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Boundary Optima and the Theory of Public Goods Supply: A Comment

by

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Campbell-Truchon (1988) provide a new generalized Samuelson condition for Pareto optimal allocations that takes care of all interior and boundary solutions in an economy with finite number of participants, one private good, and finite number of public goods. In particular, they find a rich class of Pareto optimal allocations that do not satisfy the Samuelson condition used so far, but their generalized one's. In this class, the *positive* level of some public goods and the boundary consumption of private good for some participants are observed. They show examples in which sets of non-Samuelsonian Pareto optima are open sets relative to the set of all Pareto optima. That is, they claim that sets are very "large." Hence, they say, "in models with public goods, interior optima may be limit cases in the set of all Pareto optima, rather than the other way around (Campbell-Truchon, p.242)." Although the generalized Samuelson condition seems to be slightly different from the old one's, in both topological and geometrical sense this phenomenon is found not only in public good economies but also in private good economies. This claim will be shown by using Kolm's triangle that corresponds to the Edgeworth box diagram in private good economies [see Malinvaud (1971) and Thomson (1987)].

Campbell-Truchon's result is summarized in a very simple economy with two participants (*1* and *2*), one private good (x), and one public good (y). Let the total endowment of the private good be w and no public good endowment, and let the production function of the public good be $y = f(x) = x$. Each participant has substantially minimum level of private good consumption b_i. Then the old Samuelson condition is for Pareto optimal allocation $(\hat{x}_1,\hat{x}_2,\hat{y}) \geq (b_1,b_2,0) = (b,0)$,

$$\left(MRS_1(\hat{x}_1,\hat{y}) + MRS_2(\hat{x}_2,\hat{y}) - 1\right)\hat{y} = 0,$$

where $MRS_i(\hat{x}_i,\hat{y}) = \dfrac{\partial u_i(\hat{x}_i,\hat{y})}{\partial y} \Big/ \dfrac{\partial u_i(\hat{x}_i,\hat{y})}{\partial x_i}$ and u_i is participant i's utility function on the non-negative orthant. Consider two problems;

Pareto optimal problem (POP)*:* Find an allocation (x_1^*,x_2^*,y^*), if it exists, such that it is Pareto optimal; and

Generalized Samuelson problem (GSP)*:* Find a feasible allocation $(x_1',x_2',y') \geq (b_1,b_2,0)$ and (α_1,α_2) with $0 \leq \alpha_i \leq 1$ such that

$$\left(\alpha_1 MRS_1(x_1',y') + \alpha_2 MRS_2(x_2',y') - 1\right) y' = 0 \quad \text{and} \quad (1-\alpha_i)(x_i-b_i) = 0 \text{ for all } i.$$

By using the Karush-Kuhn-Tucker theorem, Campbell-Truchon prove that if (x_1^*,x_2^*,y^*) is a solution of POP, then there exists (α_1,α_2) such that $(x_1^*,x_2^*,y^*;\alpha_1,\alpha_2)$ is a solution of GSP, and conversely with usual convexity assumptions if $(x_1',x_2',y',\alpha_1',\alpha_2')$ is a solution of GSP, then (x_1',x_2',y') is a solution of POP.

Campbell-Truchon's contribution is that they correctly derive the Samuelson condition including boundary solutions and then point out possibility of boundary solutions with *positive* public good level but zero or b_i level of private good consumption for some i. In this case, α_i is not always one. That is, the old Samuelson condition does not hold.

Kolm's triangle is explained briefly. Take any point in the equilateral triangle in Figure 1. The perpendicular distance from that point to line O_1-T measures the consumption of x_1, the perpendicular distance from that point to line 0_2-T measures x_2, and the distance from the point to the bottom side measures y. Then, the sum of three distances is always constant and it measures the height of the triangle, which is also the total amount of endowments. The situation in Figure a is translated to the triangle in the following manner: C in Figure a becomes C'. The height of C' is equal to 0_1-$\hat{y}$ and the perpendicular distance from C' to line 0_1- F' is $\hat{x}_1$. Tangent line A-F to indifferent curve I at C in Figure a becomes A'-F' in Figure b. In a similar manner, indifference curve I becomes I'. Repeat the same procedure to participant 2, take the mirror image of participant 2's picture, and then construct a triangle with height w. Consider allocation E in Figure 1. Since $MRS_1(\hat{x}_1,\hat{y}) = \dfrac{x_1 - \hat{x}_1}{\hat{y}}$ in Figures a and b and $x_1 + x_2 = w$,

