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ALTERNATIVE PROOF OF PANDORA'S RULE IN OPTIMAL SEARCH FOR THE BEST ALTERNATIVE

by

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The present paper provides an alternative proof of Pandora's Rule in optimal search problem that was verified by M.L. Weitzman [1], whose proof is quite technical as he himself states in his paper. As compared with it, the proof that we are going to present here is well enough systematic to be successfully applied to more general search problems.

INTRODUCTION

M.L. Weitzman posed the following interesting search problem [1]. Suppose there exist N closed boxes. In each box i, $1 \leqq i \leqq N$, is contained a certain amount of reward x_i. It is the following three that can be known about each box i before opening it:

1. probability distribution function $F_i(x_i)$ of reward x_i in it,
2. cost c_i to open and learn reward x_i in it, and
3. time lag t_i after which since opening it reward x_i in it becomes known.

Assume that both cost and reward are evaluated in the same monetary value and that all costs and rewards are continuously discounted by the discount rate r (By being continuously discounted, we means that one unit of monetary value at time t is equivalent to e^{-rt} at time 0). Furthermore, assume that an initial amount of reward y_0 is available before starting search.

The search process proceeds as follows. First, you must decide either to accept the initial reward y_0 with quitting the search process or to open any one of N boxes with paying cost. If deciding to open a box, say box 2, with paying cost c_2, then you can know reward x_2 in it after t_2 unit times, and then you must again decide either to accept the maximum reward $y_1 = \max\{y_0, x_2\}$ or to further open any one of $N-1$ boxes remaining. If you have opened all the boxes, then you can get the maximum of rewards in boxes opened. For example, suppose search was terminated after opening boxes 2, 5, and 3 in this order. Then the present discounted value obtained is

$$(1) \qquad -c_2 e^{-rt_2} - c_5 e^{-r(t_2+t_5)} + (\max\{y_0, x_2, x_5, x_3\} - c_3) e^{-r(t_2+t_5+t_3)}$$

Objective is here to maximize the expected present discounted value; i.e., the expectation of the present value of the maximum reward obtained after terminating search minus the total present value of costs incurred to open boxes.

Weitzman proved that a decision strategy called Pandora's Rule becomes optimal for the search problem. Unfortunately, his proof is quite technical as he himself states in his paper. The objective of the current paper is to provide well enough systematic proof to be successfully applied to more general search problems.

In the next section, the functional equation for the search problem is given, and in the section that follows, Pandora's Rule is explained with the outline of his proof. In the forth section, an alternative proof of Pandora's Rule is stated, and in the last section, by applying the idea of the alternative proof, a necessary condition is proved on which Pandora's Rule is still optimal for the problem of opening only M of N boxes available, $N > M \geqq 2$.

FUNCTIONAL EQUATION

For any real number y, define

$$(2) \qquad K_i(y) = \beta_i \int \max\{w,y\}\, dF_i(w) - y - c_i, \quad 1 \leqq i \leqq N$$

Then it can be easily shown that

$$(3) \qquad dK_i(y)/dy = \beta_i F_i(y) - 1 \leqq 0.$$

Here we shall define

$$(4) \qquad h_i = \sup\{y | K_i(y) > 0\},$$

called a *reservation price* of box i, where $K_i(y) > (\leqq)\ 0$ for $y < (\geqq)\ h_i$ because $K_i(y)$ is nonincreasing in y from (3). Now, let us denote a set of N boxes available by $S = \{1,2,\ldots,N\}$ and any subset of it by L. Define $v(L,y)$ as the maximum expected present discounted value starting with boxes L and the maximum reward y. Then, from the principle of optimality in dynamic programming, we have

$$(5) \qquad v(L,y) = \max\{y, \max_{i \in L} \{-c_i + \beta_i \int v(L_i, \max\{y,w\})\, dF_i(w)\}\}$$

where

$$(6) \qquad v(\Phi,y) = y$$

$$(7) \qquad \beta_i = e^{-rt_i}$$

$$(8) \qquad L_i = L - \{i\}.$$

For convenience of later discussions, let us write (5) as

(9) $$v(L,y) = y + \max\{0, \max_{i \in L} K_i(L,y)\}$$

where

(10) $$K_i(L,y) = \int v(L_i, \max\{y,w\})\, dG_i(w) - y - c_i$$

(11) $$G_i(w) = \beta_i F_i(w),$$

Throughout the paper, without loss in generality, let

(12) $$h_1 \geqq h_2 \geqq \cdots \geqq h_N$$

PANDORA'S RULE

Weitzman proved that the optimal decision strategy for the search problem is given by the following rule, called **Pandora's Rule**:

Selection Rule : *If a box is to be opened, it should be that closed box with highest reservation price.*

Stopping Rule : *Terminate search whenever the maximum sampled reward exceeds the reservation price of every closed box.*

In other words, it is optimal to open boxes in the order of the height of reservation price and terminate search if the maximum sampled reward so far exceeds the reservation price of the box just opened. The outline of his proof, based on induction on the number of closed boxes, is as follows.

First it is clear from (9) that $v_k(\{k\},y) = y + \max\{0, K_k(y)\}$ for any $k \in S$. Therefore, if $y < h_k$, then opening box k is optimal due to $K_k(y) > 0$; otherwise, not opening box k is optimal due to $K_k(y) \leqq 0$.

Next assume Pandora's Rule is optimal with m closed boxes remaining and any maximum reward y, and let search start with any $m+1$ closed boxes, say $L = \{1,2,\ldots,m+1\}$, and with the maximum reward y. Then, by O_0 we shall denote the expected present discounted value from not opening any box in L; clearly $O_0 = y$.

Suppose $y \geqq h_1$. Then, since $\max\{y,w\} \geqq h_1 \geqq h_2$ for any w, the expected present discounted value from opening any box k becomes $O_k = \beta_k \int \max\{y,w\}\, dF_k(w) - c_k$ from the induction hypothesis, yielding $O_k - O_0 = K_k(y) \leqq 0$ due to $y \geqq h_1 \geqq h_k$. Therefore, in the case, it follows that not opening any box, or stopping search with accepting the current maximum reward y, is optimal.

Suppose $y < h_1$, and let O_1 denote the expected present discounted value from opening box j and then stopping. Then $O_1 = \beta_1 \int \max\{y,w\}\, dF_1(w) - c_1$, we have $O_J - O_0 = K_1(y) > 0$, implying that not opening any box is not optimal. Accordingly, a question to be answered here is which box to be opened; opening box 1 is its answer as Pandora's Rule claims. Roughly speaking, his line of argument is as follows: first define the expected discounted present values of opening box 1 and of opening any other box k be designated by, respectively, A and B, next express them in a form into which some appropriately defined terms are assembled into in a dexterous fashion, and finally arrange the difference $A - B$ in a form which is capable of making us ascertaine its positivity. In this discussions, in addition to the fact that the forms of A and B become very complicated, the last step is remarkably technical.

ALTERNATIVE PROOF

In order to prove Pandora's Rule, it suffices to show the next two: for all L where let $k = \min L$, the smallest element in L,

i. $v(L,y) = y + \max\{0, K_k(L,y)\}$

ii. $K_k(L,y) >(\leqq)\ 0$ on $y <(\geqq) h_k$

Instead of directly verifying the above two, we shall prove the following general theorem including them.

THEOREM 1. For any $L = \{j_1,j_2,\cdots,j_m\}$ with $1 \leqq j_1 < j_2 < \cdots < j_m \leqq N$,
(a) $v(L,y) = y + \max\{0, K_k(L,y)\}$, $k = j_1$ (= min L)
(b) $K_k(L,y)$ is nonincreasing in y with $K_k(L,h_k) = 0$
(c) $K_{j_1}(\{j_1,j_2,\cdots,j_m\},y) = K_{j_1}(\{j_1,j_2,\cdots,j_{m-1}\},y)$ on $y \geqq h_{j_m}$, $2 \leqq m \leqq M$.

