem, (3.63) means that there exists a vector  $x \in P(f_1) \cap P(f_2)$  such that  $x(E) \geq f_2(E)$ , and such a vector  $x$  is a common base since  $f_1(E) = f_2(E)$ . In this way Theorem 3.12 can also be shown by the intersection theorem.

The common base problem can be solved by finding a maximum common subbase  $x \in P(f_1) \cap P(f_2)$  through the algorithm shown in Section 3.1. We shall also give an algorithm for the common base problem which deals only with bases in  $B(f_1)$  and  $B(f_2)$ .

Given bases  $b_1 \in B(f_1)$  and  $b_2 \in B(f_2)$ , we denote  $\beta = (b_1, b_2)$  and define the auxiliary network  $\mathcal{N}_\beta = (G_\beta = (V, A_\beta), T_\beta^+, T_\beta^-, c_\beta)$  as follows.  $G_\beta$  is the underlying graph with the vertex set  $V = E$  and the arc set  $A_\beta$  defined by

$$A_\beta = A_\beta^1 \cup A_\beta^2, \quad (3.64)$$

$$A_\beta^1 = \{(u, v) \mid u, v \in V, u \in \text{dep}_1(b_1, v) - \{v\}\}, \quad (3.65)$$

$$A_\beta^2 = \{(v, u) \mid u, v \in V, u \in \text{dep}_2(b_2, v) - \{v\}\}, \quad (3.66)$$

where  $\text{dep}_1$  and  $\text{dep}_2$  are the dependence functions associated with  $(\mathcal{D}_1, f_1)$  and  $(\mathcal{D}_2, f_2)$ , respectively.  $c_\beta: A_\beta \rightarrow \mathbb{R}$  is the capacity function defined by

$$c_\beta(a) = \begin{cases} \tilde{c}_1(b_1, v, u) & (a = (u, v) \in A_\beta^1), \\ \tilde{c}_2(b_2, u, v) & (a = (u, v) \in A_\beta^2). \end{cases} \quad (3.67)$$

### 3.3. THE COMMON BASE PROBLEM

Here,  $\tilde{c}_1$  and  $\tilde{c}_2$  are the exchange capacities associated with  $(\mathcal{D}_1, f_1)$  and  $(\mathcal{D}_2, f_2)$ , respectively. Also,  $T_\beta^+$  and  $T_\beta^-$  are subsets of  $V$  defined by

$$T_\beta^+ = \{v \mid v \in V, b_1(v) > b_2(v)\}, \quad (3.68)$$

$$T_\beta^- = \{v \mid v \in V, b_1(v) < b_2(v)\}. \quad (3.69)$$

$T_\beta^+$  is the set of entrances and  $T_\beta^-$  the set of exits in  $\mathcal{N}_\beta$ .

Now, an algorithm for finding a common base of  $B(f_1)$  and  $B(f_2)$  is given as follows.

#### An algorithm for finding a common base

**Input:** Submodular systems  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ) on  $E$  with  $f_1(E) = f_2(E)$  and initial bases  $b_i$  of  $(\mathcal{D}_i, f_i)$  ( $i = 1, 2$ ).

**Output:** A common base  $b_1 (= b_2)$  if any exists.

**Step 1:** While  $b_1 \neq b_2$ , do the following (a)–(c):

(a) Construct the auxiliary network  $\mathcal{N}_\beta = (G_\beta = (V, A_\beta), T_\beta^+, T_\beta^-, c_\beta)$  associated with  $\beta = (b_1, b_2)$ . If there is no directed path from  $T_\beta^+$  to  $T_\beta^-$ , then stop (there is no common base in  $B(f_1)$  and  $B(f_2)$ ).

(b) Let  $P$  be a directed path from  $T_\beta^+$  to  $T_\beta^-$  in  $G_\beta$  having the smallest number of arcs and put

$$\alpha \leftarrow \min \{ \min \{ c_\beta(a) \mid a \text{ is an arc on } P \}, \\ b_1(\partial^+ P) - b_2(\partial^+ P), b_2(\partial^- P) - b_1(\partial^- P) \}.$$

(c) For each arc  $a \in A_\beta$ ,  
if  $a = (u, v) \in A_\beta^1$ , then put

$$b_1(u) \leftarrow b_1(u) - \alpha, \quad b_1(v) \leftarrow b_1(v) + \alpha,$$

if  $a = (u, v) \in A_\beta^2$ , then put

$$b_2(u) \leftarrow b_2(u) + \alpha, \quad b_2(v) \leftarrow b_2(v) - \alpha.$$

**Step 2:** The current  $b_1$  is a common base of  $B(f_1)$  and  $B(f_2)$  and the algorithm terminates.

(End)

Because of the way of choosing a directed path  $P$  in (b) of Step 1, the vectors  $b_1$  and  $b_2$  remain bases in  $B(f_1)$  and  $B(f_2)$ , respectively, due to Lemma 3.5.

### III. NEOFLOWS

If the algorithm terminates at (a) of Step 1, then let  $U$  be the set of vertices in  $G_\beta$  which are reachable by directed paths from  $T_\beta^+$ . It follows from the definition of the auxiliary network  $\mathcal{N}_\beta$  that

$$b_1(U) = f_1(E) - f_1(E - U), \quad (3.70)$$

$$b_2(U) = f_2(U). \quad (3.71)$$

Since  $b_1(U) > b_2(U)$ , we have from (3.70) and (3.71)

$$f_1(E)(= f_2(E)) > f_1(E - U) + f_2(U). \quad (3.72)$$

From (3.72) (and Theorem 3.9) we see that there exists no common base in  $B(f_1)$  and  $B(f_2)$ .

When  $f_1$  and  $f_2$  are integer-valued and initial bases  $b_1$  and  $b_2$  are integral, bases  $b_1$  and  $b_2$  obtained during the execution of the above algorithm are integral and hence the algorithm terminates after repeating (a)–(c) of Step 1 at most  $b_1(T_\beta^+) - b_2(T_\beta^+)$  times.

For general rank functions  $f_1$  and  $f_2$  we adopt the lexicographic ordering described in Section 3.1.c ([Schönsleben80], [Lawler+ Martel82]). When finding a shortest path from  $T_\beta^+$  to  $T_\beta^-$  by the breadth-first search, for each  $u \in V$  search arc  $(u, v)$  in  $A_\beta^1$  (or  $A_\beta^2$ ) earlier than arc  $(u, v')$  in  $A_\beta^1$  (or  $A_\beta^2$ ) if  $\pi(v) < \pi(v')$ , for a fixed numbering  $\pi: V \rightarrow \{1, 2, \dots, |V|\}$  of  $V$ . By this modification the algorithm terminates after repeating Cycle (a)–(c) of Step 1  $O(|E|^3)$  times.

## 4.1. NEOFLOWS

### 4. Neoflows

In this section we consider the submodular flow problem, the independent flow problem and the polymatroidal flow problem, which we call *neoflow problems*. We discuss the equivalence among these neoflow problems and give algorithms for solving them.

#### 4.1. Neoflows

We first give definitions of the submodular flow problem, the independent flow problem and the polymatroidal flow problem.

##### (a) Submodular flows

Let  $G = (V, A)$  be a graph with a vertex set  $V$  and an arc set  $A$ . Also let  $\bar{c}: A \rightarrow \mathbb{R} \cup \{+\infty\}$  be an *upper capacity function* and  $\underline{c}: A \rightarrow \{-\infty\}$  be a *lower capacity function*. A function  $\gamma: A \rightarrow \mathbb{R}$  is a *cost function*. Let  $\mathcal{F} \subseteq 2^V$  be a crossing family with  $\emptyset, V \in \mathcal{F}$  and  $f: \mathcal{F} \rightarrow \mathbb{R}$  be a crossing-submodular function on the crossing family  $\mathcal{F}$  with  $f(\emptyset) = f(V) = 0$ . (See Section 1.3 for the definition of crossing-submodular function on a crossing family.) Denote this network by  $\mathcal{N}_S = (G = (V, A), \underline{c}, \bar{c}, \gamma, (\mathcal{F}, f))$ .

The *submodular flow problem* considered by Edmonds and Giles [Edm + Giles77] is described as follows.

$$P_S: \text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.1a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.1b)$$

$$\partial \varphi \in B(f). \quad (4.1c)$$

Here,  $\partial \varphi$  is the boundary of  $\varphi$  with respect to  $G$  (see (1.23)) and  $B(f)$  is the base polyhedron associated with  $f$  (see Theorem 1.4), where we assume  $B(f) \neq \emptyset$ .

A feasible  $\varphi: A \rightarrow \mathbb{R}$  satisfying (4.1b) and (4.1c) is called a *submodular flow* in  $\mathcal{N}_S$  and an optimal solution of the submodular flow problem  $P_S$  is called an *optimal submodular flow* in  $\mathcal{N}_S$ . (The term, submodular flow, was introduced by Zimmermann [Zimmermann82].)

##### (b) Independent flows

Let  $G = (V, A; S^+, S^-)$  be a graph with a vertex set  $V$ , an arc set  $A$ , a set  $S^+$  of *entrances* and a set  $S^-$  of *exits* such that  $S^+, S^- \subseteq V$  and

### III. NEOFLOWS

$S^+ \cap S^- = \emptyset$ . Also let  $\bar{c}: A \rightarrow \mathbf{R} \cup \{+\infty\}$  be an *upper capacity function*,  $\underline{c}: A \rightarrow \mathbf{R} \cup \{-\infty\}$  be a *lower capacity function*, and  $\gamma: A \rightarrow \mathbf{R}$  be a *cost function*. Moreover, let  $(\mathcal{D}^+, f^+)$  be a submodular system on  $S^+$  and  $(\mathcal{D}^-, f^-)$  be a submodular system on  $S^-$ . Denote this network by  $\mathcal{N}_I = (G = (V, A; S^+, S^-), \underline{c}, \bar{c}, \gamma, (\mathcal{D}^+, f^+), (\mathcal{D}^-, f^-))$ .

The *independent flow problem* [Fuji78a] is given as follows. For a given  $v^* \in \mathbf{R}$ ,

$$P_I: \text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.2a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.2b)$$

$$\partial \varphi(v) = 0 \quad (v \in V - (S^+ \cup S^-)), \quad (4.2c)$$

$$(\partial \varphi)^{S^+} \in P(f^+), \quad (4.2d)$$

$$-(\partial \varphi)^{S^-} \in P(f^-), \quad (4.2e)$$

$$\partial \varphi(S^+) = v^*. \quad (4.2f)$$

Here,  $(\partial \varphi)^{S^+}$  (or  $(\partial \varphi)^{S^-}$ ) is the restriction of  $\partial \varphi: V \rightarrow \mathbf{R}$  to  $S^+$  (or  $S^-$ ) and  $P(f^+)$  and  $P(f^-)$  are, respectively, the submodular polyhedra associated with  $(\mathcal{D}^+, f^+)$  and  $(\mathcal{D}^-, f^-)$ . (The original independent flow problem considered by the author [Fuji78a] is described in terms of polymatroids and is slightly generalized here to submodular systems.)

It should also be noted that if the system of (4.2b)–(4.2f) is feasible, we have  $\min\{f^+(S^+), f^-(S^-)\} \geq v^*$  and that letting  $(\mathcal{D}^+, \hat{f}^+)$  and  $(\mathcal{D}^-, \hat{f}^-)$  be, respectively, the  $(f^+(S^+) - v^*)$ -truncation of  $(\mathcal{D}^+, f^+)$  and the  $(f^-(S^-) - v^*)$ -truncation of  $(\mathcal{D}^-, f^-)$ , we can replace (4.2d)–(4.2f) by

$$(\partial \varphi)^{S^+} \in B(\hat{f}^+), \quad (4.2g)$$

$$-(\partial \varphi)^{S^-} \in B(\hat{f}^-). \quad (4.2h)$$

Here, note that  $\partial \varphi(S^+) = -\partial \varphi(S^-)$  due to (4.2c).

A feasible flow  $\varphi: A \rightarrow \mathbf{R}$  satisfying (4.2b)–(4.2f) is called an *independent flow* of value  $v^*$  in  $\mathcal{N}_I$  and an optimal solution of the independent flow problem  $P_I$  is called an *optimal independent flow* of value  $v^*$  in  $\mathcal{N}_I$ .

#### (c) Polymatroidal flows

Let  $G = (V, A)$  be a graph with a vertex set  $V$  and an arc set  $A$ . For each vertex  $v \in V$  we consider distributive lattices  $\mathcal{D}_v^+ \subseteq 2^{\delta^+ v}$  and  $\mathcal{D}_v^- \subseteq 2^{\delta^- v}$  and submodular functions  $f_v^+: \mathcal{D}_v^+ \rightarrow \mathbf{R}$  and  $f_v^-: \mathcal{D}_v^- \rightarrow \mathbf{R}$ . Here, we assume  $\emptyset \in \mathcal{D}_v^+$ ,  $\emptyset \in \mathcal{D}_v^-$  but not necessarily  $\delta^+ v \in \mathcal{D}_v^+$ ,  $\delta^- v \in \mathcal{D}_v^-$ . Also let  $\underline{c}: A \rightarrow \mathbf{R} \cup$



## 4.2. THE EQUIVALENCE OF THE NEOFLOW PROBLEMS

$\{-\infty\}$  be a lower capacity function and  $\gamma: A \rightarrow \mathbf{R}$  be a cost function. Denote this network by  $\mathcal{N}_P = (G = (V, A), \underline{c}, \gamma, (\mathcal{D}_v^+, f_v^+)(v \in V), (\mathcal{D}_v^-, f_v^-)(v \in V))$ .

The *polymatroidal flow problem* [Hassin82], [Lawler + Martel82] is given as follows.

$$P_P: \text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.3a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \quad (a \in A), \quad (4.3b)$$

$$\partial \varphi(v) = 0 \quad (v \in V), \quad (4.3c)$$

$$\varphi^{\delta^+ v} \in P(f_v^+) \quad (v \in V), \quad (4.3d)$$

$$\varphi^{\delta^- v} \in P(f_v^-) \quad (v \in V), \quad (4.3e)$$

where  $\partial \varphi$  is the boundary of  $\varphi$  with respect to  $G$ ,  $\varphi^{\delta^+ v}$  (or  $\varphi^{\delta^- v}$ ) is the restriction of  $\varphi: A \rightarrow \mathbf{R}$  to  $\delta^+ v$  (or  $\delta^- v$ ), and

$$P(f_v^+) = \{x \mid x \in \mathbf{R}^{\delta^+ v}, \forall X \in \mathcal{D}_v^+: x(X) \leq f_v^+(X)\}, \quad (4.4)$$

$$P(f_v^-) = \{x \mid x \in \mathbf{R}^{\delta^- v}, \forall X \in \mathcal{D}_v^-: x(X) \leq f_v^-(X)\}, \quad (4.5)$$

(The problem is originally defined in terms of polymatroids and slightly generalized here.)

A feasible flow  $\varphi: A \rightarrow \mathbf{R}$  satisfying (4.3b)–(4.3e) is called a *polymatroidal flow* in  $\mathcal{N}_P$  and an optimal solution of Problem  $P_P$  is called an *optimal polymatroidal flow* in  $\mathcal{N}_P$ .

### 4.2. The Equivalence of the Neoflow problems

We show the equivalence of the submodular flow problem, the independent flow problem and the polymatroidal flow problem. Here, the equivalence is with respect to the capability of modeling flow problems. Different models may require different oracles for algorithms but we will not go into this matter here.

#### (a) From submodular flows to independent flows

Consider the submodular flow problem  $P_S$  defined by (4.1). From Theorem 1.4 and the definition of the submodular flow problem we can assume that  $f$  appearing in (4.1) is a submodular function on a distributive lattice  $\mathcal{D} \subseteq 2^V$  such that  $\emptyset, V \in \mathcal{D}$  and  $f(\emptyset) = f(V) = 0$ . Let  $s^-$  be a new vertex not in  $G = (V, A)$  and define

$$S^+ = V, \quad S^- = \{s^-\}. \quad (4.6)$$

### III. NEOFLOWS

Then the submodular flow problem  $P_S$  is rewritten as

$$\text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.7a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.7b)$$

$$(\partial \varphi)^{S^+} \in B(f), \quad (4.7c)$$

$$-(\partial \varphi)^{S^-} \in \{0\}. \quad (4.7d)$$

This is an independent flow problem with the underlying graph  $G' = (V \cup \{s^-\}, A; S^+, S^-)$ . (See (4.2b), (4.2c), (4.2g) and (4.2h), where constraint (4.2c) is void.)

It is interesting to see that the submodular flow problem looks like a very special case of the independent flow problem (see Fig. 4.1).

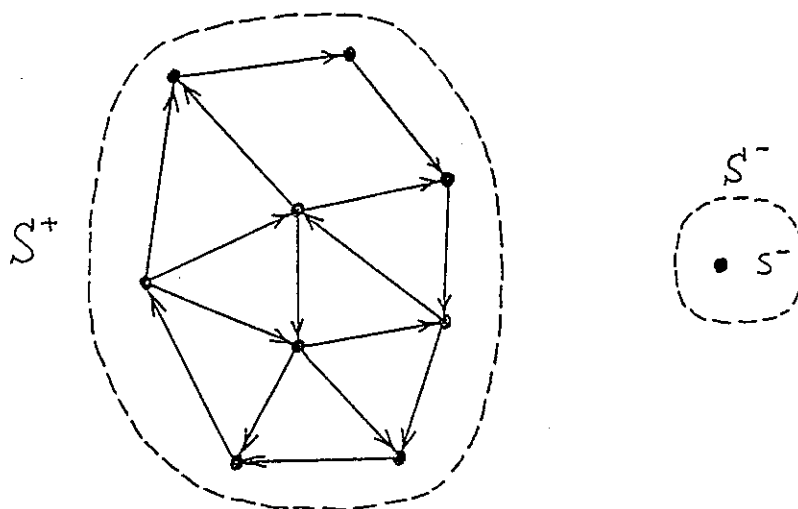


Figure 4.1.

#### (b) From independent flows to polymatroidal flows

Consider the independent flow problem  $P_I$  described by (4.2). Without loss of generality we assume that there is no arc entering  $S^+$  or leaving  $S^-$ .

## 4.2. THE EQUIVALENCE OF THE NEOFLOW PROBLEMS

Let  $s^+$  and  $s^-$  be new vertices not in  $G = (V, A; S^+, S^-)$ . Construct a graph  $\hat{G} = (\hat{V}, \hat{A})$  with vertex set  $\hat{V}$  and arc set  $\hat{A}$  given by

$$\hat{V} = V \cup \{s^+, s^-\}, \quad (4.8)$$

$$\hat{A} = A \cup A^+ \cup A^- \cup A^0, \quad (4.9)$$

$$A^+ = \{(s^+, v) \mid v \in S^+\}, \quad (4.10)$$

$$A^- = \{(v, s^-) \mid v \in S^-\}, \quad (4.11)$$

$$A^0 = \{(s^-, s^+)\}. \quad (4.12)$$

Also define a lower capacity function  $\hat{c}: \hat{A} \rightarrow \mathbb{R} \cup \{-\infty\}$  and a cost function  $\hat{\gamma}: \hat{A} \rightarrow \mathbb{R}$  by

$$\hat{c}(a) = \begin{cases} c(a) & (a \in A), \\ f^+(S^+) & (a = (s^-, s^+)), \\ -\infty & (a \in A^+ \cup A^-), \end{cases} \quad (4.13)$$

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & (a \in A), \\ 0 & (a \in A^+ \cup A^- \cup A^0). \end{cases} \quad (4.14)$$

Moreover, for each vertex  $v \in V$  define polyhedron  $P_v^+ \subseteq \mathbb{R}^{\delta^+ v}$  and  $P_v^- \subseteq \mathbb{R}^{\delta^- v}$  by

$$P_v^+ = \begin{cases} P(f^+) & (v = s^+), \\ (-\infty, +\infty) & (v \in \{s^-\} \cup S^-), \\ \{x \mid x \in \mathbb{R}^{\delta^+ v}, \forall a \in \delta^+ v: x(a) \leq \bar{c}(a)\} & (v \in V - S^-), \end{cases} \quad (4.15)$$

$$P_v^- = \begin{cases} P(f^-) & (v = s^-), \\ (-\infty, +\infty) & (v \in \{s^+\} \cup S^+), \\ \{x \mid x \in \mathbb{R}^{\delta^- v}, \forall a \in \delta^- v: x(a) \leq \bar{c}(a)\} & (v \in V - S^+), \end{cases} \quad (4.16)$$

where  $P(f^+)$  and  $P(f^-)$  should, respectively, be regarded as polyhedra in  $\mathbb{R}^{A^+}$  and  $\mathbb{R}^{A^-}$  under the natural correspondence between  $S^+$  and  $A^+$  and between  $S^-$  and  $A^-$ .

Now, the independent flow problem  $P_I$  is rewritten as

$$\text{Minimize } \sum_{a \in \hat{A}} \hat{\gamma}(a) \varphi(a) \quad (4.17a)$$

$$\text{subject to } \hat{c}(a) \leq \varphi(a) \quad (a \in \hat{A}), \quad (4.17b)$$

$$\partial \varphi(v) = 0 \quad (v \in \hat{V}), \quad (4.17c)$$

$$\varphi^{\delta^+ v} \in P_v^+ \quad (v \in \hat{V}), \quad (4.17d)$$

$$\varphi^{\delta^- v} \in P_v^- \quad (v \in \hat{V}). \quad (4.17e)$$

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We can easily see that for each  $v \in V$  there exist distributive lattices  $\mathcal{D}_v^+ \subseteq 2^{\delta^+ v}$  and  $\mathcal{D}_v^- \subseteq 2^{\delta^- v}$  and submodular functions  $f_v^+ : \mathcal{D}_v^+ \rightarrow \mathbb{R}$  and  $f_v^- : \mathcal{D}_v^- \rightarrow \mathbb{R}$  such that  $\emptyset \in \mathcal{D}_v^+ \cap \mathcal{D}_v^-$ ,  $f_v^+(\emptyset) = f_v^-(\emptyset)$  and

$$P_v^+ = P(f_v^+), \quad P_v^- = P(f_v^-). \quad (4.18)$$

Therefore, the independent flow problem is reduced to a polymatroidal flow problem (see Fig. 4.2).

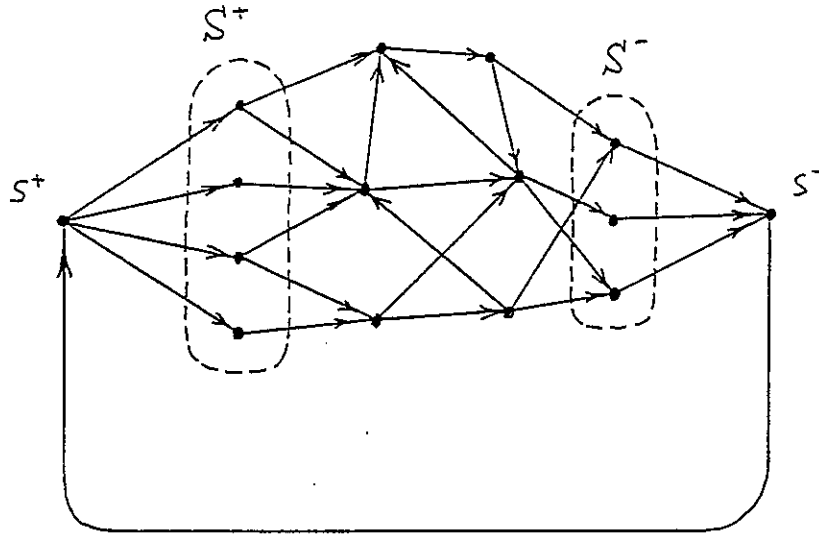


Figure 4.2.

#### (c) From polymatroidal flows to submodular flows

Consider the polymatroidal flow problem  $P_P$  defined by (4.3). From the underlying graph  $G = (V, A)$  we construct a graph  $G' = (W, A)$  with the same arc set, where the vertex set  $W$  is given by

$$W = \{w_a^+ \mid a \in A\} \cup \{w_a^- \mid a \in A\} \quad (4.19)$$

and we define  $\partial^+ a = w_a^+$  and  $\partial^- a = w_a^-$  for each  $a \in A$  in  $G'$  (see Fig. 4.3). Each arc of  $G'$  forms a connected component of  $G'$ . Define an upper capacity function  $\bar{c}' : A \rightarrow \mathbb{R}$  and a lower capacity function  $\underline{c}' : A \rightarrow \mathbb{R}$  by

$$\bar{c}'(a) = +\infty \quad (a \in A), \quad \underline{c}'(a) = \underline{c}(a) \quad (a \in A). \quad (4.20)$$

## 4.2. THE EQUIVALENCE OF THE NEOFLOW PROBLEMS

Also define

$$W_v^+ = \{w_a^+ \mid a \in \delta^+ v\} \quad (4.21)$$

$$W_v^- = \{w_a^- \mid a \in \delta^- v\} \quad (4.22)$$

where  $\delta^+$  and  $\delta^-$  are with respect to  $G$ . Under the natural correspondences between  $W_v^+$  and  $\delta^+ v$  and between  $W_v^-$  and  $\delta^- v$  for each  $v \in V$  we regard  $P(f_v^+)$  and  $P(f_v^-)$  as polyhedra in  $\mathbf{R}^{W_v^+}$  and  $\mathbf{R}^{W_v^-}$ , respectively.

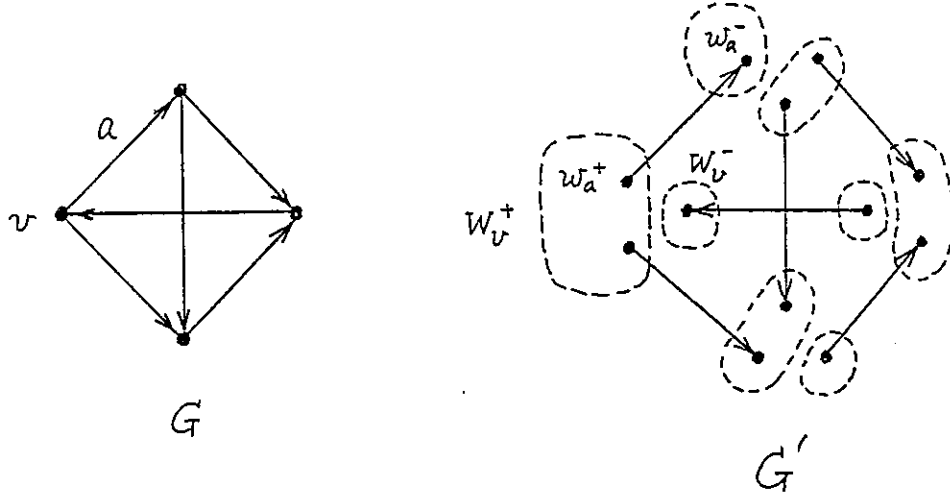


Figure 4.3.

Define for each  $v \in V$

$$B_v = \{y \mid y \in \mathbf{R}^{W_v^+ \cup W_v^-}, y^{W_v^+} \in P(f_v^+), \\ -y^{W_v^-} \in P(f_v^-), y(W_v^+) + y(W_v^-) = 0\} \quad (4.23)$$

and let  $B \subseteq \mathbf{R}^W$  be the direct sum of  $B_v$  ( $v \in V$ ):

$$B = \bigoplus_{v \in V} B_v. \quad (4.24)$$

Then, the polymatroidal flow problem is rewritten as

$$\text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.25a)$$

$$\text{subject to } \underline{c}'(a) \leq \varphi(a) \leq \bar{c}'(a) \quad (a \in A), \quad (4.25b)$$

$$\partial \varphi \in B. \quad (4.25c)$$

### III. NEOFLOWS

Note that  $y \in B_v$  if and only if

$$\forall X \in \mathcal{D}_v^+ : y(X) \leq f_v^+(X), \quad (4.26)$$

$$\forall X \in \mathcal{D}_v^- : -y(X) \leq f_v^-(X), \quad (4.27)$$

$$y(W_v^+) + y(W_v^-) = 0, \quad (4.28)$$

where we regard  $\mathcal{D}_v^+ \subseteq 2^{\delta^+ v}$  and  $\mathcal{D}_v^- \subseteq 2^{\delta^- v}$ . The system of (4.26)–(4.28) is equivalent to that of (4.26), (4.28) and

$$\forall X \in \mathcal{D}_v^- : y(W_v^+) + y(W_v^- - X) \leq f_v^-(X). \quad (4.29)$$

For each  $v \in V$  let  $\mathcal{F}_v$  be the crossing family defined by

$$\mathcal{F}_v = \mathcal{D}_v^+ \cup \overline{\mathcal{D}_v^-} \cup \{W_v^+ \cup W_v^-\}, \quad (4.30)$$

where  $\overline{\mathcal{D}_v^-} = \{W_v^- - X \mid X \in \mathcal{D}_v^-\}$ , and let  $f_v: \mathcal{F}_v \rightarrow \mathbb{R}$  be defined by

$$f_v(X) = \begin{cases} f_v^+(X) & (X \in \mathcal{D}_v^+), \\ f_v^-(W_v^- - X) & (X \in \overline{\mathcal{D}_v^-}), \\ 0 & (X = W_v^+ \cup W_v^-). \end{cases} \quad (4.31)$$

Then,  $f_v$  is a crossing-submodular function on the crossing family  $\mathcal{F}_v$  and we have  $B_v = B(f_v)$ , a base polyhedron. Therefore, it follows from (4.24) and (4.25) that the polymatroidal flow problem  $P_P$  is a submodular flow problem. It is interesting to see again that the polymatroidal flow problem looks like a very special case of the submodular flow problem.

#### 4.3. Feasibility for Submodular flows

In the previous section we have shown the equivalence of the submodular flow problem, the independent flow problem and the polymatroidal flow problem. Since the submodular flow problem is simple to describe, let us consider as a neoflow problem the submodular flow problem

$$P_S: \text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.1a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \overline{c}(a) \quad (a \in A), \quad (4.1b)$$

$$\partial \varphi \in B(f). \quad (4.1c)$$

Due to Theorem 1.4 we assume for simplicity that  $f$  is a submodular function on a distributive lattice  $\mathcal{D} \subseteq 2^V$  with  $\emptyset, V \in \mathcal{D}$  and  $f(\emptyset) = f(V) = 0$ .

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Recall (1.62), where we have shown

$$\begin{aligned}\partial\Phi &\equiv \{\partial\varphi \mid \varphi: A \rightarrow \mathbb{R}, \forall a \in A: \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a)\} \\ &= B(\kappa_{\underline{c}, \bar{c}}).\end{aligned}\tag{4.32}$$

Here,  $\kappa_{\underline{c}, \bar{c}}: 2^V \rightarrow \mathbb{R} \cup \{+\infty\}$  is the cut function associated with the network  $\mathcal{N} = (G = (V, A), \underline{c}, \bar{c})$  and  $\partial\Phi$  is a base polyhedron. Hence, there exists a feasible flow  $\varphi$  for the submodular flow problem if and only if

$$\partial\Phi \cap B(f) (= B(\kappa_{\underline{c}, \bar{c}}) \cap B(f)) \neq \emptyset,\tag{4.33}$$

i.e., there exists a common base in  $B(\kappa_{\underline{c}, \bar{c}})$  and  $B(f)$ .

From Theorem 3.12 we have

**Theorem 4.1** [Frank84]: There exists a feasible flow for the submodular flow problem  $P_S$  satisfying (4.1b) and (4.1c) if and only if

$$\forall X \in \mathcal{D}: (\kappa_{\underline{c}, \bar{c}})^\#(X) \leq f(X)\tag{4.34}$$

or

$$\forall X \in \mathcal{D}: \bar{c}(\Delta^- X) - \underline{c}(\Delta^+ X) + f(X) \geq 0,\tag{4.35}$$

where for each  $X \subseteq V$   $\Delta^+ X = \{a \mid a \in A, \partial^+ a \in X, \partial^- a \in V - X\}$  and  $\Delta^- X = \{a \mid a \in A, \partial^- a \in X, \partial^+ a \in V - X\}$ .

Moreover, if  $\bar{c}$ ,  $\underline{c}$  and  $f$  are integer-valued and  $P_S$  is feasible, there exists an integral feasible flow.

(Proof) Immediate from Theorem 3.12.

Q.E.D.

A feasible flow for the submodular flow problem can be obtained by the use of the algorithm shown in Section 3.3.

Frank [Frank84] showed feasibility theorems for the cases where  $f$  is an intersecting-submodular function on an intersecting family and where  $f$  is a crossing-submodular function on a crossing family. We can give theorems for these cases by combining Theorems 4.1 and 1.5.

Since the description of the algorithm for the common base problem given in Section 3.3 depends on the base polyhedron  $B(f)$  but not on the submodular function  $f$  or the system of linear inequalities expressing  $B(f)$ , the algorithm also works even if  $f$  is a crossing-submodular function on a crossing family, provided that an initial base in  $B(f)$  and an oracle for exchange capacities are available. For such an  $f$ , however, finding a base in  $B(f)$  together with determining the nonemptiness of  $B(f)$  is itself a nontrivial problem. See [Frank84], [Fuji87] and [Frank + Tardos88] for algorithms for finding a base in  $B(f)$  when  $f$  is a crossing-submodular function on a crossing family.

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#### 4.4. Optimality for Submodular flows

Consider the submodular flow problem  $P_S$  described by (4.1), where  $(\mathcal{D}, f)$  is a submodular system on  $V$  with  $f(V) = 0$ .

We show the following optimality theorem.

**Theorem 4.2:** A submodular flow  $\varphi: A \rightarrow \mathbb{R}$  for Problem  $P_S$  is optimal if and only if there exists a function  $p: V \rightarrow \mathbb{R}$  such that, defining  $\gamma_p: A \rightarrow \mathbb{R}$  by

$$\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A), \quad (4.36)$$

we have for each  $a \in A$

$$\gamma_p(a) > 0 \implies \varphi(a) = \underline{c}(a), \quad (4.37)$$

$$\gamma_p(a) < 0 \implies \varphi(a) = \bar{c}(a) \quad (4.38)$$

and such that the boundary  $\partial\varphi: V \rightarrow \mathbb{R}$  is a maximum-weight base of  $B(f)$  with respect to the weight function  $p$ .

(Proof) *The "if" part:* By an elementary calculation we have

$$\sum_{a \in A} \gamma(a) \varphi(a) = \sum_{a \in A} \gamma_p(a) \varphi(a) - \sum_{v \in V} p(v) \partial\varphi(v). \quad (4.39)$$

The "if" part immediately follows from (4.39). We postpone the proof of the "only if" part. Q.E.D.

To prove the "only if" part of Theorem 4.2 we need some preliminaries.

Any function  $p: V \rightarrow \mathbb{R}$  is called a *potential*. A potential  $p$  satisfying the conditions of Theorem 4.2 is called an *optimal potential*.

Given a submodular flow  $\varphi$ , we define the *auxiliary network*  $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi, \gamma_\varphi)$ , where  $G_\varphi$  is the graph with vertex set  $V$  and arc set  $A_\varphi$  given by

$$A_\varphi = A_\varphi^* \cup B_\varphi^* \cup C_\varphi, \quad (4.40)$$

$$A_\varphi^* = \{a \mid a \in A, \varphi(a) < \bar{c}(a)\}, \quad (4.41)$$

$$B_\varphi^* = \{\bar{a} \mid a \in A, \underline{c}(a) < \varphi(a)\} \quad (\bar{a}: \text{a reorientation of } a), \quad (4.42)$$

$$C_\varphi = \{(u, v) \mid u, v \in V, u \in \text{dep}(\partial\varphi, v) - \{v\}\}, \quad (4.43)$$

$c_\varphi: A_\varphi \rightarrow \mathbb{R}$  is the capacity function given by

$$c_\varphi(a) = \begin{cases} \bar{c}(a) - \varphi(a) & (a \in A_\varphi^*) \\ \varphi(\bar{a}) - \underline{c}(\bar{a}) & (a \in B_\varphi^*, \bar{a} \in A: \text{a reorientation of } a) \\ \tilde{c}(\partial\varphi, v, u) & (a = (u, v) \in C_\varphi), \end{cases} \quad (4.44)$$



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and  $\gamma_\varphi: A_\varphi \rightarrow \mathbf{R}$  is the length function given by

$$\gamma_\varphi(a) = \begin{cases} \gamma(a) & (a \in A_\varphi^*) \\ -\gamma(\bar{a}) & (a \in B_\varphi^*, \bar{a} \in A): \text{ a reorientation of } a \\ 0 & (a = (u, v) \in C_\varphi). \end{cases} \quad (4.45)$$

We call a directed cycle of negative length a *negative cycle*.

**Lemma 4.3:** Let  $\varphi$  be an optimal submodular flow for Problem  $P_S$ . Then there exists no negative cycle, relative to the length function  $\gamma_\varphi$ , in the auxiliary network  $\mathcal{N}_\varphi$ .

(Proof) Suppose, on the contrary, that there exists a negative cycle in  $\mathcal{N}_\varphi$ . Let  $Q$  be a negative cycle having the smallest number of arcs in  $\mathcal{N}_\varphi$  and let the arcs in  $C_\varphi$  lying on  $Q$  be given by  $(u_i, v_i)$  ( $i \in I$ ).

We show that by an appropriate numbering of arcs  $(u_i, v_i)$  ( $i \in I$ ) the assumption of Lemma 3.5 is satisfied. Suppose, on the contrary, that the assumption of Lemma 3.5 cannot be satisfied by any numbering of arcs  $(u_i, v_i)$  ( $i \in I$ ), i.e., there are arcs  $(u_{i_k}, v_{i_k})$  ( $k = 1, 2, \dots, p$ ) such that  $i_k \in I$  ( $k = 1, 2, \dots, p$ ) and  $(u_{i_k}, v_{i_{k+1}}) \in C_\varphi$  ( $k = 1, 2, \dots, p$ ) with  $i_{p+1} = i_1$ . Then for each  $k = 1, 2, \dots, p$  let  $Q_k$  be the directed cycle formed by arc  $(u_{i_k}, v_{i_{k+1}})$  and the path in  $Q$  from  $v_{i_{k+1}}$  to  $u_{i_k}$  (see Fig. 4.4). Since  $\gamma((u_{i_k}, v_{i_{k+1}})) = \gamma_\varphi((u_{i_k}, v_{i_{k+1}})) = 0$  ( $k = 1, 2, \dots, p$ ), we see that

$$\sum_{k=1}^p \gamma_\varphi(Q_k) = (p - q)\gamma_\varphi(Q) < 0 \quad (4.46)$$

for some integer  $q$  such that  $1 \leq q < p$ , where  $\gamma_\varphi(Q_k)$  and  $\gamma_\varphi(Q)$  are the lengths of  $Q_k$  and  $Q$  relative to the length function  $\gamma_\varphi$ . It follows from (4.46) that there is at least one  $Q_k$  ( $k = 1, 2, \dots, p$ ) having the negative length and such a directed cycle  $Q_k$  has a smaller number of arcs than  $Q$ . This contradicts the definition of  $Q$ , so that by an appropriate numbering of arcs  $(u_i, v_i)$  ( $i \in I$ ) the assumption of Lemma 3.5 is satisfied.