$$MRS_1(\hat{x}_1,\hat{y}) + MRS_2(\hat{x}_2,\hat{y}) = \frac{x_1 - \hat{x}_1}{\hat{y}} + \frac{x_2 - \hat{x}_2}{\hat{y}} = \frac{w - (\hat{x}_1 + \hat{x}_2)}{\hat{y}} = \frac{\hat{y}}{\hat{y}} = 1,$$

which satisfies the interior Samuelson condition. That is, the double tangency of two participants' indifference curves in the interior of Kolm's triangle is a necessary condition for interior Pareto optimality.

Example 0 corresponds to the case with zero public good. The rest are the same examples as in Campbell-Truchon. Let $w = 1$ for all the following examples.

<u>Example</u> <u>0</u>. $u_i(x_i,y) = 4x_i+y$ for each i and $b = (0,0)$. Then in GSP, $\alpha_i = 1$ and $y = 0$. See Figure 0.

<u>Example</u> <u>1</u>. $u_i(x_i,y) = x_iy$ for each i and $b = (0,0)$. Then in GSP, $\alpha_i = 1$ and $y = 1/2$. See Figure 1.

Example 2. $u_1(x_1,y) = x_1y$, $u_2(x_2,y) = x_2y^2$, and $b = (0,0)$. Then in GSP, $\alpha_i = 1$ and the set of Pareto optimal allocations is $\{(2-3y,2y-1,y):1/2\le y\le 2/3\}$. See Figure 2.

Example 3. The same as Example 2, except that $b = (1/8,0)$. Let $S = \{(2-3y,2y-1,y):1/2\le y\le 5/8\}$ and $S' = \{(1/8,7/8-y,y):7/12\le y<5/8\}$. The set of Pareto optimal allocations is the union of S and S'. See Figure 3.

Example 4. The same as Example 2, except that $b = (1/8,1/8)$. Let $S = \{(2-3y,2y-1,y):1/2\le y\le 5/8\}$ and $S' = \{(1/8,7/8-y,y):7/12\le y<5/8\}$, and $S'' = \{(7/8-y,1/8,y):7/16\le y\le 9/16\}$. Then the set of Pareto optimal allocations is the union of S, S' and S''. See Figure 4.

The following example is for an economy with two participants, two private goods, no public good, and $w = (1,1)$. Let x_i^j be participant i's consumption of private good j and let $A = \{((x_1^1,x_1^2),(x_2^1,x_2^2)\colon\ x_1^1+x_2^1=1,\ x_1^2+x_2^2=1 \text{ and } x_i^j\ge 0\}$.

Example 5. $u_1(x_1^1,x_1^2) = x_1^1+x_1^2$ and $u_2(x_2^1,x_2^2) = x_2^1(x_2^2+1/2)$. Let $S = \{((x_1^1,x_1^2),(x_2^1,x_2^2) \in A: x_1^2=x_1^1-1/2\}$, $S' = \{((x_1^1,x_1^2),(x_2^1,x_2^2) \in A\colon\ 0<x_1^1<1/2,\ x_1^2=0\}$ and $S'' = \{((x_1^1,x_1^2),(x_2^1,x_2^2) \in A\colon\ x_2^1=0,\ 0<x_2^1<1/2\}$. Then the set of Pareto optimal allocations is the union of S, S' and S''. See Figure 5.

In Example 5, S is the set in which usual double tangency occurs, and both S' and S'' are open relative to the union of S, S' and S''. We can regard Example 5 as a counterpart in private good economy to Example 4.