Proof: First consider any L with $m = 1$, say $L = \{1\}$. Then (9) and (10) become, respectively,

$$(13)\qquad v(\{1\},y) = y + \max\{0, K_1(\{1\},y)\}$$

$$(14)\qquad K_1(\{1\},y) = K_1(y)$$

Accordingly, (a) and (b) become true for $\{1\}$, hence for any L with $m = 1$. Next, consider any L with $m = 2$, say $L = \{1,2\}$. Then, arranging (10) by using (13) and (14) for $\{2\}$, instead of $\{1\}$, yields

$$(15)\qquad K_1(\{1,2\},y) = K_1(y) + \int \max\{0, K_2(\max\{y,w\})\}\, dG_1(w)$$

For $y \geqq h_2$, clearly $K_1(\{1,2\},y) = K_1(y) = K_1(\{1\},y)$ due to $\max\{y,w\} \geqq h_2$ for any w. Therefore, (c) holds for $\{1,2\}$, hence for any L with $m = 2$.

Assume that (a) and (b) hold for any $L \in \mathcal{L}_n$, $1 \leqq n \leqq m-1$. Then, for any $L \in \mathcal{L}_m$, say $L = \{1,2,\ldots,m\}$, we have, from the assumption,

$$(16) \qquad v(L_1,y) = y + \max\{0,\ K_2(L_1,y)\}$$

$$(17) \qquad v(L_i,y) = y + \max\{0,\ K_i(L_i,y)\} \quad 2 \leqq i \leqq N$$

Arranging (10) by substituting (16) and (17) into yields

$$(18) \qquad K_1(L,y) = K_1(y) + \int \max\{0,\ K_2(L_1,\max\{y,w\})\}\,dG_1(w)$$

$$(19) \qquad K_i(L,y) = K_i(y) + \int \max\{0,\ K_1(L_i,\max\{y,w\})\}\,dG_i(w) \qquad 2 \leqq i \leqq N$$

It can be easily seen from (18) and the induction hypothesis that (b) holds for $\{1,2,\cdots,m\}$, hence for any $L \in L_m$. Now, if $y \geqq h_m$, then since $K_2(\{2,3,\cdots,m\},y) = K_2(\{2,3,\cdots,m-1\},y)$ from the induction hypothesis, we have

$$\begin{aligned}(20) \qquad K_1(\{1,2,\cdots,m\},y) &= K_1(y) + \int \max\{0,\ K_2(\{2,3,\cdots,m\},\max\{y,w\})\}\,dG_1(w)\\ &= K_1(y) + \int \max\{0,\ K_2(\{2,3,\cdots,m-1\},\max\{y,w\})\}\,dG_1(w)\\ &= K_1(\{1,2,\cdots,m-1\},y)\end{aligned}$$

Thus, (c) was proved for $\{1,2,\cdots,m\}$, hence for any $L \in \mathcal{L}_m$.

If $h_1 < y$, then (a) is true because $K_i(L,y) \leqq 0$ for all $i \geqq 1$ from (18) and (19). Below, let $y \leqq h_1$. Then (18) and (19) can be arranged as

$$(21) \qquad K_1(L,y) = K_1(y) + (K_2(L_1,y)G_1(y) + \int_y^{h_2} K_2(L_1,w)\,dG_1(w))I(y \leqq h_2)$$

$$(22) \qquad K_i(L,y) = K_i(y) + K_1(L_i,y)G_i(y) + \int_y^{h_1} K_1(L_i,w)\,dG_i(w) \qquad 2 \leqq i \leqq N$$

where $I(S)$ is an indicator function in which S represents a statement, either true or false. Then, differentiating the above with respect to y produces

$$(23) \qquad dK_1(L,y)/dy = G_1(y) - 1 + dK_2(L_1,y)/dy\ G_1(y)I(y \leqq h_2)$$

$$(24) \qquad dK_i(L,y)/dy = G_i(y) - 1 + dK_1(L_i,y)/dy\ G_i(y) \qquad 2 \leqq i \leqq N$$