Define

$$\alpha = \min\{c_\varphi(a) \mid a \text{ lies on } Q\} \quad (4.47)$$

and modify the flow  $\varphi$  as

$$\varphi'(a) = \begin{cases} \varphi(a) + \alpha & (a \in A_\varphi^* \cap A_\varphi(Q)) \\ \varphi(a) - \alpha & (\bar{a} \in B_\varphi^* \cap A_\varphi(Q), \bar{a}: \text{ a reorientation of } a) \\ \varphi(a) & (\text{otherwise}) \end{cases} \quad (4.48)$$

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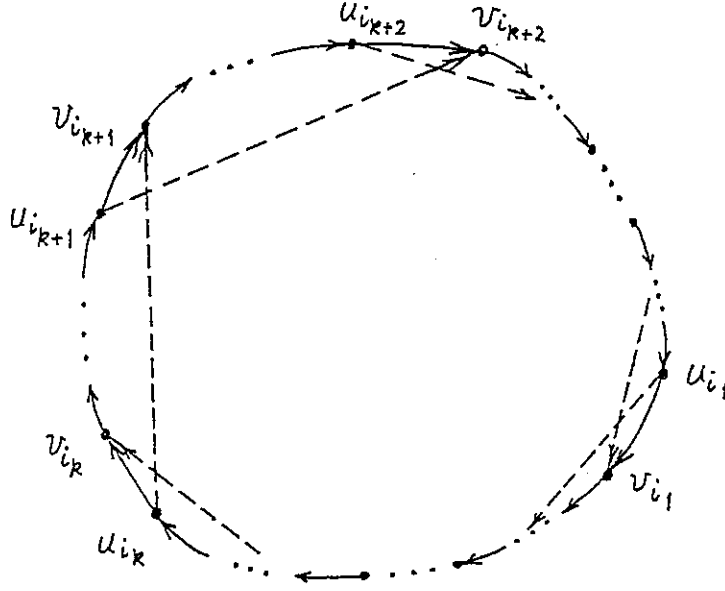


Figure 4.4.

for each  $a \in A$ , where  $A_\varphi(Q)$  denotes the set of arcs lying on  $Q$ . Then  $\varphi'$  is a submodular flow due to Lemma 3.5 and

$$\sum_{a \in A} \gamma(a) \varphi'(a) = \sum_{a \in A} \gamma(a) \varphi(a) + \alpha \cdot \gamma_\varphi(Q) < \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.49)$$

This contradicts the assumption that  $\varphi$  is an optimal submodular flow. Consequently, there is no negative cycle in  $\mathcal{N}_\varphi$ . Q.E.D.

The proof technique concerning (4.46) was originated by the author [Fuji77a, 77b] for matroids and may be interesting in its own right (also see [Zimmermann82]).

(Proof of the "only if" part of Theorem 4.2) Let  $\varphi$  be an optimal submodular flow for Problem  $P_S$ . Then, from Lemma 4.3 there exists no negative cycle in the auxiliary network  $\mathcal{N}_\varphi$  relative to the length function  $\gamma_\varphi$ . This implies that there exists a potential  $p: V \rightarrow \mathbb{R}$  such that

$$\gamma_{\varphi,p}(a) \equiv \gamma_\varphi(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \quad (4.50)$$

for each  $a \in A_\varphi^*$ . (Such a potential  $p$  can be found by a shortest path computation from a fixed origin  $s$  outside  $\mathcal{N}_\varphi$ , where the origin  $s$  is connected with

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each vertex of  $V$  by a new arc of any finite length, say, zero.) We can easily see that (4.50) for  $a \in A_\varphi^* \cup B_\varphi^*$  implies (4.37) and (4.38) and that (4.50) for  $a \in C_\varphi$  implies

$$p(u) \geq p(v) \quad (u \in \text{dep}(\partial\varphi, v) - \{v\}). \quad (4.51)$$

From (4.51) and Theorem 2.13  $\partial\varphi(\in B(f))$  is a maximum-weight base of  $B(f)$  with respect to the weight function  $p$ . Q.E.D.

It should be emphasized here that Theorem 4.2 is independent of how the base polyhedron  $B(f)$  is represented by a system of linear inequalities.

The proof of Lemma 4.3 suggests an algorithm for solving the submodular flow problem  $P_S$  as follows:

Starting from an arbitrary submodular flow  $\varphi$ ,

- (i) find a negative cycle  $Q$  having the smallest number of arcs in the auxiliary network  $\mathcal{N}_\varphi$ ,
- (ii) modify the present flow  $\varphi$  along the negative cycle  $Q$  as in (4.47) and (4.48), where if  $\alpha = +\infty$ , the problem is unbounded, and
- (iii) repeat this process until there is no negative cycle in  $\mathcal{N}_\varphi$ .

This is the primal algorithm given in [Fuji78a] and [Zimmermann82]. When  $\bar{c}$ ,  $\underline{c}$ ,  $f$  and an initial  $\varphi$  are integer-valued, this algorithm terminates after a finite number of steps and finds an optimal submodular flow if any exists. We shall discuss more efficient algorithms in the next section.

A necessary and sufficient condition for the existence of an optimal submodular flow is given as follows.

**Theorem 4.4:** For the submodular flow problem  $P_S$  define a network  $\hat{\mathcal{N}} = (\hat{G} = (V, \hat{A}), \hat{\gamma})$ , where  $\hat{G}$  is a graph with vertex set  $V$  and arc set  $\hat{A}$  given by

$$\hat{A} = A^* \cup B^* \cup C, \quad (4.52)$$

$$A^* = \{a \mid a \in A, \bar{c}(a) = +\infty\}, \quad (4.53)$$

$$B^* = \{\bar{a} \mid a \in A, \underline{c}(a) = -\infty\} \quad (\bar{a}: \text{a reorientation of } a), \quad (4.54)$$

$$C = \{(u, v) \mid u, v \in V, \forall X \in \mathcal{D}: (v \in X \Rightarrow u \in X)\} \quad (4.55)$$

and  $\hat{\gamma}: \hat{A} \rightarrow \mathbb{R}$  is the length function defined by

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & (a \in A^*) \\ -\gamma(\bar{a}) & (a \in B^*, \bar{a} \in A): \text{a reorientation of } a \\ 0 & (a \in C). \end{cases} \quad (4.56)$$

Suppose that there exists a submodular flow for Problem  $P_S$ . Then, there exists an optimal submodular flow for Problem  $P_S$  if and only if there is no negative cycle in  $\hat{\mathcal{N}}$  relative to the length function  $\hat{\gamma}$ .

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(Proof) *The "if" part:* Suppose that there is no negative cycle in  $\hat{\mathcal{N}}$  relative to  $\hat{\gamma}$ . Then there is a potential  $p: V \rightarrow \mathbb{R}$  such that for each  $\hat{a} \in \hat{A}$

$$\hat{\gamma}_p(\hat{a}) \equiv \hat{\gamma}(\hat{a}) + p(\partial^+ \hat{a}) - p(\partial^- \hat{a}) \geq 0, \quad (4.57)$$

where  $\partial^+$  and  $\partial^-$  are with respect to  $\hat{G}$ . (4.57) implies

$$\gamma(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \quad (a \in A), \quad \bar{c}(a) = +\infty, \quad (4.58)$$

$$\gamma(a) + p(\partial^+ a) - p(\partial^- a) \leq 0 \quad (a \in A), \quad \underline{c}(a) = -\infty, \quad (4.59)$$

$$p(u) \geq p(v) \quad ((u, v) \in C). \quad (4.60)$$

Since (4.1c) is expressed as

$$\forall X \in \mathcal{D}: \partial\varphi(X) \leq f(X), \quad (4.61)$$

$$\partial\varphi(V) = f(V)(= 0), \quad (4.62)$$

where (4.62) is void, and since

$$\partial\varphi(X) = \varphi(\Delta^+ X) - \varphi(\Delta^- X), \quad (4.63)$$

the linear-programming dual of Problem  $P_S$  with (4.1c) being replaced by (4.61) is described as

$$P_S^*: \text{Maximize} \quad \sum_{a \in A} \underline{\xi}(a) - \sum_{a \in A} \bar{\xi}(a) \bar{c}(a) - \sum_{X \in \mathcal{D}} \eta(X) f(X) \quad (4.64a)$$

$$\text{subject to} \quad \forall a \in A: \underline{\xi}(a) - \bar{\xi}(a) - \sum \{ \eta(X) \mid X \in \mathcal{D}, a \in \Delta^+ X \} \\ + \sum \{ \eta(X) \mid X \in \mathcal{D}, a \in \Delta^- X \} = \gamma(a), \quad (4.64b)$$

$$\underline{\xi}(a) = 0 \quad (a \in A, \underline{c}(a) = -\infty), \quad (4.64c)$$

$$\bar{\xi}(a) = 0 \quad (a \in A, \bar{c}(a) = +\infty), \quad (4.64d)$$

$$\underline{\xi}, \bar{\xi}, \eta \geq 0, \quad (4.64e)$$

where  $\underline{\xi}, \bar{\xi}: A \rightarrow \mathbb{R}$ ,  $\eta: \mathcal{D} \rightarrow \mathbb{R}$  and we regard (4.64a) as the objective function with the terms  $\underline{\xi}(a)\underline{c}(a)$  ( $a \in A, \underline{c}(a) = -\infty$ ) and  $\bar{\xi}(a)\bar{c}(a)$  ( $a \in A, \bar{c}(a) = +\infty$ ) being suppressed.

Because of (4.60) there is a maximal chain

$$C: \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = V \quad (4.65)$$

of  $\mathcal{D}$  such that  $p$  is constant in each quotient  $S_k - S_{k-1}$  ( $k = 1, 2, \dots, n$ ). Using this chain  $C$  and the potential  $p$ , define

$$\eta(S_k) = p_{k+1} - p_k \quad (k = 1, 2, \dots, n-1), \quad (4.66)$$

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where  $p_k$  is the value of  $p$  taken in  $S_{k+1} - S_k$  and note that  $\eta(S_k) \geq 0$ . Also define  $\eta(X) = 0$  for other  $X \in \mathcal{D}$ . Moreover, define

$$\underline{\xi}(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A, \bar{c}(a) = +\infty), \quad (4.67)$$

$$\bar{\xi}(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A, \underline{c}(a) = -\infty), \quad (4.68)$$

and for each arc  $a \in A$  with  $\bar{c}(a) < +\infty$  and  $\underline{c}(a) > -\infty$  define  $\underline{\xi}$  and  $\bar{\xi}$  such that  $\underline{\xi}, \bar{\xi} \geq 0$  and (4.64b) holds. We can easily see that thus defined  $\underline{\xi}, \bar{\xi}, \eta$  satisfy (4.64b)–(4.64e), where note that for each arc  $a \in A$  with  $\bar{c}(a) = +\infty$  and  $\underline{c}(a) = -\infty$  we have  $\gamma(a) + p(\partial^+ a) - p(\partial^- a) = 0$  due to (4.58) and (4.59).

Since the dual of Problem  $P_S$  has a feasible solution and the feasibility of the primal problem  $P_S$  is assumed, there exists an optimal solution of Problem  $P_S$ .

*The “only if” part:* Suppose that there is a negative cycle in  $\hat{\mathcal{N}}$  relative to the length function  $\hat{\gamma}$ , and let  $Q$  be such a negative cycle in  $\hat{\mathcal{N}}$ . Then for any positive  $\alpha$ , if we define  $\varphi': A \rightarrow \mathbb{R}$  by (4.48),  $\varphi'$  is feasible for Problem  $P_S$  because of the definition of  $\hat{\mathcal{N}}$  and we have (4.49). Since  $\alpha (> 0)$  is arbitrary and  $\gamma_\varphi(Q) < 0$ , Problem  $P_S$  is unbounded. This completes the proof. Q.E.D.

We also have

**Theorem 4.5** [Edm+Giles77]: The system of inequalities

$$\underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.69)$$

$$\partial\varphi(X) (= \varphi(\Delta^+ X) - \varphi(\Delta^- X)) \leq f(X) \quad (X \in \mathcal{D}) \quad (4.70)$$

for the submodular flow problem  $P_S$  is totally dual integral.

(Proof) Consider a submodular flow problem  $P_S$  such that the coefficients  $\gamma(a)$  ( $a \in A$ ) of the objective function are integers and that there exists an optimal submodular flow  $\varphi$ . From the proof of the “only if” part of Theorem 4.2 there exists an integral potential  $p: V \rightarrow \mathbb{Z}$  such that for each  $a \in A$

$$\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) > 0 \implies \varphi(a) = \underline{c}(a), \quad (4.71)$$

$$\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a) < 0 \implies \varphi(a) = \bar{c}(a) \quad (4.72)$$

and for each  $u \in V$  and  $v \in \text{dep}(\partial\varphi, u) - \{u\}$

$$p(u) \leq p(v). \quad (4.73)$$

Let  $p_1 > p_2 > \dots > p_l$  be the distinct values of  $p(u)$  ( $u \in V$ ) and define

$$W_i = \{u \mid u \in V, p(u) \geq p_i\} \quad (i = 1, 2, \dots, l). \quad (4.74)$$

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From (4.73) we have

$$\partial\varphi(W_i) = f(W_i) \quad (i = 1, 2, \dots, l). \quad (4.75)$$

For the dual problem  $P_S^*$  in (4.64) of  $P_S$  define

$$\underline{\xi}(a) = \max\{0, \gamma_p(a)\} \quad (a \in A), \quad (4.76)$$

$$\bar{\xi}(a) = \max\{0, -\gamma_p(a)\} \quad (a \in A), \quad (4.77)$$

$$\eta(W_i) = p_i - p_{i+1} \quad (i = 1, 2, \dots, l-1), \quad (4.78)$$

$$\eta(X) = 0 \quad \text{for other } X \in \mathcal{D}. \quad (4.79)$$

We can easily see that these  $\underline{\xi}$ ,  $\bar{\xi}$  and  $\eta$  satisfy (4.64b)–(4.64e). Moreover, from (4.71), (4.72) and (4.74),

$$\begin{aligned} & \sum_{a \in A} \underline{\xi}(a) \underline{c}(a) - \sum_{a \in A} \bar{\xi}(a) \bar{c}(a) - \sum_{X \in \mathcal{D}} \eta(X) f(X) \\ &= \sum_{a \in A} \gamma_p(a) \varphi(a) - \sum_{i=1}^{l-1} (p_i - p_{i+1}) f(W_i) \\ &= \sum_{a \in A} \gamma_p(a) \varphi(a) - \sum_{i=1}^{l-1} (p_i - p_{i+1}) \partial\varphi(W_i) \\ &= \sum_{a \in A} \gamma_p(a) \varphi(a) - \sum_{i=1}^l p_i \partial\varphi(W_i - W_{i-1}) \\ &= \sum_{a \in A} \gamma_p(a) \varphi(a) - \sum_{v \in V} p(v) \partial\varphi(v) \\ &= \sum_{a \in A} \gamma(a) \varphi(a), \end{aligned} \quad (4.80)$$

where  $W_0 = \emptyset$  and note that  $\partial\varphi(W_l) = \partial\varphi(V) = 0$ . Therefore,  $\underline{\xi}$ ,  $\bar{\xi}$  and  $\eta$  defined by (4.76)–(4.79) form an integral optimal solution of the dual problem  $P_S^*$ . This completes the proof. Q.E.D.

It follows from Theorems 4.5 and 1.5 that if we replace  $f$  in (4.70) by a crossing-submodular function on a crossing family, the system of (4.69) and (4.70) is totally dual integral.

**Corollary 4.6** [Edm+Giles77]: For a crossing-submodular function  $f$  on a crossing family  $\mathcal{F} \subseteq 2^V$ , the system of inequalities

$$\underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.81)$$

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$$\partial\varphi(X) \leq f(X) \quad (X \in \mathcal{F}) \quad (4.82)$$

is totally dual integral.

The proof of Theorem 4.5 shows the way of constructing an optimal dual solution  $\xi$ ,  $\bar{\xi}$  and  $\eta$  from an optimal potential  $p$  when  $f$  is a crossing-submodular function on the distributive lattice  $\mathcal{D}$ . When  $f$  is a submodular function on a crossing family  $\mathcal{F} \subseteq 2^V$  with  $\emptyset, V \in \mathcal{F}$  and  $f(\emptyset) = f(V) = 0$ , suppose that  $B(f) = B(f_2)$  for a submodular system  $(\mathcal{D}, f_2)$  (see Theorem 1.5). Also suppose that for each  $W_i$  in (4.75) we have the expression of  $f_2(W_i)$  in terms of  $f$  as follows (see Theorem 1.5).

$$f_2(W_i) = \sum_{j \in J_i} \sum_{k \in K_{ij}} f(X_{ijk}) \quad (i = 1, 2, \dots, l-1), \quad (4.83)$$

where  $V - (\sum_{k \in K_{ij}} X_{ijk})$  ( $j \in J_i$ ) form a partition of  $V - W_i$  for each  $i = 1, 2, \dots, l-1$  and  $X_{ijk}$  ( $k \in K_{ij}$ ) are disjoint sets (as subsets of  $V$ ) in  $\mathcal{F}$  for each  $i = 1, 2, \dots, l-1$  and  $j \in J_i$ . Here, note that  $f_2(V) = 0$ . Then define for each  $X \in \mathcal{F}$

$$\eta(X) = \sum \{p_i - p_{i+1} \mid X = X_{ijk}, i \in \{1, \dots, p-1\}, j \in J_i, k \in K_{ij}\}, \quad (4.84)$$

where the summation over the empty set is equal to zero. Note that if all the  $X_{ijk}$  are distinct, we have  $\eta(X_{ijk}) = p_i - p_{i+1}$  and  $\eta(X) = 0$  for other  $X \in \mathcal{F}$ .  $\xi, \bar{\xi}$  defined by (4.76) and (4.77) and this  $\eta$  form an optimal dual solution.

We show the way of finding an expression in (4.83) for each  $i = 1, 2, \dots, l-1$ . Put  $x = \partial\varphi$  and define

$$\mathcal{F}(x) = \{X \mid X \in \mathcal{F}, x(X) = f(X)\}. \quad (4.85)$$

We assume an oracle which, for each ordered pair  $(u, v)$  of distinct vertices  $u, v \in V$ , gives a  $u$ - $v$  cut  $X \in \mathcal{F}(x)$  or answers that there is no  $u$ - $v$  cut  $X \in \mathcal{F}(x)$ , where a  $u$ - $v$  cut is a set  $X \subseteq V$  such that  $u \in X$  and  $v \notin X$ . Let  $G = (V, A(x))$  be the graph defined by

$$A(x) = \{(u, v) \mid u \in V, v \in \text{dep}(x, u) - \{u\}\}, \quad (4.86)$$

where note that  $(u, v) \in A(x)$  if and only if  $u \neq v$  and there is no  $u$ - $v$  cut  $X \in \mathcal{F}(x)$ . Choose any  $W_i$  ( $i = 1, 2, \dots, l-1$ ). Let  $G_i$  be the graph obtained from  $G$  by deleting all the vertices in  $W_i$  together with the arcs incident to  $W_i$ , and let  $U_{ij}$  ( $j \in J_i$ ) be the vertex sets of the connected components of  $G_i$ . Note that

$$\Delta^- U_{ij} = \emptyset \quad (j \in J_i) \quad (4.87)$$

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in  $G$ , so that

$$f_2(V - U_{ij}) = x(V - U_{ij}) \quad (j \in J_i). \quad (4.88)$$

Therefore, for each  $u \in V - U_{ij}$  and  $v \in U_{ij}$  there exists a  $u$ - $v$  cut  $X \in \mathcal{F}(x)$ . If  $X \cap U_{ij} \neq \emptyset$ , choose  $v' \in X \cap U_{ij}$  such that for some  $w \in U_{ij} - X$  we have  $(v', w) \in A(x)$ . Such a  $v'$  exists since  $U_{ij} - X \neq \emptyset$  and  $U_{ij}$  is the vertex set of a connected component of  $G_i$ . There exists a  $u$ - $v'$  cut  $X' \in \mathcal{F}(x)$  and either  $X$  and  $X'$  cross or  $X' \subseteq X$ , due to the way of choosing  $v'$ . Since  $\mathcal{F}(x)$  is a crossing family, we have  $X \cap X' \in \mathcal{F}(x)$  and  $u \in X \cap X' \subseteq X'$ . Put  $X \leftarrow X \cap X'$ . If  $X \cap U_{ij} \neq \emptyset$ , then repeat this process until we have  $X \cap U_{ij} = \emptyset$ . After repeating this process at most  $|U_{ij}|$  time we have  $X \in \mathcal{F}(x)$  such that  $u \in X \subseteq V - U_{ij}$ . Denote this  $X$  by  $X_*$ .

In this way, for each  $u \in V - U_{ij}$  we can find  $X_*$  such that  $u \in X_* \subseteq V - U_{ij}$ . Consider the hypergraph  $H = (V - U_{ij}, \{X_* \mid u \in V - U_{ij}\})$  and let  $X_{ijk}$  ( $k \in K_{ij}$ ) be the vertex sets of the connected components of  $H$ . Since  $\mathcal{F}(x)$  is a crossing family of subsets of  $V$ , we have  $X_{ijk} \in \mathcal{F}(x)$  ( $k \in K_{ij}$ ). Consequently,

$$\begin{aligned} f_2(W_i) &= x(W_i) = \sum_{j \in J_i} \sum_{k \in K_{ij}} x(X_{ijk}) - (|J_i| - 1)x(V) \\ &= \sum_{j \in J_i} \sum_{k \in K_{ij}} f(X_{ijk}) \end{aligned} \quad (4.89)$$

since  $x(V) = 0$ .

When  $f$  is an intersecting-submodular function on an intersecting family  $\mathcal{F} \subseteq 2^V$  with  $\emptyset, V \in \mathcal{F}$  and  $f(\emptyset) = f(V) = 0$ , for each  $u \in W_i$  we can find  $X_* \in \mathcal{F}(x)$  such that  $u \in X_* \subseteq W_i$ . (Note that for any  $v, v' \in V - W_i$  there exist a  $u$ - $v$  cut  $X \in \mathcal{F}(x)$  and a  $u$ - $v'$  cut  $X' \in \mathcal{F}(x)$  and that  $u \in X \cap X' \in \mathcal{F}(x)$  since  $\mathcal{F}(x)$  is an intersecting family.) Let  $X_{ij}$  ( $j \in J_i$ ) be the vertex sets of the connected components of the hypergraph  $H = (W_i, \{X_* \mid u \in W_i\})$ . Then  $X_{ij} \in \mathcal{F}(x)$  and we have

$$f_1(W_i) = \sum_{j \in J_i} f(X_{ij}), \quad (4.90)$$

where  $f_1$  is the submodular function appearing in (i) of Theorem 1.5.

#### 4.5. Algorithms for Neoflows

We consider maximum flow and minimum-cost flow problems and show algorithms for these problems.



#### 4.5. ALGORITHMS FOR NEOFLOWS

##### (a) Maximum independent flows

The maximum flow problem for neoflows seems to be easily formulated by the independent flow problem. The *maximum independent flow problem* is:

$$P_{MI}: \text{Maximize } \partial\varphi(S^+) \quad (4.91a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.91b)$$

$$\partial\varphi(v) = 0 \quad (v \in V - (S^+ \cup S^-)), \quad (4.91c)$$

$$(\partial\varphi)^{S^+} \in P(f^+), \quad (4.91d)$$

$$-(\partial\varphi)^{S^-} \in P(f^-), \quad (4.91e)$$

where we consider the same network described in Section 4.1.b. We denote  $(\partial\varphi)^{S^+}$  and  $-(\partial\varphi)^{S^-}$  by  $\partial^+\varphi$  and  $\partial^-\varphi$ , respectively, in the following.

We suppose there exists a feasible flow. A feasible flow, if any exists, can be found by adapting the algorithm in Section 3.3 as discussed in Section 4.3 for submodular flows. Given a feasible flow  $\varphi$ , we define an auxiliary network  $\mathcal{N}_\varphi = (G_\varphi = (V \cup \{s^+, s^-\}, A_\varphi), c_\varphi, s^+, s^-)$  with source  $s^+$  and sink  $s^-$  as follows.  $G_\varphi$  is the underlying graph with vertex set  $V \cup \{s^+, s^-\}$  and arc set  $A_\varphi$  defined by

$$A_\varphi = S_\varphi^+ \cup S_\varphi^- \cup A_\varphi^+ \cup A_\varphi^- \cup A_\varphi^* \cup B_\varphi^*, \quad (4.92)$$

$$S_\varphi^+ = \{(s^+, v) \mid v \in S^+ - \text{sat}^+(\partial^+\varphi)\}, \quad (4.93)$$

$$S_\varphi^- = \{(v, s^-) \mid v \in S^- - \text{sat}^-(\partial^-\varphi)\}, \quad (4.94)$$

$$A_\varphi^+ = \{(u, v) \mid v \in \text{sat}^+(\partial^+\varphi), u \in \text{dep}^+(\partial^+\varphi, v) - \{v\}\}, \quad (4.95)$$

$$A_\varphi^- = \{(v, u) \mid v \in \text{sat}^-(\partial^-\varphi), u \in \text{dep}^-(\partial^-\varphi, v) - \{v\}\}, \quad (4.96)$$

$$A_\varphi^* = \{a \mid a \in A, \varphi(a) < \bar{c}(a)\}, \quad (4.97)$$

$$B_\varphi^* = \{\bar{a} \mid a \in A, \varphi(a) > \underline{c}(a)\}, \quad (4.98)$$

where  $\bar{a}$  denotes a reorientation of  $a$ ,  $\text{sat}^+$  ( $\text{sat}^-$ ) is the saturation function with respect to  $(\mathcal{D}^+, f^+)$  ( $(\mathcal{D}^-, f^-)$ ) and  $\text{dep}^+$  ( $\text{dep}^-$ ) is the dependence function with respect to  $(\mathcal{D}^+, f^+)$  ( $(\mathcal{D}^-, f^-)$ ). Also,  $c_\varphi: A_\varphi \rightarrow \mathbb{R}$  is defined by

$$c_\varphi(a) = \begin{cases} \hat{c}^+(\partial^+\varphi, v) & (a = (s^+, v) \in S_\varphi^+) \\ \hat{c}^-(\partial^-\varphi, v) & (a = (v, s^-) \in S_\varphi^-) \\ \tilde{c}^+(\partial^+\varphi, v, u) & (a = (u, v) \in A_\varphi^+) \\ \tilde{c}^-(\partial^-\varphi, v, u) & (a = (v, u) \in A_\varphi^-) \\ \bar{c}(a) - \varphi(a) & (a \in A_\varphi^*) \\ \varphi(\bar{a}) - \underline{c}(\bar{a}) & (a \in B_\varphi^*, \bar{a} \in A): \text{ a reorientation of } a, \end{cases} \quad (4.99)$$

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where  $\hat{c}^+$  (or  $\hat{c}^-$ ) denotes the saturation capacity associated with  $(\mathcal{D}^+, f^+)$  (or  $(\mathcal{D}^-, f^-)$ ) and  $\tilde{c}^+$  (or  $\tilde{c}^-$ ) the exchange capacity associated with  $(\mathcal{D}^+, f^+)$  (or  $(\mathcal{D}^-, f^-)$ ).

We suppose that a feasible flow (an independent flow)  $\varphi$  is given. An algorithm for finding a maximum independent flow is furnished as follows (cf. [Fuji78a], [Lawler + Martel82], [Schönsleben80]). We assume oracles for saturation capacities and exchange capacities.

#### An algorithm for finding a maximum independent flow

**Input:** an independent flow  $\varphi$  in network  $\mathcal{N} = (G = (V, A; S^+, S^-), \underline{c}, \bar{c}, (\mathcal{D}^+, f^+), (\mathcal{D}^-, f^-))$ ; a fixed numbering  $\pi: V \rightarrow \{1, 2, \dots, |V|\}$ .

**Output:** a maximum independent flow  $\varphi$  in  $\mathcal{N}$ .

**Step 1:** While there exists a directed path from  $s^+$  to  $s^-$  in the auxiliary network  $\mathcal{N}_\varphi = (G_\varphi = (V \cup \{s^+, s^-\}, A_\varphi), c_\varphi, s^+, s^-)$ , do the following.

(\*) Find a lexicographically shortest path  $P$  from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  and put

$$\alpha \leftarrow \min\{c_\varphi(a) \mid a: \text{an arc lying on } P\}, \quad (4.100)$$

$$\varphi(a) \leftarrow \begin{cases} \varphi(a) + \alpha & (a \in A_\varphi^* \cap A_\varphi(P)) \\ \varphi(a) - \alpha & (\bar{a} \in B_\varphi^* \cap A_\varphi(P), \\ & a(\in A): \text{a reorientatin of } \bar{a}). \end{cases} \quad (4.101)$$

(End)

In this algorithm a lexicographically shortest path from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  is a directed path from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  which has the minimum number of arcs among directed paths from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  and whose vertex sequence  $(s^+, v_1, \dots, v_p, s^-)$ , say, gives the lexicographically minimum sequence  $(\pi(v_1), \dots, \pi(v_p))$  among directed paths from  $s^+$  to  $s^-$  in  $\mathcal{N}_\varphi$  having the minimum number of arcs. Also,  $A_\varphi(P)$  is the set of arcs, in  $A_\varphi$ , lying on  $P$ .

The validity of the above algorithm can be shown almost in the same manner as in the proof of the validity of the algorithm for the intersection theorem in Sections 3.1.b and 3.1.c. The algorithm described above finds a maximum independent flow after repeating (\*) at most  $|V|^3$  times.

The maximum independent flow problem naturally includes the intersection problem, where the underlying graph is the bipartite graph representing the bijection between  $S^+$  and  $S^-$ .

**Theorem 4.7** (The maximum independent flow-minimum cut theorem) (cf. [McDiarmid75], [Fuji78a]): Suppose that the maximum independent flow problem  $P_{MI}$  has a feasible flow. Then we have

$$\begin{aligned} & \max\{\partial\varphi(S^+) \mid \varphi \text{ is an independent flow (satisfying (4.70)-(4.73))}\} \\ &= \min\{f^+(S^+ - U) + \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U) + f^-(S^- \cap U) \mid U \subseteq V\}, \end{aligned} \quad (4.102)$$

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where  $\Delta^+$  and  $\Delta^-$  are with respect to  $G$  and we regard  $f^+(X) = +\infty$  ( $f^-(Y) = +\infty$ ) if  $X \notin \mathcal{D}^+$  ( $Y \notin \mathcal{D}^-$ ).

Moreover, if  $\bar{c}$ ,  $\underline{c}$ ,  $f^+$  and  $f^-$  are integer-valued and Problem  $P_{MI}$  is feasible, then there exists an integral maximum independent flow for Problem  $P_{MI}$ .

(Proof) Let  $\varphi^*$  be a maximum independent flow for Problem  $P_{MI}$ , and consider the auxiliary network  $\mathcal{N}_{\varphi^*}$  associated with  $\varphi^*$ . Let  $U^*$  be the set of vertices in  $V$  which are reachable by directed paths from  $s^+$  in the auxiliary network  $\mathcal{N}_{\varphi^*}$ . Then we have

$$S^+ - U^* \subseteq \text{sat}^+(\partial^+ \varphi^*), \quad (4.103)$$

$$S^- \cap U^* \subseteq \text{sat}^-(\partial^- \varphi^*) \quad (4.104)$$

since there is no directed path from  $s^+$  to  $s^-$  in  $\mathcal{N}_{\varphi^*}$ . Hence, by the definition of  $U^*$ ,

$$\text{dep}^+(\partial^+ \varphi^*, v) \subseteq S^+ - U^* \quad (v \in S^+ - U^*), \quad (4.105)$$

$$\text{dep}^-(\partial^- \varphi^*, v) \subseteq S^- \cap U^* \quad (v \in S^- \cap U^*), \quad (4.106)$$

$$\varphi^*(a) = \bar{c}(a) \quad (a \in \Delta^+ U^*), \quad (4.107)$$

$$\varphi^*(a) = \underline{c}(a) \quad (a \in \Delta^- U^*) \quad (4.108)$$

From (4.105) and (4.106),

$$\partial^+ \varphi^*(S^+ - U^*) = f^+(S^+ - U^*), \quad (4.109)$$

$$\partial^- \varphi^*(S^- \cap U^*) = f^-(S^- \cap U^*) \quad (4.110)$$

due to Lemma 1.1. From (4.107) and (4.108),

$$\varphi^*(\Delta^+ U^*) = \bar{c}(\Delta^+ U^*), \quad \varphi^*(\Delta^- U^*) = \underline{c}(\Delta^- U^*). \quad (4.111)$$

It follows from (4.109)–(4.111) that

$$\begin{aligned} \partial \varphi(S^+) &= \partial^+ \varphi^*(S^+ - U^*) + \varphi^*(\Delta^+ U^*) - \varphi^*(\Delta^- U^*) + \partial^- \varphi^*(S^- \cap U^*) \\ &= f^+(S^+ - U^+) + \bar{c}(\Delta^+ U^*) - \underline{c}(\Delta^- U^*) + f^-(S^- \cap U^*). \end{aligned} \quad (4.112)$$

On the other hand, for any independent flow  $\varphi$  and any subset  $U$  of  $V$  we have

$$\begin{aligned} \partial \varphi(U) &= \partial^+ \varphi(S^+ - U) + \varphi(\Delta^+ U) - \varphi(\Delta^- U) + \partial^- \varphi(S^- \cap U) \\ &\leq f^+(S^+ - U) + \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U) + f^-(S^- \cap U). \end{aligned} \quad (4.113)$$

(4.102) follows from (4.112) and (4.113).

Moreover, if  $\bar{c}$ ,  $\underline{c}$ ,  $f^+$  and  $f^-$  are integer-valued and Problem  $P_{MI}$  is feasible, then there exists an integral feasible flow due to Theorem 4.1. Starting with

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an integral feasible flow, the algorithm finds an integral maximum independent flow in finitely many steps. Q.E.D.

We call any  $U \subseteq V$  a *cut* and the value  $f^+(S^+ - U) + \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U) + f^-(S^- \cap U)$  the *capacity* of the cut  $U$ . Theorem 4.7 is a generalization of the classical max-flow min-cut theorem for ordinary capacitated networks [Ford + Fulkerson62].

Theorem 4.7 was shown non-algorithmically by McDiarmid [McDiarmid75] and algorithmically by the author [Fuji78a].

Now, consider a supermodular system  $(\mathcal{D}^+, g^+)$  on  $S^+$  instead of submodular system  $(\mathcal{D}^+, f^+)$  and also consider the following system of inequalities.

$$\underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.114)$$

$$\partial\varphi(v) = 0 \quad (v \in V - (S^+ \cap S^-)), \quad (4.115)$$

$$(\partial\varphi)^{S^+} \in P(g^+), \quad (4.116)$$

$$-(\partial\varphi)^{S^-} \in P(f^-), \quad (4.117)$$

where  $P(g^+)$  is the supermodular polyhedron associated with  $(\mathcal{D}^+, g^+)$ . Then we have the following theorem.

**Theorem 4.8:** There exists a feasible flow  $\varphi$  satisfying (4.114)–(4.117) if and only if we have for each  $U \subseteq V$  such that  $S^+ \cap U \in \mathcal{D}^+$  and  $S^- \cap U \in \mathcal{D}^-$

$$g^+(S^+ \cap U) - f^-(S^- \cap U) \leq \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U) \quad (4.118)$$

and for each  $U \supseteq S^+ \cup S^-$

$$0 \leq \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U). \quad (4.119)$$

Moreover, if there exists a feasible flow and  $\bar{c}$ ,  $\underline{c}$ ,  $g^+$  and  $f^-$  are integer-valued, then there exists an integral feasible flow.

(Proof) Define  $\mathcal{D} \subseteq 2^V$  and  $g: \mathcal{D} \rightarrow \mathbb{R}$  by

$$\mathcal{D} = \{U \mid U \subseteq V, S^+ \cap U \in \mathcal{D}^+, S^- \cap U \in \mathcal{D}^-\}, \quad (4.120)$$

$$g(U) = \begin{cases} g^+(S^+ \cap U) - f^-(S^- \cap U) & (U \in \mathcal{D}, (S^+ \cup S^-) - U \neq \emptyset) \\ 0 & (U \in \mathcal{D}, S^+ \cup S^- \subseteq U). \end{cases} \quad (4.121)$$

If there is a feasible flow, we must have

$$g^+(S^+) - f^-(S^-) \leq 0. \quad (4.122)$$

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Under condition (4.122), the function  $g: \mathcal{D} \rightarrow \mathbb{R}$  defined by (4.121) is a supermodular function on the distributive lattice  $\mathcal{D}$  with  $\emptyset, V \in \mathcal{D}$  and  $g(\emptyset) = g(V) = 0$ . We have  $x \in B(g)$  if and only if

$$x^{S^+} \in P(g^+), \quad -x^{S^-} \in P(f^-), \quad x^{V-(S^+ \cup S^-)} = 0, \quad (4.123)$$

$$x(V) = 0. \quad (4.124)$$

It follows from (1.62) and Theorem 3.12 that there exists a feasible flow satisfying (4.114)–(4.117) if and only if

$$g(U) \leq \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U) \quad (U \in \mathcal{D}), \quad (4.125)$$

where note that (4.122) is included in (4.125). We see that (4.125) is equivalent to (4.118) and (4.119).

The integrality part of the present theorem follows from the counterpart of Theorem 3.12. Q.E.D.

Theorem 4.8, where  $\underline{c} = 0$ ,  $\bar{c} \geq 0$ ,  $f^-$  is a polymatroid rank function and  $g^+$  is the dual supermodular function of a polymatroid rank function, is shown in [Fuji78d].

The discrete separation theorem also follows from Theorem 4.8. The readers may deduce Theorem 4.8 from the feasibility theorem for submodular flows (Theorem 4.1).

##### (b) Maximum submodular flows

For the submodular flow problem with the network  $\mathcal{N}_S = (G = (V, A), \underline{c}, \bar{c}, \gamma, (\mathcal{D}, f))$ , disregarding the cost function  $\gamma$ , let us consider a “max-flow” problem  $P_{MS}$  given as follows [Cunningham + Frank85]. Let  $a_0$  be a fixed reference arc in  $A$ .

$$P_{MS}: \text{Maximize } \varphi(a_0) \quad (4.126a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.126b)$$

$$\partial\varphi \in B(f). \quad (4.126c)$$

We assume that there is a feasible flow in  $\mathcal{N}_S$  and define the auxiliary network  $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), c_\varphi)$  associated with  $\varphi$  as follows.  $G_\varphi$  is the graph with vertex set  $V$  and arc set  $A_\varphi$  defined by (4.40)–(4.43) except that

$$B_\varphi^* = \{\bar{a} \mid a \in A - \{a_0\}, \underline{c}(a) < \varphi(a)\} \quad (\bar{a}: \text{a reorientation of } a) \quad (4.127)$$

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and  $c_\varphi: A_\varphi \rightarrow \mathbb{R}$  is defined by (4.44).

**An algorithm for finding a maximum submodular flow**

**Input:** a feasible flow  $\varphi$  in  $\mathcal{N}_S$  with reference arc  $a_0$  and a vertex numbering  $\pi: V \rightarrow \{1, 2, \dots, |V|\}$  which defines the lexicographic ordering among directed paths from  $\partial^- a_0$  to  $\partial^+ a_0$  in  $\mathcal{N}_\varphi$ .

**Output:** a maximum submodular flow  $\varphi$  in  $\mathcal{N}_S$  with reference arc  $a_0$ .

**Step 1:** While  $\varphi(a_0) < \bar{c}(a_0)$  and there exists a directed path from  $\partial^- a_0$  to  $\partial^+ a_0$  in the auxiliary network  $\mathcal{N}_\varphi$ , do the following.

(\*) Find the lexicographically shortest path  $P$  from  $\partial^- a_0$  to  $\partial^+ a_0$  in  $\mathcal{N}_\varphi$  and let  $Q$  be the directed cycle formed by  $P$  and reference arc  $a_0$ . Put

$$\begin{aligned} \alpha &\leftarrow \min\{c_\varphi(a) \mid a \text{ lies on } Q\}, \\ \varphi(a) &\leftarrow \begin{cases} \varphi(a) + \alpha & (a \in A_\varphi^* \cap A_\varphi(Q)) \\ \varphi(a) - \alpha & (\bar{a} \in B_\varphi^* \cap A_\varphi(Q), \bar{a}: \text{a reorientation of } a \in A), \end{cases} \end{aligned}$$

where if  $\alpha = +\infty$ , then stop (the flow value  $\varphi(a_0)$  can be made arbitrarily large).

(End)

Here,  $A_\varphi(Q)$  is the set of the arcs in  $A_\varphi$  lying on  $Q$  and the lexicographically shortest path  $P$  is the directed path from  $\partial^- a_0$  to  $\partial^+ a_0$  in  $\mathcal{N}_\varphi$  which has the minimum number of arcs among directed paths from  $\partial^- a_0$  to  $\partial^+ a_0$  in  $\mathcal{N}_\varphi$  and whose vertex sequence  $(\partial^- a_0, v_1, \dots, v_p, \partial^+ a_0)$ , say, gives the lexicographically minimum sequence  $(\pi(v_1), \dots, \pi(v_p))$  among directed paths from  $\partial^- a_0$  to  $\partial^+ a_0$  in  $\mathcal{N}_\varphi$  having the minimum number of arcs.

The algorithm terminates after repeating (\*)  $O(|V|^3)$  times. The analysis is by the same technique as shown in Section 3.1.c.

**Theorem 4.9:** For the maximum submodular flow problem  $P_{MS}$  described by (4.126),

$$\begin{aligned} &\max\{\varphi(a_0) \mid \varphi \text{ is a feasible flow in } \mathcal{N}_S\} \\ &= \min\{\bar{c}(a_0), \min\{\bar{c}(\Delta^- X) - \underline{c}(\Delta^+ X - \{a_0\}) \\ &\quad + f(X) \mid X \in \mathcal{D}, a_0 \in \Delta^+ X\}\}. \end{aligned} \quad (4.128)$$

Moreover, if  $\underline{c}$ ,  $\bar{c}$  and  $f$  are integer-valued and there exists a maximum submodular flow, then there exists an integral maximum submodular flow in  $\mathcal{N}_S$ .

(Proof) The maximum flow value is equal to the maximum value of  $\underline{c}(a_0)$  under the constraint that the network  $\mathcal{N}_S$  has a feasible flow, where  $\underline{c}(a_0)$  is regarded

#### 4.5. ALGORITHMS FOR NEOFLOWS

as a variable. Therefore, relation (4.128) is deduced from Theorem 4.1. The integrality property also follows from Theorem 4.1. Q.E.D.

Theorem 4.9 can also be shown algorithmically. When the algorithm terminates with a maximum flow  $\varphi^*$  such that  $\varphi^*(a_0) < \bar{c}(a_0)$ , let  $U$  be the set of vertices which are reachable by directed paths from  $\partial^- a_0$  in the then obtained  $\mathcal{N}_{\varphi^*}$ . It follows from the definition of  $\mathcal{N}_{\varphi^*}$  that

$$\varphi^*(a) = \bar{c}(a) \quad (a \in \Delta^+ U), \quad (4.129)$$

$$\varphi^*(a) = \underline{c}(a) \quad (a \in \Delta^- U - \{a_0\}), \quad (4.130)$$

$$\text{dep}(\partial\varphi^*, v) \subseteq V - U \quad (v \in V - U), \quad (4.131)$$

where  $\Delta^+$  and  $\Delta^-$  are with respect to  $G$ . From (4.131) we have

$$\partial\varphi^*(V - U) = f(V - U). \quad (4.132)$$

and from (4.129) and (4.130)

$$\partial\varphi^*(V - U) = \varphi(a_0) + \underline{c}(\Delta^- U - \{a_0\}) - \bar{c}(\Delta^+ U). \quad (4.133)$$

Combining (4.132) with (4.133), we get

$$\varphi^*(a_0) = \bar{c}(\Delta^+ U) - \underline{c}(\Delta^- U - \{a_0\}) + f(V - U). \quad (4.134)$$

On the other hand, we can easily see that for any feasible flow  $\varphi$  in  $\mathcal{N}_S$  and any  $X \in \mathcal{D}$  with  $a_0 \in \Delta^+ X$ ,

$$\varphi(a_0) \leq \bar{c}(\Delta^+ X) - \underline{c}(\Delta^- X - \{a_0\}) + f(X). \quad (4.135)$$

From (4.134) and (4.135) we have (4.128), where the case when  $\varphi^+(a_0) = \bar{c}(a_0)$  is taken into account.

Moreover, the integrality part of Theorem 4.9 follows from the fact that if Problem  $P_{MS}$  is feasible and  $\underline{c}$ ,  $\bar{c}$  and  $f$  are integer-valued, there exists an integral feasible flow and, starting from such an integral feasible flow, we get an integral maximum submodular flow by the algorithm if a maximum submodular flow exists.

When  $\varphi^*(a_0) < \bar{c}(a_0)$ , the above defined  $U$  is called a *minimum cut* in  $\mathcal{N}_S$  with reference arc  $a_0$ .

We can also consider the *minimum submodular flow problem* which is to minimize  $\varphi(a_0)$  subject to (4.126b) and (4.126c). Note that this problem is a maximum submodular flow problem when we consider the dual order  $\leq^*$  among  $\mathbb{R}$ . Hence an algorithm for the minimum submodular flow problem is given *mutatis mutandis*.

### III. NEOFLOWS

#### (c) Minimum-cost submodular flows

As a minimum-cost flow problem for neoflows, consider the submodular flow problem  $P_S$ , described by (4.1), in network  $\mathcal{N}_S = (G = (V, A), \underline{c}, \bar{c}, \gamma, (\mathcal{D}, f))$ .

$$P_S: \text{Minimize } \sum_{a \in A} \gamma(a) \varphi(a) \quad (4.1a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (4.1b)$$

$$\partial \varphi \in B(f), \quad (4.1c)$$

where  $(\mathcal{D}, f)$  is a submodular system on  $V$ , the vertex set of the underlying graph  $G = (V, A)$ . We suppose that Problem  $P_S$  has a feasible flow.

We shall show an algorithm which tries to find a feasible flow  $\varphi: A \rightarrow \mathbb{R}$  and a potential  $p: V \rightarrow \mathbb{R}$  satisfying the optimality condition of Theorem 4.2.

Suppose that we are given a feasible flow  $\varphi$  in  $\mathcal{N}_S$ . Choose a potential  $p: V \rightarrow \mathbb{R}$  such that  $\partial \varphi \in B(f)$  is a maximum-weight base of  $B(f)$  with respect to the weight function  $p$ , i.e.,  $p(u) \geq p(v)$  for each  $u, v \in V$  with  $u \in \text{dep}(\partial \varphi, v) - \{v\}$ . For example,  $p = 0$  (the zero function) satisfies this requirement. We define the auxiliary network  $\mathcal{N}_{\varphi, p} = (G_{\varphi} = (V, A), c_{\varphi}, \gamma_{\varphi, p})$  as follows.  $G_{\varphi}$  is the graph with vertex set  $V$  and arc set  $A_{\varphi}$  defined by (4.40)–(4.43) and  $c_{\varphi}: A_{\varphi} \rightarrow \mathbb{R}$  is defined by (4.44). We define  $\gamma_{\varphi, p}: A_{\varphi} \rightarrow \mathbb{R}$  by

$$\gamma_{\varphi, p}(a) = \gamma_{\varphi}(a) + p(\partial^+ a) - p(\partial^- a) \quad (a \in A_{\varphi}), \quad (4.136)$$

where  $\gamma_{\varphi}$  is given by (4.45).

Now, an algorithm based on Cunningham and Frank's [Cunningham + Frank85] is given as follows. Also, compare it with an out-of-kilter method given in [Fuji87].

#### An algorithm for finding an optimal submodular flow

**Input:** a feasible flow  $\varphi$  in  $\mathcal{N}_S$  and a potential  $p: V \rightarrow \mathbb{R}$  such that  $p(u) \geq p(v)$  ( $v \in V, u \in \text{dep}(\partial \varphi, v) - \{v\}$ ).

**Output:** an optimal submodular flow  $\varphi$  and an optimal potential  $p$ .

**Step 1:** While  $\gamma_{\varphi, p}(a) < 0$  for some  $a \in A_{\varphi}$ , choose an arc  $a^0 \in A_{\varphi}$  such that  $\gamma_{\varphi, p}(a^0) < 0$  and do (1-1) or (1-2) according as  $a^0 \in A$  or  $a^0 \notin A$ :

(1-1) If  $a^0 \in A$ , then, while  $\gamma_p(a^0) (= \gamma(a^0) + p(\partial^+ a^0) - p(\partial^- a^0)) \neq 0$ , do the following:

(\*) Starting with  $\varphi$ , find a maximum submodular flow  $\varphi^0$  in the modified network  $\mathcal{N}^0 = (G = (V, A), \underline{c}^0, \bar{c}^0, (\mathcal{D}^0, f^0))$  with the reference arc



#### 4.5. ALGORITHMS FOR NEOFLOWS

$a^0$ , where the lower and upper capacity functions  $\underline{c}^0, \bar{c}^0: A \rightarrow \mathbb{R}$  are defined by

$$\underline{c}^0(a) = \begin{cases} \varphi(a) & (a \in A, \gamma_p(a) < 0) \\ \underline{c}(a) & (a \in A, \gamma_p(a) \geq 0), \end{cases} \quad (4.137)$$

$$\bar{c}^0(a) = \begin{cases} \varphi(a) & (a \in A, \gamma_p(a) > 0) \\ \bar{c}(a) & (a \in A, \gamma_p(a) \leq 0), \end{cases} \quad (4.138)$$

where  $\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a)$  with  $\partial^+$  and  $\partial^-$  being defined with respect to  $G$ , and, letting  $p_1 > p_2 > \dots > p_t$  be the distinct values of  $p(v)$  ( $v \in V$ ) and defining

$$S_i = \{v \mid v \in V, p(v) \geq p_i\} \quad (i = 1, 2, \dots, t), \quad (4.139)$$

$$S_0 = \emptyset, \quad (4.140)$$

$(\mathcal{D}^0, f^0)$  is the submodular system given by the direct sum

$$\bigoplus_{i=1}^t (\mathcal{D}, f) \cdot S_i / S_{i-1} \quad (4.141)$$

of the set minors  $(\mathcal{D}, f) \cdot S_i / S_{i-1}$  ( $i = 1, 2, \dots, t$ ).

If the maximum flow value is equal to  $+\infty$ , then stop (Problem  $P_S$  is unbounded). Otherwise put  $\varphi \leftarrow \varphi^0$ .

If  $\varphi^0(a^0) < \bar{c}(a^0)$ , then let  $U$  be a minimum cut of  $\mathcal{N}^0$  with reference arc  $a^0$ , define

$$H_1 = \{a \mid a \in \Delta^- U, \gamma_p(a) < 0\}, \quad (4.142)$$

$$H_2 = \{a \mid a \in \Delta^+ U, \gamma_p(a) > 0\}, \quad (4.143)$$

( $\Delta^+, \Delta^-$  in (4.142) and (4.143) are with respect to  $G$ .)

$$H_3 = \{(u, v) \mid u \in U, v \in V - U, u \in \text{dep}(\partial\varphi, v) - \{v\}\}, \quad (4.144)$$

$$p^* = \min\{\min\{|\gamma_p(a)| \mid a \in H_1 \cup H_2\}, \min\{p(u) - p(v) \mid (u, v) \in H_3\}\}, \quad (4.145)$$

and put

$$p(u) \leftarrow p(u) - p^* \quad (u \in U). \quad (4.146)$$

- (1-2) If  $a^0 \notin A$ , let  $\bar{a}^0 \in A$  be the reorientation of  $a^0$  and, starting with  $\varphi$ , find a minimum submodular flow  $\varphi^0$  in the modified network  $\mathcal{N}^0$  as defined in Step (1-1). Carry out Step (1-1) *mutatis mutandis*.

(End)

For a feasible flow  $\varphi$  and a potential  $p$ , if an arc  $a \in A$  satisfies (i)  $\gamma_p(a) > 0$  and  $\underline{c}(a) < \varphi(a)$  or (ii)  $\gamma_p(a) < 0$  and  $\varphi(a) < \bar{c}(a)$ , then arc  $a$  is said to be *out*

### III. NEOFLOWS

of *kilter* and otherwise in *kilter* with respect to  $\varphi$  and  $p$ . Denote the set of all the out-of-kilter arcs by  $A^O(\varphi, p)$  and that of all the in-kilter arcs by  $A^I(\varphi, p)$ .

During the execution of the algorithm,

- (1) in-kilter arcs remain in kilter,
- (2) the value of  $|\gamma_p(a)|$  of each out-of-kilter arc  $a$  is monotone non-increasing,
- (3)  $|\gamma_p(a^0)|$  decreases every time the potential  $p$  is modified by (4.146) in Step (1-1) or (1-2), and
- (4)  $\varphi$  is a submodular flow in  $\mathcal{N}_S$  and  $\partial\varphi$  is a maximum-weight base of  $B(f)$  with respect to weight function  $p$ .

Therefore, when the cost function  $\gamma$  is integer-valued, the algorithm terminates after at most  $\sum_{a \in A} |\gamma(a)|$  maximum- and minimum-flow computations in Step (1-1) and Step (1-2) if we start with the initial potential  $p = 0$ . If the algorithm terminates with  $\varphi$  and  $p$  such that  $\gamma_{\varphi, p}(a) \geq 0$  for all  $a \in A_{\varphi}$ , then the obtained  $\varphi$  is an optimal submodular flow and  $p$  is an optimal potential due to Theorem 4.2.

Moreover, if the maximum flow value for the modified network  $\mathcal{N}^0$  in Step (1-1) is equal to  $+\infty$ , then there exists a directed cycle  $Q$ , containing  $a_0$ , in the auxiliary network  $\mathcal{N}_{\varphi}^0 = (G_{\varphi}^0 = (V, A_{\varphi}), c_{\varphi}^0)$  associated with the current  $\varphi$  such that  $c_{\varphi}^0(a) = +\infty$  for any arc  $a$  in  $Q$ . By the definition of  $\mathcal{N}^0$ ,  $Q$  is also a directed cycle in the auxiliary network  $\mathcal{N}_{\varphi} = (G_{\varphi}, c_{\varphi}, \gamma_{\varphi})$  for the original submodular flow problem  $P_S$  and the length of  $Q$  relative to the length function  $\gamma_{\varphi}$  is negative. Consequently, Problem  $P_S$  is unbounded, i.e., the value of the objective function of  $P_S$  is made arbitrarily small. Also, the same argument is valid *mutatis mutandis* for Step (1-2).

When the cost function  $\gamma$  is integer-valued and an oracle for exchange capacities is assumed, the above algorithm requires time polynomial in  $|V|$  and  $\max\{|\gamma(a)| \mid a \in A\}$ , i.e., it is a pseudopolynomial algorithm. If we apply the cost scaling technique for the ordinary min-cost flows of Röck [Röck80], we obtain a polynomial algorithm for the submodular flow problem, provided that an oracle for exchange capacities is available (see [Cunningham + Frank85]).

Moreover, if we apply the cost-rounding and tree-projection technique of the author [Fuji86] (which is an improved version of Tardos's algorithm [Tardos85]) for the ordinary minimum-cost flows, we can get a strongly polynomial algorithm for the submodular flow problem (see [Fuji + Röck + Zimmermann89]), where an oracle for exchange capacities is also assumed. The first strongly polynomial algorithm for the submodular flow problem was given by Frank and Tardos [Frank + Tardos85] by the use of the simultaneous approximation algorithm of A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász [Lenstra + Lenstra + Lovász82].

Given an optimal submodular flow  $\varphi$  and an optimal potential  $p$ , we can find an optimal dual solution of  $P_S$  by the procedure given in Section 4.4 when the base polyhedron is expressed in terms of a submodular function on a

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distributive lattice, or an intersecting- or crossing-submodular function on an intersecting or crossing family.

### 4.6. Matroid Optimization

We show some specializations of the results obtained in the previous sections to matroids, which is a retrospective view of matroid optimization.

#### (a) Maximum independent matchings

Let  $G = (V^+, V^-; A)$  be a bipartite graph with the left and right end-vertex sets  $V^+$  and  $V^-$  and the arc set  $A$ . Also let  $M^+ = (V^+, \mathcal{I}^+)$  and  $M^- = (V^-, \mathcal{I}^-)$ , respectively, be matroids on  $V^+$  and  $V^-$  with families  $\mathcal{I}^+ \subseteq 2^{V^+}$  and  $\mathcal{I}^- \subseteq 2^{V^-}$  of independent sets. Denote  $\mathcal{N} = (G = (V^+, V^-; A), M^+, M^-)$ .

An *independent matching*  $M \subseteq A$  in  $\mathcal{N}$  is a matching in  $G$  such that

$$\partial^+ M \in \mathcal{I}^+, \quad \partial^- M \in \mathcal{I}^-, \quad (4.150)$$

where  $\partial^+ M$  ( $\partial^- M$ ) is the set of end-vertices in  $V^+$  ( $V^-$ ) of arcs in  $M$ . (We assume that for each arc  $a \in A$  we have  $\partial^+ a \in V^+$  and  $\partial^- a \in V^-$ .) The *maximum independent matching problem* is to find a maximum independent matching (i.e., an independent matching of maximum cardinality) in  $\mathcal{N}$ .

The maximum independent matching problem can naturally be reduced to a maximum independent flow problem as follows. Consider a network  $\tilde{\mathcal{N}} = (G = (V^+, V^-; A), \underline{c}, \bar{c}, (2^{V^+}, \rho^+), (2^{V^-}, \rho^-))$ , where  $V^+$  ( $V^-$ ) is the set of entrances (exits),  $(2^{V^+}, \rho^+)$  ( $(2^{V^-}, \rho^-)$ ) is the submodular system on  $V^+$  ( $V^-$ ) with  $\rho^+$  ( $\rho^-$ ) being the rank function of matroid  $M^+$  ( $M^-$ ), and

$$\underline{c}(a) = 0 \quad (a \in A), \quad (4.151)$$

$$\bar{c}(a) = +\infty \quad (a \in A). \quad (4.152)$$

We see that any integral independent flow  $\varphi$  in  $\tilde{\mathcal{N}}$  is  $\{0, 1\}$ -valued and that  $\{a \mid a \in A, \varphi(a) = 1\}$  is an independent matching in  $\mathcal{N}$ . From Theorem 4.7 an integral maximum independent flow in  $\tilde{\mathcal{N}}$  gives a maximum independent matching in  $\mathcal{N}$  and we have the following min-max theorem.

**Theorem 4.10** (The maximum independent matching-minimum covering rank theorem) [Edm70], [Rado42], [Welsh70]: For network  $\mathcal{N} = (G = (V^+, V^-; A), M^+, M^-)$  we have

$$\begin{aligned} & \max\{|M| \mid M \text{ is an independent matching in } \mathcal{N}\} \\ &= \min\{\rho^+(U^+) + \rho^-(U^-) \mid (U^+, U^-) \text{ is a cover of } G\}. \end{aligned} \quad (4.153)$$

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(Proof) The present theorem immediately follows from Theorem 4.7, where a cut  $U$  of finite capacity of  $\tilde{N}$  corresponds to a cover  $(U^+, U^-)$  of  $G$  such that  $U^+ = V^+ - U$  and  $U^- = V^- \cap U$ ; this gives a one-to-one correspondence between the set of cuts of finite capacities of  $\tilde{N}$  and that of covers of  $G$  and the capacity of a cut  $U$  of finite capacity of  $\tilde{N}$  is equal to  $\rho^+(U^+) + \rho^-(U^-)$ , the rank of the cover  $(U^+, U^-)$ . Q.E.D.

#### The transformation of matroids by bipartite graphs

Let  $G = (V^+, V^-; A)$  be a bipartite graph and  $M^+ = (V^+, \mathcal{I}^+)$  be a matroid with a family  $\mathcal{I}^+$  of independent sets. Define

$$\mathcal{I}^- = \{\partial^- M \mid M: \text{a matching in } G, \partial^+ M \in \mathcal{I}^+\}, \quad (4.154)$$

where  $\partial^- M = \{\partial^- a \mid a \in M\}$ . We can easily see that  $\mathcal{I}^-$  is a family of independent sets of a matroid. From Theorem 4.10 the rank function  $\rho^-$  of the matroid  $M^- = (V^-, \mathcal{I}^-)$  is given by

$$\rho^-(U^-) = \min\{\rho^+(X^+) + |Y^-| \mid (X^+, Y^- \cup (V^- - U^-)) \text{ is a cover of } G\} \quad (4.155)$$

for each  $U^- \subseteq V^-$ . We call  $M^-$  the matroid *induced from  $M^+$  by the bipartite graph  $G$* .

#### The matroid intersection problem

When  $V^-$  is a copy of  $V^+$  and a bipartite graph  $G = (V^+, V^-; A)$  represents the natural bijection between  $V^+$  and  $V^-$ , the maximum independent matching problem for the network  $\mathcal{N} = (G = (V^+, V^-; A), M^+, M^-)$  becomes the problem of finding a maximum common independent set of the two matroid  $M^+$  and  $M^-$ , where  $V^+$  is identified with  $V^-$ . This problem is called the *matroid intersection problem*. From Theorem 4.10 we have

**Theorem 4.11** (The matroid intersection theorem) [Edm70]: For two matroids  $M^+ = (E, \mathcal{I}^+)$  and  $M^- = (E, \mathcal{I}^-)$  with  $\mathcal{I}^+$  and  $\mathcal{I}^-$  being families of independent sets,

$$\begin{aligned} & \max\{|I| \mid I \in \mathcal{I}^+ \cap \mathcal{I}^-\} \\ &= \min\{\rho^+(X) + \rho^-(E - X) \mid X \subseteq E\}, \end{aligned} \quad (4.156)$$

where  $\rho^+$  ( $\rho^-$ ) is the rank function of  $M^+$  ( $M^-$ ).

Theorem 4.11 also follows from Theorem 3.9.

The matroid intersection problem is a special case of the maximum independent matching problem in a natural way. Conversely, we can show that

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the maximum independent matching problem is reduced to a matroid intersection problem. Consider the maximum independent matching problem for a network  $\mathcal{N} = (G = (V^+, V^-; A), M^+, M^-)$ . From  $G$  define a bipartite graph  $\hat{G} = (\hat{W}^+, \hat{W}^-; \hat{A})$ , where

$$\hat{W}^+ = \{w_a^+ \mid a \in A\}, \quad \hat{W}^- = \{w_a^- \mid a \in A\}, \quad (4.157)$$

$$\hat{A} = \{(w_a^+, w_a^-) \mid a \in A\} \quad (4.158)$$

(see Fig. 4.5).

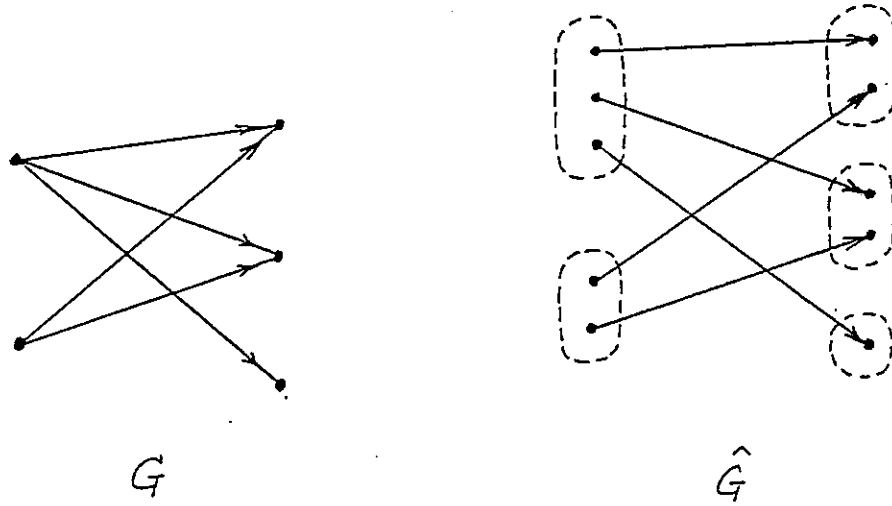


Figure 4.5.

Moreover, define

$$\begin{aligned} \hat{I}^+ = \{I^+ \mid I^+ \subseteq \hat{W}^+, \forall v \in V^+: |\{a \mid a \in \delta^+ v, w_a^+ \in I^+\}| \leq 1, \\ \{\partial^+ a \mid a \in A, w_a^+ \in I^+\} \in \mathcal{I}^+\}, \end{aligned} \quad (4.159)$$

$$\begin{aligned} \hat{I}^- = \{I^- \mid I^- \subseteq \hat{W}^-, \forall v \in V^-: |\{a \mid a \in \delta^- v, w_a^- \in I^-\}| \leq 1, \\ \{\partial^- a \mid a \in A, w_a^- \in I^-\} \in \mathcal{I}^-\}, \end{aligned} \quad (4.160)$$

where  $\partial^+$ ,  $\partial^-$ ,  $\delta^+$  and  $\delta^-$  are with respect to  $G$ . We can easily show that  $\hat{M}^+ = (\hat{W}^+, \hat{I}^+)$  and  $\hat{M}^- = (\hat{W}^-, \hat{I}^-)$  are matroids with families  $\hat{I}^+$  and  $\hat{I}^-$  of independent sets. The matroid  $\hat{M}^+$  ( $\hat{M}^-$ ) is regarded as the one induced

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from  $M^+$  ( $M^-$ ) by a bipartite graph, or as a composition of  $M^+$  ( $M^-$ ) and the direct sum of rank-one uniform matroids on  $\delta^+v$  ( $v \in V^+$ ) ( $\delta^-v$  ( $v \in V^-$ )). The maximum independent matching problem for network  $\mathcal{N} = (G = (V^+, V^-; A), M^+, M^-)$  is thus reduced to a maximum independent matching problem for the new network  $\hat{\mathcal{N}} = (\hat{G} = (\hat{W}^+, \hat{W}^-; \hat{A}), \hat{M}^+, \hat{M}^-)$ , which is a matroid intersection problem.

#### The matroid union

Let  $M_i = (E_i, \mathcal{I}_i)$  ( $i = 1, 2$ ) be two matroids. Define

$$\mathcal{I}_{1 \vee 2} = \{I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}. \quad (4.161)$$

Then  $M_{1 \vee 2} = (E_1 \cup E_2, \mathcal{I}_{1 \vee 2})$  is a matroid, which is the one induced from the direct sum  $M_1 \oplus M_2$  of the two by the bipartite graph  $G = (V^+, V^-; A)$ , where  $V^+ = E_1 \oplus E_2$ ,  $V^- = E_1 \cup E_2$  and the arc set  $A$  consists of the natural bijections between  $E_1 \subseteq V^+$  and  $E_1 \subseteq V^-$  and between  $E_2 \subseteq V^+$  and  $E_2 \subseteq V^-$  (see Fig. 4.6). Therefore, from (4.155) we have

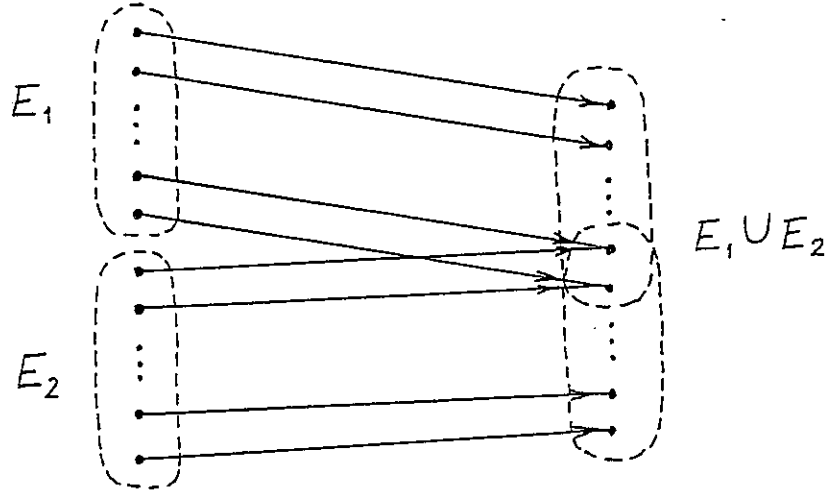


Figure 4.6

**Theorem 4.12:** The rank function  $\rho_{1 \vee 2}$  of  $M_{1 \vee 2}$  is given by

$$\rho_{1 \vee 2}(X) = \min\{\rho_1(Y \cap E_1) + \rho_2(Y \cap E_2) + |X - Y| \mid Y \subseteq X\} \quad (4.162)$$

for each  $X \subseteq E_1 \cup E_2$ .

The matroid  $M_{1 \vee 2}$  is called the *union* (or *sum*) of  $M_i$  ( $i = 1, 2$ ).

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Note that if  $E_1 = E_2$ , we have  $\rho_{1 \vee 2} = (\rho_1 + \rho_2)^1$ , which is the rank function of the reduction of the sum  $(2^E, \rho_1 + \rho_2)$  of submodular systems  $(2^E, \rho_i)$  ( $i = 1, 2$ ) by vector  $1 = (1(e) = 1: e \in E)$  (see (2.6)).

A base of the union  $M_{1 \vee 2}$  of  $M_1$  and  $M_2$  is given in the form of the union  $B_1 \cup B_2$  of some bases  $B_i$  of matroids  $M_i$  ( $i = 1, 2$ ). When  $E_1 = E_2$ , for bases  $B_i$  ( $i = 1, 2$ ) of matroids  $M_i$ ,  $B_1 \cup B_2$  is a base of  $M_{1 \vee 2}$  if and only if  $B_1 \cup (E_2 - B_2)$  is a maximum common independent set of  $M_1$  and the dual  $M_2^*$  of  $M_2$ . In this sense the problem of finding a base of the union of two matroids is equivalent to the matroid intersection problem and hence to the maximum independent matching problem.

When we are given matroids  $M_i = (E_i, \mathcal{I}_i)$  ( $i = 1, 2, \dots, k$ ) ( $k \geq 2$ ), we can define the union of  $M_i$  ( $i = 1, 2, \dots, k$ ) in the same way as in the case of  $k = 2$ . That is, an independent set of the union of  $M_i$  ( $i = 1, 2, \dots, k$ ) is the union of independent sets  $I_i \in \mathcal{I}_i$  ( $i = 1, 2, \dots, k$ ).

**Theorem 4.13:** There exist disjoint bases of  $M_i = (E, \mathcal{I}_i)$  ( $i = 1, 2, \dots, k$ ), one from each  $M_i$ , if and only if

$$\rho_1(E) + \dots + \rho_k(E) = \min\{\rho_1(X) + \dots + \rho_k(X) + |E - X| \mid X \subseteq E\}. \quad (4.163)$$

(Proof) The present theorem follows from the fact that the rank of the union of  $M_i$  ( $i = 1, 2, \dots, k$ ) is equal to the right-hand side of (4.163). Q.E.D.

##### The matroid partitioning

For matroids  $M_i = (E, \mathcal{I}_i)$  with rank functions  $\rho_i$  ( $i = 1, 2, \dots, k$ ), the *matroid partitioning problem* is to find  $k$  disjoint subsets  $I_i$  of  $E$  such that  $I_1 \cup I_2 \cup \dots \cup I_k = E$  and  $I_i \in \mathcal{I}_i$  ( $i = 1, 2, \dots, k$ ). We can easily see that such  $k$  subsets  $I_i$  ( $i = 1, 2, \dots, k$ ) exist if and only if the union of  $M_i$  ( $i = 1, 2, \dots, k$ ) is the free matroid on  $E$ , i.e., from Theorem 4.12

$$|E| = \min\{\rho_1(X) + \dots + \rho_k(X) + |E - X| \mid X \subseteq E\}. \quad (4.164)$$

In other words,

**Theorem 4.14 [Edm70]:** There exists a base  $B_i$  of  $M_i$  for each  $i = 1, 2, \dots, k$  such that the union of  $B_i$  ( $i = 1, 2, \dots, k$ ) is  $E$  if and only if

$$\rho_1(X) + \dots + \rho_k(X) \geq |X| \quad (X \subseteq E). \quad (4.165)$$

When matroids  $M_i = (E, \mathcal{I}_i)$  ( $i = 1, 2, \dots, k$ ) are the same matroid  $M = (E, \mathcal{I})$  with the rank function  $\rho$ ,  $E$  is partitioned into independent sets  $I_i \in \mathcal{I}$  ( $i = 1, 2, \dots, k$ ) if and only if

$$k\rho(X) \geq |X| \quad (X \subseteq E). \quad (4.166)$$

### III. NEOFLOWS

From (4.166) we have

**Corollary 4.15** [Edm65] (see [Tutte61], [Nash-Williams61] for graphs): The minimum number  $k$  for which  $E$  is partitioned into  $k$  disjoint independent sets of  $M = (E, \mathcal{I})$  is equal to

$$\lceil \max\{|X|/\rho(X) \mid \emptyset \neq X \subseteq E\} \rceil, \quad (4.167)$$

where we assume  $M$  does not have any selfloop, i.e., we assume  $\rho(\{e\}) > 0$  ( $e \in E$ ).

Note that (4.166) is equivalent to that a uniform vector  $(1/k)1$  is an independent vector of the matroidal polymatroid  $P = (E, \rho)$  corresponding to  $M$ .

If the matroidal polymatroid  $P = (E, \rho)$  has a uniform base  $(l/k)1$  for positive integers  $k, l$  such that  $l|E| = k\rho(E)$ , then there exist  $k$  bases  $B_i$  ( $i = 1, 2, \dots, k$ ) of  $M$  such that each  $e \in E$  is uniformly covered by  $B_i$  ( $i = 1, 2, \dots, k$ )  $l$  times, i.e.,

$$|\{i \mid i \in \{1, 2, \dots, k\}, e \in B_i\}| = l, \quad (4.168)$$

due to the fact that  $B(\rho) + \dots + B(\rho)$  ( $k$  times)  $= B(k\rho)$  with  $\mathbb{Z}$  as the underlying totally ordered additive group (see Section 2.1.c). Such a family of bases  $B_i$  ( $i = 1, 2, \dots, k$ ) is called a *complete family of bases* of  $M$  and a matroid having a complete family of bases is called *irreducible* [Tomi75], [Tomi76]. The concepts of irreducible matroid was introduced by N. Tomizawa for the analysis of *principal partition* of a matroid (see Sections 6.2.b.1 and 8.2). The same concept was also independently introduced by H. Narayanan [Narayanan74] and was called a *molecule* (also see [Narayanan + Vartak81]).

An algorithm for the maximum independent matching problem by using auxiliary graphs is given by Tomizawa and Iri [Tomi + Iri74]. An algorithm for the matroid intersection problem is given by Edmonds [Edm79]. For other related algorithms see [Edm + Fulkerson65] for matroid partitionings and [Bruno + Weinberg71] for matroid unions.

#### (b) Optimal independent assignments

Consider a bipartite graph  $G = (V^+, V^-; A)$ , matroids  $M^+ = (V^+, \mathcal{I}^+)$  and  $M^- = (V^-, \mathcal{I}^-)$  and a weight function  $w: A \rightarrow \mathbb{R}$ . Denote such a network by  $\mathcal{N} = (G = (V^+, V^-; A), M^+, M^-, w)$ . For a positive integer  $k$ ,



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a  $k$ -independent matching  $M$  in  $\mathcal{N}$  is an independent matching of cardinality  $k$  in  $\mathcal{N}^0 = (G = (V^+, V^-; A), M^+, M^-)$ . An *optimal  $k$ -independent assignment* in  $\mathcal{N}$  is a  $k$ -independent matching  $M$  having the minimum weight  $w(M) = \sum_{e \in E} w(e)$  among all the  $k$ -independent matchings in  $\mathcal{N}$ . The *independent assignment problem* is to find an optimal  $k$ -independent assignment in  $\mathcal{N}$ . The independent assignment problem is a special case of a neoflow problem, especially of the independent flow problem in a natural way.

The independent assignment problem is equivalent to the weighted matroid intersection problem, which is to find a minimum-weight common independent set, of two matroids, having cardinality  $k$  for a given positive integer  $k$ .

For an independent matching  $M$  define the auxiliary network  $\mathcal{N}_M = (G_M = (V^*, A_M), w_M)$  as follows.  $G_M$  is the graph with vertex set  $V^* = V^+ \cup V^- \cup \{s^+, s^-\}$  and arc set  $A_M = S_M^+ \cup A_M^+ \cup \hat{A}_M \cup \tilde{M} \cup A_M^- \cup S_M^-$ , where

$$S_M^+ = \{(s^+, v) \mid v \in V^+ - \text{cl}^+(\partial^+ M)\} \cup \{(v, s^+) \mid v \in \partial^+ M\}, \quad (4.169)$$

$$A_M^+ = \{(u, v) \mid v \in \text{cl}^+(\partial^+ M) - \partial^+ M, u \in C^+(\partial^+ M|v) - \{v\}\}, \quad (4.170)$$

$$\hat{A}_M = A - M, \quad (4.171)$$

$$\tilde{M} = \{\bar{a} \mid a \in M\} \quad (\bar{a}: \text{a reorientation of } a), \quad (4.172)$$

$$A_M^- = \{(u, v) \mid v \in \text{cl}^-(\partial^- M) - \partial^- M, u \in C^-(\partial^- M|v) - \{v\}\}, \quad (4.173)$$

$$S_M^- = \{(s^-, v) \mid v \in V^- - \text{cl}^-(\partial^- M)\} \cup \{(v, s^-) \mid v \in \partial^- M\}. \quad (4.174)$$

Here,  $\text{cl}^+$  and  $\text{cl}^-$  are, respectively, the closure functions of  $M^+$  and  $M^-$ ,  $C^+(\partial^+ M|v)$  for  $v \in \text{cl}^+(\partial^+ M) - \partial^+ M$  is the fundamental circuit associated with  $\partial^+ M \in \mathcal{I}^+$  and  $v$  which is the unique circuit of  $M^+$  contained in  $\partial^+ M \cup \{v\}$ , and  $C^-(\partial^- M|v)$  is similarly defined for  $M^-$ . Also,  $w_M: A_M \rightarrow \mathbb{R}$  is the length function defined by

$$w_M(a) = \begin{cases} w(a) & (a \in \hat{A}_M) \\ -w(\bar{a}) & (a \in \tilde{M}, \bar{a} \in M \text{ is a reorientation of } a) \\ 0 & (a \in S_M^+ \cup A_M^+ \cup A_M^- \cup S_M^-). \end{cases} \quad (4.175)$$

**A primal-dual algorithm for the independent assignment problem**  
[Iri+Tomi76]

**Step 1:** Put  $M \leftarrow \emptyset$ .

**Step 2:** While  $|M| < k$ , do the following.

(2-1) Find a shortest path  $P$ , relative to the length function  $w_M$ , from  $s^+$  to  $s^-$  in the auxiliary network  $\mathcal{N}_M$  having the minimum number of arcs.