Let $\Gamma_k(y) = G_1(y)G_2(y)\cdots G_k(y)$. Then expressing (23) in an expanded form yields

$$(25) \qquad dK_1(L,y)/dy = \Gamma_1(y)I(h_2<y) + \sum_{k=2}^{m-1} \Gamma_k(y)I(h_{k+1}<y \leqq h_k) + \Gamma_m(y)I(y \leqq h_m) - 1$$

From this, we have

$$(26)\qquad dK_1(L_i,y)/dy = \Gamma_1(y)I(h_2<y)+\sum_{k=2}^{i-1}\Gamma_k(y)I(h_{k+1}<y\leqq h_k)$$
$$+\sum_{k=i}^{m-1}\Gamma_k(y)G_i(y)^{-1}I(h_{k+1}<y\leqq h_k)+\Gamma_m(y)G_i(y)^{-1}I(y\leqq h_m)-1 \qquad 2\leqq i\leqq m-1$$

$$(27)\qquad dK_1(L_m,y)/dy = \Gamma_1(y)I(h_2<y)+\sum_{k=2}^{m-1}\Gamma_k(y)I(h_{k+1}<y\leqq h_k)+\Gamma_m(y)G_m(y)^{-1}(y\leqq h_m)-1$$

Arranging (24) by substituting (26) and (27) into produces

$$(28)\qquad dK_i(L,y)/dy = \Gamma_1(y)G_i(y)I(h_2<y)+\sum_{k=2}^{i-1}\Gamma_k(y)G_i(y)I(h_{k+1}<y\leqq h_k)$$
$$+\sum_{k=i}^{m-1}\Gamma_k(y)I(h_{k+1}<y\leqq h_k)+\Gamma_m(y)I(y\leqq h_m)-1 \qquad 2\leqq i\leqq m-1$$

$$(29)\qquad dK_m(L,y)/dy = \Gamma_1(y)G_m(y)I(h_2<y)+\sum_{k=2}^{m-1}\Gamma_k(y)G_m(y)I(h_{k+1}<y\leqq h_k)$$
$$+\Gamma_m(y)I(y\leqq h_m)-1$$

From (25), (28), and (29), we have

$$(30)\qquad dK_1(L,y)/dy - dK_i(L,y)/dy$$
$$= \Gamma_1(y)(1-G_i(y))I(h_2<y)+\sum_{k=2}^{i-1}\Gamma_k(y)(1-G_i(y))I(h_{k+1}<y\leqq h_k),\quad 2\leqq i\leqq m$$

This is nonnegative for all y with being equal to 0 for $y \leq h_i$; in other words, $K_1(L,y) - K_i(L,y)$ is constant on $y \leqq h_i$ and increasing on $h_i < y \leqq h_1$. Consequently, it follows that it suffices to show $K_1(L,h_i) - K_i(L,h_i) \geqq 0$ for all $i \geqq 2$ in order to prove (a). By sequentially applying the induction hypothesis (c) with noticing $\max\{h_i,w\} \geqq h_i \geqq h_{i+1} \geqq \cdots \geqq h_m$, we obtain

$$(31)\qquad K_2(\{2,3,\cdots,m\},\max\{h_i,w\}) = K_2(\{2,3,\cdots,i-1\},\max\{h_i,w\})$$

$$(32)\qquad K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},h_i) = K_i(\{1,2,\cdots,i-1\},h_i)$$

Using this, we have

$$(33)\qquad K_1(\{1,2,\cdots,m\},h_i) = K_1(h_i) + \int \max\{0,\ K_2(\{2,3,\cdots,m\},\max\{h_i,w\})\}\,dG_1(w)$$
$$= K_1(h_i) + \int \max\{0,\ K_2(\{2,3,\cdots,i-1\},\max\{h_i,w\})\}\,dG_1(w)$$
$$= K_1(\{1,2,\cdots,i-1\},h_i)$$

$$(34)\qquad K_i(\{1,2,\cdots,m\},h_i) = \int \max\{0,\ K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},\max\{h_i,w\})\}\,dG_i(w)$$
$$\leqq K_i(\{1,2,\cdots,i-1,i+1,\cdots,m\},h_i) = K_i(\{1,2,\cdots,i-1\},h_i). \qquad 2\leqq i\leqq N$$