If there exists no directed path from  $s^+$  to  $s^-$  in  $\mathcal{N}_M$ , then stop (there is no  $k$ -independent matching).

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(2-2) Put  $M \leftarrow M \Delta A(P)$  ( $\Delta$ : the symmetric difference), where

$$\begin{aligned} A(P) = \{a \mid a \in A, a \text{ lies on } P\} \\ \cup \{a \mid a \in M, a \text{ reorientation of } a \text{ lies on } P\}. \end{aligned} \quad (4.176)$$

(End)

If we define in Step (2-1) a potential  $p: V^* \rightarrow \mathbb{R}$  in such a way that for each  $v \in V^+$   $p(v)$  is equal to the length of a shortest path from  $s^+$  to  $v$  in  $\mathcal{N}_M$ , then using the potential  $p$  we can replace  $w_M$  in the next execution of Step (2-1) by  $w_{M,p}$  defined by

$$w_{M,p}(a) = w_M(a) + p(\partial^+ a) - p(\partial^- a). \quad (4.177)$$

Here, we disregard those vertices which are not reachable from  $s^+$  in  $\mathcal{N}_M$ , since vertices which are not reachable from  $s^+$  will not become reachable from  $s^+$  in  $\mathcal{N}_M$  for new  $M$ . We can show that thus defined  $w_{M,p}$  is nonnegative for those arcs reachable from  $s^+$ , so that we can reduce the complexity of finding a shortest path in  $\mathcal{N}_M$  (see [Iri + Tomi76]). This is an adaptation of the technique for the classical min-cost flows developed by Tomizawa [Tomi71] and also independently by Edmonds and Karp [Edm + Karp72].

It should be noted that arcs entering  $s^+$  or leaving  $s^-$  in  $\mathcal{N}_M$  play no rôle in the primal-dual algorithm. These arcs are for the primal algorithm given as follows.

**A primal algorithm for the independent assignment problem [Fuji77a]**

**Step 1:** Find a  $k$ -independent matching  $M$  in  $\mathcal{N}$ .

**Step 2:** While there exists a negative cycle, relative to the length function  $w_M$ , in the auxiliary network  $\mathcal{N}_M$ , do the following.

(2-1) Find a negative cycle  $Q$  in  $\mathcal{N}_M$  having the minimum number of arcs.

(2-2) Put  $M \leftarrow M \Delta A(Q)$ , where  $A(Q)$  is defined by (2.176) with  $P$  replaced by  $Q$ .

(End)

For other related algorithms, see [Edm79], [Lawler75], [Fuji77b], [Iri78], [Frank81a].

The problem of finding a minimum weight directed spanning tree in a graph is an example of the independent assignment problem or the weighted matroid intersection problem. Consider a graph  $G = (V, A)$  and a weight function  $w: A \rightarrow \mathbb{R}$ . Let  $M_1$  be the 1-elongation of the graphic matroid  $M(G)$  represented by  $G$  (see Section 2.1.d) and  $M_2$  be the direct sum of the rank-one

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uniform matroids  $M^-(v)$  on  $\delta^-v$  ( $v \in V$ ), i.e.,  $M_2 = \oplus_{v \in V} M^-(v)$ . We see that  $B \subseteq A$  with  $|B| = |V|$  is a common independent set of  $M_1$  and  $M_2$  if and only if  $B$  forms a directed spanning tree of  $G$ . Hence the problem of finding a minimum weight directed spanning tree is a weighted matroid intersection problem. If we want to find a minimum weight spanning tree with a fixed root  $v_0 \in V$ , then replace  $M^-(v_0)$  by the trivial matroid (rank-zero matroid) on  $\delta^-v_0$ . For other applications, see [Iri83], [Iri + Fuji81] and [Recski].

Finally, it should be noted that the problem of finding a common independent set of maximum cardinality for *three* matroids is NP-hard. For, consider a graph  $G = (V, A)$ , and let  $M_1$  and  $M_2$  be the matroids defined as above, where  $M^-(v)$  ( $v \in V$ ) are all rank-one uniform matroids. Also, let  $M_3$  be the direct sum of rank-one uniform matroids  $M^+(v)$  on  $\delta^+v$  ( $v \in V$ ). We can easily see that the maximum cardinality of common independent sets of  $M_i$  ( $i = 1, 2, 3$ ) is equal to  $|V|$  if and only if there exists a directed Hamiltonian cycle in  $G$ , and that a common independent set of  $M_i$  ( $i = 1, 2, 3$ ) of cardinality  $|V|$  forms a directed Hamiltonian cycle in  $G$ .



## Chapter IV. Submodular Analysis

Submodular (or supermodular) functions on distributive lattices share similar structures with convex (or concave) functions on convex sets. In this chapter we develop a theory of submodular and supermodular functions from the point of view of the duality in convex analysis.

### 5. Submodular Functions and Convexity

We define the convex (concave) conjugate function of a submodular (supermodular) function and show a Fenchel-type duality theorem for submodular and supermodular functions. We also define the subgradients and subdifferentials of a submodular function and examine the relationship among these concepts and the polyhedra such as the submodular and supermodular polyhedra and the base polyhedron associated with the submodular function. The reason for the analogy between a submodular function and a convex function is nicely explained by the Lovász extension of a submodular function.

#### 5.1. Conjugate functions and a Fenchel-type min-max theorem for submodular and supermodular functions

##### (a) Conjugate functions

Let  $f: \mathcal{D} \rightarrow \mathbb{R}$  be a submodular function and  $g: \mathcal{D} \rightarrow \mathbb{R}$  be a supermodular function on a distributive lattice  $\mathcal{D} \subseteq 2^E$ . Here, we do not necessarily assume that  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = g(\emptyset) = 0$ .

Define the function  $f^*: \mathbb{R}^E \rightarrow \mathbb{R}$  by

$$f^*(x) = \max\{x(X) - f(X) \mid X \in \mathcal{D}\} \quad (x \in \mathbb{R}^E) \quad (5.1)$$

and also, in a dual form, the function  $g^*: \mathbb{R}^E \rightarrow \mathbb{R}$  by

$$g^*(x) = \min\{x(X) - g(X) \mid X \in \mathcal{D}\} \quad (x \in \mathbb{R}^E). \quad (5.2)$$

By the definition  $f^*$  is a convex function and  $g^*$  is a concave function. We call  $f^*$  the *convex conjugate function* of the submodular function  $f$  and  $g^*$  a *concave conjugate function* of the supermodular function  $g$ .

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It may be noted that the terms  $x(X)$  appearing in (5.1) and (5.2) can be regarded as the inner product of  $x$  and  $\chi_X$ , the characteristic vector of  $X$ . Notice the analogy between the definition (5.1) and that of a convex conjugate function of an ordinary convex function (see [Rockafellar70], [Stoer + Witzgall70]).

**Theorem 5.1:** For a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  and a supermodular function  $g: \mathcal{D} \rightarrow \mathbb{R}$  we have for any  $X \in \mathcal{D}$

$$f(X) = \max\{x(X) - f^*(x) \mid x \in \mathbb{R}^E\}, \quad (5.3)$$

$$g(X) = \min\{x(X) - g^*(x) \mid x \in \mathbb{R}^E\}. \quad (5.4)$$

Moreover, for any  $X \in 2^E - \mathcal{D}$   $x(X) - f^*(x)$  (or  $x(X) - g^*(x)$ ) as a function of  $x \in \mathbb{R}^E$  can be made arbitrarily large (or small).

(Proof) Without loss of generality we assume that  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = g(\emptyset) = 0$ . From (5.1),

$$f(X) \geq x(X) - f^*(x) \quad (5.5)$$

for any  $X \in \mathcal{D}$  and  $x \in \mathbb{R}^E$ . We show that for any  $X \in \mathcal{D}$  there exists a vector  $x \in \mathbb{R}^E$  such that (5.5) holds with equality, from which (5.3) follows. From Lemma 2.2, for any  $X \in \mathcal{D}$  there exists a subbase  $\hat{x} \in P(f)$  such that  $\hat{x}(X) = f(X)$ . Since  $\hat{x} \in P(f)$ , we have from (5.1)

$$f^*(\hat{x}) = \hat{x}(X) - f(X) (= 0). \quad (5.6)$$

This implies (5.3).

Relation (5.4) is the same as (5.3) by considering the dual order on  $\mathbb{R}$ .

Moreover, it follows from Lemma 2.2 that for any  $X \in 2^E - \mathcal{D}$  we can make  $x(X)$  arbitrarily large (or small) subject to  $x \in B(f)$  (or  $x \in B(g)$ ). Since  $f^*(x) = 0$  for  $x \in B(f)$  and  $g^*(x) = 0$  for  $x \in B(g)$ , this completes the proof. Q.E.D.

We see from Theorem 5.1 that the correspondence between a submodular (or supermodular) function  $f$  (or  $g$ ) and its convex (or concave) conjugate function  $f^*$  (or  $g^*$ ) is one to one.

The convex conjugate function  $f^*$  is closely related to the vector rank function  $r_f$  of the submodular system  $(\mathcal{D}, f)$  when  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = 0$ . Recall that for each  $x \in \mathbb{R}^E$

$$r_f(x) = \min\{f(X) + x(E - X) \mid X \in \mathcal{D}\} \quad (5.7)$$

(see (2.6) or (2.17)).

### 5.1. CONJUGATE FUNCTIONS AND A FENCHEL-TYPE THEOREM

**Lemma 5.2:** For a submodular system  $(\mathcal{D}, f)$  on  $E$  and a vector  $x \in \mathbb{R}^E$  we have

$$f^*(x) = x(E) - r_f(x). \quad (5.8)$$

(Proof) Immediate from (5.1) and (5.7).

Q.E.D.

Note that since  $r_f$  is submodular on the vector lattice  $\mathbb{R}^E$  (see [Welsh76] when  $\mathcal{D} = 2^E$ ), the convex function  $f^*$  is supermodular on  $\mathbb{R}^E$ , i.e., for any  $x, y \in \mathbb{R}^E$

$$f^*(x) + f^*(y) \leq f^*(x \vee y) + f^*(x \wedge y), \quad (5.9)$$

where  $(x \vee y)(e) = \max(x(e), y(e))$ ,  $(x \wedge y)(e) = \min(x(e), y(e))$  ( $e \in E$ ). (See [Topkis84] for submodular functions on general lattices.) Here, it may also be noted that vector lattice  $\mathbb{R}^E$  is a distributive lattice.

#### (b) A Fenchel-type min-max theorem

We show a Fenchel-type min-max theorem for submodular and supermodular functions, which relates the difference between a submodular function  $f$  and a supermodular function  $g$  to that between the concave conjugate function  $g^*$  and the convex conjugate function  $f^*$ . (For Fenchel's duality theorem for ordinary convex and concave functions, see [Rockafellar70] and [Stoer + Witzgall70].)

**Theorem 5.3** (A Fenchel-type min-max theorem): For a submodular function  $f: \mathcal{D}_1 \rightarrow \mathbb{R}$  and a supermodular function  $g: \mathcal{D}_2 \rightarrow \mathbb{R}$  on distributive lattices  $\mathcal{D}_1, \mathcal{D}_2 \subseteq 2^E$  with  $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$  we have

$$\begin{aligned} & \min\{f(X) - g(X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \max\{g^*(x) - f^*(x) \mid x \in \mathbb{R}^E\}. \end{aligned} \quad (5.10)$$

Moreover, if  $f$  and  $g$  are integer-valued, the maximum in the right-hand side of (5.10) can be attained by an integral vector  $x$ .

(Proof) Without loss of generality we assume that  $\emptyset, E \in \mathcal{D}_1 \cap \mathcal{D}_2$  and  $f(\emptyset) = g(\emptyset) = 0$ . Hence  $(\mathcal{D}_1, f)$  is a submodular system and  $(\mathcal{D}_2, g)$  is a supermodular system, both on  $E$ . The min-max relation (5.10) is equivalent to

$$\begin{aligned} & \min\{f(X) + g^\#(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \max\{g^*(x) + g(E) - f^*(x) \mid x \in \mathbb{R}^E\}, \end{aligned} \quad (5.11)$$

where recall that  $g^\#$  is the dual submodular function of  $g$ . From Lemma 5.2, (5.2) and (5.7),

$$f^*(x) = x(E) - r_f(x), \quad (5.12)$$

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$$\begin{aligned} g^*(x) + g(E) &= \min\{x(X) + g^\#(E - X) \mid X \in \mathcal{D}_2\} \\ &= r_{g^\#}(x). \end{aligned} \quad (5.13)$$

Substituting (5.12) and (5.13) into (5.11) yields

$$\begin{aligned} &\min\{f(X) + g^\#(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \max\{r_f(x) + r_{g^\#}(x) - x(E) \mid x \in \mathbb{R}^E\}, \end{aligned} \quad (5.14)$$

which is equivalent to (5.10).

For any  $x \in \mathbb{R}^E$  let  $y$  and  $z$  be bases of the reductions of  $(\mathcal{D}_1, f)$  and  $(\mathcal{D}_2, g^\#)$  by  $x$ , respectively, i.e.,

$$y \in P(f), \quad y \leq x, \quad y(E) = r_f(x), \quad (5.15)$$

$$z \in P(g^\#), \quad z \leq x, \quad z(E) = r_{g^\#}(x). \quad (5.16)$$

Also define

$$w = y \wedge z \quad (\equiv (\min\{y(e), z(e)\} : e \in E)). \quad (5.17)$$

Then,

$$w \in P(f) \cap P(g^\#). \quad (5.18)$$

Because of (5.15)–(5.18),

$$\begin{aligned} &r_f(x) + r_{g^\#}(x) - x(E) \\ &= y(E) + z(E) - x(E) \\ &\leq w(E) + w(E) - w(E) \\ &= r_f(w) + r_{g^\#}(w) - w(E) (= w(E)). \end{aligned} \quad (5.19)$$

It follows from (5.18) and (5.19) that (5.14) (or (5.11)) is also equivalent to

$$\begin{aligned} &\min\{f(X) + g^\#(E - X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &= \max\{x(E) \mid x \in P(f) \cap P(g^\#)\}. \end{aligned} \quad (5.20)$$

The min-max relation (5.20) is exactly the intersection theorem (Theorem 3.9), so that (5.10) holds.

The integrality part of the present theorem follows from the counterpart of the intersection theorem. Q.E.D.

Note that if  $x \in \mathbb{R}^E$  is a maximizer of the right-hand side of (5.10), then  $w$  given by (5.15)–(5.17) is a maximizer of the right-hand side of (5.20). If  $f$  and  $g$  are integer-valued functions and  $x$  is an integral vector, then such a



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vector  $w$  can also be integral. Conversely, any maximizer  $x$  of the right-hand side of (5.20) is a maximizer of the right-hand side of (5.10).

The above proof shows the equivalence between the Fenchel-type min-max theorem and the intersection theorem. Combining this with the results in Sections 3.1 and 3.2, we see that the three theorems, the intersection theorem (Theorem 3.9), the discrete separation theorem (Theorem 3.11) and the Fenchel-type min-max theorem (Theorem 5.3), are equivalent.

The Fenchel-type min-max theorem motivates further investigation of submodular and supermodular functions from the point of view of the duality theory in convex analysis [Rockafellar70], [Stoer + Witzgall70].

### 5.2. Subgradients of Submodular Functions

#### (a) Subgradients and subdifferentials

Consider a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  on a distributive lattice  $\mathcal{D} \subseteq 2^E$ . For a vector  $x \in \mathbb{R}^E$  and a set  $X \in \mathcal{D}$ , if

$$x(Y) - x(X) \leq f(Y) - f(X) \quad (5.21)$$

holds for each  $Y \in \mathcal{D}$ , then we call  $x$  a *subgradient* of  $f$  at  $X$ . We denote by  $\partial f(X)$  the set of all the subgradients of  $f$  at  $X$  and call  $\partial f(X)$  the *subdifferential* of  $f$  at  $X$ . Previously we employed symbol  $\partial$  as the boundary operator for flows in networks. Here we use the same symbol for subdifferentials, following the convention in convex analysis, because there seems to be no possibility of confusion.

Figure 5.1 shows a two-dimensional example of subdifferentials of  $f: \mathcal{D} \rightarrow \mathbb{R}$  with  $\mathcal{D} = 2^{\{1,2\}}$ .

In general,  $\mathbb{R}^E$  is divided into  $|\mathcal{D}|$  nonempty unbounded polyhedra  $\partial f(X)$  ( $X \in \mathcal{D}$ ) and for distinct  $X, Y \in \mathcal{D}$  the subdifferentials  $\partial f(X)$  and  $\partial f(Y)$  may have common faces but not common interior points.

It may be noted that a subgradient of any set function can be defined by (5.21). Some of the arguments in the following are valid for any set function.

**Lemma 5.4:** For a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  and a set  $X \in \mathcal{D}$ , we have  $x \in \partial f(X)$  if and only if  $x \in \mathbb{R}^E$  satisfies

$$x(Y) - x(X) \leq f(Y) - f(X) \quad (5.22)$$

for each  $Y \in [\emptyset, X]_{\mathcal{D}} \cup [X, E]_{\mathcal{D}}$ , where

$$[\emptyset, X]_{\mathcal{D}} = \{Y \mid Y \in \mathcal{D}, Y \subseteq X\}, \quad (5.23)$$

$$[X, E]_{\mathcal{D}} = \{Y \mid Y \in \mathcal{D}, X \subseteq Y \subseteq E\}. \quad (5.24)$$

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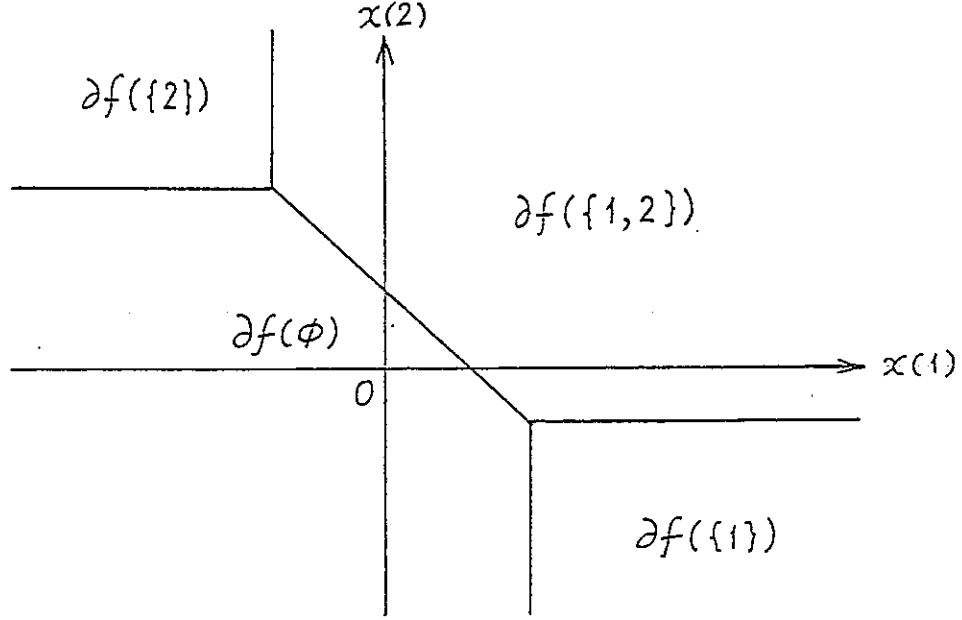


Figure 5.1.

(Proof) It is sufficient to show the "if" part alone. Suppose that (5.22) holds for each  $Y \in [\emptyset, X]_{\mathcal{D}} \cup [X, E]_{\mathcal{D}}$ . Then for any  $Z \in \mathcal{D}$ ,

$$x(X \cup Z) - x(X) \leq f(X \cup Z) - f(X), \quad (5.25)$$

$$x(X \cap Z) - x(X) \leq f(X \cap Z) - f(X). \quad (5.26)$$

From (5.25), (5.26) and the submodularity of  $f$ ,

$$\begin{aligned} x(Z) - x(X) &= x(X \cup Z) - x(X) + x(X \cap Z) - x(X) \\ &\leq f(X \cup Z) + f(X \cap Z) - 2f(X) \\ &\leq f(Z) - f(X). \end{aligned}$$

Q.E.D.

**Lemma 5.5:** For a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  and a set  $X \in \mathcal{D}$  we have

$$\partial f(X) = \partial f^X(X) \times \partial f_X(\emptyset), \quad (5.27)$$

where  $\partial f^X(X) \subseteq \mathbb{R}^X$ ,  $\partial f_X(\emptyset) \subseteq \mathbb{R}^{E-X}$ ,  $\times$  denotes the direct product, and  $f^X$  and  $f_X$  are, respectively, the submodular functions on  $[\emptyset, X]_{\mathcal{D}}$  and  $[X, E]_{\mathcal{D}}/X = \{Y - X \mid Y \in [X, E]_{\mathcal{D}}\}$  defined by

$$f^X(Y) = f(Y) \quad (Y \in [\emptyset, X]_{\mathcal{D}}), \quad (5.28)$$

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$$f_X(Y) = f(X \cup Y) - f(X) \quad (Y \in [X, E]_{\mathcal{D}/X}). \quad (5.29)$$

(Proof) From Lemma 5.4, we have  $x \in \partial f(X)$  if and only if

$$\begin{aligned} x(Y) - x(X) &\leq f(Y) - f(X) \\ &= f^X(Y) - f^X(X) \quad (Y \in [\emptyset, X]_{\mathcal{D}}), \end{aligned} \quad (5.30)$$

$$\begin{aligned} x(Z) - x(\emptyset) &= x(Z \cup X) - x(X) \\ &\leq f(Z \cup X) - f(X) \\ &= f_X(Z) - f_X(\emptyset) \quad (Z \in [X, E]_{\mathcal{D}/X}). \end{aligned} \quad (5.31)$$

(5.30) means  $x^X (= (x(e): e \in X)) \in \partial f^X(X)$  and (5.31) means  $x^{E-X} (= (x(e): e \in E - X)) \in \partial f_X(\emptyset)$ . We thus have (5.27). Q.E.D.

**Lemma 5.6** (cf. [Rockafellar70, Theorem 23.5]): For a (submodular) function  $f: \mathcal{D} \rightarrow \mathbb{R}$ , a vector  $x \in \mathbb{R}^E$  and a set  $X \in \mathcal{D}$ , the following three are equivalent:

$$(i) \quad x \in \partial f(X), \quad (5.32)$$

$$(ii) \quad \min\{f(Y) + x(E - Y) \mid Y \in \mathcal{D}\} = f(X) + x(E - X), \quad (5.33)$$

$$(iii) \quad f(X) + f^*(x) = x(X). \quad (5.34)$$

(Proof) We can easily see that each of the above three is equivalent to  $\min\{f(Y) - x(Y) \mid Y \in \mathcal{D}\} = f(X) - x(X)$ . Q.E.D.

Lemma 5.6 holds for any set function.

**Lemma 5.7:** For a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  with  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = 0$ ,

$$(a) \quad \partial f(\emptyset) = P(f), \quad (5.35)$$

$$(b) \quad \partial f(E) = P(f^\#), \quad (5.36)$$

$$(c) \quad \partial f(X) \cap B(f) \neq \emptyset \quad (X \in \mathcal{D}). \quad (5.37)$$

(Proof) (a) and (b) immediately follow from the definition of subdifferential. We prove (c). For any  $X \in \mathcal{D}$  there is a base  $x$  of the submodular system  $(\mathcal{D}, f)$  such that  $x(X) = f(X)$  (see Lemma 2.2). Since  $x \in B(f)$  and  $x(X) = f(X)$ , it easily follows from (5.21) that  $x \in \partial f(X)$ . Hence,  $\partial f(X) \cap B(f) \neq \emptyset$ . Q.E.D.

The following theorem is a submodular analogue of [Rockafellar70, Theorem 23.8] in convex analysis.

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**Theorem 5.8:** Let  $f_1: \mathcal{D}_1 \rightarrow \mathbb{R}$  and  $f_2: \mathcal{D}_2 \rightarrow \mathbb{R}$  be submodular functions, where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are distributive lattices with  $\emptyset, E \in \mathcal{D}_1 \cap \mathcal{D}_2$ . Then, for each  $X \in \mathcal{D}_1 \cap \mathcal{D}_2$  we have

$$\partial(f_1 + f_2)(X) = \partial f_1(X) + \partial f_2(X). \quad (5.38)$$

Also, for each  $\lambda > 0$  and  $X \in \mathcal{D}_1$ ,

$$\partial(\lambda f_1)(X) = \lambda \partial f_1(X). \quad (5.39)$$

Moreover, for  $\lambda = 0$  and  $X \in \mathcal{D}_1$ ,

$$\begin{aligned} \partial(0 \cdot f_1)(X) &= \{x \mid x \in \mathbb{R}^E, \forall Y \in \mathcal{D}_1: x(Y) - x(X) \leq 0\} \\ &= 0^+ \partial f_1(X), \end{aligned} \quad (5.40)$$

where  $0^+ \partial f_1(X)$  is the characteristic cone (or recession cone) of  $\partial f_1(X)$  (see [Rockafellar70, Section 8]).

(Proof) From Lemma 5.5, for each  $i = 1, 2$  and  $X \in \mathcal{D}_i$   $\partial f_i(X)$  is the direct product of  $\partial f_i^X(X)$  and  $\partial f_{iX}(\emptyset)$ . It follows from (2.27) (and its dual counterpart for supermodular functions) and from Lemma 5.7 that for each  $X \in \mathcal{D}_1 \cap \mathcal{D}_2$  we have

$$\begin{aligned} \partial(f_1 + f_2)(X) &= \partial(f_1 + f_2)^X(X) \times \partial(f_1 + f_2)_X(\emptyset) \\ &= \partial(f_1^X + f_2^X)(X) \times \partial(f_{1X} + f_{2X})(\emptyset) \\ &= (\partial f_1^X(X) + \partial f_2^X(X)) \times (\partial f_{1X}(\emptyset) + \partial f_{2X}(\emptyset)) \\ &= (\partial f_1^X(X) \times \partial f_{1X}(\emptyset)) + (\partial f_2^X(X) \times \partial f_{2X}(\emptyset)) \\ &= \partial f_1(X) + \partial f_2(X). \end{aligned} \quad (5.41)$$

Relations (5.39) and (5.40) are immediate from the definition of subdifferential. Q.E.D.

Note that we have used the intersection theorem through (2.27) to prove Theorem 5.8.

For a convex conjugate function  $f^*$  of a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  and a vector  $x \in \mathbb{R}^E$ , define the set  $\partial_2 f^*(x)$  of subsets of  $E$  as follows:

$$X \in \partial_2 f^*(x) \quad (5.42)$$

if and only if  $X \subseteq E$  and

$$y(X) - x(X) \leq f^*(y) - f^*(x) \quad (5.43)$$

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for each  $y \in \mathbb{R}^E$ . We call  $\partial_2 f^*(x)$  the *binary subdifferential* of  $f^*$  at  $x$ . Note that from (5.43) and Theorem 5.1 each  $X \in \partial_2 f^*(x)$  belongs to  $\mathcal{D}$ .

**Theorem 5.9** (cf. [Rockafellar70, Theorem 23.5]): Consider a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$ . For any vector  $x \in \mathbb{R}^E$  and any set  $X \in \mathcal{D}$  the following two are equivalent.

$$(i) \ x \in \partial f(X), \quad (5.44)$$

$$(ii) \ X \in \partial_2 f^*(x). \quad (5.45)$$

Moreover, the binary subdifferential  $\partial_2 f^*(x)$  is a sublattice of  $\mathcal{D}$ .

(Proof) From Theorem 5.1, Lemma 5.6 and the definition (5.43) of binary subdifferential of  $f^*$ , we see that both (i) and (ii) are equivalent to

$$f(X) + f^*(x) = x(X). \quad (5.46)$$

Moreover, (5.45) or equivalently (5.46) implies

$$x(X) - f(X) = \max\{x(Y) - f(Y) \mid Y \in \mathcal{D}\}. \quad (5.46)$$

Therefore,  $\partial_2 f^*(x)$  is the set of the maximizers of the supermodular function  $x - f$  on  $\mathcal{D}$  and hence is a sublattice of  $\mathcal{D}$ . Q.E.D.

For two submodular functions  $f_i: \mathcal{D}_i \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) define the *convolution*  $f_1^* \circ f_2^*: \mathbb{R}^E \rightarrow \mathbb{R}$  of the convex conjugate functions  $f_i^*$  ( $i = 1, 2$ ) by

$$(f_1^* \circ f_2^*)(x) = \min\{f_1^*(x_1) + f_2^*(x_2) \mid x_1 + x_2 = x\} \quad (5.47)$$

for each  $x \in \mathbb{R}^E$ . Q.E.D.

**Theorem 5.10** (cf. [Rockafellar70, Theorem 16.4]): For two submodular functions  $f_i: \mathcal{D}_i \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) with  $\emptyset \in \mathcal{D}_1 \cap \mathcal{D}_2$  we have

$$f_1^* \circ f_2^* = (f_1 + f_2)^*. \quad (5.48)$$

(Proof) For any  $x \in \mathbb{R}^E$ ,

$$\begin{aligned} (f_1^* \circ f_2^*)(x) &= \max\{x(X) - f_1(X) - f_2(X) \mid X \in \mathcal{D}_1 \cap \mathcal{D}_2\} \\ &\leq \min\{\max\{x_1(X) - f_1(X) + x_2(Y) - f_2(Y) \mid X \in \mathcal{D}_1, Y \in \mathcal{D}_2\} \\ &\quad \mid x_1 + x_2 = x\} \\ &= \min\{f_1^*(x_1) + f_2^*(x_2) \mid x_1 + x_2 = x\} \\ &= (f_1^* \circ f_2^*)(x). \end{aligned} \quad (5.49)$$

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On the other hand, let  $x$  be any vector in  $\mathbb{R}^E$ . There exists  $X \in \mathcal{D}_1 \cap \mathcal{D}_2$  such that  $x \in \partial(f_1 + f_2)(X)$ . Furthermore, from Theorem 5.8 there exist  $x_1 \in \partial f_1(X)$  and  $x_2 \in \partial f_2(X)$  such that  $x_1 + x_2 = x$ . Consequently, from Lemma 5.6

$$\begin{aligned} (f_1 + f_2)^*(x) &= x(X) - f_1(X) - f_2(X) \\ &= x_1(X) - f_1(X) + x_2(X) - f_2(X) \\ &= f_1^*(x_1) + f_2^*(x_2) \\ &\geq (f_1^* \circ f_2^*)(x). \end{aligned} \tag{5.50}$$

This completes the proof.

Q.E.D.

#### (b) Structures of subdifferentials

Suppose that  $\mathcal{D} \subseteq 2^E$  is a simple distributive lattice, i.e.,  $\mathcal{D} = 2^{\mathcal{P}}$  for a poset  $\mathcal{P} = (E, \preceq)$ . Consider a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$ .

We first give a characterization of the extreme points of the subdifferential  $\partial f(A)$  for  $A \in \mathcal{D}$ .

**Theorem 5.11:** For each  $A \in \mathcal{D}$ ,  $x \in \mathbb{R}^E$  is an extreme point of  $\partial f(A)$  if and only if there exists a maximal chain

$$C: \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E \tag{5.51}$$

of  $\mathcal{D}$ , including  $A$  in it, such that

$$x(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \dots, n). \tag{5.52}$$

(Proof) We assume, without loss of generality, that  $f(\emptyset) = 0$ . From Lemma 5.7 we have  $\partial f(\emptyset) = P(f)$ . Note that  $P(f)$  and  $B(f)$  have the same set of extreme points. Therefore, the present theorem for  $A = \emptyset$  follows from Theorem 2.18 and, similarly, the present theorem for  $A = E$  follows from 2.18, since  $\partial f(E) = P(f^\#)$  due to Lemma 5.7 and  $P(f^\#)$  and  $B(f^\#)$  ( $= B(f)$ ) have the same set of extreme points. Furthermore, for any  $A \in \mathcal{D}$  we have  $\partial f(A) = \partial f^A(A) \times \partial f_A(\emptyset)$  due to Lemma 5.5. Hence the extreme points of  $\partial f(A)$  are given by the direct product of extreme points of  $\partial f^A(A)$  and those of  $\partial f_A(\emptyset)$ . Since  $A$  is the unique maximal element of the domain  $[\emptyset, A]_{\mathcal{D}}$  of  $f^A$  and  $\emptyset$  is the unique minimal element of the domain  $[A, E]_{\mathcal{D}}/A$  of  $f_A$ , the present theorem for  $A \in \mathcal{D}$  with  $\emptyset \subset A \subset E$  follows from that for  $A = E$  and  $A = \emptyset$  for  $f$  with domain  $\mathcal{D}$ .

Q.E.D.

## 5.2. SUBGRADIENTS OF SUBMODULAR FUNCTIONS

The characteristic cone (or the recession cone) of a subdifferential  $\partial f(A)$  ( $A \in \mathcal{D}$ ) is given by

$$\begin{aligned} C_f(A) &= \{x \mid x \in \mathbb{R}^E, \forall X \in \mathcal{D}: x(X) - x(A) \leq 0\} \\ &= \{x \mid x \in \mathbb{R}^E, \forall X \in [\emptyset, A]_{\mathcal{D}} \cup [A, E]_{\mathcal{D}}: x(X) - x(A) \leq 0\}. \end{aligned} \quad (5.53)$$

Note that  $C_f(A)$  depends only on  $\mathcal{D}$  and  $A$ . We next give a characterization of extreme rays of  $C_f(A)$ .

Let  $G(\mathcal{P}) = (E, B^*(\mathcal{P}))$  be the directed graph with the vertex set  $E$  and the arc set  $B^*(\mathcal{P})$  which represents the Hasse diagram of the poset  $\mathcal{P} = (E, \preceq)$ , i.e.,  $(e, e') \in B^*(\mathcal{P})$  if and only if  $e' \prec e$  and there exists no element  $e''$  such that  $e' \prec e'' \prec e$ . Denote by  $E^+$  and  $E^-$ , respectively, the set of all the maximal elements of  $\mathcal{P}$  and the set of all the minimal elements of  $\mathcal{P}$ . Note that  $E^+ \cap E^-$  may be nonempty. For each  $A \in \mathcal{D}$  denote by  $\Delta^-(A)$  the set of all the arcs entering  $A$  in  $G(\mathcal{P})$ . Define vectors  $\xi_{p^+}$  ( $p^+ \in E^+$ ),  $\eta_{p^-}$  ( $p^- \in E^-$ ) and  $\zeta_a$  ( $a \in B^*(\mathcal{P})$ ) in  $\mathbb{R}^E$  by

$$\xi_{p^+}(e) = \begin{cases} -1 & (e = p^+) \\ 0 & (e \in E - \{p^+\}) \end{cases} \quad (p^+ \in E^+), \quad (5.54)$$

$$\eta_{p^-}(e) = \begin{cases} 1 & (e = p^-) \\ 0 & (e \in E - \{p^-\}) \end{cases} \quad (p^- \in E^-), \quad (5.55)$$

$$\zeta_a(e) = \begin{cases} 1 & (e = e') \\ -1 & (e = e'') \\ 0 & (e \in E - \{e', e''\}) \end{cases} \quad (a = (e', e'') \in B^*(\mathcal{P})). \quad (5.56)$$

Also define for each  $A \in \mathcal{D}$

$$\begin{aligned} \text{ER}(A) &= \{\xi_{p^+} \mid p^+ \in E^+ - A\} \cup \{\eta_{p^-} \mid p^- \in E^- \cap A\} \\ &\quad \cup \{\zeta_a \mid a \in B^*(\mathcal{D}) - \Delta^-(A)\}. \end{aligned} \quad (5.57)$$

Now, we are ready to show a theorem characterizing extreme rays of  $C_f(A)$  ( $A \in \mathcal{D}$ ).

**Theorem 5.12:** For each  $A \in \mathcal{D}$ , the set of all the extreme rays of the characteristic cone  $C_f(A)$  of the subdifferential  $\partial f(A)$  is given by  $\text{ER}(A)$  in (5.57).

(Proof) We can easily see from (5.53) that

$$\text{ER}(A) \subseteq C_f(A). \quad (5.58)$$

Since no vector in  $\text{ER}(A)$  can be expressed as a nonnegative linear combination of the other vectors in  $\text{ER}(A)$ , it suffices to prove that every vector in  $C_f(A)$  can be expressed as a nonnegative linear combination of vectors in  $\text{ER}(A)$ .

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Let  $v$  be an arbitrary vector in  $C_f(A)$ . From (5.53),

$$v(A - X) \geq 0 \quad (A \supseteq X \in \mathcal{D}), \quad (5.59)$$

$$v(X - A) \leq 0 \quad (A \subseteq X \in \mathcal{D}). \quad (5.60)$$

Suppose that each arc of  $B^*(\mathcal{D}) - \Delta^-(A)$  has the infinite upper capacity and the zero lower capacity and that each arc of  $\Delta^-(A)$  has the zero upper and lower capacities. Then it easily follows from (5.59), (5.60) and the feasibility theorem for network flows [Hoffman60] ([Ford + Fulkerson62]) that there exist a nonnegative flow  $\varphi: B^*(\mathcal{P}) \rightarrow \mathbb{R}_+$  in  $G(\mathcal{P})$  with  $\varphi(a) = 0$  ( $a \in \Delta^-(A)$ ), a nonpositive vector  $x \in \mathbb{R}_-^E$  with  $x(e) = 0$  ( $e \notin E^+ - A$ ) and a nonnegative vector  $y \in \mathbb{R}_+^E$  with  $y(e) = 0$  ( $e \notin E^- \cap A$ ) such that

$$v = \partial\varphi + x + y, \quad (5.61)$$

where  $\partial\varphi$  is the boundary of  $\varphi$  in  $G(\mathcal{P})$ . (5.61) gives an expression of  $v$  as a nonnegative linear combination of vectors in  $\text{ER}(A)$ . Q.E.D.

It should be noted that if  $v$  in  $C_f(A)$  satisfies  $v(A) = 0$ , then  $y = \mathbf{0}$  in (5.61) and that if  $v$  satisfies  $v(E - A) = 0$ , then  $x = \mathbf{0}$  in (5.61). Theorem 2.21 (the extreme ray theorem for base polyhedra) also follows from this theorem.

#### 5.3. The Lovász Extensions of Submodular Functions

Consider a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  on a simple distributive lattice  $\mathcal{D} = 2^{\mathcal{P}}$  with  $\mathcal{P} = (E, \preceq)$ . We assume  $f(\emptyset) = 0$ .

Define the function  $\hat{f}: \mathbb{R}^E \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$\hat{f}(c) = \max\{(c, x) \mid x \in P(f)\} \quad (5.62)$$

for each  $c \in \mathbb{R}^E$ , where

$$(c, x) = \sum_{e \in E} c(e)x(e). \quad (5.63)$$

Here,  $\hat{f}$  is called the *support function* of  $P(f)$  and is a positively homogeneous function [Rockafellar70], [Stoer + Witzgall70]. We see from Corollary 2.11 that  $\hat{f}(c) < +\infty$  if and only if  $c: E \rightarrow \mathbb{R}$  is a nonnegative monotone nonincreasing function from  $\mathcal{P} = (E, \preceq)$  to  $(\mathbb{R}, \leq)$ . Therefore, for any  $c \in \mathbb{R}^E$  such that  $\hat{f}(c) < +\infty$  there uniquely exist a chain

$$A_1 \subset A_2 \subset \cdots \subset A_k \quad (5.64)$$



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of nonempty  $A_i \in \mathcal{D}$  ( $i = 1, 2, \dots, k$ ) and positive numbers  $\lambda_i \in \mathbb{R}$  ( $i = 1, 2, \dots, k$ ) such that

$$c = \sum_{i=1}^k \lambda_i \chi_{A_i} \quad (5.65)$$

where  $k \geq 0$ ,  $\chi_{A_i} \in \mathbb{R}^E$  is the characteristic vector of  $A_i \subseteq E$  ( $i = 1, 2, \dots, k$ ) and if  $k = 0$  (i.e.,  $c = \mathbf{0}$ ), the empty sum is defined to be zero vector  $\mathbf{0}$  in  $\mathbb{R}^E$ .

Moreover, we have

$$\hat{f}(c) = \sum_{i=1}^k \lambda_i f(A_i) \quad (5.66)$$

since the value of  $\hat{f}(c)$  defined by (5.62) can be obtained by the greedy algorithm (see Section 2.2.b) and a maximizer  $x$  of (5.62) satisfies

$$x(A_i - A_{i-1}) = f(A_i) - f(A_{i-1}) \quad (i = 1, 2, \dots, k) \quad (5.67)$$

with  $A_0 = \emptyset$ . If the right-hand side of (5.66) is the empty sum, it is defined to be zero.

Formula (5.66) was introduced by L. Lovász [Lovász83] for  $\mathcal{D} = 2^E$ . The construction of  $\hat{f}$  through (5.64)–(5.66) can be applied to any function  $f$  on  $\mathcal{D}$  with  $f(\emptyset) = 0$  and  $\hat{f}$  is an extension of  $f$ . We call such an extension  $\hat{f}$  the *Lovász extension* of  $f$ .

**Theorem 5.13** [Lovász83]: A function  $f: \mathcal{D} \rightarrow \mathbb{R}$  is submodular if and only if the Lovász extension  $\hat{f}$  of  $f$  is convex.

(Proof) If  $f$  is a submodular function, then its extension  $\hat{f}$  is given by (5.62) and hence is a convex function. Conversely, suppose that the extension  $\hat{f}$  of  $f$  is a convex function. By definition, for any  $X, Y \in \mathcal{D}$

$$\begin{aligned} \hat{f}(\chi_X + \chi_Y) &= \hat{f}(\chi_{X \cup Y} + \chi_{X \cap Y}) \\ &= f(X \cup Y) + f(X \cap Y). \end{aligned} \quad (5.68)$$

Since  $\hat{f}$  is a positively homogeneous convex function, we also have

$$\hat{f}(\chi_X + \chi_Y) \leq \hat{f}(\chi_X) + \hat{f}(\chi_Y) = f(X) + f(Y). \quad (5.69)$$

From (5.67) and (5.68)  $f$  is a submodular function on  $\mathcal{D}$ . Q.E.D.

Theorem 5.13 shows the close relationship between the submodularity and the convexity. The results in Sections 5.1 and 5.2 can be viewed from the theory of convex functions through this theorem. However, the integrality result in

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Theorem 5.3 does not follow directly from the ordinary convex analysis; it is truly a combinatorial deep result.

Define

$$P(\mathcal{D}) = \text{the convex hull of vectors } \chi_A \ (A \in \mathcal{D}). \quad (5.70)$$

**Lemma 5.14** [Lovász83]: For a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  we have

$$\min\{f(X) \mid X \in \mathcal{D}\} = \min\{\hat{f}(c) \mid c \in P(\mathcal{D})\}. \quad (5.71)$$

(Proof) Immediate from Theorem 5.13 and (5.66), the positive homogeneity of  $\hat{f}$ . Q.E.D.

**Lemma 5.15:** For any  $c \in P(\mathcal{D})$  there uniquely exists a nonempty chain

$$B_1 \subset B_2 \subset \cdots \subset B_p \quad (5.72)$$

of  $\mathcal{D}$  such that  $c$  is expressed as a convex combination

$$c = \sum_{i=1}^p \mu_i \chi_{B_i} \quad (5.73)$$

with  $\mu_i > 0$  ( $i = 1, 2, \dots, p$ ) and  $\sum_{i=1}^p \mu_i = 1$ .

(Proof) Any  $c \in P(\mathcal{D})$  is a nonnegative monotone nonincreasing function from  $\mathcal{P} = (E, \leq)$  to  $(\mathbb{R}, \leq)$ . Therefore, there uniquely exists a chain (5.64) of nonempty  $A_i \in \mathcal{D}$  ( $i = 1, 2, \dots, k$ ) and  $\lambda_i > 0$  ( $i = 1, 2, \dots, k$ ) such that (5.65) holds. Since  $c \in P(\mathcal{D})$ , we have

$$\sum_{i=1}^k \lambda_i \leq 1. \quad (5.74)$$

If  $\sum_{i=1}^k \lambda_i = 1$ , then (5.65) is the desired unique expression. Otherwise put  $\lambda_0 = 1 - \sum_{i=1}^k \lambda_i$  and  $A_0 = \emptyset$ . This yields a desired unique expression

$$c = \sum_{i=0}^k \lambda_i \chi_{A_i} \quad (5.75)$$

with  $\lambda_i > 0$  ( $i = 1, 2, \dots, k$ ) and  $\sum_{i=0}^k \lambda_i = 1$ . Q.E.D.

### 5.3. THE LOVÁSZ EXTENSIONS OF SUBMODULAR FUNCTIONS

Now, let  $f: \mathcal{D} \rightarrow \mathbb{R}$  be a submodular function and define  $\tilde{f}: \mathbb{R}^E \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$\tilde{f}(c) = \begin{cases} \hat{f}(c) & (c \in P(\mathcal{D})) \\ +\infty & (c \in \mathbb{R}^E - P(\mathcal{D})), \end{cases} \quad (5.76)$$

where  $\hat{f}$  is the Lovász extension of  $f$ . Call  $\tilde{f}$  the *truncated Lovász extension* of  $f$ .

**Theorem 5.16:** For each  $A \in \mathcal{D}$  we have

$$\partial f(A) = \partial \tilde{f}(\chi_A), \quad (5.77)$$

where  $\partial \tilde{f}(\chi_A)$  denotes the subdifferential of the convex function  $\tilde{f}$  at  $\chi_A$  in an ordinary sense of convex analysis [Rockafellar70].

(Proof) By definition, we have  $x \in \partial \tilde{f}(\chi_A)$  if and only if

$$\forall c \in \mathbb{R}^E: (c - \chi_A, x) \leq \tilde{f}(c) - \tilde{f}(\chi_A). \quad (5.78)$$

Since  $\tilde{f}(\chi_A) = f(A)$ , (5.78) is rewritten as

$$\begin{aligned} f(A) - x(A) &\leq \min\{\tilde{f}(c) - (c, x) \mid c \in \mathbb{R}^E\} \\ &= \min\{\hat{f}(c) - (c, x) \mid c \in P(\mathcal{D})\} \\ &= \min\{f(X) - x(X) \mid X \in \mathcal{D}\}, \end{aligned} \quad (5.79)$$

where the last equality follows from Lemma 5.14 with  $f$  replaced by  $f - x$ . (5.79) is equivalent to  $x \in \partial f(A)$ . Q.E.D.

**Theorem 5.17:** Let  $c$  be an arbitrary vector in  $P(\mathcal{D})$  and suppose that  $c$  is expressed as (5.73) with (5.72). Then, we have

$$\partial \tilde{f}(c) = \bigcap \{\partial f(B_i) \mid i = 1, 2, \dots, p\}. \quad (5.80)$$

(Proof) We have  $x \in \partial \tilde{f}(c)$  if and only if

$$\forall b \in P(\mathcal{D}): (b - c, x) \leq \hat{f}(b) - \hat{f}(c). \quad (5.81)$$

From (5.72) and (5.73), (5.81) is rewritten as

$$\begin{aligned} \sum_{i=1}^p \mu_i (f(B_i) - x(B_i)) &\leq \min\{\hat{f}(b) - (b, x) \mid b \in P(\mathcal{D})\} \\ &= \min\{f(X) - x(X) \mid X \in \mathcal{D}\} \end{aligned} \quad (5.82)$$

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due to Lemma 5.14. Furthermore, since  $\sum_{i=1}^p \mu_i = 1$  and  $\mu_i > 0$  ( $i = 1, 2, \dots, p$ ), (5.82) is equivalent to

$$f(B_i) - x(B_i) = \min\{f(X) - x(X) \mid X \in \mathcal{D}\} \\ (i = 1, 2, \dots, p) \quad (5.83)$$

or

$$x \in \bigcap \{\partial f(B_i) \mid i = 1, 2, \dots, p\}. \quad (5.84)$$

Q.E.D.

For any maximal chain  $\mathcal{C}: \emptyset = S_0 \subset S_1 \subset \dots \subset S_n = E$  of  $\mathcal{D}$ , denote by  $P(\mathcal{C})$  the  $n$ -simplex with vertices  $\chi_{S_i}$  ( $i = 1, 2, \dots, n$ ).

**Lemma 5.18:** The collection of  $P(\mathcal{C})$ 's for all the maximal chains  $\mathcal{C}$  of  $\mathcal{D}$  forms a simplicial subdivision of  $P(\mathcal{D})$ . Moreover, for two maximal chains  $\mathcal{C}^i: \emptyset = S_0^i \subset S_1^i \subset \dots \subset S_n^i = E$  ( $i = 1, 2$ ) the  $n$ -simplices  $P(\mathcal{C}^i)$  ( $i = 1, 2$ ) have a common facet if and only if for some  $k$  with  $1 \leq k \leq n-1$  we have

$$S_j^1 = S_j^2 \quad (0 \leq j \leq n, j \neq k). \quad (5.85)$$

(Proof) The first half of this lemma follows from the uniqueness property of Lemma 5.15. Moreover, any facet of the  $n$ -simplex  $P(\mathcal{C}^i)$  corresponds to a subchain of  $\mathcal{C}^i$  with length  $n-1$ . From this follows the second half of the lemma. Q.E.D.

**Lemma 5.19:** For any maximal chain  $\mathcal{C}: \emptyset = S_0 \subset S_1 \subset \dots \subset S_n = E$  of  $\mathcal{D}$  and any interior point  $c$  of  $P(\mathcal{C})$ ,  $\tilde{f}$  has a unique subgradient  $x$  at  $c$  given by

$$x(S_i - S_{i-1}) = f(S_i) - f(S_{i-1}) \quad (i = 1, 2, \dots, n). \quad (5.86)$$

(Proof) From Theorem 5.17 the vector  $x$  given by (5.86) is a unique subgradient of  $\tilde{f}$  at  $c$ . Q.E.D.

Note that the subgradient  $x$  of  $\tilde{f}$  given by (5.86) is an extreme point of the base polyhedron  $B(f)$ .

We can see that for a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$ , the convex conjugate function  $f^*$  and the truncated Lovász extension  $\tilde{f}$  of  $f$  are the convex conjugate functions of each other in an ordinary sense of convex analysis (see [Rockafellar70]). Consequently, the Fenchel-type min-max theorem for submodular and supermodular functions (Theorem 5.3), except for the integrality property, follows from Fenchel's duality theorem for ordinary convex and concave functions.

## 6. Submodular Programs

In this section we consider optimization problems with objective functions and constraints described by submodular functions, which we call *submodular programs*.

### 6.1. Submodular Programs – Unconstrained Optimization

Let  $f: \mathcal{D} \rightarrow \mathbb{R}$  be a submodular function on a simple distributive lattice  $\mathcal{D} \subseteq 2^E$  with  $\emptyset, E \in \mathcal{D}$  and  $f(\emptyset) = 0$ . We consider the problem of minimizing the submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  without any constraints. It should, however, be noted that the underlying distributive lattice  $\mathcal{D}$  itself may be regarded as a constrained feasible region.

#### (a) Minimizing submodular functions

From the definition of subdifferential of a submodular function we have the following trivial but fundamental lemma.

**Lemma 6.1:** A set  $A \in \mathcal{D}$  is a minimizer of  $f: \mathcal{D} \rightarrow \mathbb{R}$  if and only if

$$0 \in \partial f(A), \quad (6.1)$$

where  $0$  is the zero vector in  $\mathbb{R}^E$ .

From Lemmas 5.4 and 6.1 we get the following theorem which means that some “local” or “partial” minimality implies “global” minimality.

**Theorem 6.2:** A set  $A \in \mathcal{D}$  is a minimizer of  $f: \mathcal{D} \rightarrow \mathbb{R}$  if and only if  $A \in \mathcal{D}$  minimizes  $f$  restricted to the sublattice  $[\emptyset, A]_{\mathcal{D}} \cup [A, E]_{\mathcal{D}}$ .

Grötschel, Lovász and Schrijver [Grötschel + Lovász + Schrijver88] have devised a strongly polynomial algorithm for minimizing a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  which requires time polynomial in  $|E|$ . Their algorithm heavily depends on the so-called ellipsoid method [Khachian79, 80] for linear programs and is not a combinatorial one. A combinatorial but pseudopolynomial algorithm is proposed by Cunningham [Cunningham85], where the submodular function  $f$  is integer-valued and the required running time is polynomial in  $|E|$  and  $\max |f(X)|$  but not  $\log(\max |f(X)|)$ .

For special classes of submodular functions the problem of minimizing a submodular function can be reduced to polynomial-time solvable ones such

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as the minimum cut problem [Picard76] and the maximum-weight stable-set problem in a bipartite graph [Billionnet + Minoux85].

We show a "practical" algorithm for minimizing submodular functions based on the following lemma (see (2.18)).

**Lemma 6.3:**

$$\min\{f(X) \mid X \in \mathcal{D}\} = \max\{y(E) \mid y \in P(f), y \leq 0\}. \quad (6.2)$$

Consider a special quadratic programming problem over the base polyhedron  $B(f)$  described as

$$\text{Minimize } \|x\|^2 (= \sum_{e \in E} x(e)^2) \quad (6.3a)$$

$$\text{subject to } x \in B(f) \quad (6.3b)$$

(see [Fuji80b]).

In the present subsection we assume that the underlying totally ordered additive group  $\mathbb{R}$  is the set of reals or rationals. (In Chapter V we will consider in detail a class of nonlinear optimization problems over the base polyhedron, including Problem (6.3).)

Let  $x^*$  be the optimal solution of Problem (6.3) and define

$$y^* = x^* \wedge 0 (= (\min\{x^*(e), 0\} : e \in E)), \quad (6.4)$$

$$A_- = \{e \mid e \in E, x^*(e) < 0\}, \quad (6.5)$$

$$A_0 = \{e \mid e \in E, x^*(e) \leq 0\}. \quad (6.6)$$

**Lemma 6.4:**  $y^*$  in (6.4) is a maximizer of the right-hand side of (6.2). Also,  $A_-$  is the unique minimal maximizer of  $f$  and  $A_0$  is the unique maximal maximizer of  $f$ .

(Proof) Because of the optimality of  $x^*$  we have

$$\forall e \in A_- : \text{dep}(x^*, e) \subseteq A_-, \quad (6.7)$$

$$\forall e \in A_0 : \text{dep}(x^*, e) \subseteq A_0. \quad (6.8)$$

From (6.4)–(6.8),  $A_-$ ,  $A_0 \in \mathcal{D}$  and

$$f(A_-) = f(A_0) = y^*(E) (= x^*(A_-) = x^*(A_0)). \quad (6.9)$$

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Since for any  $X \in \mathcal{D}$  and any  $y \in P(f)$  with  $y \leq 0$  we have  $f(X) \geq y(E)$ , it follows from (6.9) that  $y^*$  is a maximizer of the right-hand side of (6.2) and  $A_-$  and  $A_0$  are minimizers of  $f$ . Moreover, for any  $X \in \mathcal{D}$  such that  $X \subset A_-$ ,

$$f(X) \geq x^*(X) > x^*(A_-) = f(A_-) \quad (6.10)$$

and for any  $X \in \mathcal{D}$  such that  $A_0 \subset X$ ,

$$f(X) \geq x^*(X) > x^*(A_0) = f(A_0). \quad (6.11)$$

From (6.9)–(6.11)  $A_-$  is the unique minimal minimizer of  $f$  and  $A_0$  is the unique maximal minimizer of  $f$ . Q.E.D.

Let  $c_1 < c_2 < \cdots < c_p$  be the distinct values of  $x^*(e)$  ( $e \in E$ ) and define

$$B_i = \{e \mid e \in E, x^*(e) \leq c_i\} \quad (i = 1, 2, \dots, p), \quad (6.12)$$

$$B_0 = \emptyset. \quad (6.13)$$

By a similar argument as (6.7)–(6.9) we see that

$$\emptyset = B_0 \subset B_1 \subset \cdots \subset B_p = E \quad (6.14)$$

is a chain of  $\mathcal{D}$  and that

$$f(B_i) = x^*(B_i) \quad (i = 0, 1, \dots, p). \quad (6.15)$$

Consequently, we have

$$c_i = \frac{f(B_i) - f(B_{i-1})}{|B_i - B_{i-1}|} \quad (i = 1, 2, \dots, p). \quad (6.16)$$

Problem (6.3) is to find the minimum-norm point in  $B(f)$ . For simplicity, let us suppose  $B(f)$  is bounded, i.e.,  $\mathcal{D} = 2^E$ . A solution algorithm for the minimum norm-point problem is proposed by P. Wolfe [Wolfe76]. We can adopt Wolfe's algorithm for Problem (6.3). We express a simplex  $S$  in  $\mathbb{R}^E$  by the set of its extreme points.

**An algorithm for finding the minimum-norm point in  $B(f)$**

**Step 1:** Let  $x^*$  be any extreme point of  $B(f)$ . Put  $S \leftarrow \{x^*\}$ .

**Step 2:** Using the greedy algorithm, find a minimum-weight base  $\hat{x}$  of  $B(f)$  with respect to the weight function  $x^*$ . If  $(x^*, \hat{x} - x^*) = 0$ , then stop. Otherwise put  $S \leftarrow S \cup \{\hat{x}\}$ .

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**Step 3:** Find the minimum-norm point  $x_0$  in the affine space generated by  $S$ . If  $x_0$  is in the relative interior of the simplex  $S$ , then put  $x^* \leftarrow x_0$  and go to Step 2. Otherwise let  $S' \subset S$  be the unique minimal face of simplex  $S$  which has the nonempty intersection with the line segment  $[x^*, x_0]$ . Put  $x^* \leftarrow$  the intersection point of the face  $S'$  and the line segment  $[x^*, x_0]$  and  $S \leftarrow S'$ . Go to the beginning of Step 3.  
(End)

Step 3 is consecutively repeated at most  $|E|$  times. Each simplex  $S$  available when executing Step 2 uniquely determines  $x^*$  lying in the relative interior of simplex  $S$  and the norm  $\|x^*\|$  monotonically decreases every time  $x^*$  is renewed. Therefore, all the simplices  $S$  which we encounter during the execution of the algorithm are different, so that the algorithm terminates after a finite number of steps.

**Example:** As an illustrative example, consider the minimum-cut problem for the two-terminal network  $\mathcal{N} = (G = (V, A), s^+, s^-, c)$  shown in Fig. 6.1, where  $V = \{s^+, s^-, 1, 2, 3\}$ . The numbers attached to the arcs denote the capacities  $c(a)$  ( $a \in A$ ). The problem is to find a cut  $U \subseteq V$  with  $s^+ \in U$  and  $s^- \notin U$  which minimizes its capacity  $\sum_{a \in \Delta^+ U} c(a)$ , where  $\Delta^+ U$  is the set of arcs leaving  $U$ . Putting  $V^* = V - \{s^+, s^-\}$ , we define a submodular function  $f: 2^{V^*} \rightarrow \mathbb{R}$  as follows. For each  $W \subseteq V^*$ ,

$$f(W) = \sum_{a \in \Delta^+(W \cup \{s^+\})} c(a) - \sum_{a \in \Delta^+\{s^+\}} c(a). \quad (6.17)$$

The second constant term is for the normalization,  $f(\emptyset) = 0$ .

Now, the minimum-cut problem is reduced to the problem of minimizing the submodular function  $f: 2^{V^*} \rightarrow \mathbb{R}$ .

Let us start with the extreme point of  $B(f)$  corresponding to the sequence  $(1, 2, 3)$  of the vertices in  $V^*$ , i.e.,

$$x^* (= (x^*(1), x^*(2), x^*(3))) = (0, 10, -20). \quad (6.18)$$

Next, in Step 2 we find by the greedy algorithm the minimum-weight base  $\hat{x}$  of  $B(f)$  with respect to weight  $x^*$ , which is the extreme point of  $B(f)$  corresponding to the vertex sequence  $(3, 1, 2)$ , i.e.,

$$\hat{x} = (-30, 0, 20). \quad (6.19)$$

The minimum-norm point  $x_0$  in the line through the two points (6.18) and (6.19) is given by

$$\begin{aligned} x_0 &= \frac{17}{26}(0, 10, -20) + \frac{9}{26}(-30, 0, 20) \\ &= \frac{1}{26}(-270, 170, -160). \end{aligned} \quad (6.20)$$



## 6.1. CONSTRAINED SUBMODULAR PROGRAMS

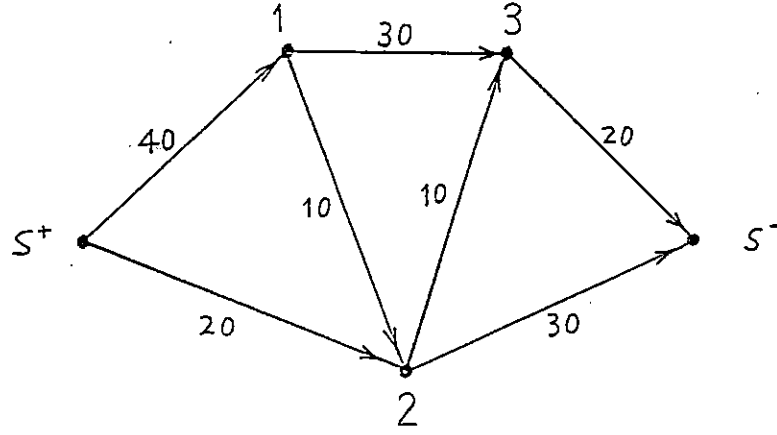


Figure 6.1.

Since  $0 < \frac{17}{26}, \frac{9}{26}$ , we put  $x^* \leftarrow x_0$  and find a new extreme point, denoted by  $\hat{x}$ , of  $B(f)$  corresponding to the vertex sequence  $(1, 3, 2)$  determined by  $x^*$  ( $= x_0$  in (6.20)).  $\hat{x}$  is given by

$$\hat{x} = (0, 0, -10). \quad (6.21)$$

The minimum-norm point  $x_0$  in the two-dimensional affine space generated by the three points of (6.18), (6.19) and (6.21) is given by

$$x_0 = -\frac{1}{3}(0, 10, -20) + \frac{1}{9}(-30, 0, 20) + \frac{11}{9}(0, 0, -10). \quad (6.22)$$

Since  $-\frac{1}{3} < 0 < \frac{1}{9}, \frac{11}{9}$ , the minimum-norm intersection point of the present simplex formed by points of (6.18), (6.19) and (6.21) and the line segment between points of (6.20) and (6.22) lies in the face formed by points of (6.19) and (6.21). Hence, we discard point (6.18) from the present simplex  $S$  and find the minimum-norm point, denoted by  $x_0$  again, in the line through points of (6.19) and (6.21).

$$\begin{aligned} x_0 &= \frac{5}{6}(0, 0, -10) + \frac{1}{6}(-30, 0, 20) \\ &= (-5, 0, -5). \end{aligned} \quad (6.23)$$

Since the present  $x_0$  is in the relative interior of the simplex formed by points of (6.19) and (6.21), we put  $x^* \leftarrow x_0$  and go back to Step 2. Now, we find

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a minimum-weight base  $\hat{x}$  with respect to the weight given by (6.23). Such a base  $\hat{x}$  is given by (6.19) (or (6.21)) and the algorithm terminates.

From the minimum-norm base (6.23) we see that  $U_0 = \{s^+, 1, 2, 3\}$  is the unique maximal minimum-cut and  $U_- = \{s^+, 1, 3\}$  is the unique minimal minimum-cut of the network, due to Lemma 6.4.

When the algorithm for finding the minimum-norm point in  $B(f)$  terminates, we are given the minimum-norm point  $x^*$  and a set of extreme points  $x_i$  ( $i \in I$ ) of  $B(f)$  such that  $x^*$  is expressed as a convex combination

$$x^* = \sum_{i \in I} \lambda_i x_i \quad (6.24)$$

of  $x_i$  ( $i \in I$ ). For each  $i \in I$  let  $\mathcal{P}_i = (E, \preceq_i)$  be the poset which corresponds to the distributive lattice

$$\mathcal{D}(x_i) = \{X \mid X \in \mathcal{D}, x_i(X) = f(X)\} \quad (6.25)$$

with  $\mathcal{D}(x_i) = 2^{\mathcal{P}_i}$ , and define a distributive lattice  $\hat{\mathcal{D}}$  by

$$\hat{\mathcal{D}} = \bigcap_{i \in I} 2^{\mathcal{P}_i}. \quad (6.26)$$

Then,

$$\hat{\mathcal{D}} = \mathcal{D}(x^*) (= \{X \mid X \in \mathcal{D}, x^*(X) = f(X)\}). \quad (6.27)$$

(An algorithm for finding the poset  $\mathcal{P}_i = (E, \preceq_i)$  for each extreme point  $x_i \in B(f)$  is proposed by Bixby, Cunningham and Topkis [Bixby + Cunningham + Topkis85]. The poset  $\hat{\mathcal{P}} = (\hat{E}, \preceq)$  on a partition of  $E$  which corresponds to the distributive lattice  $\hat{\mathcal{D}}$  is easily obtained by superimposing the posets  $\mathcal{P}_i$  ( $i \in I$ ).)

For any maximal chain

$$\emptyset = \hat{A}_0 \subset \hat{A}_1 \subset \cdots \subset \hat{A}_q = E \quad (6.28)$$

of  $\hat{\mathcal{D}}$ , the minimum-norm point  $x^*$  satisfies

$$x^*(e) = \frac{f(\hat{A}_j) - f(\hat{A}_{j-1})}{|\hat{A}_j - \hat{A}_{j-1}|} \quad (6.29)$$

for each  $e \in \hat{A}_j - \hat{A}_{j-1}$  ( $j = 1, 2, \dots, q$ ) (cf. (6.16)). Furthermore, all the minimizers of  $f$  are given by the interval  $[A_-, A_0]_{\hat{\mathcal{D}}}$  of  $\hat{\mathcal{D}}$ , where  $A_-$  and  $A_0$  are defined by (6.5) and (6.6).

#### (b) Minimizing modular functions

## 6.1. CONSTRAINED SUBMODULAR PROGRAMS

Let  $\mu: \mathcal{D} \rightarrow \mathbb{R}$  be a modular function on a simple distributive lattice  $\mathcal{D} = 2^{\mathcal{P}}$  with  $\mathcal{P} = (E, \preceq)$  and  $\mu(\emptyset) = 0$ .

**Lemma 6.5:** There exists a unique vector  $\nu \in \mathbb{R}^E$  such that for each  $X \in \mathcal{D}$

$$\mu(X) = \sum_{e \in X} \nu(e). \quad (6.30)$$

(Proof) Choose any maximal chain  $\mathcal{C}: \emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = E$  of  $\mathcal{D}$  and define a vector  $\nu \in \mathbb{R}^E$  by

$$\nu(e_i) = \mu(S_i) - \mu(S_{i-1}) \quad (i = 1, 2, \dots, n), \quad (6.31)$$

where  $\{e_i\} = S_i - S_{i-1}$  ( $i = 1, 2, \dots, n$ ). For any  $X \in \mathcal{D}$  with  $|X| = p$  there exist integers  $1 \leq j_1 < \cdots < j_p \leq n$  such that

$$X = \{e_{j_k} \mid \{e_{j_k}\} = S_{j_k} - S_{j_k-1}, k = 1, 2, \dots, p\}. \quad (6.32)$$

Since  $\mu$  is a modular function and  $S_{j_1} \subset S_{j_2} \subset \cdots \subset S_{j_p}$ , we have

$$\begin{aligned} \mu(X) + \sum_{k=1}^p \mu(S_{j_k-1}) \\ &= \mu(X \cup S_{j_p-1}) + \sum_{k=2}^p \mu((X \cap S_{j_k-1}) \cup S_{j_{k-1}-1}) + \mu(X \cap S_{j_1-1}) \\ &= \sum_{k=1}^p \mu(S_{j_k}), \end{aligned} \quad (6.33)$$

where use is made of the fact that  $X \cup S_{j_p-1} = S_{j_p}$ ,  $X \cap S_{j_k-1} = X \cap S_{j_{k-1}}$  ( $k = 2, \dots, p$ ) and  $X \cap S_{j_1-1} = \emptyset$ . (6.30) follows from (6.31) and (6.33). The uniqueness of  $\nu$  is clear from (6.30). Q.E.D.

We see from this lemma that the problem of minimizing the modular function  $\mu: \mathcal{D} \rightarrow \mathbb{R}$  is equivalent to the problem of finding a minimum-weight (lower) ideal of the poset  $\mathcal{P} = (E, \preceq)$  with respect to the weight function  $\nu: E \rightarrow \mathbb{R}$ . The latter problem was solved by J. C. Picard [Picard76] by reducing it to a minimum-cut problem. Conversely, the reducibility of the minimum-cut problem to a problem of minimizing a modular function was shown by W. H. Cunningham [Cunningham85a].

We first show Picard's approach. Let  $G(\mathcal{P}) = (E, B^*(\mathcal{P}))$  be the graph representing the Hasse diagram of  $\mathcal{P} = (E, \preceq)$ , i.e.,  $e_1$  covers  $e_2$  in  $\mathcal{P}$  if and only if  $(e_1, e_2) \in B^*(\mathcal{P})$ . (The following procedure works if we take as  $G(\mathcal{P})$

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any graph  $G'$  whose transitive closure coincides with that of  $G(\mathcal{P})$  and a (transitively) closed set of  $G'$  is an ideal of  $\mathcal{P}$ .) Consider new vertices  $s^+$  and  $s^-$  and sets of new arcs

$$S^+ = \{(s^+, e) \mid e \in E, \nu(e) < 0\}, \quad (6.34)$$

$$S^- = \{(e, s^-) \mid e \in E, \nu(e) > 0\}. \quad (6.35)$$

Also define the capacities  $c(a)$  of arcs  $a$  in  $B^*(\mathcal{P}) \cup S^+ \cup S^-$  by

$$c(a) = \begin{cases} +\infty & (a \in A(\mathcal{P})) \\ -\nu(e) & ((s^+, e) \in S^+) \\ \nu(e) & ((e, s^-) \in S^-). \end{cases} \quad (6.36)$$

Denote this network by  $\mathcal{N} = (\hat{G} = (\hat{E}, \hat{A}), s^+, s^-, c)$ , where  $\hat{E} = E \cup \{s^+, s^-\}$  and  $\hat{A} = B^*(\mathcal{P}) \cup S^+ \cup S^-$  (see Fig. 6.2).

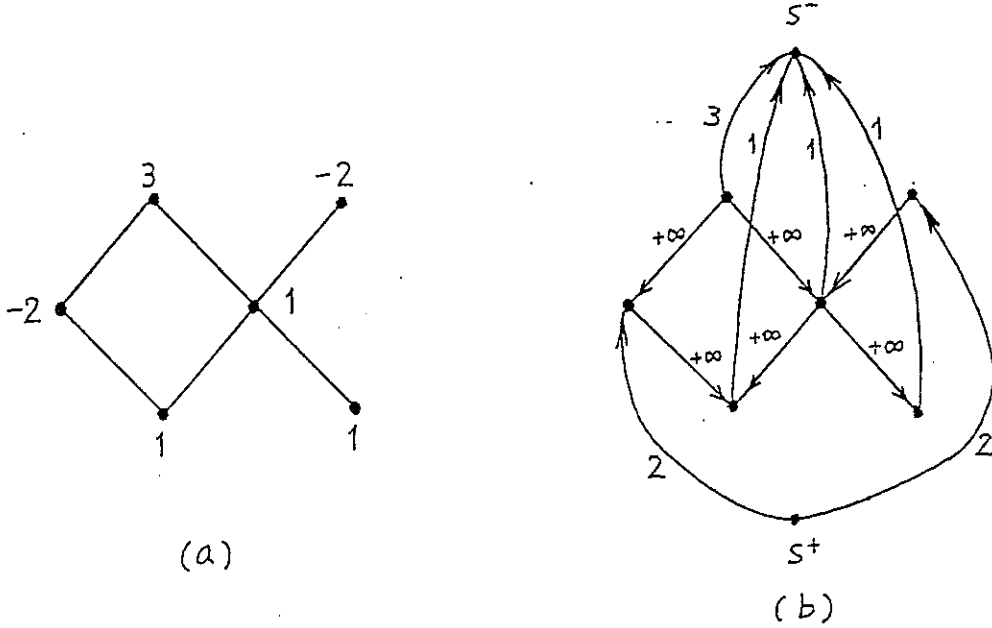


Figure 6.2. (a) A poset  $\mathcal{P}$  with weights. (b) Network  $\mathcal{N}$ .

For any cut  $U \subseteq \hat{E}$  of the network  $\mathcal{N}$  (i.e.,  $s^+ \in U$  and  $s^- \notin U$ ), the capacity of the cut  $U$  is given by

$$\begin{aligned} \kappa_c(U) = & \sum \{-\nu(e) \mid e \in E - U, \nu(e) < 0\} \\ & + \sum \{\nu(e) \mid e \in E \cap U, \nu(e) > 0\} \\ & + \sum \{c(a) \mid a \in B^*(\mathcal{P}), \partial^+ a \in U, \partial^- a \in E - U\} \end{aligned} \quad (6.37)$$

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Note that for any cut  $U$  of finite capacity  $\kappa_c(U)$ , the third term in (6.37) vanishes and  $U - \{s^+\}$  is an ideal of  $\mathcal{P} = (E, \preceq)$  and, conversely, that for any ideal  $I \subseteq E$  of  $\mathcal{P}$   $I \cup \{s^+\}$  is a cut of finite capacity in  $\mathcal{N}$ .

Furthermore, for any ideal  $I$  of  $\mathcal{P}$  we have

$$\begin{aligned}\kappa_c(I \cup \{s^+\}) &= \sum \{-\nu(e) \mid e \in E - I, \nu(e) < 0\} \\ &\quad + \sum \{\nu(e) \mid e \in I, \nu(e) > 0\} \\ &= \nu(I) + \sum \{-\nu(e) \mid e \in E, \nu(e) < 0\}.\end{aligned}\quad (6.38)$$

Therefore, minimizing  $\kappa_c(I \cup \{s^+\})$  for cuts  $I \cup \{s^+\}$  of  $\mathcal{N}$  is equivalent to minimizing  $\nu(I)$  for ideals  $I$  of  $\mathcal{P}$ .

See [Picard + Queyranne82] for practical applications of the problem of finding a maximum-weight ideal of a poset or a maximum-weight closed set of a graph.

Next, we consider the problem of minimizing a modular function  $\mu: \mathcal{D} \rightarrow \mathbb{R}$  from the point of view of submodular analysis.

Given a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  and a vector  $x \in \mathbb{R}^E$ , consider the following problem. It includes the problem of minimizing the submodular function  $f$ .

(\*) Find  $A \in \mathcal{D}$  such that  $x \in \partial f(A)$  and then find an expression

$$x = x_1 + x_2 \quad (6.39)$$

such that  $x_1$  is a convex combination of extreme points of  $\partial f(A)$  and  $x_2$  is a nonnegative linear combination of extreme vectors of  $C_f(A)$ , the characteristic cone of  $\partial f(A)$ .

Note that the problem of minimizing  $f$  is equivalent to that of finding a set  $A \in \mathcal{D}$  such that  $0 \in \partial f(A)$ . We consider the above problem (\*) in the special case when  $f$  is a modular function  $\mu: \mathcal{D} \rightarrow \mathbb{R}$  with  $\mu(\emptyset) = 0$ .

For modular function  $\mu$ , the subdifferential  $\partial\mu(A)$  for each  $A \in \mathcal{D}$  has a unique extreme point  $\nu$  due to Theorem 5.11 and Lemma 6.5, where  $\nu$  is the vector appearing in Lemma 6.5. Hence Problem (\*) is reduced to the problem of finding  $A \in \mathcal{D}$  such that  $x - \nu \in C_f(A)$  and of finding a nonnegative linear combination, of  $\text{ER}(A)$  given by (5.57), which expresses  $x - \nu$ .

The proof of Theorem 5.12 suggests an algorithm as follows. The graph  $G(\mathcal{P}) = (E, B^*(\mathcal{P}))$  represents the Hasse diagram of the poset  $\mathcal{P} = (E, \preceq)$ .

**An algorithm for Problem (\*)**

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**Step 1:** Find a nonnegative flow  $\varphi: B^*(\mathcal{P}) \rightarrow \mathbb{R}_+$  in  $G(\mathcal{P})$  and nonnegative coefficients  $\alpha(p^+)$  ( $p^+ \in E^+$ ) and  $\beta(p^-)$  ( $p^- \in E^-$ ) such that

$$\sum_{p^+ \in E^+} \alpha(p^+) \xi_{p^+} + \sum_{p^- \in E^-} \beta(p^-) \eta_{p^-} + \partial\varphi = x - \nu. \quad (6.40)$$

(Here,  $\xi_{p^+}$  ( $p^+ \in E^+$ ) and  $\eta_{p^-}$  ( $p^- \in E^-$ ) are vectors in  $\mathbb{R}^E$  defined by (5.54) and (5.55).) For each  $p^+$  and  $p^-$  such that  $p^+ = p^- \in E^+ \cap E^-$  we choose the values of  $\alpha(p^+)$  and  $\beta(p^-)$  so that  $\alpha(p^+) \beta(p^-) = 0$ .

**Step 2:** Construct a network  $\hat{\mathcal{N}} = (\hat{G}(\mathcal{P}), \hat{c})$  with an underlying graph  $\hat{G}(\mathcal{P}) = (E, \hat{B}(\mathcal{P}))$  and a capacity function  $\hat{c}: \hat{B}(\mathcal{P}) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  defined as follows. The arc set  $\hat{B}(\mathcal{P})$  of  $\hat{G}(\mathcal{P})$  is given by

$$\hat{B}(\mathcal{P}) = B^*(\mathcal{P}) \cup \{(e, e') \mid (e', e) \in B^*(\mathcal{P})\} \quad (6.41)$$

and the capacity function  $\hat{c}$  by

$$\hat{c}(a) = \begin{cases} \varphi(a) & (a \in B^*(\mathcal{P})) \\ +\infty & (a = (e, e'), (e', e) \in B^*(\mathcal{P})). \end{cases} \quad (6.42)$$

Then find a maximum flow  $\psi: \hat{B}(\mathcal{P}) \rightarrow \mathbb{R}_+$  in  $\hat{\mathcal{N}}$  from the entrance vertex set  $E^+ - E^-$  to the exit vertex set  $E^- - E^+$  such that

$$0 \leq \psi(a) \leq \hat{c}(a) \quad (a \in \hat{B}(\mathcal{P})), \quad (6.43)$$

$$\partial\psi(e) = 0 \quad (e \in E - (E^+ \cup E^-)), \quad (6.44)$$

$$\partial\psi(p^+) \leq \alpha(p^+) \quad (p^+ \in E^+ - E^-), \quad (6.45)$$

$$-\partial\psi(p^-) \leq \beta(p^-) \quad (p^- \in E^- - E^+). \quad (6.46)$$

(Here, the boundary operator  $\partial$  is defined with respect to  $\hat{G}(\mathcal{P})$ .)