Accordingly it follows that $K_1(\{1,2,\cdots,m\},h_1) - K_1(\{1,2,\cdots,m\},h_1) \geqq 0$; hence, $K_1(L,h_1) - K_1(L,h_1) \geqq 0$ for any $L \in \mathcal{L}_m$. Thus (a) was proved. □

PROBLEM OF ONLY OPENING M OF THE N BOXES

The section considers a more general case of only opening M of N boxes available, $N > M \geqq 2$. Here, by $\mathcal{L}_n$, $1 \leqq n \leqq N$, we shall denote a family of sets consisting of n elements in S. Let $v_m(L,y)$, $1 \leqq m \leqq M$, be the maximum expected present discounted value starting with any $L \in \mathcal{L}_{N-M+m}$ and the maximum reward y. Then (5) becomes

$$(35)\qquad v_m(L,y) = \max\{y, \max_{i \in L}\{-c_i + \int v_{m-1}(L_i, \max\{y,w\})\, dG_i(w)\}\}$$

where $L_i \in \mathcal{L}_{N-M+m-1}$ and $v_0(L,y) = y$ for any $L \in \mathcal{L}_{N-M}$, and (9) and (10) becomes

$$(36)\qquad v_m(L,y) = y + \max\{0, \max_{i \in L} K^m{}_i(L,y)\}$$

where

$$(37)\qquad K^m{}_i(L,y) = \int v_{m-1}(L_i, \max\{y,w\})\, dG_i(w) - y - c_i$$

Now, does Pandora's Rule determine an optimal ordering for the search problem ? Unfortunately, its answer is "No" : a counter-example is shown in [1]. However, it is easily realized that, if the following two requirments are satisfied for any $L \in \mathcal{L}_{N-M+m}$ with $1 \leqq m \leqq M$, then Pandora's Rule determines an optimal ordering:

i. $v_m(L,y) = \max\{0, K^m{}_k(L,y)\}$

ii. $K^m{}_k(L,y)$ is nonincreasing in y with $K^m{}_k(L,y) >(\leqq) 0$ on $y <(\geqq) h_k$

In Proposition 2 below, we shall verify that inequalites

$$(38)\qquad \beta_1 F_1(w) \leqq \beta_2 F_2(w) \leqq \cdots \leqq \beta_N F_N(w) \quad \text{for all } w$$

is a necessary condition on which the above two requirements are met. Now, under (38), it must be that

$$(39)\qquad \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_N$$

If (39) holds as well as if

$$(40)\qquad F_1(w) \leqq F_2(w) \leqq \cdots \leqq F_N(w) \quad \text{for all } w,$$

then clearly (38) is satisfied. Here remenber that the assumption (12) was made throughout the paper. Therefore, a question arises whether or not there exists such cases that all of (12), (39), and (40) are satisfied. Examples of such cases include one in which (42) is satisfied, $\beta_1 = \beta_2 = \cdots = \beta_N$, and $c_1 = c_2 = \cdots = c_N$; under the conditions, $K_i(y)$, a nonincreasing function of y from (3), is nondecreasing in i for all y (Lehman [2]).

LEMMA 1. If (38) holds, then, for all y,

$$(41)\qquad \max\{0, K_1(y)\} \geq \max\{0, K_2(y)\} \geq \cdots \geq \max\{0, K_N(y)\}$$

Proof: The difference $K_i(y) - K_{i+1}(y)$, $1 \leq i \leq N-1$, is nonincreasing in y because $dK_i(y)/dy - dK_{i+1}(y)/dy = G_i(y) - G_{i+1}(y) \leq 0$ for all y. In addition, since $K_i(h_{i+1}) - K_{i+1}(h_{i+1}) = K_i(h_{i+1}) \geq 0$, it follows that, on $y \leq h_{i+1}$, $K_i(y) - K_{i+1}(y) \geq 0$, hence $\max\{0,K_i(y)\} \geq \max\{0,K_{i+1}(y)\}$. On $h_{i+1} < y \leq h_i$, $K_i(y) \geq 0 \geq K_{i+1}(y)$, hence $\max\{0,K_i(y)\} \geq \max\{0,K_{i+1}(y)\}$. On $h_i < y$, $0 \geqq K_i(y)$ and $0 \geqq K_{i+1}(y)$, implying $\max\{0,K_i(y)\} = \max\{0,K_{i+1}(y)\} = 0$. Thus, the above inequalities hold for all y. □