**Step 3:** Put

$$\varphi((e, e')) \leftarrow \varphi((e, e')) - \psi((e, e')) + \psi((e', e)) \quad ((e', e) \in B^*(\mathcal{P})), \quad (6.47)$$

$$\alpha(p^+) \leftarrow \alpha(p^+) - \partial\psi(p^+) \quad (p^+ \in E^+ - E^-), \quad (6.48)$$

$$\beta(p^-) \leftarrow \beta(p^-) + \partial\psi(p^-) \quad (p^- \in E^- - E^+). \quad (6.49)$$

Then find  $A \in \mathcal{D}$  such that

$$(i) \quad \varphi(a) = 0 \quad (a \in \Delta^-(A)), \quad (6.50)$$

$$(ii) \quad \alpha(p^+) = 0 \quad (p^+ \in E^+ \cap A), \quad (6.51)$$

$$(iii) \quad \beta(p^-) = 0 \quad (p^- \in E^- - A). \quad (6.52)$$

(End)

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For any  $A \in \mathcal{D}$  satisfying (6.50)–(6.52) we have  $x \in \partial\mu(A)$  and  $x$  is expressed as

$$x = \nu + \sum_{p^+ \in E^+} \alpha(p^+) \xi_{p^+} + \sum_{p^- \in E^-} \beta(p^-) \eta_{p^-} + \sum_{a \in B^*(\mathcal{P})} \varphi(a) \zeta_a, \quad (6.53)$$

using the obtained  $\alpha$ ,  $\beta$  and  $\varphi$ .

Step 1 can be carried out by the breadth-first search and requires linear time. Step 2 is performed by any maximum flow algorithm. Any minimum cut  $U \subseteq E$  obtained by the maximum flow algorithm in Step 2 gives a desired  $A \in \mathcal{D}$  to be found in Step 3 as  $A = E - U$ . From (6.50)–(6.52),  $x - \nu$  in (6.53) is a nonnegative linear combination of  $\text{ER}(A)$ .

When  $x = 0$ , the above algorithm solves the problem of minimizing the modular function  $\mu$ . In this case  $A \in \mathcal{D}$  found in Step 3 is a minimizer of  $\mu$  since  $0 \in \partial\mu(A)$ .

Let us consider the problem of minimizing a modular function  $\mu: \mathcal{D} \rightarrow \mathbb{R}$  from a polyhedral point of view. Denote by  $P(\mathcal{P})$  the convex hull of all the characteristic vectors  $\chi_X$  ( $X \in \mathcal{D}$ ).

**Lemma 6.6:** For any  $A \in \mathcal{D}$ , the inequalities of all the facets of the polyhedron  $P(\mathcal{D})$  which include the vertex  $\chi_A$  are given by

$$x(e) \geq 0 \quad (e \in E^+ - A), \quad (6.54)$$

$$x(e) - x(e') \leq 0 \quad (e \text{ covers } e' \text{ in } \mathcal{P} \text{ and either } e, e' \in A \text{ or } e, e' \notin A), \quad (6.55)$$

$$x(e) \leq 1 \quad (e \in E^- \cap A). \quad (6.56)$$

(Proof) The lemma follows from the polarity between  $P(\mathcal{D})$  with  $\chi_A$  regarded as the origin and the characteristic cone  $C_\mu(A)$  of  $\partial\mu(A)$ . Notice the one-to-one correspondence between the set of facets (6.54)–(6.56) and the set  $\text{ER}(A)$  of the extreme rays of  $C_\mu(A)$  given by (5.57) with (5.54)–(5.56). Q.E.D.

**Corollary 6.7:** All the facet inequalities of  $P(\mathcal{D})$  are given by

$$x(e) \geq 0 \quad (e \in E^+), \quad (6.57)$$

$$x(e) - x(e') \leq 0 \quad (e \text{ covers } e' \text{ in } \mathcal{P}), \quad (6.58)$$

$$x(e) \leq 1 \quad (e \in E^-). \quad (6.59)$$

The problem of minimizing the modular function  $\mu: \mathcal{D} \rightarrow \mathbb{R}$  is reduced to the problem of minimizing the linear function

$$(\nu, x) = \sum_{e \in E} \nu(e) x(e) \quad (6.60)$$

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over  $P(\mathcal{D})$ , where  $\nu$  is the vector appearing in Lemma 6.5. Therefore, it follows from Lemma 6.6 that  $A \in \mathcal{D}$  is a minimizer of  $\mu$  (or  $\chi_A$  is a minimizer of  $(\nu, x)$  over  $P(\mathcal{D})$ ) if and only if the vector  $-\nu$  is expressed as a nonnegative linear combination of vectors  $\xi_{p^+}$  ( $p^+ \in E^+ - A$ ),  $\zeta_a$  ( $a \in B^*(\mathcal{P}) - \Delta^-(A)$ ) and  $\eta_{p^-}$  ( $p^- \in E^- \cap A$ ) which are coefficient vectors of (6.54)–(6.56), where inequalities (6.54) should be considered in the form of  $-x(e) \leq 0$  ( $e \in E^+ - A$ ).

#### 6.2. Submodular Programs – Constrained Optimization

We consider the problem of minimizing a submodular function  $f: \mathcal{D} \rightarrow \mathbb{R}$  with constraints on the domain  $\mathcal{D}$  of  $f$  and discuss some other related problems.

##### (a) Lagrangian functions and optimality conditions

Suppose that a sublattice  $\mathcal{D}_0$  of  $\mathcal{D}$  is given. We say that a vector  $a \in \mathbb{R}^E$  is *normal to  $\mathcal{D}_0$  at  $A \in \mathcal{D}_0$*  if for each  $X \in \mathcal{D}_0$

$$a(X) - a(A) \leq 0. \quad (6.61)$$

The following theorem characterizes the minimizers of  $f$  when the domain of  $f$  is restricted to a sublattice of  $\mathcal{D}$ .

**Theorem 6.8** (cf. [Rockafellar70, Theorem 27.4]): Let  $f: \mathcal{D} \rightarrow \mathbb{R}$  be a submodular function and  $\mathcal{D}_0$  be a sublattice of  $\mathcal{D}$ . Then, for  $A \in \mathcal{D}_0$  we have

$$f(A) = \min\{f(X) \mid X \in \mathcal{D}_0\} \quad (6.62)$$

if and only if there exists a subgradient  $a \in \partial f(A)$  such that  $-a$  is normal to  $\mathcal{D}_0$  at  $A$ .

(Proof) *The “if” part:* From the assumption, we have for any  $X \in \mathcal{D}_0$

$$0 \leq a(X) - a(A) \leq f(X) - f(A), \quad (6.63)$$

from which (6.62) follows.

*The “only if” part:* Define a modular function  $\mu_0: \mathcal{D}_0 \rightarrow \mathbb{R}$  by

$$\mu_0(X) = 0 \quad (X \in \mathcal{D}_0). \quad (6.64)$$

Then, by the assumption  $A$  is a minimizer of  $f_0 \equiv f + \mu_0: \mathcal{D}_0 \rightarrow \mathbb{R}$ . It follows from Theorem 5.8 and Lemma 6.1 that

$$0 \in \partial f_0(A) = \partial f(A) + \partial \mu_0(A). \quad (6.65)$$



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Hence, there exists a vector  $a \in \partial f(A)$  such that

$$-a \in \partial \mu_0(A). \quad (6.66)$$

From (6.64), (6.66) implies that  $-a$  is normal to  $\mathcal{D}_0$  at  $A$ .

Q.E.D.

Next, consider a constrained minimization problem for which the "feasible region"  $\mathcal{D}_0$  in (6.62) is defined by a set of equations. Suppose that we are given submodular functions  $f_i: \mathcal{D} \rightarrow \mathbb{R}$  ( $i = 0, 1, \dots, m$ ) and that the minimum value of  $f_i$  for each  $i = 1, 2, \dots, m$  is known and equal to  $\alpha_i$ . The feasible region  $\mathcal{D}_0$  is given by

$$\mathcal{D}_0 = \{X \mid X \in \mathcal{D}, f_i(X) = \alpha_i \ (i = 1, 2, \dots, m)\}. \quad (6.67)$$

Here, we assume  $\mathcal{D}_0 \neq \emptyset$ . From Lemma 1.1,  $\mathcal{D}_0$  is a sublattice of  $\mathcal{D}$ .

Let us consider the following constrained minimization problem:

$$\text{Minimize } f_0(X) \quad (2.68a)$$

$$\text{subject to } f_i(X) = \alpha_i \ (i = 1, 2, \dots, m). \quad (6.68b)$$

Define a function  $L: \mathbb{R}_+^m \times \mathcal{D} \rightarrow \mathbb{R}$  by

$$L(\lambda, X) = f_0(X) + \sum_{i=1}^m \lambda_i (f_i(X) - \alpha_i) \quad (6.69)$$

for  $\lambda \in \mathbb{R}_+^m$  and  $X \in \mathcal{D}$ . We call  $L$  the *Lagrangian function* associated with Problem (6.68). Also, we call  $\lambda \in \mathbb{R}_+^m$  an *optimal Lagrange multiplier* if

$$\min\{L(\lambda, X) \mid X \in \mathcal{D}\} \quad (6.70)$$

is equal to the optimal value of the objective function of Problem (6.68).

**Theorem 6.9** (cf. [Rockafellar70, Theorem 28.3]): For Problem (6.68),

(1)  $\hat{\lambda} = (\hat{\lambda}_1, \dots, \hat{\lambda}_m)$  is an optimal Lagrange multiplier and

(2)  $\hat{X}$  is an optimal solution of Problem (6.68)

if and only if

(i)  $\hat{\lambda} \in (\hat{\lambda}_1, \dots, \hat{\lambda}_m) \in \mathbb{R}_+^m$ ,

(ii)  $\hat{X}$  is a feasible solution of Problem (6.68) and

(iii)  $0 \in \partial f_0(\hat{X}) + \hat{\lambda}_1 \partial f_1(\hat{X}) + \dots + \hat{\lambda}_m \partial f_m(\hat{X})$ .

(Proof) First, note that for each  $i = 1, \dots, m$ ,  $\partial f_i(X)$  and  $\partial f_0(X)$  has the same characteristic cone, since the characteristic cone is determined by the underlying distributive lattice  $\mathcal{D}$  alone. Therefore,

$$\partial f_0(X) = \partial f_0(X) + 0^+ \partial f_i(X) \ (i = 1, 2, \dots, m). \quad (6.71)$$

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Hence, from Theorem 5.8 and Lemma 6.1, (iii) is equivalent to

$$0 \in \partial_X L(\hat{\lambda}, \hat{X}), \quad (6.72)$$

where  $\partial_X L(\hat{\lambda}, \hat{X})$  denotes the subdifferential of the submodular function  $L(\hat{\lambda}, \cdot): \mathcal{D} \rightarrow \mathbb{R}$  at  $\hat{X}$ .

*The "if" part:* From (i)–(iii) and (6.72) we have

$$\min\{L(\hat{\lambda}, X) \mid X \in \mathcal{D}\} = L(\hat{\lambda}, \hat{X}) = f_0(\hat{X}), \quad (6.73)$$

while we have for any feasible solution  $Y$

$$\min\{L(\hat{\lambda}, X) \mid X \in \mathcal{D}\} \leq L(\hat{\lambda}, Y) = f_0(Y). \quad (6.74)$$

Hence (1) and (2) follow.

*The "only if" part:* From (1) and (2),

$$\hat{X} \in \mathcal{D}_0, \quad (6.75)$$

$$\min\{L(\hat{\lambda}, X) \mid X \in \mathcal{D}\} = f_0(\hat{X}) = L(\hat{\lambda}, \hat{X}). \quad (6.76)$$

Therefore, we have (6.72) or (iii). (i) and (ii) trivially follow from (1) and (2).  
Q.E.D.

The above proof almost parallels that of the corresponding theorem for ordinary convex functions ([Rockafellar70, Theorem 28.3]). It should, however, be noted that the above proof heavily depends on the results in Section 5.2, especially Theorem 5.8.

Define a function  $p: \mathbb{R}_+^m \rightarrow \mathbb{R} \cup \{+\infty\}$  by

$$p(u) = \min\{f_0(X) \mid X \in \mathcal{D}, \forall i \in \{1, \dots, m\}: f_i(X) - \alpha_i \leq u_i\}, \quad (6.77)$$

where  $u = (u_1, \dots, u_m) \in \mathbb{R}_+^m$ . We define  $p(u) = +\infty$  for  $u$  for which there exists no  $X \in \mathcal{D}$  such that  $f_i(X) - \alpha_i \leq u_i$  for all  $i = 1, \dots, m$ . We call  $p$  the *perturbation function* associated with Problem (6.68).

**Theorem 6.10:** For the perturbation function  $p$  associated with Problem (6.68) we have for each  $\lambda \in \mathbb{R}_+^m$

$$\begin{aligned} & \min\{p(u) + \lambda_1 u_1 + \dots + \lambda_m u_m \mid u \in \mathbb{R}_+^m\} \\ &= \min\{L(\lambda, X) \mid X \in \mathcal{D}\}. \end{aligned} \quad (6.78)$$

(Proof) Suppose that  $\hat{X} \in \mathcal{D}$  is a minimizer of  $L(\lambda, X)$  in  $X \in \mathcal{D}$ . Then, from the definition of  $p(u)$ ,

$$L(\lambda, \hat{X}) = p(\bar{u}) + \lambda_1 \bar{u}_1 + \dots + \lambda_m \bar{u}_m, \quad (6.79)$$

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where  $\bar{u}_i = f_i(\hat{X}) - \alpha_i$  ( $i = 1, \dots, m$ ). Hence we have

$$\begin{aligned} & \min\{p(u) + \lambda_1 u_1 + \dots + \lambda_m u_m \mid u \in \mathbb{R}_+^m\} \\ & \leq \min\{L(\lambda, X) \mid X \in \mathcal{D}\}. \end{aligned} \quad (6.80)$$

On the other hand, suppose that  $\hat{u} \in \mathbb{R}_+^m$  is a minimizer of  $p(u) + \lambda_1 u_1 + \dots + \lambda_m u_m$  in  $u \in \mathbb{R}_+^m$ . Then, there exists an  $X_0 \in \mathcal{D}$  such that

$$p(\hat{u}) = f_0(X_0), \quad (6.81)$$

$$f_i(X_0) - \alpha_i \leq \hat{u}_i \quad (i = 1, \dots, m). \quad (6.82)$$

Since  $\lambda \in \mathbb{R}_+^m$ , we have from (6.81) and (6.82)

$$p(\hat{u}) + \lambda_1 \hat{u}_1 + \dots + \lambda_m \hat{u}_m \geq L(\lambda, X_0). \quad (6.83)$$

Therefore,

$$\begin{aligned} & \min\{p(u) + \lambda_1 u_1 + \dots + \lambda_m u_m \mid u \in \mathbb{R}_+^m\} \\ & \geq \min\{L(\lambda, X) \mid X \in \mathcal{D}\}. \end{aligned} \quad (6.84)$$

The present theorem follows from (6.80) and (6.84). Q.E.D.

In the proof of Theorem 6.10 we have already shown the following.

**Theorem 6.11:** For each  $\lambda \in \mathbb{R}_+^m$ ,

(a) if  $\hat{X} \in \mathcal{D}$  is a minimizer of  $L(\lambda, X)$  in  $X \in \mathcal{D}$ , then  $\hat{u} = (\hat{u}_1, \dots, \hat{u}_m)$  given by

$$\hat{u}_i = f_i(\hat{X}) - \alpha_i \quad (i = 1, \dots, m) \quad (6.85)$$

is a minimizer of  $p(u) + \lambda_1 u_1 + \dots + \lambda_m u_m$  in  $u \in \mathbb{R}_+^m$ ;

(b) if  $\hat{u} = (\hat{u}_1, \dots, \hat{u}_m)$  is a minimizer of  $p(u) + \lambda_1 u_1 + \dots + \lambda_m u_m$  in  $u \in \mathbb{R}_+^m$ , then there exists an  $\hat{X} \in \mathcal{D}$  such that

$$p(\hat{u}) = f_0(\hat{X}), \quad (6.86)$$

$$f_i(\hat{X}) - \alpha_i \leq \hat{u}_i \quad (i = 1, \dots, m), \quad (6.87)$$

and  $\hat{X}$  is a minimizer of  $L(\lambda, X)$  in  $X \in \mathcal{D}$ . Here, for each  $i = 1, \dots, m$ , (6.87) holds with equality if  $\lambda_i > 0$ , while  $\lambda_i = 0$  if (6.87) for  $i$  holds with strict inequality.

Furthermore, we have

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**Theorem 6.12** (cf. [Rockafellar70, Theorem 29.1]): A vector  $\hat{\lambda} \in \mathbb{R}_+^m$  is an optimal Lagrange multiplier of Problem (6.68) if and only if

$$\min\{p(u) + \hat{\lambda}_1 u_1 + \cdots + \hat{\lambda}_m u_m \mid u \in \mathbb{R}_+^m\} = p(0). \quad (6.88)$$

(Proof) (6.88) means that the zero vector  $0 \in \mathbb{R}_+^m$  is a minimizer of  $p(u) + \hat{\lambda}_1 u_1 + \cdots + \hat{\lambda}_m u_m$  in  $u \in \mathbb{R}_+^m$ . Hence, the present theorem follows from Theorems 6.9 and 6.11. Q.E.D.

We see from Theorems 6.11 and 6.12 that if  $\hat{\lambda} \in \mathbb{R}_+^m$  is an optimal Lagrange multiplier, then any  $\lambda \in \mathbb{R}_+^m$  such that  $\hat{\lambda} \leq \lambda$  is also an optimal one. Note that (6.87) for  $i$  such that  $\hat{u}_i = 0$ , holds with equality since  $f_i(X) - \alpha_i \geq 0$  for any  $X \in \mathcal{D}$ .

An algorithm for finding an optimal solution and an optimal Lagrange multiplier for Problem (6.68) is given as follows. For simplicity, we consider the case when  $m = 1$ .

**An algorithm for solving Problem (6.68) (with  $m = 1$ )**

**Step 1:** Let  $a_0$  be an upper bound of  $f_0$  such that  $a_0 > f_0(X)$  for all  $X \in \mathcal{D}$  (with strict inequality) and let  $\alpha_0$  be a lower bound of  $f_0$ . Also let  $a_1$  be an upper bound of  $f_1$  with  $a_1 > \alpha_1$ . Put  $\lambda \leftarrow (a_0 - \alpha_0)/(a_1 - \alpha_1)$ .

**Step 2:** Put  $\hat{X} \leftarrow$  a minimizer of  $L(\lambda, X) = f_0(X) + \lambda(f_1(X) - \alpha_1)$  in  $X \in \mathcal{D}$ .

**Step 3:** If  $f_1(\hat{X}) - \alpha_1 = 0$ , then stop ( $\hat{X}$  is an optimal solution and  $\lambda$  is an optimal Lagrange multiplier). Otherwise, put  $\lambda \leftarrow (a_0 - f_0(\hat{X}))/(f_1(\hat{X}) - \alpha_1)$  and go back to Step 2.

(End)

The validity of the above algorithm follows from Theorems 6.8, 6.11 and 6.12. The case when  $m > 1$  can also be treated by the algorithm by setting  $f_1 \leftarrow f_1 + f_2 + \cdots + f_m$ . If  $\min\{f_1(X) + f_2(X) + \cdots + f_m(X) \mid X \in \mathcal{D}\} \neq \alpha_1 + \alpha_2 + \cdots + \alpha_m$ , then there is no feasible solution.

The upper bounds of the submodular functions required in Step 1 are obtained by adapting (2.78). When  $f_i$  ( $i = 0, 1$ ) are integer-valued, the above algorithm terminates after repeating the cycle of Steps 2 and 3 at most  $2 \min\{a_0 - \alpha_0, a_1 - \alpha_1\}$ , where  $a_i$  ( $i = 0, 1$ ) are integral upper bounds.

For each  $\lambda \in \mathbb{R}_+^m$  denote by  $\mathcal{L}(\lambda)$  the set of minimizers of the Lagrangian function

$$L(\lambda, X) = f_0(X) + \lambda_1 f_1(X) + \cdots + \lambda_m f_m(X) \quad (6.89)$$

in  $X \in \mathcal{D}$ .  $\mathcal{L}(\lambda)$  is a sublattice of  $\mathcal{D}$ . Because of the finiteness character of the problem there are a finite number of distinct  $\mathcal{L}(\lambda)$ 's ( $\lambda \in \mathbb{R}_+^m$ ). The structure

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of these  $\mathcal{L}(\lambda)$ 's is closely related to the *principal partition* ([Kishi + Kajitani68], [Iri71], [Tomi76], [Fuji78c], [Iri79], [Nakamura + Iri81], [Iri84], [Nakamura88], [Tomi + Fuji82]), which will be discussed in the subsequent subsection.

### (b) Related problems

We discuss other problems related to submodular programs.

#### (b.1) The principal partition [Iri79], [Nakamura + Iri81], [Tomi + Fuji82]

For simplicity let us consider the Lagrangian function  $L(\lambda, X)$  of (6.89) for the case when  $m = 1$  and  $\alpha_1 = 0$ .  $(\mathcal{D}, f_i)$  ( $i = 1, 2$ ) are submodular systems on  $E$ . We also suppose that  $f_1$  is monotone nondecreasing. This is an essential assumption in the following argument.

Consider the Lagrangian function

$$L(\lambda, X) = f_0(X) + \lambda f_1(X) \quad (6.90)$$

for  $\lambda \geq 0$  and  $X \in \mathcal{D}$ . We also define  $L(\lambda, X)$  for  $\lambda < 0$  as

$$L(\lambda, X) = f_0(X) + \lambda f_1^\#(X), \quad (6.91)$$

where  $f_1^\#$  is the dual supermodular function of  $f_1$ , i.e.,  $f_1^\#(X) = f_1(E) - f_1(E - X)$  ( $X \in \mathcal{D}$ ). For each  $\lambda \in \mathbb{R}$  let  $\mathcal{L}(\lambda)$  be the set of minimizers of  $L(\lambda, X)$  in  $X \in \mathcal{D}$ .  $\mathcal{L}(\lambda)$  is a sublattice of  $\mathcal{D}$ .

**Theorem 6.13** ([Tomi + Fuji82]): Define

$$\mathcal{L}^* = \bigcup_{\lambda \in \mathbb{R}} \mathcal{L}(\lambda). \quad (6.92)$$

$\mathcal{L}^*$  is a sublattice of  $\mathcal{D}$ . More precisely, for any  $\lambda$  and  $\lambda'$  with  $\lambda \leq \lambda'$  and for any  $X \in \mathcal{L}(\lambda)$  and  $X' \in \mathcal{L}(\lambda')$ , we have

$$X \cup X' \in \mathcal{L}(\lambda), \quad X \cap X' \in \mathcal{L}(\lambda'). \quad (6.93)$$

(Proof) If  $\lambda = \lambda'$ , (6.93) holds.

*Case 1:*  $0 \leq \lambda < \lambda'$

For any  $X \in \mathcal{L}(\lambda)$  and  $X' \in \mathcal{L}(\lambda')$  we have

$$\begin{aligned} & L(\lambda, X) + L(\lambda', X') \\ &= f_0(X) + \lambda f_1(X) + f_0(X') + \lambda' f_1(X') \\ &\geq f_0(X \cup X') + \lambda f_1(X \cup X') + f_0(X \cap X') + \lambda' f_1(X \cap X') \\ &\quad + (\lambda' - \lambda)(f_1(X') - f_1(X \cap X')) \\ &\geq L(\lambda, X \cup X') + L(\lambda', X \cap X') \end{aligned} \quad (6.94)$$

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due to the submodularity of  $f_0$  and  $f_1$  and the monotonicity of  $f_1$ . It follows from (6.94) that

$$L(\lambda, X \cup X') = L(\lambda, X), \quad (6.95)$$

$$L(\lambda', X \cap X') = L(\lambda', X') \quad (6.96)$$

(and  $f_1(X') = f_1(X \cap X')$ ). We thus have (6.93).

*Case 2:*  $\lambda < 0 < \lambda'$

Similarly as (6.94), we have

$$\begin{aligned} & L(\lambda, X) + L(\lambda', X') \\ &= f_0(X) + \lambda f_1^\#(X) + f_0(X') + \lambda' f_1(X') \\ &\geq f_0(X \cup X') + \lambda f_1^\#(X \cup X') + f_0(X \cap X') + \lambda' f_1(X \cap X') \\ &\quad + \lambda(f_1^\#(X \cap X') - f_1^\#(X')) + \lambda'(f_1(X \cup X') - f_1(X)) \\ &\geq L(\lambda, X \cup X') + L(\lambda', X \cap X'). \end{aligned} \quad (6.97)$$

From this we have (6.93).

*Case 3:*  $\lambda < \lambda' \leq 0$

Similarly, we have

$$\begin{aligned} & L(\lambda, X) + L(\lambda', X') \\ &= f_0(X) + \lambda f_1^\#(X) + f_0(X') + \lambda' f_1^\#(X') \\ &\geq f_0(X \cup X') + \lambda f_1^\#(X \cup X') + f_0(X \cap X') + \lambda' f_1^\#(X \cap X') \\ &\quad + (\lambda' - \lambda)(f_1^\#(X \cup X') - f_1^\#(X)) \\ &\geq L(\lambda, X \cup X') + L(\lambda', X \cap X'). \end{aligned} \quad (6.98)$$

Hence we have (6.93). Q.E.D.

Denote by  $S^+(\lambda)$  the unique maximal element of  $\mathcal{L}(\lambda)$  and by  $S^-(\lambda)$  the unique minimal element of  $\mathcal{L}(\lambda)$  for each  $\lambda \in \mathbb{R}$ .

**Theorem 6.14:** For any  $\lambda$  and  $\lambda'$  with  $\lambda < \lambda'$  we have

$$S^+(\lambda) \subseteq S^+(\lambda'), \quad S^-(\lambda) \subseteq S^-(\lambda'). \quad (6.99)$$

(Proof) For any  $X \in \mathcal{L}(\lambda)$  and  $X' \in \mathcal{L}(\lambda')$  we have from Theorem 6.13

$$X \cup S^+(\lambda') \in \mathcal{L}(\lambda'), \quad S^-(\lambda) \cap X' \in \mathcal{L}(\lambda). \quad (6.100)$$

This implies  $X \subseteq S^+(\lambda')$  ( $X \in \mathcal{L}(\lambda)$ ) and  $S^-(\lambda) \subseteq X'$  ( $X' \in \mathcal{L}(\lambda')$ ). Hence we have (6.99). Q.E.D.

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We further suppose that  $f_1$  is monotone increasing. Then, for a sufficiently large  $\lambda > 0$  we have  $\mathcal{L}(\lambda) = \{\emptyset\}$  and for a sufficiently small  $\lambda < 0$ ,  $\mathcal{L}(\lambda) = \{E\}$ .

**Theorem 6.15:** Suppose that  $f_1$  is monotone increasing. Then, there exists a finite sequence of reals

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p \quad (6.101)$$

such that the distinct  $\mathcal{L}(\lambda)$  ( $\lambda \in \mathbb{R}$ ) are given by

$$\mathcal{L}(\lambda_i) \quad (i = 1, 2, \dots, p), \quad (6.102)$$

$$\{S^+(\lambda_i)\} (= \{S^-(\lambda_{i+1})\}) \quad (i = 2, \dots, p-1), \quad (6.103)$$

$$\{S^+(\lambda_1)\} (= \{E\}), \quad \{S^-(\lambda_p)\} (= \{\emptyset\}). \quad (6.104)$$

For any  $i \in \{1, 2, \dots, p-1\}$  and any  $\lambda$  such that  $\lambda_i < \lambda < \lambda_{i+1}$  we have

$$\mathcal{L}(\lambda) = \{S^+(\lambda_i)\} = \{S^-(\lambda_{i+1})\}. \quad (6.105)$$

Also,

$$\mathcal{L}(\lambda) = \begin{cases} \{E\} & (\lambda < \lambda_1) \\ \{\emptyset\} & (\lambda_p < \lambda). \end{cases} \quad (6.106)$$

(Proof) Because of the finiteness character there exist finitely many distinct  $\mathcal{L}(\lambda)$  ( $\lambda \in \mathbb{R}$ ). Choose any  $\hat{\lambda} \in \mathbb{R}$ . If  $\mathcal{L}(\hat{\lambda})$  contains more than two elements, there exist some  $X, X' \in \mathcal{L}$  with  $X \subset X'$  and

$$f_0(X) + \hat{\lambda} f_1(X) = f_0(X') + \hat{\lambda} f_1(X'). \quad (6.107)$$

Since  $f_1(X) < f_1(X')$  by the assumption, the value of  $\hat{\lambda}$  is uniquely determined from (6.107). If  $\mathcal{L}(\hat{\lambda})$  contains only one element, then from the finiteness character there exists an open interval  $(\lambda', \lambda'')$  such that  $\hat{\lambda} \in (\lambda', \lambda'')$  and

$$\mathcal{L}(\lambda) = \mathcal{L}(\hat{\lambda}) \quad (\lambda \in (\lambda', \lambda'')). \quad (6.108)$$

It follows that there exists a finite sequence of reals

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p \quad (6.109)$$

such that distinct  $\mathcal{L}(\lambda)$  ( $\lambda \in \mathbb{R}$ ) are given by

$$\mathcal{L}(\lambda_i) \quad (i = 1, 2, \dots, p), \quad (6.110)$$

$$\mathcal{L}(\lambda) \quad (\lambda \in (\lambda_i, \lambda_{i+1}), i = 0, 1, \dots, p), \quad (6.111)$$

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where  $\lambda_0 \equiv -\infty$ ,  $\lambda_{p+1} \equiv +\infty$ ,  $\mathcal{L}(\lambda)$  are the same in each interval  $(\lambda_i, \lambda_{i+1})$  ( $i = 0, 1, \dots, p$ ),  $|\mathcal{L}(\lambda_i)| \geq 2$  ( $i = 1, 2, \dots, p$ ) and  $|\mathcal{L}(\lambda)| = 1$  ( $\lambda \in (\lambda_i, \lambda_{i+1})$ ,  $i = 0, 1, \dots, p$ ). Moreover, for each  $i = 1, 2, \dots, p$ , because of the finiteness character there exists a (sufficiently small) positive number  $\epsilon$  such that

$$\mathcal{L}(\lambda_i - \epsilon) \subseteq \mathcal{L}(\lambda_i), \quad (6.112)$$

$$\mathcal{L}(\lambda_i + \epsilon) \subseteq \mathcal{L}(\lambda_i). \quad (6.113)$$

Since  $f_1$  is monotone increasing, we have from (6.112) and (6.113)

$$S^+(\lambda_i) \in \mathcal{L}(\lambda_i - \epsilon), \quad (6.114)$$

$$S^-(\lambda_i) \in \mathcal{L}(\lambda_i + \epsilon). \quad (6.115)$$

From (6.114) and (6.115) we have (6.103)–(6.106). Q.E.D.

The  $\lambda_i$  ( $i = 1, 2, \dots, p$ ) in (6.101) are called *critical values* for the pair of the submodular systems  $(\mathcal{D}, f_0)$  and  $(\mathcal{D}, f_1)$ . Denote  $S_0 = (\mathcal{D}, f_0)$  and  $S_1 = (\mathcal{D}, f_1)$ . The submodular systems  $S_i$  ( $i = 0, 1$ ) are decomposed according to the distributive lattice  $\mathcal{L}^* = \bigcup_{\lambda \in \mathbb{R}} \mathcal{L}(\lambda)$  as follows. Choose any maximal chain

$$C: \emptyset = A_0 \subset A_1 \subset \dots \subset A_k = E \quad (6.116)$$

of  $\mathcal{L}^*$  and then decompose  $S_i$  ( $i = 0, 1$ ) into their minors

$$S_i \cdot A_j / A_{j-1} \quad (j = 1, 2, \dots, k), \quad (6.117)$$

where  $S_i \cdot A_j / A_{j-1}$  is the set minor of  $S_i$  obtained by restricting  $S_i$  to  $A_j$  and contracting  $A_{j-1}$ . Such a set of decompositions of  $S_i$  ( $i = 0, 1$ ) is called the *principal partition* of the pair of  $S_i$  ( $i = 0, 1$ ). By the poset on the partition  $\{A_j - A_{j-1} \mid j = 1, 2, \dots, k\}$  of  $E$  which is uniquely determined by  $\mathcal{L}^*$  (see Section 2.2.a) the corresponding poset structure is defined on the set of minors (6.117) for each  $i = 0, 1$ . We can show that the decompositions (6.117) do not depend on the choice of a maximal chain in  $\mathcal{L}^*$  ([Nakamura + Iri81], [Tomi + Fuji82]).

The concept of principal partition was originated by G. Kishi and Y. Kajitani [Kishi + Kajitani68] for graphs, where  $S_0$  is a graphic matroid and  $S_1 = (2^E, 1)$  with uniform modular function  $1(X) = |X|$  ( $X \subseteq E$ ). It was generalized to a pair of graphs [Ozawa74]; to a pair of a matrix and a uniform modular function [Iri71]; to a pair of a matroid and a uniform modular function [Narayanan74], [Tomi76] (also [Bruno + Weinberg71]); to a pair of a polymatroid and a positive modular function [Fuji78c], [Fuji80b]; to a pair of polymatroids [Iri79]; to a pair of a submodular system and a modular function



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[Fuji80c]; and to a pair of submodular systems [Nakamura + Iri81], [Tomi80d], [Tomi + Fuji81]. The concept has effectively been applied to electrical network analysis [Iri + Tomi75], [Narayanan74], network flows [Fuji80b], structure and scene analyses [Sugihara82] etc. (Also see [Iri79], [Iri83], [Iri + Fuji81], [Tomi + Fuji82]).

The concept of principal partition is closely related to convex optimization problems over base polyhedra, which will be treated in Chapter V.

Finally, it should also be noted that the above argument is also valid with an appropriate modification if  $f_1$  is a negative submodular function except that  $f_1(\emptyset) = 0$ .

### (b.2) The minimum-ratio problem

Suppose that  $f: \mathcal{D} \rightarrow \mathbb{R}$  is a nonnegative submodular function with  $f(\emptyset) = 0$  and that  $g: \mathcal{D} \rightarrow \mathbb{R}$  is a supermodular function with  $g(\emptyset) = 0$  and  $g(X) > 0$  ( $X \in \mathcal{D} - \{\emptyset\}$ ).

Consider the following problem:

$$\text{Minimize } f(X)/g(X) \quad (6.118a)$$

$$\text{subject to } X \in \mathcal{D} - \{\emptyset\}. \quad (6.118b)$$

This is called a *minimum-ratio problem*. Special cases of the problem have been treated in [Brown79] and [Ichimori + Ishii + Nishida82] as a sharing problem in a network (also see [Megiddo74], [Fuji80b]), in [Cunningham85b] as a minimum-cost problem of disconnecting a network, and in [Goldberg83] as a maximum-density subgraph problem. The minimum-ratio problem given by (6.118) is closely related to the principal partition discussed in the preceding subsection (b.1).

Define a Lagrangian function for  $f$  and  $-g$  associated with Problem (6.118) by

$$L(\lambda, X) = f(X) - \lambda g(X) \quad (6.119)$$

for  $\lambda \geq 0$  and  $X \in \mathcal{D}$ .

**Theorem 6.16:** A nonnegative  $\hat{\lambda}$  is the minimum value of the ratio of Problem (6.118) if and only if

$$\min\{L(\lambda, X) \mid X \in \mathcal{D}\} = 0 \quad (0 \leq \lambda \leq \hat{\lambda}), \quad (6.120)$$

$$\min\{L(\lambda, X) \mid X \in \mathcal{D}\} < 0 \quad (\hat{\lambda} < \lambda). \quad (6.121)$$

(Proof) Suppose that  $\hat{\lambda}$  is the minimum value of the ratio of Problem (6.118). Then we have

$$L(\hat{\lambda}, X) = f(X) - \hat{\lambda}g(X) \geq 0 \quad (X \in \mathcal{D} - \{\emptyset\}), \quad (6.122)$$

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where (6.122) holds with equality for some  $\hat{X} \in \mathcal{D} - \{\emptyset\}$ . It follows that

$$L(\lambda, X) \geq L(\hat{\lambda}, X) \geq 0 \quad (0 \leq \lambda \leq \hat{\lambda}, X \in \mathcal{D}), \quad (6.123)$$

$$L(\lambda, \hat{X}) < L(\hat{\lambda}, \hat{X}) = 0 \quad (\hat{\lambda} < \lambda). \quad (6.124)$$

Since  $L(\lambda, \emptyset) = 0$  for any  $\lambda \geq 0$ , from (6.123) and (6.124) we have (6.120) and (6.121).

Conversely, suppose (6.120) and (6.121) hold. Since  $\hat{\lambda}$  is uniquely determined by the Lagrangian function  $L(\lambda, X)$ , it follows from the previous argument that  $\hat{\lambda}$  must be the minimum value of the ratio of Problem (6.118).

Q.E.D.

It should be noted that Theorem 6.16 holds for  $f$  and  $g$  without submodularity or supermodularity, though the problem would become hard without submodularity or supermodularity. A minimum-ratio problem for general set functions has also been discussed by Cunningham [Cunningham83].

From Theorem 6.16, the minimum-ratio problem (6.118) is reduced to the problem of finding the minimum critical value  $\lambda_1 (= \hat{\lambda})$  for  $L(\lambda, X)$ . The minimum critical value  $\lambda_1$  can be obtained by a binary search in  $\mathbb{R}_+$  based on Theorem 6.16 (cf. [Goldberg83], [Imai83] for special cases and also see [Cunningham85b]). Also a dichotomy works for finding an element of  $\mathcal{L}(\lambda_1)$  from among  $\mathcal{L}^* = \cup\{\mathcal{L}(\lambda) \mid \lambda \geq 0\}$  by searching a chain of  $\mathcal{L}^*$  (cf. [Fuji80b], [Nakamura + Iri81], [Tomi80d], [Tomi + Fuji82]).

## Chapter V.

### Nonlinear Optimization with Submodular Constraints

#### 7. Separable Convex Optimization

Let  $(\mathcal{D}, f)$  be a submodular system on  $E$  with a real-valued (or rational-valued) rank function  $f$ . The underlying totally ordered additive group is assumed to be the set  $\mathbb{R}$  of reals (or the set  $\mathbb{Q}$  of rationals) unless otherwise stated.