If $F_i(w) = 0$ on $w \leq 0$ for all i, then (41) yields

$$(42)\qquad \max\{0, \beta_1\mu_1 - c_1\} \geq \max\{0, \beta_2\mu_2 - c_2\} \geq \cdots \geq \max\{0, \beta_N\mu_N - c_N\}$$

Hence, in the case, it follows that at least the above inequalities must be satisfied if you wants to assume (12) and (38).

PROPOSITION 2. Suppose (38) holds. Then, for any $L = \{j_1, j_2, \cdots, j_{N-M+m}\}$, $1 \leq m \leq M$, with $j_1 < j_2 < \cdots < j_{N-M+m}$ where let $k = j_1$ (= min L),

(a) $v_m(L,y) = y + \max\{0, K^m{}_k(L,y)\}$

(b) $K^m{}_k(L,y)$ is nonincreasing in y with $K^m{}_k(L,h_k) = 0$

(c) $K^m{}_{j_1}(\{j_1,j_2,\cdots,j_{N-M+m}\},y)$ is independent of $\{j_m,j_{m+1},\cdots,j_{N-M+m}\},y)$ for $y \geq h_{j_m}$, $m \geqq 2$.

(d) $K^m{}_{j_1}(\{j_1,j_2,\cdots,j_{N-M+m}\},y) = K^{m-1}{}_{j_1}(\{j_1,j_2,\cdots,j_{N-M+m-1}\},y)$ for $y \geq h_{j_m}$, $m \geq 2$.

Proof: First, for any $L \in L_{N-M+1}$, (a) can be immediately verified from (41), and (b) is also clear. For any $L \in L_{N-M+2}$, say $L = \{1,2,\cdots,N-M+2\}$, we have, for any $y \geq h_2$,

$$(43)\qquad K^2{}_1(\{1,2,\cdots,N-M+2\},y) = K_1(y) + \int \max\{0, K^1{}_2(\{2,3,\cdots,N-M+2\},\max\{y,w\})\}\, dG_1(w)$$

$$= K_1(y)$$

$$= K^1{}_1(\{1,2,\cdots,N-M+1\},y)$$

Therefore, (c) and (d) hold for $\{1,2,\cdots,N-M+2\}$, hence for any $L \in L_{N-M+2}$.

Suppose (a) to (d) are true for any $L \in L_{N-M+n}$, $1 \leqq n \leqq m-1$. Then, for any $L \in L_{N-M+m}$, say $L = \{1,2,\cdots,N-M+m\}$, we have

(44) $$K^m{}_1(L,y) = K_1(y) + \int \max\{0,\ K^{m-1}{}_2(L_1,\max\{y,w\})\}\, dG_1(w)$$

(45) $$K^m{}_i(L,y) = K_i(y) + \int \max\{0,\ K^{m-1}{}_1(L_i,\max\{y,w\})\}\, dG_i(w), \quad 2 \leqq i \leqq N-M+m$$

Here note that, from the induction hypothesis, for $y \geqq h_m$,

1. $K^{m-1}{}_2(\{2,3,\cdots,N-M+m\},\max\{y,w\})$ is independent of $\{m,m+1,\cdots,N-M+m\}$
2. $K^{m-1}{}_2(\{2,3,\cdots,N-M+m\},\max\{y,w\}) = K^{m-2}{}_2(\{2,3,\cdots,N-M+m-1\},\max\{y,w\})$

Accordingly, for $y \geqq h_m$, the expression

(46) $$K^m{}_1(\{1,2,\cdots,N-M+m\},y) = K_1(y) + \int \max\{0,\ K^{m-1}{}_2(\{2,3,\cdots,N-M+m\},\max\{y,w\})\}\, dG_1(w)$$

becomes independent of $\{m,m+1,\cdots,N-M+m\}$ and can be rewritten as

(47) $$\begin{aligned} K^m{}_1(\{1,2,\cdots,N-M+m\},y) &= K_1(y) + \int \max\{0,\ K^{m-2}{}_2(\{2,3,\cdots,N-M+m-1\},\max\{y,w\})\}\, dG_1(w) \\ &= K^{m-1}{}_1(\{1,2,\cdots,N-M+m-1\},y) \end{aligned}$$

Thus, (c) and (d) were proved for $\{1,2,\cdots,N-M+m\}$, hence for any $L \in L_{N-M+m}$. Now, if $h_1 < y$, then since $K^m{}_i(L,y) \leqq 0$ for $1 \leqq i \leqq M$ from (44) and (45), (a) holds for $\{1,2,\cdots,N-M+m\}$, hence for any $L \in L_{N-M+m}$. Below, let $y \leqq h_1$. Then differentiating (44) and (45) with respect to y produces

(48) $$dK^m{}_1(L,y)/dy = G_1(y) - 1 + G_1(y)\, dK^{m-1}{}_2(L_1,y)/dy\, I(y \leqq h_2)$$

(49) $$dK^m{}_i(L,y)/dy = G_i(y) - 1 + G_i(y)\, dK^{m-1}{}_1(L_i,y)/dy \quad 2 \leqq i \leqq N-M+m$$

Expressing (48) in an expanded form yields the same expression as (25); i.e.,

(50) $$dK^m{}_1(L,y)/dy = \text{the right side of (25)}$$

where notice that there exists the relation of

(51) $$dK^{m-1}{}_1(L,y)/dy = dK^m{}_1(L,y)/dy - \Gamma_{m-1}(y)(G_m(y)-1)I(y \leqq h_m)$$

From this, we have

(52) $$dK^{m-1}{}_1(L_i,y)/dy = \text{the right side of (26)} \qquad 2 \leqq i \leqq m-1$$

(53) $$dK^{m-1}{}_1(L_m,y)/dy = \text{the right side of (27)}$$

(54) $\quad dK^{m-1}{}_{1}(L_{i},y)/dy$ = the right side of (25) with replacement of m by $m-1$ $\quad m+1 \leq i \leq N-M+m$

Arranging (49) by substituting (52), (53), and (54) into produces

(55) $\quad dK^{m}{}_{i}(L,y)/dy$ = the right side of (28) $\quad 2 \leq i \leq m-1$

(56) $\quad dK^{m}{}_{m}(L,y)/dy$ = the right side of (29)

(57) $\quad dK^{m}{}_{i}(L,y)/dy = G_{i}(y) + G_{i}(y)\,dK^{m-1}{}_{1}(L,y)/dy - 1 \quad m+1 \leq i \leq N-M+m$

Then we have

(58)
$$dK^{m}{}_{1}(L,y)/dy - dK^{m}{}_{i}(L,y)/dy$$
$$=\begin{cases} \text{the right side of (30)} & 2 \leq i \leq m \\ (1-G_{i}(y))\,dK^{m}{}_{1}(L,y)/dy - G_{i}(y)(1+\Gamma_{m-1}(y)(1-G_{m}(y))I(y \leq h_{m})) + 1 & \\ = \begin{cases} \Gamma_{m-1}(y)(G_{m}(y)-G_{i}(y)) \leq 0 & y \leq h_{m} \\ (1-G_{i}(y))(dK^{m}(L,y)/dy+1) \geq 0 & y > h_{m} \end{cases} & m+1 \leq i \leq N-M+m \end{cases}$$