For each  $e \in E$  let  $w_e: \mathbb{R} \rightarrow \mathbb{R}$  be a real-valued convex function on  $\mathbb{R}$ , and consider the following problem

$$P_1: \text{Minimize } \sum_{e \in E} w_e(x(e)) \quad (7.1a)$$

$$\text{subject to } x \in B(f). \quad (7.1b)$$

Problem  $P_1$  was first considered by the author [Fuji80b] for the case where for each  $e \in E$   $w_e(x(e))$  is a quadratic function given by  $x(e)^2/w(e)$  with a positive real weight  $w(e)$  and  $f$  is a polymatroid rank function. H. Groenevelt [Groenevelt85] also considered Problem  $P_1$  where each  $w_e$  is a convex function and  $f$  is a polymatroid rank function.

##### 7.1. Optimality Conditions

It is almost straightforward to generalize the result of [Fuji80b] and [Groenevelt85] to Problem  $P_1$  for a general submodular system.

Optimal solutions of Problem  $P_1$  are characterized as follows.

**Theorem 7.1** ([Groenevelt85]; also see [Fuji80b]): A base  $x \in B(f)$  is an optimal solution of Problem  $P_1$  if and only if for each exchangeable pair  $(e, e')$  associated with base  $x$  (i.e.,  $e \in E$  and  $e' \in \text{dep}(x, e) - \{e\}$ ), we have

$$w_e^+(x(e)) \geq w_{e'}^-(x(e')), \quad (7.2)$$

where  $w_e^+$  denotes the right derivative of  $w_e$  and  $w_{e'}^-$  the left derivative of  $w_{e'}$ .

(Proof) *The "if" part:* Denote by  $A_x$  the set of all the exchangeable pairs  $(e, e')$  associated with  $x$ . Suppose that (7.2) holds for each  $(e, e') \in A_x$ . From

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Theorem 2.23, for any base  $z \in B(f)$  there exist some nonnegative coefficients  $\lambda(e, e')$   $((e, e') \in A_x)$  such that

$$z = x + \sum \{ \lambda(e, e')(\chi_e - \chi_{e'}) \mid (e, e') \in A_x \}. \quad (7.3)$$

For each  $e \in E$  define

$$\bar{w}_e(x(e)) = \max \{ w_{e'}^-(x(e')) \mid e' \in \text{dep}(x, e) \}. \quad (7.4)$$

We see from (7.2) and (7.4) that

$$w_e^-(x(e)) \leq \bar{w}_e(x(e)) \leq w_e^+(x(e)) \quad (e \in E), \quad (7.5)$$

$$\bar{w}_e(x(e)) \geq \bar{w}_{e'}(x(e')) \quad ((e, e') \in A_x). \quad (7.6)$$

From (7.3)–(7.6) and the convexity of  $w_e$  ( $e \in E$ ) we have

$$\begin{aligned} \sum_{e \in E} w_e(z(e)) &= \sum_{e \in E} w_e(x(e) + \partial\lambda(e)) \\ &\geq \sum_{e \in E} \{ w_e(x(e)) + \partial\lambda(e) \cdot \bar{w}_e(x(e)) \} \\ &= \sum_{e \in E} w_e(x(e)) + \sum_{(e, e') \in A_x} \lambda(e, e') (\bar{w}_e(x(e)) - \bar{w}_{e'}(x(e'))) \\ &\geq \sum_{e \in E} w_e(x(e)), \end{aligned} \quad (7.7)$$

where  $\partial\lambda: E \rightarrow \mathbb{R}$  is defined by

$$\partial\lambda(e) = \sum_{(e, e') \in A_x} \lambda(e, e') - \sum_{(e', e) \in A_x} \lambda(e', e) \quad (7.8)$$

for each  $e \in E$ . (7.7) shows the optimality of  $x$ .

*The “only if” part:* Suppose that for a base  $x \in B(f)$  there exists an exchangeable pair  $(e, e')$  associated with  $x$  such that

$$w_e^+(x(e)) < w_{e'}^-(x(e')). \quad (7.9)$$

Then for a sufficiently small  $\alpha > 0$  we have

$$w_e(x(e)) + w_{e'}(x(e')) > w_e(x(e) + \alpha) + w_{e'}(x(e') - \alpha), \quad (7.10)$$

$$x + \alpha(\chi_e - \chi_{e'}) \in B(f). \quad (7.11)$$

Therefore,  $x$  is not an optimal solution.

Q.E.D.

## 7.1. OPTIMALITY CONDITIONS

Theorem 7.1 generalizes Theorem 2.13 for the minimum-weight base problem to that with a convex weight function.

For each  $e \in E$  and  $\xi \in \mathbb{R}$  define the interval

$$J_e(\xi) = [w_e^-(\xi), w_e^+(\xi)]. \quad (7.12)$$

$J_e(\xi)$  is the subdifferential of  $w_e$  at  $\xi$ . Conversely, for each  $e \in E$  and  $\eta \in \mathbb{R}$  define

$$I_e(\eta) = \{\xi \mid \xi \in \mathbb{R}, \eta \in J_e(\xi)\}. \quad (7.13)$$

Because of the convexity of  $w_e$ ,  $I_e(\eta)$ , if nonempty, is an interval in  $\mathbb{R}$  and we express it as

$$I_e(\eta) = [i_e^-(\eta), i_e^+(\eta)]. \quad (7.14)$$

Note that  $\eta \in J_e(\xi)$  if and only if  $\xi \in I_e(\eta)$ .

**Theorem 7.2:** A base  $x \in B(f)$  is an optimal solution of Problem  $P_1$  if and only if there exists a chain

$$C: \emptyset = A_0 \subset A_1 \subset \cdots \subset A_k = E \quad (7.15)$$

of  $\mathcal{D}$  such that

$$(i) \quad x(A_i) = f(A_i) \quad (i = 0, 1, \dots, k), \quad (7.16)$$

$$(ii) \quad \text{for each } i = 1, \dots, k,$$

$$\bigcap \{J_e(x(e)) \mid e \in A_i - A_{i-1}\} \neq \emptyset, \quad (7.17)$$

(iii) for each  $i, j = 1, \dots, k$  such that  $i < j$ , we have

$$w_e^-(x(e)) \leq w_{e'}^+(x(e')) \quad (7.18)$$

for any  $e \in A_i - A_{i-1}$  and  $e' \in A_j - A_{j-1}$ .

(Proof) *The "if" part:* This easily follows from Theorem 7.1.

*The "only if" part:* Let  $x \in B(f)$  be an optimal solution of Problem  $P_1$ . The sublattice

$$\mathcal{D}(x) = \{X \mid X \in \mathcal{D}, x(X) = f(X)\} \quad (7.19)$$

defines a poset  $\mathcal{P}(\mathcal{D}(x)) = (\Pi(\mathcal{D}(x)), \preceq_{\mathcal{D}(x)})$  (see Section 2.2.a). Suppose that  $\Pi(\mathcal{D}(x)) = \{E_1, E_2, \dots, E_k\}$ . For each  $i = 1, 2, \dots, k$  define

$$J_i = \bigcap \{J_e(x(e)) \mid e \in E_i\}. \quad (7.20)$$

For each distinct  $e, e' \in E_i$  ( $e, e'$ ) is an exchangeable pair associated with  $x$  due to the definition of  $E_i$ . Hence, because of the optimality of  $x$  we have

$$w_e^-(x(e)) \leq w_{e'}^+(x(e')) \quad (7.21)$$

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for each  $e, e' \in E_i$  due to Theorem 7.1. Therefore,

$$J_i \neq \emptyset \quad (i = 1, 2, \dots, k). \quad (7.22)$$

For each  $i = 1, 2, \dots, k$  define

$$J_i = [\eta_i^-, \eta_i^+]. \quad (7.23)$$

Also define for each  $i = 1, 2, \dots, k$

$$\bar{\eta}_i = \max\{\eta_j^- \mid j: E_j \preceq_{\mathcal{D}(x)} E_i\}. \quad (7.24)$$

For any  $i, j$  such that  $E_j \preceq_{\mathcal{D}(x)} E_i$  we have

$$\eta_j^- \leq \eta_i^+ \quad (7.25)$$

due to Theorem 7.1. From (7.24) and (7.25),

$$\bar{\eta}_i \in [\eta_i^-, \eta_i^+] \quad (i = 1, 2, \dots, k) \quad (7.26)$$

and we have the following monotonicity:

$$E_j \preceq_{\mathcal{D}(x)} E_i \implies \bar{\eta}_j \leq \bar{\eta}_i. \quad (7.27)$$

We assume without loss of generality that

$$\bar{\eta}_1 \leq \bar{\eta}_2 \leq \dots \leq \bar{\eta}_k, \quad (7.28)$$

and define

$$A_i = E_1 \cup E_2 \cup \dots \cup E_i \quad (i = 1, 2, \dots, k), \quad (7.29)$$

$$A_0 = \emptyset. \quad (7.30)$$

We see from (7.27) that

$$\mathcal{C}: \emptyset = A_0 \subset A_1 \subset \dots \subset A_k = E \quad (7.31)$$

is a (maximal) chain of  $\mathcal{D}(x)$ . In particular, (i) holds. Also, (ii) is exactly (7.22). Moreover, from (7.26) and (7.28) we have (iii). Q.E.D.

Theorem 7.2 generalizes the greedy algorithm given in Section 2.2.b.

### 7.2. A Decomposition Algorithm

## 7.2. A DECOMPOSITION ALGORITHM

In the following we assume that  $I_e(\eta) \neq \emptyset$  for every  $\eta \in \mathbb{R}$ , to simplify the following argument. It should also be noted that this assumption guarantees the existence of an optimal solution even if  $B(f)$  is unbounded. When  $B(f)$  is bounded, there is no loss of generality with this assumption.

We first describe an algorithm for Problem  $P_1$  in (7.1). Here,  $x^*$  is the output vector giving an optimal solution.

### A decomposition algorithm

**Step 1:** Choose  $\eta \in \mathbb{R}$  such that

$$\sum_{e \in E} i_e^-(\eta) \leq f(E) \leq \sum_{e \in E} i_e^+(\eta) \quad (7.32)$$

(see (7.14)).

**Step 2:** Find a base  $x \in B(f)$  such that for each  $e, e' \in E$

- (1) if  $w_e^+(x(e)) < \eta$  and  $w_{e'}^-(x(e')) > \eta$ , then we have  $e' \notin \text{dep}(x, e)$ ,
- (2) if  $w_e^+(x(e)) < \eta$ ,  $w_{e'}^-(x(e')) = \eta$  and  $e' \in \text{dep}(x, e)$ , then for any  $\alpha > 0$  we have  $w_{e'}^-(x(e') - \alpha) < \eta$ , i.e.,  $x(e') = i_{e'}^-(\eta)$ ,
- (3) if  $w_e^+(x(e)) = \eta$ ,  $w_{e'}^-(x(e')) > \eta$  and  $e' \in \text{dep}(x, e)$ , then for any  $\alpha > 0$  we have  $w_{e'}^+(x(e) + \alpha) > \eta$ , i.e.,  $x(e) = i_e^+(\eta)$ .

Put

$$E_- = \bigcup \{ \text{dep}(x, e) \mid e \in E, w_e^+(x(e)) < \eta \}, \quad (7.33)$$

$$E_+ = \bigcup \{ \text{dep}^\#(x, e) \mid e \in E, w_e^-(x(e)) > \eta \}, \quad (7.34)$$

$$E_0 = E - (E_+ \cup E_-), \quad (7.35)$$

where  $\text{dep}^\#$  is the *dual dependence function* defined by

$$\text{dep}^\#(x, e) = \bigcap \{ X \mid e \in X \in \overline{\mathcal{D}}, x(X) = f^\#(X) \}. \quad (7.36)$$

Put  $x^*(e) = x(e)$  for each  $e \in E_0$ .

**Step 3:** If  $E_- \neq \emptyset$ , then apply the present algorithm recursively to the problem with  $E$  and  $f$ , respectively, replaced by  $E_-$  and  $f^{E_-}$  and with the base polyhedron associated with the reduction  $(\mathcal{D}, f) \cdot E_-$ . Also, if  $E_+ \neq \emptyset$ , then apply the present algorithm recursively to the problem with  $E$  and  $f$ , respectively, replaced by  $E_+$  and  $f_{E_+}$  and with the base polyhedron associated with the contraction  $(\mathcal{D}, f) \times E_+$ .

(End)

The present algorithm is adapted from the algorithms in [Fuji80b] and [Groenevelt85]. Here, it is described in a self-dual form.

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It should be noted that for the dual dependence function  $\text{dep}^\#$  we have  $e' \in \text{dep}^\#(x, e)$  if and only if  $e \in \text{dep}(x, e')$ , for  $x \in B(f)$  ( $= B(f^\#)$ ).

Now, we show the validity of the decomposition algorithm described above. First, note that  $E_- \cap E_+ = \emptyset$  in Step 2 since otherwise there exist  $e, e' \in E$  such that  $w_e^+(x(e)) < \eta$ ,  $w_{e'}^-(x(e')) > \eta$  and  $e' \in \text{dep}(x, e)$ , which contradicts (1) of Step 2.

Suppose we have  $E_- = E$  in Step 2. Then, from (7.33)

$$x(e) < i_e^-(\eta) \quad (e \in E). \quad (7.37)$$

Therefore,

$$f(E) = x(E) < \sum_{e \in E} i_e^-(\eta), \quad (7.38)$$

which contradicts (7.32). Hence  $E_- \neq E$ . Similarly, we have  $E_+ \neq E$ . Consequently, the total number of executions of Step 2 is at most  $|E| - 1$ . When the algorithm terminates, then obtained vector  $x^*$  is an optimal solution of Problem  $P_1$  due to Theorem 7.2. Note that in Step 2  $x(E_-) = f(E_-)$  and  $x(E - E_+) = f(E - E_+)$  from (7.33) and (7.34) and that  $E_-$  and  $E - E_+$ , if nonempty proper subsets of  $E$ , will be members of the chain  $\mathcal{C}$  in (7.15).

In Step 1 a desired  $\eta$  may be found by a binary search but Step 1 heavily depends on the structure of the given functions  $w_e$  ( $e \in E$ ). A base  $x \in B(f)$  satisfying (1)–(3) of Step 2 is obtained by  $O(|E|^2)$  elementary transformations if an oracle for exchange capacities for  $B(f)$  is available. If an oracle for saturation capacities for  $P(f)$  is available, Step 2 can be executed in  $O(|E|)$  time, where calling the oracle is assumed to require unit time.

**Remark 7.1:** In Step 3 the original problem on  $E$  is decomposed into two problems, one being on  $E_-$  and the other on  $E_+$ , and the values of  $x^*(e)$  ( $e \in E_0$ ) are fixed. The decomposition relation obtained through the algorithm recursively defines a binary tree in such a way that  $E_-$  is the left child and  $E_+$  the right child of  $E$ . The decomposition algorithm described above traverses the binary tree by the depth-first search where the search of the left child is prior to that of its sibling, the right child. The above algorithm is also valid if we modify Step 3 according to any efficient way of traversing the binary tree from the root.

**Remark 7.2:** If  $w_e$  is strictly convex for each  $e \in E$ , then conditions (2) and (3) in Step 2 are always satisfied, so that we have only to consider condition (1). Moreover, if  $w_e$  is strictly convex and differentiable for each  $e \in E$ , then the above algorithm will further be simplified. This is the case to be treated in Section 8.

The decomposition algorithm for the separable convex optimization problem  $P_1$  lays a basis for the algorithms for the other problems to be considered in Sections 8–10.



### 7.3. DISCRETE OPTIMIZATION

#### 7.3. Discrete Optimization

Suppose that the rank function  $f$  of the submodular system  $(\mathcal{D}, f)$  is integer-valued. We consider a discrete optimization problem which is Problem  $P_1$  with variables being restricted to integers.

For each  $e \in E$  let  $\hat{w}_e$  be a real-valued function on  $\mathbb{Z}$  such that the piecewise linear extension, denoted by  $w_e$ , of  $\hat{w}_e$  on  $\mathbb{R}$  is a convex function, where  $w_e(\xi) = \hat{w}_e(\xi)$  for  $\xi \in \mathbb{Z}$  and  $w_e$  restricted on each unit interval  $[\xi, \xi + 1]$  ( $\xi \in \mathbb{Z}$ ) is a linear function. Consider a discrete optimization problem described as

$$IP_1: \text{Minimize } \sum_{e \in E} \hat{w}_e(x(e)) \quad (7.39a)$$

$$\text{subject to } x \in B_Z(f), \quad (7.39b)$$

where

$$B_Z(f) = \{x \mid x \in \mathbb{Z}^E, \forall X \in \mathcal{D}: x(X) \leq f(X), x(E) = f(E)\}. \quad (7.40)$$

$B_Z(f)$  is the base polyhedron associated with  $(\mathcal{D}, f)$  where the underlying totally ordered additive group is the set  $\mathbb{Z}$  of integers. For the same  $f$  we also denote by  $B_R(f)$  the base polyhedron  $B(f)$  associated with  $(\mathcal{D}, f)$  where the underlying totally ordered additive group is the set  $\mathbb{R}$  of reals.

Also consider the continuous version of  $IP_1$ :

$$P_1: \text{Minimize } \sum_{e \in E} w_e(x(e)) \quad (7.41a)$$

$$\text{subject to } x \in B_R(f). \quad (7.41b)$$

Recall that  $w_e$  is the piecewise linear extension of  $\hat{w}_e$  for each  $e \in E$ .

**Theorem 7.3** (cf. [Groenevelt85]): If there exists an optimal solution for Problem  $P_1$  of (7.41), there exists an integral optimal solution for Problem  $P_1$ .

(Proof) Suppose that  $x^*$  is an optimal solution of Problem  $P_1$ . Define vectors  $l, u \in \mathbb{R}^E$  by

$$l(e) = \lfloor x^*(e) \rfloor \quad (e \in E), \quad (7.42)$$

$$u(e) = \lceil x^*(e) \rceil \quad (e \in E). \quad (7.43)$$

Consider the following problem

$$P_1': \text{Minimize } \sum_{e \in E} w_e(x(e)) \quad (7.44a)$$

$$\text{subject to } x \in B_R(f)_l^*, \quad (7.44b)$$

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where  $B_R(f)_l^*$  is the base polyhedron of the submodular system  $(\mathcal{D}, f)_l^*$  which is the vector minor of  $(\mathcal{D}, f)$  obtained by the restriction by  $u$  and the contraction by  $l$ . Note that an optimal solution of  $P_1'$  is an optimal solution of  $P_1$ . Since  $f$  is integer-valued and  $l$  and  $u$  are integral,  $B_R(f)_l^*$  is an integral base polyhedron. Also, the objective function in (7.44a) is linear on  $B_R(f)_l^*$ . Therefore, by the greedy algorithm given in Section 2.2.b we can find an integral optimal solution for Problem  $P_1'$ , which is also optimal for Problem  $P_1$ . Q.E.D.

We can also prove Theorem 7.3 by using the decomposition algorithm given in Section 7.2. From the assumption, for each  $e \in E$  and  $\eta \in \mathbb{R}$   $i_e^-(\eta)$  and  $i_e^+(\eta)$  are integers. We can choose an integral base  $x \in B(f)$  as the base  $x$  required in Step 2 of the decomposition algorithm. Note that  $w_e^+(x(e)) < \eta$  ( $w_e^-(x(e)) > \eta$ ) is equivalent to  $x(e) < i_e^-(\eta)$  ( $x(e) > i_e^+(\eta)$ ).

An integral optimal solution of Problem  $P_1$  of (7.41) is an optimal solution of Problem  $IP_1$  of (7.39), and vice versa. An incremental algorithm is also given in [Federgruen + Groenevelt86].

## 8. The Lexicographically Optimal Base Problem

We consider a submodular system  $(\mathcal{D}, f)$  on  $E$  with the set  $\mathbb{R}$  of reals (or the set  $\mathbb{Q}$  of rationals) as the underlying totally ordered additive group. Throughout this section  $\mathbb{R}$  may be replaced by  $\mathbb{Q}$  but not by the set  $\mathbb{Z}$  of integers.

### 8.1. Nonlinear Weight Functions

For each  $e \in E$  let  $h_e$  be a continuous and monotone increasing function from  $\mathbb{R}$  onto  $\mathbb{R}$ . For any vector  $x \in \mathbb{R}^E$  we denote by  $T(x)$  the sequence of the components  $x(e)$  ( $e \in E$ ) of  $x$  arranged in order of increasing magnitude, i.e.,  $T(x) = (x(e_1), x(e_2), \dots, x(e_n))$  with  $x(e_1) \leq x(e_2) \leq \dots \leq x(e_n)$ , where  $|E| = n$  and  $E = \{e_1, e_2, \dots, e_n\}$ .

Consider the following problem.

$$P_2: \text{Lexicographically maximize } T((h_e(x(e)): e \in E)) \quad (8.1a)$$

$$\text{subject to } x \in B(f). \quad (8.1b)$$

We call an optimal solution of Problem  $P_2$  a *lexicographically optimal base* of  $(\mathcal{D}, f)$  with respect to nonlinear weight functions  $h_e$  ( $e \in E$ ). Informally, Problem  $P_2$  is to find a base  $x$  which is as close as possible to a vector which equalizes the values of  $h_e(x(e))$  ( $e \in E$ ) on the basis of the lexicographic ordering in (8.1a).

### 8.1. LEXICOGRAPHICALLY OPTIMAL BASE: NONLINEAR WEIGHTS

**Theorem 8.1** (cf. [Fuji80b]): Let  $x$  be a base in  $B(f)$ . Define a vector  $\eta \in \mathbb{R}^E$  by

$$\eta(e) = h_e(x(e)) \quad (e \in E) \quad (8.2)$$

and let the distinct numbers of  $\eta(e)$  ( $e \in E$ ) be given by

$$\eta_1 < \eta_2 < \cdots < \eta_p. \quad (8.3)$$

Also, define

$$A_i = \{e \mid e \in E, \eta(e) \leq \eta_i\} \quad (i = 1, 2, \dots, p). \quad (8.4)$$

Then the following four statements are equivalent:

- (i)  $x$  is a lexicographically optimal base of  $(\mathcal{D}, f)$  with respect to  $h_e$  ( $e \in E$ ).
- (ii)  $A_i \in \mathcal{D}$  and  $x(A_i) = f(A_i)$  ( $i = 1, 2, \dots, p$ ).
- (iii)  $\text{dep}(x, e) \subseteq A_i$  ( $e \in A_i$ ,  $i = 1, 2, \dots, p$ ).
- (iv)  $x$  is an optimal solution of Problem  $P_1$  in (7.1) where for each  $e \in E$  the derivative of  $w_e$  coincides with  $h_e$ .

(Proof) The equivalence, (ii)  $\iff$  (iii), immediately follows from the definition of dependence function. Also, the equivalence, (ii) (or (iii))  $\iff$  (iv), follows from Theorem 7.2. Therefore, (ii)–(iv) are equivalent. We show the equivalence, (i)  $\iff$  (ii)–(iv).

(i)  $\implies$  (iii): Suppose (i). If there exist  $i \in \{1, 2, \dots, p\}$  and  $e \in A_i$  such that  $e' \in \text{dep}(x, e) - A_i$ , then for a sufficiently small  $\alpha > 0$  the vector given by

$$y = x + \alpha(\chi_e - \chi_{e'}) \quad (8.5)$$

is a base in  $B(f)$  and  $T((h_e(y(e)) : e \in E))$  is lexicographically greater than  $T((h_e(x(e)) : e \in E))$ . This contradicts (i). So, (iii) holds.

(ii), (iii)  $\implies$  (i): Suppose (ii) (and (iii)). Let  $\bar{x}$  be an arbitrary base such that  $T((h_e(\bar{x}(e)) : e \in E))$  is lexicographically greater than or equal to  $T((h_e(x(e)) : e \in E))$ . Define a vector  $\bar{\eta} \in \mathbb{R}^E$  by

$$\bar{\eta}(e) = h_e(\bar{x}(e)) \quad (e \in E). \quad (8.6)$$

Also define  $A_0 = \emptyset$ . We show by induction on  $i$  that

$$x(e) = \bar{x}(e) \quad (e \in A_i) \quad (8.7)$$

for  $i = 0, 1, \dots, p$ , from which the optimality of  $x$  follows. For  $i = 0$  (8.7) trivially holds. So, suppose that (8.7) holds for some  $i = i_0 < p$ . Since  $T(\bar{\eta})$  is lexicographically greater than or equal to  $T(\eta)$ , we have from (8.3) and (8.4)

$$\bar{\eta}(e) \geq \eta(e) = \eta_{i_0+1} \quad (e \in A_{i_0+1} - A_{i_0}). \quad (8.8)$$

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From (8.8) and the monotonicity of  $h_e$  ( $e \in E$ ),

$$\bar{x}(e) \geq x(e) \quad (e \in A_{i_0+1} - A_{i_0}). \quad (8.9)$$

Since  $\bar{x} \in B(f)$ , it follows from (8.7) with  $i = i_0$ , (8.9) and assumption (ii) that

$$f(A_{i_0+1}) \geq \bar{x}(A_{i_0+1}) \geq x(A_{i_0+1}) = f(A_{i_0+1}). \quad (8.10)$$

From (8.9) and (8.10) we have  $\bar{x}(e) = x(e)$  ( $e \in A_{i_0+1}$ ). Q.E.D.

We see from Theorem 8.1 that the lexicographically optimal base is unique and that the problem can be solved by the decomposition algorithm given in Section 7.2.

For a vector  $x \in \mathbb{R}^E$  denote by  $T^*(x)$  the sequence of the components  $x(e)$  ( $e \in E$ ) of  $x$  arranged in order of decreasing magnitude. We call a base  $x \in B(f)$  which lexicographically minimizes  $T^*((h_e(x(e)): e \in E))$  a *co-lexicographically optimal base* of  $(\mathcal{D}, f)$  with respect to  $h_e$  ( $e \in E$ ).

**Theorem 8.2:**  $x$  is a lexicographically optimal base of  $(\mathcal{D}, f)$  with respect to  $h_e$  ( $e \in E$ ) if and only if it is a co-lexicographically optimal base of  $(\mathcal{D}, f)$  with respect to  $h_e$  ( $e \in E$ ).

(Proof) Using  $\eta(e)$  ( $e \in E$ ) and  $\eta_i$  ( $i = 1, 2, \dots, p$ ) appearing in Theorem 8.1, define

$$A_i^* = \{e \mid e \in E, \eta(e) \geq \eta_{p-i+1}\} \quad (i = 1, 2, \dots, p). \quad (8.11)$$

Also define  $A_0 = \emptyset = A_0^*$ . Since  $A_i^* = E - A_{p-i}$  ( $i = 0, 1, \dots, p$ ) and  $x(E) = f(E)$ , we can easily see that for a base  $x \in B(f)$   $x$  satisfies (ii) of Theorem 8.1 if and only if  $x$  satisfies

$$(ii^*) \quad A_i^* \in \bar{\mathcal{D}} \text{ and } x(A_i^*) = f^\#(A_i^*) \quad (i = 1, 2, \dots, p).$$

Consequently, the present theorem follows from Theorem 8.1 and the duality shown in Lemma 1.3. Q.E.D.

### 8.2. Linear Weight Functions

Let us consider Problem  $P_2$  in (8.1) for the case when  $h_e(x(e))$  is a linear function expressed as  $x(e)/w(e)$  with  $w(e) > 0$  for each  $e \in E$ . Such a lexicographically optimal base is called a *lexicographically optimal base with respect to the weight vector*  $w = (w(e): e \in E)$  (see [Fuji80b]). This is a generalization of the concept of (lexicographically) optimal flow introduced by N. Megiddo [Megiddo74] concerning multiple-source multiple-sink networks. A polymatroid induced on the set of sources (or sinks) is considered in [Megiddo74] (see Section 1.2).

## 8.2 LEXICOGRAPHICALLY OPTIMAL BASE: LINEAR WEIGHTS

From Theorem 8.1 the lexicographically optimal base problem with respect to the weight vector  $w$  is equivalent to the following separable quadratic optimization problem (see [Fuji80b]).

$$P_1^w: \text{Minimize } \sum_{e \in E} x(e)^2 / w(e) \quad (8.12a)$$

$$\text{subject to } x \in B(f). \quad (8.12b)$$

This is a minimum-norm point problem for  $B(f)$  (see Section 6.1.a, where  $w(e) = 1$  ( $e \in E$ )). Problem  $P_1^w$  can be solved by the decomposition algorithm shown in Section 7.2.

The following procedure was also given in [Fuji80b] for polymatroids.

### A Greedy Algorithm

Step 1: Put  $i \leftarrow 1$  and  $F \leftarrow E$ .

Step 2: Compute

$$\lambda^* = \max\{\lambda \mid \lambda w^F \in P(f)\}. \quad (8.13)$$

Put  $E_i \leftarrow \text{sat}(\lambda^* w^F)$  and  $x^*(e) \leftarrow \lambda^* w(e)$  for each  $e \in E_i$ .

Step 3: If  $\lambda^* w^F \in B(f)$  (or  $E_i = F$ ), then stop. Otherwise put  $(\mathcal{D}, f) \leftarrow (\mathcal{D}, f)/E_i$  and  $F \leftarrow F - E_i$ . Put  $i \leftarrow i + 1$  and go to Step 2.

(End)

Note that  $w^F$  appearing in Step 2 is the restriction of  $w$  to  $F \subseteq E$ .

The above greedy algorithm can also be viewed as follows. Start from a subbase  $x = \lambda w \in P(f)$  for a sufficiently small  $\lambda$ , where one such  $\lambda$  can be given in terms of the greatest lower bound  $\underline{\alpha}$  defined by (2.82) and (2.83) when  $(\mathcal{D}, f)$  is simple, since  $\lambda w \leq \underline{\alpha}$  implies  $\lambda w \in P(f)$ ; or take any subbase  $y \in P(f)$  and choose  $\lambda$  such that  $\lambda w \leq y$ . Increase all the components of  $x$  ( $= \lambda w$ ) proportionally to  $w$  as far as  $x$  belongs to  $P(f)$ . Then fix the saturated components of  $x$  and increase all the other components of  $x$  proportionally to  $w$  as far as  $x$  belongs to  $P(f)$ . Repeat this process until  $x$  becomes a base in  $B(f)$ .

The above greedy algorithm can also be adapted to Problem  $P_2$  with nonlinear weight functions  $h_e$  ( $e \in E$ ).

**Theorem 8.3** ([Fuji80b]): Let  $x^*$  be the base in  $B(f)$  obtained by the greedy algorithm described above. Then  $x^*$  is the lexicographically optimal base with respect to the weight vector  $w$ .

(Proof) We see from the greedy algorithm that for any  $e, e' \in E$  such that

$$x^*(e)/w(e) < x^*(e')/w(e') \quad (8.14)$$

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$(e, e')$  is not an exchangeable pair associated with  $x^*$ . The optimality of  $x^*$  follows from Theorems 7.1 and 8.1. Q.E.D.

It should be noted that the above-described greedy algorithm traverses the leaves, from the leftmost to the rightmost one, of the binary tree mentioned in Remark 7.1 in Section 7.2. For a general submodular system  $(\mathcal{D}, f)$ , computing  $\lambda^*$  in (8.13) is not easy even if we are given oracles for saturation capacities and exchange capacities for  $P(f)$ . One exception is the case where  $\mathcal{D}$  is induced by a laminar family of subsets of  $E$  expressed as a tree structure.

Also, in case of multi-source multi-sink networks considered in [Megiddo74] and [Fuji80b], the problem can be solved in time proportional to that required for finding a maximum flow in the same network (see [Gallo + Grigoriadis + Tarjan89], where use is made of the distance labelling introduced by A. V. Goldberg ([Goldberg + Tarjan88])).

**Lemma 8.4** (see [Fuji79]): Let  $x^*$  be the lexicographically optimal base with respect to the weight vector  $w$ . Then, for any  $\lambda \in \mathbb{R}$   $x^* \wedge (\lambda w) = (\min\{x^*(e), \lambda w(e)\} : e \in E)$  is a base of  $\lambda w$  (i.e.,  $\lambda x^*$  is a maximal vector in  $P(f)^{\lambda w}$ ).

(Proof) Since the lexicographically optimal base  $x^*$  is unique and the above greedy algorithm finds it, the present lemma follows from the algorithm. Q.E.D.

In the sense of Lemma 8.4 we also call  $x^*$  the *universal base* with respect to  $w$  (cf. [Nakamura + Iri81]).

Now, let us examine the relationship between the lexicographically optimal base and the subdifferentials of  $f$ .

**Theorem 8.4:** Let  $x^*$  be the lexicographically optimal base of  $(\mathcal{D}, f)$  with respect to  $w$ . For  $\lambda \in \mathbb{R}$  and  $A \in \mathcal{D}$  we have  $\lambda w \in \partial f(A)$  if and only if, defining

$$A_\lambda^+ = \{e \mid e \in E, x^*(e) \leq \lambda w(e)\}, \quad (8.15)$$

$$A_\lambda^- = \{e \mid e \in E, x^*(e) < \lambda w(e)\}, \quad (8.16)$$

we have  $A_\lambda^- \subseteq A \subseteq A_\lambda^+$  and  $A \in \mathcal{D}(x^*)$  (i.e.,  $x^*(A) = f(A)$ ).

(Proof) *The "if" part:* Suppose that  $A_\lambda^- \subseteq A \subseteq A_\lambda^+$  and  $A \in \mathcal{D}(x^*)$ . Then for any  $X \in \mathcal{D}$  such that  $X \subseteq A$  or  $A \subseteq X$ , we have

$$\begin{aligned} \lambda w(X) - \lambda w(A) &\leq x^*(X) - x^*(A) \\ &\leq f(X) - f(A). \end{aligned} \quad (8.17)$$

From Lemma 5.4 we have  $\lambda w \in \partial f(A)$ .

## 8.2 LEXICOGRAPHICALLY OPTIMAL BASE: LINEAR WEIGHTS

The “only if” part: Suppose  $\lambda w \in \partial f(A)$ . From the greedy algorithm  $x^*$  satisfies

$$x^*(A_\lambda^-) = f(A_\lambda^-). \quad (8.18)$$

From (8.18) and the assumption we have

$$\begin{aligned} \lambda w(A_\lambda^-) - \lambda w(A) &\leq f(A_\lambda^-) - f(A) \\ &\leq x^*(A_\lambda^-) - x^*(A) \end{aligned} \quad (8.19)$$

or

$$\lambda w(A_\lambda^-) - x^*(A_\lambda^-) \leq \lambda w(A) - x^*(A). \quad (8.20)$$

Since from (8.16)  $\lambda w(X) - x^*(X)$  is maximized at  $X = A_\lambda^-$ , it follows from (8.20) that  $X = A$  also maximizes  $\lambda w(X) - x^*(X)$ . This implies  $A_\lambda^- \subseteq A \subseteq A_\lambda^+$  because of (8.15) and (8.16). Since we have

$$\lambda w(A_\lambda^-) - x^*(A_\lambda^-) = \lambda w(A) - x^*(A), \quad (8.21)$$

from (8.18), (8.21) and the fact that  $x^*(A) \leq f(A)$  we have

$$\lambda w(A_\lambda^-) - \lambda w(A) \geq f(A_\lambda^-) - f(A). \quad (8.22)$$

Since  $\lambda w \in \partial f(A)$ , we must have from (8.22)

$$x^*(A) = f(A). \quad (8.23)$$

This completes the proof. Q.E.D.

**Corollary 8.6:** Under the same assumption as in Theorem 8.5,  $\lambda w$  belongs to the interior of  $\partial f(A)$  if and only if  $A_\lambda^- = A_\lambda^+$ .

(Proof) Suppose that  $\lambda w$  is in the interior of  $\partial f(A)$ . If  $A \neq A_\lambda^-$ , then

$$\begin{aligned} \lambda w(A_\lambda^-) - \lambda w(A) &< f(A_\lambda^-) - f(A) \\ &\leq x^*(A_\lambda^-) - x^*(A). \end{aligned} \quad (8.24)$$

This contradicts the fact that  $X = A_\lambda^-$  maximizes  $\lambda w(X) - x^*(X)$ . Therefore, we have  $A = A_\lambda^-$ . Similarly, we have  $A = A_\lambda^+$  and hence  $A_\lambda^- = A_\lambda^+$ .

Conversely, suppose  $A_\lambda^- = A_\lambda^+$ . Then  $X = A (= A_\lambda^- = A_\lambda^+)$  is the unique maximizer of  $\lambda w(X) - x^*(X)$ , so that

$$\lambda w(X) - x^*(X) < \lambda w(A) - x^*(A) \quad (X \in \mathcal{D}, X \neq A). \quad (8.25)$$

From (8.25),

$$\lambda w(X) - \lambda w(A) < x^*(X) - x^*(A) \leq f(X) - f(A) \quad (X \in \mathcal{D}, X \neq A), \quad (8.26)$$

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where note that  $x^*(A) = f(A)$ . Hence  $\lambda w$  belongs to the interior of  $\partial f(A)$ .  
Q.E.D.

Let

$$\partial f(A_0) = \partial f(\emptyset), \partial f(A_1), \dots, \partial f(A_p) = \partial f(E) \quad (8.27)$$

be the subdifferentials of  $f$  whose interiors have nonempty intersections with the line  $L = \{\lambda w \mid \lambda \in \mathbb{R}\}$ , where the subdifferentials in (8.27) are arranged in order of increasing  $\lambda \in \mathbb{R}$ . Suppose that for each  $i = 0, 1, \dots, p$   $(\lambda_i, \lambda_{i+1})$  is the open interval consisting of those  $\lambda$  for which  $\lambda w$  belongs to the interior of  $\partial f(A_i)$ . We see from Theorem 8.5 and Corollary 8.6 that for each  $i = 1, 2, \dots, p$

$$x^*(e) = \lambda_i w(e) \quad (e \in A_i - A_{i-1}). \quad (8.28)$$

Compare the present results with those in Sections 6.1.a and 6.2.b.1 (also see [Fuji84c]).

The concept of lexicographically optimal base is generalized by M. Nakamura [Nakamura81] and N. Tomizawa [Tomi80d]. Suppose that we are given two (polymatroid) base polyhedra  $B(f_i)$  ( $i = 1, 2$ ) such that every base of  $B(f_i)$  ( $i = 1, 2$ ) consists of positive components. If  $b_1$  is the lexicographically optimal base of  $B(f_1)$  with respect to a weight vector  $b_2 \in B(f_2)$  and  $b_2$  is the lexicographically optimal base of  $B(f_2)$  with respect to  $b_1$ , then the pair  $(b_1, b_2)$  is called a *universal pair of bases* (the original definition in [Nakamura81] and [Tomi80d] is different from but equivalent to the present one). Some characterizations of universal pairs are given by K. Murota [Murota88].

### 9. The Weighted Max-Min and Min-Max Problems

We consider the problem of maximizing the minimum (or minimizing the maximum) of a nonlinear objective function over the base polyhedron  $B(f)$ .

#### 9.1. Continuous Variables

For each  $e \in E$  let  $h_e: \mathbb{R} \rightarrow \mathbb{R}$  be a right-continuous and monotone nondecreasing function such that  $\lim_{\xi \rightarrow +\infty} h_e(\xi) = +\infty$  and  $\lim_{\xi \rightarrow -\infty} h_e(\xi) = -\infty$ . Consider the following max-min problem with the nonlinear weight function  $h_e$  ( $e \in E$ ).

$$P_*: \text{Maximize } \min_{e \in E} h_e(x(e)) \quad (9.1a)$$

$$\text{subject to } x \in B_R(f), \quad (9.1b)$$



## 9.1. THE CONTINUOUS MAN-MIX AND MIN-MAX PROBLEMS

where  $B_{\mathbf{R}}(f)$  is the base polyhedron associated with a submodular system  $(\mathcal{D}, f)$  on  $E$  and the underlying totally ordered additive group is assumed to be the set  $\mathbf{R}$  of reals.