If $2 \leq i \leq m$, then, for the same reason as in the previous section, it suffices to show $K^{m}{}_{1}(L,h_{i}) \geq K^{m}{}_{i}(L,h_{i})$ in order to prove (a). If $m+1 \leq i \leq N-M+m$, then since $K^{m}{}_{1}(L,y) - K^{m}{}_{i}(L,y)$ becomes nonincreasing on $y \leq h_{m}$ and nondecreasing on $h_{m} < y \leq h_{i}$, it suffices to show $K_{1}(L,h_{m}) - K_{i}(L,h_{m}) \geq 0$ in order to prove (a). First suppose $2 \leq i \leq m$. Then, by sequentially applying (d), we obtain

(59) $\quad K^{m-1}{}_{2}(\{2,3,\cdots,N-M+m\},\max\{h_{i},w\}) = K^{i-1}{}_{2}(\{2,3,\cdots,N-M+i\},\max\{h_{i},w\})$

(60) $\quad K^{m-1}{}_{1}(\{1,2,\cdots,i-1,i+1,\cdots,N-M+m\},h_{i}) = K^{i}{}_{1}(\{1,2,\cdots,i-1,i+1,\cdots,N-M+i+1\},h_{i})$

Then

(61)
$$\begin{aligned} K^{m}{}_{1}(\{1,2,\cdots,N-M+m\},h_{i}) &= K_{1}(h_{i}) + \int \max\{0,\ K^{m-1}{}_{2}(\{2,3,\cdots,N-M+m\},\max\{h_{i},w\})\}\,dG_{1}(w) \\ &= K_{1}(h_{i}) + \int \max\{0,\ K^{i-1}{}_{2}(\{2,3,\cdots,N-M+i\},\max\{h_{i},w\})\}\,dG_{1}(w) \\ &= K^{i}{}_{1}(\{1,2,\cdots,N-M+i\},h_{i}) \qquad (1^{*}) \end{aligned}$$

(62)
$$\begin{aligned} K^{m}{}_{i}(\{1,2,\cdots,N-M+m\},h_{i}) &= K_{i}(h_{i}) + \int \max\{0,\ K^{m-1}{}_{1}(\{1,2,\cdots,i-1,i+1, \\ &\qquad \cdots,N-M+m\},\max\{h_{i},w\})\}\,dG_{i}(w) \\ &\leq K^{m-1}{}_{1}(\{1,2,\cdots,i-1,i+1,\cdots,N-M+m\},h_{i}) \\ &= K^{i}{}_{1}(\{1,2,,\cdots,i-1,i+1,\cdots,N-M+i+1\},h_{i}) \qquad (2^{*}) \end{aligned}$$

Next suppose $m+1 \leqq i \leqq N-M+m$. Here notice $K^{m-1}{}_1(\{j_1,j_2,\cdots,j_{N-M+m-1}\},h_m)$ is independent of j_m, j_{m+1}, $\cdots$, j_{N-M+m} from (c) and (d). Then, from (d)

$$(63) \qquad K^m{}_1(\{1,2,\cdots,N-M+m\},h_m) = K^{m-1}{}_1(\{1,2,\cdots,N-M+m-1\},h_m) \qquad (3^*)$$

and from (45),

$$(64) \qquad K^m{}_i(\{1,2,\cdots,N-M+m\},h_m)$$
$$= K_i(h_m) + \int \max\{0,\ K^{m-1}{}_1(\{1,2,\cdots,i-1,i+1,\cdots,N-M+m\},\max\{h_m,w\})\}\,dG_i(w)$$
$$\leqq K^{m-1}{}_1(\{1,2,\cdots,i-1,i+1,\cdots,N-M+m\},h_m) \qquad (4^*)$$

Since both (1^*) to (2^*) are independent of $\{i,i+1,\cdots.N-M+i+1\}$ and since both (3^*) and (4^*) are independent of $\{m,m+1,\cdots,N-M+m\}$, it eventually follows that $(1^*) \geqq (2^*)$ and $(3^*) \geqq (4^*)$; that is, $K^m{}_1(\{1,2,\cdots,N-M+m\},h_m) \geqq K^m{}_i(\{1,2,\cdots,N-M+m\},h_m)$ for $2 \leqq i \leqq N-M+m$. Hence, (a) was proved for $\{1,2,\cdots,N-M+m\}$, hence for any $L \in \mathcal{L}_{N-M+m}$. □

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