For each  $e \in E$  let  $w_e: \mathbf{R} \rightarrow \mathbf{R}$  be a convex function whose right derivative  $w_e^+$  is given by  $h_e$ .

**Theorem 9.1:** Consider Problem  $P_1$  in (7.1) with  $w_e$  ( $e \in E$ ) defined as above. Let  $x$  be an optimal solution of Problem  $P_1$ . Then  $x$  is an optimal solution of Problem  $P_*$  in (9.1).

(Proof) Define

$$\eta_1 = \min\{h_e(x(e)) \mid e \in E\}, \quad (9.2)$$

$$S_1 = \{e \mid e \in E, h_e(x(e)) = \eta_1\}, \quad (9.3)$$

$$S_1^* = \bigcup \{\text{dep}(x, e) \mid e \in S_1\}. \quad (9.4)$$

We have from (9.4)

$$x(S_1^*) = f(S_1^*). \quad (9.5)$$

It follows from Theorem 7.1 that

$$w_e^-(x(e)) \leq \eta_1 \quad (e \in S_1^*). \quad (9.6)$$

If there were a base  $y \in B(f)$  such that

$$\eta_1 < \min\{h_e(y(e)) \mid e \in E\}, \quad (9.7)$$

then from (9.2)–(9.7) we would have

$$x(e) < y(e) \quad (e \in S_1), \quad (9.8)$$

$$x(e) \leq y(e) \quad (e \in S_1^* - S_1), \quad (9.9)$$

since  $h_e = w_e^+$ . Hence, from (9.5), (9.8) and (9.9),

$$f(S_1^*) = x(S_1^*) < y(S_1^*), \quad (9.10)$$

which contradicts the fact that  $y \in B_{\mathbf{R}}(f)$ .

Q.E.D.

We see from the above proof that the decomposition algorithm given in Section 7.2 can be simplified for solving Problem  $P_*$  as follows. We may put  $x^*(e) = x(e)$  for each  $e \in E_0 \cup E_+$  in Step 2 and apply the decomposition algorithm recursively to the problem on  $E_-$  but not to the one on  $E_+$  in Step 3 (cf. [Ichimori + Ishii + Nishida82]). In other words, we only go down the leftmost path in the binary decomposition tree mentioned in Remark 7.1 in Section 7.2.

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Next, consider the following min-max problem

$$P^*: \text{Minimize } \max_{e \in E} h_e(x(e)) \quad (9.11a)$$

$$\text{subject to } x \in B_R(f). \quad (9.11b)$$

Here, we assume that  $h_e$  is left-continuous rather than right-continuous for each  $e \in E$ .

For each  $e \in E$  let  $w_e: \mathbb{R} \rightarrow \mathbb{R}$  be a convex function whose left derivative  $w_e^-$  is given by  $h_e$ . Then we have

**Corollary 9.2:** Consider Problem  $P_1$  in (7.1) with  $w_e$  ( $e \in E$ ) such that  $w_e^- = h_e$ . Let  $x$  be an optimal solution of Problem  $P_1$ . Then  $x$  is an optimal solution of Problem  $P^*$  in (9.11).

The proof of Corollary 9.2 is similar to that of Theorem 9.1 by duality. An optimal solution of Problem  $P^*$  can be obtained by the decomposition algorithm given in Section 7.2, where we only go down the rightmost path of the binary decomposition tree mentioned in Remark 7.1.

Moreover, suppose that for each  $e \in E$   $h_e$  is continuous monotone nondecreasing function such that  $\lim_{\xi \rightarrow +\infty} h_e(\xi) = +\infty$  and  $\lim_{\xi \rightarrow -\infty} h_e(\xi) = -\infty$ . From Theorem 9.1 and Corollary 9.2 we have the following.

**Corollary 9.3:** Consider Problem  $P_1$  in (7.1) with  $w_e$  ( $e \in E$ ) such that the derivative of  $w_e$  is equal to  $h_e$  for each  $e \in E$ . Let  $x$  be an optimal solution of Problem  $P_1$ . Then  $x$  is an optimal solution of Problem  $P^*$  of (9.1) and, at the same time, an optimal solution of Problem  $P_*$  in (9.11).

A simultaneously optimal solution of Problems  $P_*$  and  $P^*$  can be obtained by the decomposition algorithm given in Section 7.2, where we only go down the leftmost and rightmost paths of the binary decomposition tree mentioned in Remark 7.1.

Problems  $P_*$  and  $P^*$  are sometimes called *sharing problems* in the literature ([Brown79], [Ichimori + Ishii + Nishida82]). The sharing problems with more general objective functions and feasible regions are considered by U. Zimmermann [Zimmermann86a], [Zimmermann86b].

### 9.2. Discrete Variables

For each  $e \in E$  let  $\hat{h}_e: \mathbb{Z} \rightarrow \mathbb{R}$  be a monotone nondecreasing function on  $\mathbb{Z}$  such that  $\lim_{\xi \rightarrow +\infty} \hat{h}_e(\xi) = +\infty$  and  $\lim_{\xi \rightarrow -\infty} \hat{h}_e(\xi) = -\infty$  for each  $e \in E$ . Consider

$$IP_*: \text{Maximize } \min_{e \in E} \hat{h}_e(x(e)) \quad (9.12a)$$

$$\text{subject to } x \in B_Z(f), \quad (9.12b)$$

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where  $B_Z(f)$  is the base polyhedron associated with an integral submodular system  $(\mathcal{D}, f)$  on  $E$  and the underlying totally ordered additive group is the set  $Z$  of integers.

For each  $e \in E$  let  $w_e: \mathbb{R} \rightarrow \mathbb{R}$  be a piecewise-linear convex function such that the following two hold:

$$(i) \text{ Its right derivative } w_e^+ \text{ satisfies } w_e^+(\xi) = \hat{h}_e(\xi) \quad (\xi \in Z), \quad (9.13a)$$

$$(ii) \text{ } w_e \text{ is linear on each unit interval } [\xi, \xi + 1] \quad (\xi \in Z). \quad (9.13b)$$

**Theorem 9.4:** Let  $x_*$  be an integral optimal solution of Problem  $P_1$  in (7.1) with  $w_e$  ( $e \in E$ ) defined as above. Then  $x_*$  is an optimal solution of Problem  $IP_*$  in (9.12).

(Proof) For each  $e \in E$  let  $h_e: \mathbb{R} \rightarrow \mathbb{R}$  be a right-continuous piecewise-constant nondecreasing function such that  $h_e(\eta) = \hat{h}_e(\xi)$  ( $\eta \in [\xi, \xi + 1)$ ,  $\xi \in Z$ ). It follows from Theorem 9.1 that an integral optimal solution of Problem  $P_1$  with  $w_e$  ( $e \in E$ ) defined by (9.13) is an integral optimal solution of Problem  $P_*$  in (9.1) with  $h_e$  ( $e \in E$ ) defined as above. Therefore,  $x_*$  is an optimal solution of Problem  $IP_*$ . Q.E.D.

It should be noted that there exists an integral optimal solution  $x_*$  of Problem  $P_1$  in Theorem 9.4 due to Theorem 7.3. The reduction of Problem  $IP_*$  to Problem  $P_1$  was also communicated by N. Katoh [Katoh85]. A direct algorithm for Problem  $IP_*$  is given in [Fuji + Katoh + Ichimori88].

Moreover, consider the weighted min-max problem

$$IP^*: \text{Minimize } \max_{e \in E} \hat{h}_e(x(e)) \quad (9.14a)$$

$$\text{subject to } x \in B_Z(f). \quad (9.14b)$$

For each  $e \in E$  let  $w_e: \mathbb{R} \rightarrow \mathbb{R}$  be a piecewise-linear convex function such that

$$(i) \text{ its left derivative } w_e^- \text{ satisfies } w_e^-(\xi) = \hat{h}_e(\xi) \quad (\xi \in Z), \quad (9.15a)$$

$$(ii) \text{ } w_e \text{ is linear on each unit interval } [\xi, \xi + 1] \quad (\xi \in Z). \quad (9.15b)$$

Similarly as Theorem 9.4 we have

**Corollary 9.5:** Let  $x^*$  be an integral optimal solution of Problem  $P_1$  in (7.1) with  $w_e$  ( $e \in E$ ) defined by (9.15). Then  $x^*$  is an optimal solution of Problem  $IP^*$  in (9.14).

## 10. The Fair Resource Allocation Problem

In this section we consider the problem of allocating resources in a fair manner which generalizes the max-min and min-max problems treated in the

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preceding section. The readers should also be referred to the book [Ibaraki + Katoh88] by T. Ibaraki and N. Katoh for resource allocation problems and related topics.

### 10.1. Continuous Variables

Let  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that  $g(u, v)$  is monotone nondecreasing in  $u$  and monotone nonincreasing in  $v$ . Typical examples of such a function  $g$  are the following.

$$g(u, v) = u - v, \quad (10.1)$$

$$g(u, v) = u/v \quad (u, v > 0). \quad (10.2)$$

Also, for each  $e \in E$  let  $h_e$  be a continuous monotone nondecreasing function from  $\mathbb{R}$  onto  $\mathbb{R}$ .

Consider

$$P_3: \text{Minimize } g(\max_{e \in E} h_e(x(e)), \min_{e \in E} h_e(x(e))) \quad (10.3a)$$

$$\text{subject to } x \in B_R(f). \quad (10.3b)$$

We call Problem  $P_3$  the *continuous fair resource allocation problem with sub-modular constraints*. This type of objective function was first considered by N. Katoh, T. Ibaraki and H. Mine [Katoh + Ibaraki + Mine85].

Using the same functions  $h_e$  ( $e \in E$ ) appearing in (10.3), let us consider Problems  $P_*$  and  $P^*$  described by (9.1) and (9.11), respectively. Denote the optimal values of the objective functions of Problems  $P_*$  and  $P^*$  by  $v_*$  and  $v^*$ , respectively, and define vectors  $l, u \in \mathbb{R}^E$  by

$$l(e) = \min\{\alpha \mid \alpha \in \mathbb{R}, h_e(\alpha) \geq v_*\} \quad (e \in E), \quad (10.4)$$

$$u(e) = \max\{\alpha \mid \alpha \in \mathbb{R}, h_e(\alpha) \leq v^*\} \quad (e \in E). \quad (10.5)$$

**Theorem 10.1:** Suppose that values  $v_*$  and  $v^*$  and vectors  $l$  and  $u$  are defined as above. Then we have  $v_* \leq v^*$  and  $l \leq u$ . Moreover,  $B(f)_l^*$  is nonempty and any  $x \in B(f)_l^*$  is an optimal solution of Problem  $P_3$  in (10.3), where  $B(f)_l^* = \{x \mid x \in B(f), l \leq x \leq u\}$  (see Section 2.1.b).

(Proof) Let  $x_*$  and  $x^*$ , respectively, be optimal solutions of Problems  $P_*$  and  $P^*$ : If  $v_* > v^*$ , then we have

$$x^*(e) \leq u(e) < l(e) \leq x_*(e) \quad (e \in E), \quad (10.6)$$

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which contradicts the fact that  $x^*(E) = f(E) = x_*(E)$ . Therefore, we have  $v_* \leq v^*$ . This implies  $l \leq u$ . Moreover, since  $x_* \in B(f)_l$ ,  $x^* \in B(f)^u$  and  $l \leq u$ , from Theorem 2.5 we have  $B(f)_l^* \neq \emptyset$ . For any  $x \in B(f)_l^*$  and  $y \in B(f)$  we have

$$\begin{aligned} & g(\max_{e \in E} h_e(y(e)), \min_{e \in E} h_e(y(e))) \\ & \geq g(v^*, v_*) \\ & = g(\max_{e \in E} h_e(x(e)), \min_{e \in E} h_e(x(e))), \end{aligned} \quad (10.7)$$

due to the monotonicity of  $g$ . This shows that any  $x \in B(f)_l^*$  is an optimal solution of Problem  $P_3$ . Q.E.D.

The continuous fair resource allocation problem  $P_3$  is thus reduced to Problems  $P_*$  and  $P^*$  and can be solved by the decomposition algorithm given in Section 7.2.

### 10.2. Discrete Variables

We consider the *discrete fair resource allocation problem*, which is a discrete version of the continuous fair resource allocation problem  $P_3$  treated in Section 10.1 (see [Fuji + Katoh + Ichimori88]).

Let  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that  $g(u, v)$  is monotone nondecreasing in  $u$  and monotone nonincreasing in  $v$ . Also, for each  $e \in E$  let  $\hat{h}_e: \mathbb{Z} \rightarrow \mathbb{R}$  be a monotone nondecreasing function. We assume for simplicity that  $\lim_{\xi \rightarrow +\infty} \hat{h}_e(\xi) = +\infty$  and  $\lim_{\xi \rightarrow -\infty} \hat{h}_e(\xi) = -\infty$ .

For a submodular system  $(\mathcal{D}, f)$  on  $E$  with an integer-valued rank function  $f$ , consider the problem

$$IP_3: \text{Minimize } g(\max_{e \in E} \hat{h}_e(x(e)), \min_{e \in E} \hat{h}_e(x(e))) \quad (10.8a)$$

$$\text{subject to } x \in B_{\mathbb{Z}}(f). \quad (10.8b)$$

Problem  $IP_3$  is not so easy as its continuous version  $P_3$  because of the integer constraints.

Using the same functions  $\hat{h}_e$  ( $e \in E$ ), consider the weighted integral max-min problem  $IP_*$  and the weighted integral min-max problem  $IP^*$ . Let  $\hat{v}_*$  and  $\hat{v}^*$ , respectively, be the optimal values of the objective functions of  $IP_*$  and  $IP^*$ . Define vectors  $\hat{l}, \hat{u} \in \mathbb{Z}^E$  by

$$\hat{l}(e) = \min\{\alpha \mid \alpha \in \mathbb{Z}, \hat{h}_e(\alpha) \geq \hat{v}_*\}, \quad (10.9)$$

$$\hat{u}(e) = \max\{\alpha \mid \alpha \in \mathbb{Z}, \hat{h}_e(\alpha) \leq \hat{v}^*\}. \quad (10.10)$$

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We have  $\hat{v}_* \leq \hat{v}^*$  but, unlike the continuous version of the problem, we may not have  $\hat{l} \leq \hat{u}$  in general. However, we have

$$\hat{l}(e) \leq \hat{u}(e) + 1 \quad (e \in E). \quad (10.11)$$

**Lemma 10.2:** If we have  $\hat{l} \leq \hat{u}$ , then  $B_Z(f)_i^{\hat{l}}$  is nonempty and any  $x \in B_Z(f)_i^{\hat{l}}$  is an optimal solution of Problem  $IP_3$  in (10.8).

(Proof) Since the vector minors  $B_Z(f)^{\hat{l}}$ ,  $B_Z(f)_i$  and  $B_Z(f)_i^{\hat{l}}$  are integral, the present lemma can be shown similarly as Theorem 10.1. Q.E.D.

Now, let us suppose that we do not have  $\hat{l} \leq \hat{u}$ . Define

$$D = \{e \mid e \in E, \hat{l}(e) > \hat{u}(e)\}. \quad (10.12)$$

It follows from (10.9)–(10.12) that

$$\hat{l}(e) = \hat{u}(e) + 1 \quad (e \in D), \quad (10.13)$$

$$\hat{l}(e) \leq \hat{u}(e) \quad (e \in E - D), \quad (10.14)$$

$$\hat{h}_e(\hat{u}(e)) < \hat{v}_* \leq \hat{v}^* < \hat{h}_e(\hat{l}(e)) \quad (e \in D), \quad (10.15)$$

$$\hat{v}_* \leq \hat{h}_e(\hat{l}(e)) \leq \hat{h}_e(\hat{u}(e)) \leq \hat{v}^* \quad (e \in E - D). \quad (10.16)$$

Moreover, define  $\hat{l} \wedge \hat{u} = (\min\{\hat{l}(e), \hat{u}(e)\} : e \in E)$  and  $\hat{l} \vee \hat{u} = (\max\{\hat{l}(e), \hat{u}(e)\} : e \in E)$ . Then, all the four sets  $B_Z(f)_i$ ,  $B_Z(f)^{\hat{l}}$ ,  $B_Z(f)_{i \wedge \hat{l}}$  and  $B_Z(f)^{\hat{l} \vee \hat{l}}$  are nonempty since  $B_Z(f)_{i \wedge \hat{l}} \supseteq B_Z(f)_i \neq \emptyset$  and  $B_Z(f)^{\hat{l} \vee \hat{l}} \supseteq B_Z(f)^{\hat{l}} \neq \emptyset$ . Therefore, from Theorem 2.5  $B_Z(f)_{i \wedge \hat{l}}^{\hat{l}}$  and  $B_Z(f)_i^{\hat{l} \vee \hat{l}}$  are nonempty. Choose any bases  $\hat{x} \in B_Z(f)_{i \wedge \hat{l}}^{\hat{l}}$  and  $\hat{y} \in B_Z(f)_i^{\hat{l} \vee \hat{l}}$ . From (10.12) we have

$$\hat{x}(e) = \hat{u}(e), \quad \hat{y}(e) = \hat{l}(e) \quad (e \in D). \quad (10.17)$$

Hence, from (10.13)–(10.16),

$$\hat{y}(e) = \hat{x}(e) + 1 \quad (e \in D), \quad (10.18)$$

$$\hat{h}_e(\hat{x}(e)) < \hat{v}_* \leq \hat{v}^* < \hat{h}_e(\hat{y}(e)) \quad (e \in D), \quad (10.19)$$

$$\hat{v}_* \leq \min\{\hat{h}_e(\hat{x}(e)), \hat{h}_e(\hat{y}(e))\} \leq \max\{\hat{h}_e(\hat{x}(e)), \hat{h}_e(\hat{y}(e))\} \leq \hat{v}^* \quad (e \in E - D). \quad (10.20)$$

Let the distinct values of  $\hat{h}_e(\hat{x}(e))$  ( $e \in D$ ) be given by

$$d_1 < d_2 < \cdots < d_k, \quad (10.21)$$

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where note that  $d_k < \hat{v}_*$ . Also define

$$A_i = \{e \mid e \in D, \hat{h}_e(\hat{x}(e)) \leq d_i\} \quad (i = 1, 2, \dots, k). \quad (10.22)$$

We consider a parametric problem  $IP_3^\lambda$  with a real parameter  $\lambda$  associated with the original problem  $IP_3$ .

$$IP_3^\lambda: \text{Minimize } g(\max_{e \in E} \hat{h}_e(x(e)), \lambda) \quad (10.23a)$$

$$\text{subject to } x \in B_Z(f), \quad (10.23b)$$

$$\hat{h}_e(x(e)) \geq \lambda \quad (e \in E), \quad (10.23c)$$

where  $\lambda \leq \hat{v}_*$ . Denote the minimum of the objective function (10.23a) by  $\gamma(\lambda)$ . Then, we have

**Lemma 10.3:** Suppose that  $\lambda = \lambda^*$  attains the minimum of  $\gamma(\lambda)$  for  $\lambda \leq \hat{v}_*$ . Then any optimal solution of  $IP_3^{\lambda^*}$  is an optimal solution of the original problem  $IP_3$  and the minimum of the objective function (10.8a) is equal to  $\gamma(\lambda^*)$ .

(Proof) Let  $x^*$  and  $x^0$ , respectively, be any optimal solutions of Problems  $IP_3^{\lambda^*}$  and  $IP_3$ . Denote by  $v^0$  be the minimum of the objective function (10.8a) of Problem  $IP_3$ , and define  $\lambda^0 = \min_{e \in E} \hat{h}_e(x^0(e))$ . Then

$$v^0 = g(\max_{e \in E} \hat{h}_e(x^0(e)), \lambda^0) \geq \gamma(\lambda^0) \geq \gamma(\lambda^*). \quad (10.24)$$

On the other hand, we have

$$\gamma(\lambda^*) \geq g(\max_{e \in E} \hat{h}_e(x^*(e)), \min_{e \in E} \hat{h}_e(x^*(e))) \geq v^0. \quad (10.25)$$

From (10.24) and (10.25) we have  $v^0 = \gamma(\lambda^*)$  and  $x^*$  is also an optimal solution of  $IP_3$ . Q.E.D.

We determine the function  $\gamma(\lambda)$  to solve the original problem  $IP_3$  with the help of Lemma 10.3. In the following arguments,  $\hat{x}$ ,  $\hat{y}$ ,  $d_i$ ,  $A_i$  ( $i = 1, 2, \dots, k$ ) are those appearing in (10.17)–(10.22). We consider the two cases when  $\lambda \leq d_1$  and when  $d_i < \lambda \leq d_{i+1}$  for some  $i \in \{1, 2, \dots, k\}$ , where  $d_{k+1} \equiv v_*$ .

*Case I:*  $\lambda \leq d_1$ .

Because of the definition (10.21) of  $d_1$ ,  $\hat{x}$  is a feasible solution of  $IP_3^\lambda$  for  $\lambda \leq d_1$ . Since  $\hat{x} \in B_Z(f)_{i \wedge \hat{x}}$ ,  $\hat{x}$  is also an optimal solution of  $IP^*$ . Therefore, it follows from the monotonicity of  $g$  that

$$\gamma(\lambda) = g(\hat{v}^*, \lambda). \quad (10.26)$$

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*Case II:*  $d_i < \lambda \leq d_{i+1}$  for some  $i \in \{1, 2, \dots, k\}$  ( $d_{k+1} \equiv v^*$ ).

From (10.18)–(10.22) and the monotonicity of  $g$  we have

$$\gamma(\lambda) \geq g(\max_{e \in A_i} \hat{h}_e(\hat{x}(e) + 1), \lambda). \quad (10.27)$$

It follows, from (10.18)–(10.20) for two bases  $\hat{x}$  and  $\hat{y}$ , that by repeated elementary transformations of  $\hat{x}$  we can have a base  $\hat{z}$  such that

$$\hat{z}(e) = \hat{x}(e) \quad (e \in D - A_i), \quad (10.28)$$

$$\hat{z}(e) = \hat{y}(e) (= \hat{x}(e) + 1) \quad (e \in A_i), \quad (10.29)$$

$$\min\{\hat{x}(e), \hat{y}(e)\} \leq \hat{z}(e) \leq \max\{\hat{x}(e), \hat{y}(e)\} \quad (e \in E - D). \quad (10.30)$$

We see from (10.18)–(10.20) and (10.28)–(10.30) that  $\hat{z}$  is a feasible solution of  $IP_3^\lambda$  for present  $\lambda$  and

$$\max_{e \in E} \hat{h}_e(\hat{z}(e)) = \max_{e \in A_i} \hat{h}_e(\hat{x}(e) + 1). \quad (10.31)$$

From (10.27) and (10.31)  $\hat{z}$  is an optimal solution of  $IP_3^\lambda$  and we have

$$\gamma(\lambda) = g(\max_{e \in A_i} \hat{h}_e(\hat{x}(e) + 1), \lambda). \quad (10.32)$$

The function  $\gamma(\lambda)$  is thus given by (10.26) for  $\lambda \leq d_1$  and by (10.32) for  $d_i < \lambda \leq d_{i+1}$  ( $i = 1, 2, \dots, k$ ).

**Theorem 10.4:** Suppose that we do not have  $\hat{l} \leq \hat{u}$ . Then, the minimum of the objective function (10.8a) of Problem  $IP_3$  is equal to the minimum of the following  $k + 1$  values

$$g(v^*, d_1), \quad g(\max_{e \in A_i} \hat{h}_e(\hat{u}(e) + 1), d_{i+1}) \quad (i = 1, 2, \dots, k). \quad (10.33)$$

(Proof) The present theorem follows from Lemma 10.3, (10.17), (10.26), (10.32) and the monotonicity of  $g$ . Q.E.D.

It should be noted that because of (10.17) we can employ  $\hat{u}$  instead of  $\hat{x}$  in the definitions of  $d_i$  and  $A_i$  ( $i = 1, 2, \dots, k$ ) in (10.21) and (10.22).

An algorithm for solving Problem  $IP_3$  in (10.8) is given as follows.

### An algorithm for the discrete fair resource allocation problem

**Step 1:** Solve Problems  $IP_*$  and  $IP^*$  given by (9.12) and (9.14), respectively. Let  $v_*$  and  $v^*$ , respectively, be the optimal values of the objective functions of



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$IP_*$  and  $IP^*$  and determine values  $\hat{l}$  and  $\hat{u}$  by (10.9) and (10.10). If  $\hat{l} \leq \hat{u}$ , then any base  $x \in B_Z(f)_l^{\hat{l}}$  is an optimal solution of  $IP_3$  and the algorithm terminates.

**Step 2:** Let  $D \subseteq E$  be defined by (10.12) and  $d_1 < d_2 < \dots < d_k$  be the distinct values of  $\hat{h}_e(\hat{u}(e))$  ( $e \in D$ ). Also determine sets  $A_i$  ( $i = 1, 2, \dots, k$ ) by (10.22) with  $\hat{x}$  replaced by  $\hat{u}$ .

**Step 3:** Find the minimum of the following  $k+1$  values.

$$g(\hat{v}^*, d_1), \quad g(\max_{e \in A_i} \hat{h}_e(\hat{u}(e) + 1), d_{i+1}) \quad (i = 1, 2, \dots, k). \quad (10.34)$$

(3-1) If  $g(\hat{v}^*, d_1)$  is minimum, then find any base  $\hat{x} \in B_Z(f)_{l \wedge \hat{x}}^{\hat{x}}$  and  $\hat{x}$  is an optimal solution of  $IP_3$ .

(3-2) If  $g(\max_{e \in A_{i^*}} \hat{h}_e(\hat{u}(e) + 1), d_{i^*+1})$  for  $i^* \in \{1, 2, \dots, k\}$  is minimum, then putting  $w^* = \max_{e \in A_{i^*}} \hat{h}_e(\hat{u}(e) + 1)$ , define  $l^0, u^0 \in Z^E$  by

$$l^0(e) = \min\{\alpha \mid \alpha \in Z, \hat{h}_e(\alpha) \geq d_{i^*+1}\}, \quad (10.35)$$

$$u^0(e) = \max\{\alpha \mid \alpha \in Z, \hat{h}_e(\alpha) \leq w^*\} \quad (10.36)$$

for each  $e \in E$ , and any base  $\hat{z} \in B_Z(f)_{l^0}^{u^0}$  is an optimal solution of  $IP_3$ .

(End)

It should be noted that from the arguments preceding Theorem 10.4 we have  $B_Z(f)_{l^0}^{u^0} \neq \emptyset$  in Step 3-2.

Let us consider the computational complexity of the algorithm when  $B_Z(f)$  is bounded, i.e.,  $\mathcal{D} = 2^E$ . Denote by  $T(IP_*, IP^*)$  the time required for solving Problems  $IP_*$  and  $IP^*$ . Let  $M$  be an integral upper bound of  $|f(X)|$  ( $X \subseteq E$ ). Then we have

$$-2M \leq f(E) - f(E - \{e\}) \leq x(e) \leq f(\{e\}) \leq M \quad (10.37)$$

for each  $x \in B_Z(f)$  and  $e \in E$ . Therefore, given  $\hat{v}_*$  and  $\hat{v}^*$ , we can determine the values of  $\hat{l}(e)$  and  $\hat{u}(e)$  in (10.9) and (10.10) by the binary search, which requires  $O(\log M)$  time for each  $e \in E$ . Also, finding a base in  $B_Z(f)_l^{\hat{x}}$  requires not more than  $O(T(IP_*, IP^*))$  time. Hence Step 1 runs in  $O(T(IP_*, IP^*) + |E| \log M)$  time.

Determining set  $D$  in Step 2 requires  $O(|E|)$  time. The values of  $d_1, d_2, \dots, d_k$  are found and sorted in  $O(|E| \log |E|)$  time. Instead of having sets  $A_i$  ( $i = 1, 2, \dots, k$ ) we compute differences  $\tilde{A}_i = A_i - A_{i-1}$  ( $i = 1, 2, \dots, k$ ) with  $A_0 = \emptyset$ , which requires  $O(|E|)$  time. Note that having the differences

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$\tilde{A}_i = A_i - A_{i-1}$  ( $i = 1, 2, \dots, k$ ) is enough to carry out Step 3. Then, Step 2 requires  $O(|E| \log |E|)$  time.

In Step 3 we can compute all the values

$$\max_{e \in A_i} \hat{h}_e(\hat{u}(e) + 1) \quad (i = 1, 2, \dots, k) \quad (10.38)$$

in  $O(|E|)$  time, using the differences  $\tilde{A}_i = A_i - A_{i-1}$  ( $i = 1, 2, \dots, k$ ). Moreover, for each  $e \in E$ ,  $l^0(e)$  and  $u^0(e)$  can be computed in  $O(\log M)$  time, and a base  $\hat{x} \in B_Z(f)_{i \wedge \hat{x}}^{\hat{x}}$  or  $\hat{z} \in B_Z(f)_{i \vee \hat{z}}^{\hat{z}}$  can be found in  $O(T(IP_*, IP^*))$  time. Consequently, Step 3 runs in  $O(T(IP_*, IP^*) + |E| \log M)$  time.

The overall running time of the algorithm is thus  $O(T(IP_*, IP^*) + |E|(\log M + \log |E|))$  for general (bounded) submodular constraints. When specialized to the problem of Katoh, Ibaraki and Mine [Katoh + Ibaraki + Mine85], where

$$B_Z(f) = \{x \mid x \in \mathbb{Z}_+^E, l \leq x \leq u, x(E) = c\}, \quad (10.39)$$

$f(E) (= c)$  can be chosen as  $M$ ,  $T(IP_*, IP^*)$  is  $O(|E| \log(f(E)/|E|))$  by the algorithm of G. N. Frederickson and D. B. Johnson [Frederickson + Johnson82], and hence the complexity becomes the same as the algorithm of [Katoh + Ibaraki + Mine85].

### 11. A Neoflow Problem with a Separable Convex Cost Function

In this section we consider the submodular flow problem (see Section 4.1) where the cost function is given by a separable convex function.

Let  $G = (V, A)$  be a graph with a vertex set  $V$  and an arc set  $A$ ,  $\bar{c}: A \rightarrow \mathbb{R} \cup \{+\infty\}$  be an upper capacity function and  $\underline{c}: A \rightarrow \mathbb{R} \cup \{-\infty\}$  be a lower capacity function. Also, for each arc  $a \in A$  let  $w_a: \mathbb{R} \rightarrow \mathbb{R}$  be a convex function. Suppose that  $(\mathcal{D}, f)$  is a submodular system on  $V$  such that  $f(V) = 0$ . Denote this network by  $\mathcal{N}_{SS} = (G = (V, A), \underline{c}, \bar{c}, w_a (a \in A), (\mathcal{D}, f))$ .

Consider the following flow problem in  $\mathcal{N}_{SS}$ .

$$P_{SS}: \text{Minimize } \sum_{a \in A} w_a(\varphi(a)) \quad (11.1a)$$

$$\text{subject to } \underline{c}(a) \leq \varphi(a) \leq \bar{c}(a) \quad (a \in A), \quad (11.1b)$$

$$\partial \varphi \in B_{\mathbb{R}}(f). \quad (11.1c)$$

We call a function  $\varphi: A \rightarrow \mathbb{R}$  satisfying (11.1b) and (11.1c) a *submodular flow* in  $\mathcal{N}_{SS}$ .

Optimal solutions of Problem  $P_{SS}$  in (11.1) are characterized by the following.

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**Theorem 11.1:** A submodular flow  $\varphi$  in  $\mathcal{N}_{SS}$  is an optimal solution of Problem  $P_{SS}$  in (11.1) if and only if there exists a potential  $p: V \rightarrow \mathbb{R}$  such that the following (i)–(iv) hold. Here,  $w_a^+$  and  $w_a^-$  denote, respectively, the right derivative and the left derivative of  $w_a$  for each  $a \in A$ .

(i) For each  $a \in A$  such that  $\underline{c}(a) < \varphi(a) < \bar{c}(a)$ ,

$$w_a^-(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) \leq 0 \leq w_a^+(\varphi(a)) + p(\partial^+ a) - p(\partial^- a). \quad (11.2)$$

(ii) For each  $a \in A$  such that  $\underline{c}(a) = \varphi(a) < \bar{c}(a)$ ,

$$0 \leq w_a^+(\varphi(a)) + p(\partial^+ a) - p(\partial^- a). \quad (11.3)$$

(iii) For each  $a \in A$  such that  $\underline{c}(a) < \varphi(a) = \bar{c}(a)$ ,

$$w_a^-(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) \leq 0. \quad (11.4)$$

(iv)  $\partial\varphi$  is a maximum-weight base of  $B(f)$  with respect to the weight vector  $p$ .

(Proof) *The “if” part:* Suppose that a potential  $p$  satisfies (i)–(iv) for a submodular flow  $\varphi$  in  $\mathcal{N}_{SS}$ . Then for any submodular flow  $\hat{\varphi}$  in  $\mathcal{N}_{SS}$  we have

$$\begin{aligned} \sum_{a \in A} w_a(\hat{\varphi}(a)) &\geq \sum_{a \in A} w_a(\varphi(a)) + \sum_{a \in A} (p(\partial^- a) - p(\partial^+ a))(\hat{\varphi}(a) - \varphi(a)) \\ &= \sum_{a \in A} w_a(\varphi(a)) + \sum_{v \in V} p(v)(\partial\varphi(v) - \partial\hat{\varphi}(v)) \\ &\geq \sum_{a \in A} w_a(\varphi(a)). \end{aligned} \quad (11.5)$$

Therefore,  $\varphi$  is an optimal solution of Problem  $P_{SS}$  in (11.1).

*The “only if” part:* Let  $\varphi$  be an optimal solution of Problem  $P_{SS}$  in (11.1). Construct an auxiliary network  $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi), \gamma_\varphi)$  associated with  $\varphi$  as follows. The arc set  $A_\varphi$  is defined by (4.40)–(4.43) and  $\gamma_\varphi: A_\varphi \rightarrow \mathbb{R}$  is the length function defined by

$$\gamma_\varphi(a) = \begin{cases} w_a^+(\varphi(a)) & (a \in A_\varphi^*) \\ -w_{\bar{a}}^-(\varphi(\bar{a})) & (a \in B_\varphi^*, \bar{a} \in A): \text{a reorientation of } a \\ 0 & (a \in C_\varphi), \end{cases} \quad (11.6)$$

where  $A_\varphi^*$ ,  $B_\varphi^*$  and  $C_\varphi$  are defined by (4.41)–(4.43). Since there is no directed cycle of negative length relative to the length function  $\gamma_\varphi$  due to the optimality of  $\varphi$ , there exists a potential  $p: V \rightarrow \mathbb{R}$  such that we have

$$\gamma_\varphi(a) + p(\partial^+ a) - p(\partial^- a) \geq 0 \quad (a \in A_\varphi), \quad (11.7)$$

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where  $\partial^+$ ,  $\partial^-$  are with respect to  $G_\varphi$ . (11.7) is equivalent to (i)–(iv). (iv) is due to Theorem 2.13. Q.E.D.

Theorem 11.1 generalizes Theorem 4.2. It should be noted that if  $w_a$  is differentiable for each  $a \in A$ , then (i)–(iii) in Theorem 11.1 is equivalent to the following (a) and (b).

- (a) For each  $a \in A$ ,  $w_a'(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) > 0$  implies  $\varphi(a) = \underline{c}(a)$ ;
- (b) For each  $a \in A$ ,  $w_a'(\varphi(a)) + p(\partial^+ a) - p(\partial^- a) < 0$  implies  $\varphi(a) = \bar{c}(a)$ .

Here,  $w_a'$  denotes the derivative of  $w_a$  for each  $a \in A$ .

Given a submodular flow  $\varphi$  in  $\mathcal{N}_{SS}$  and a potential  $p: V \rightarrow \mathbb{R}$ , we say that an arc  $a \in A$  is *in kilter* if one of (i)–(iii) in Theorem 11.1 is satisfied and that an ordered pair  $(u, v)$  of vertices  $u, v \in V$  ( $u \neq v$ ) is *in kilter* if (1)  $p(u) \geq p(v)$  or (2)  $p(u) < p(v)$  and  $u \notin \text{dep}(\partial\varphi, v)$ . Also, to be *out of kilter* is to be not in kilter. We see from Theorem 11.1 that if all the arcs and all the pairs of distinct vertices are in kilter, then the given submodular flow  $\varphi$  is optimal. For each arc  $a \in A$  we call the set of points  $(\varphi(a), p(\partial^- a) - p(\partial^+ a))$  in  $\mathbb{R}^2$  such that arc  $a$  is in kilter the *characteristic curve* (see Fig. 11.1).

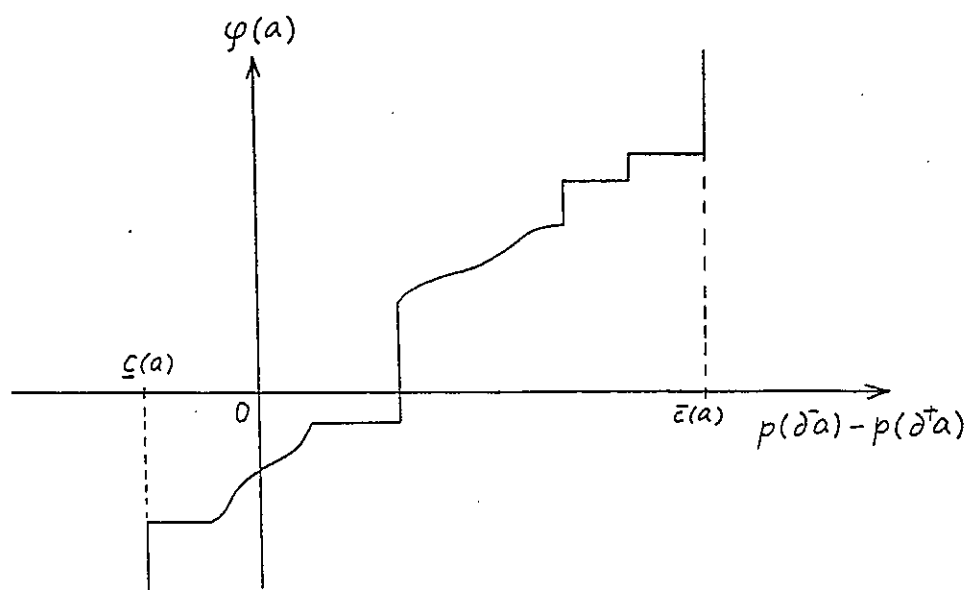


Figure 11.1.

Theorem 11.1 suggests an out-of-kilter method which keeps in-kilter arcs in kilter and monotonically decreases the *kilter number* of each out-of-kilter arc, which is a “distance” to the characteristic curve for  $a \in A$  or  $p(u) - p(v)$

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for  $(u, v)$  such that  $u \in \text{dep}(\partial\varphi, v)$ . When  $w_a$  is piecewise linear for each  $a \in A$ , The characteristic curve for each  $a \in A$  consists of vertical and horizontal line segments and the out-of-kilter method described in Section 4.5.c can easily be adapted so as to be a finite algorithm.



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