References

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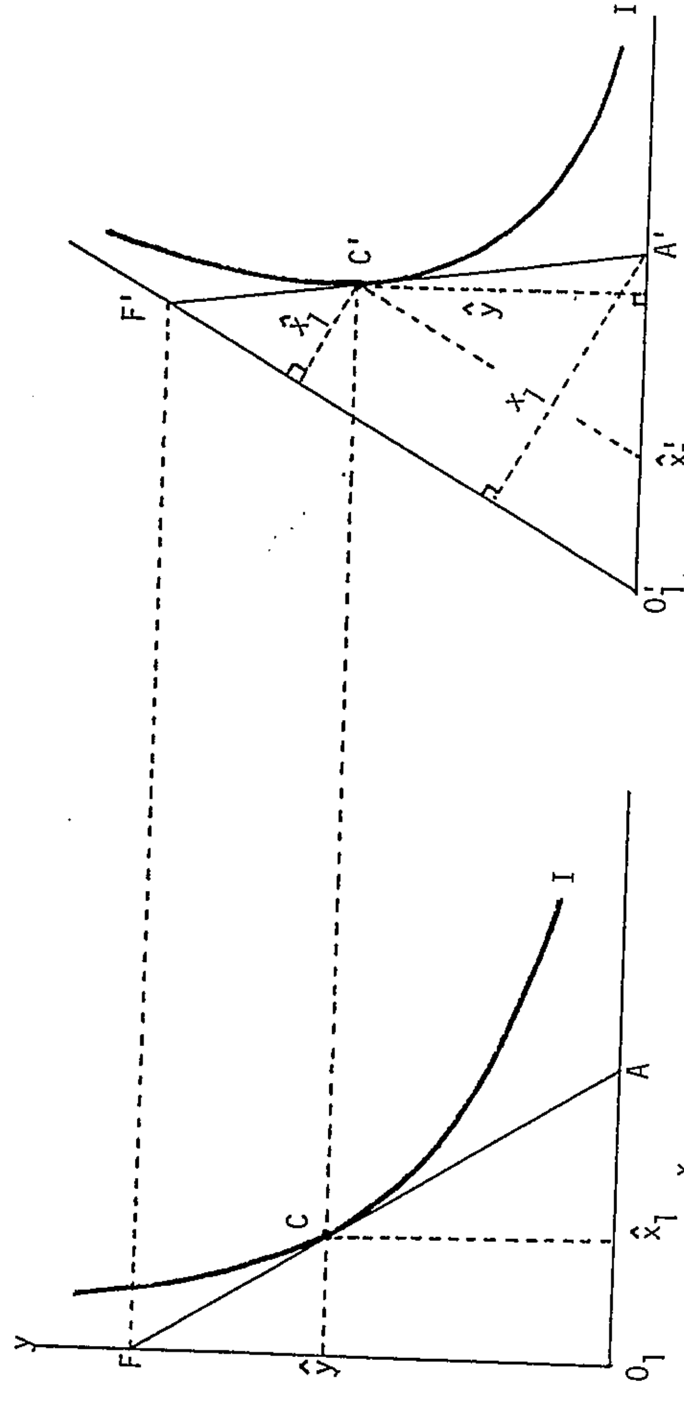

Figure b

Figure a

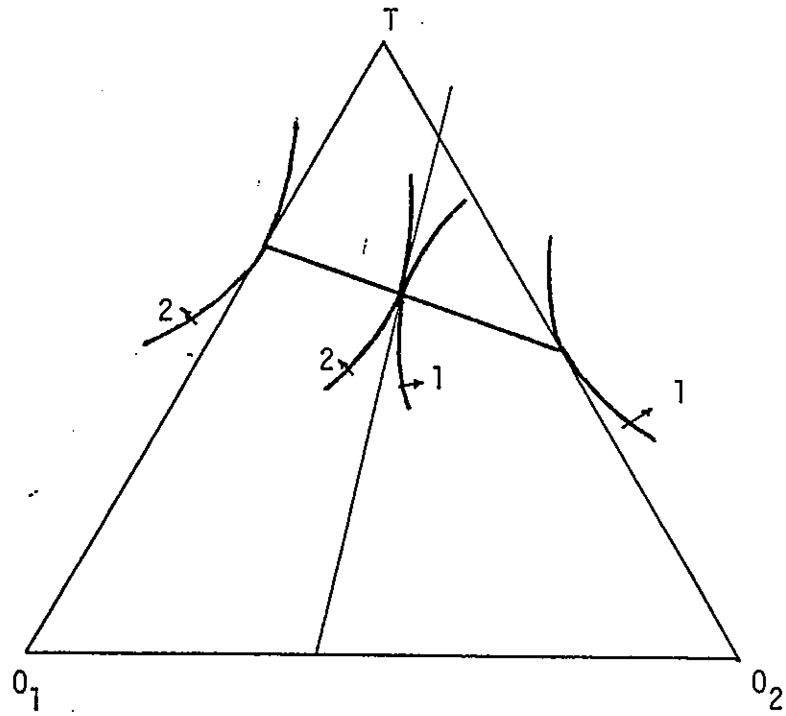

Figure 2

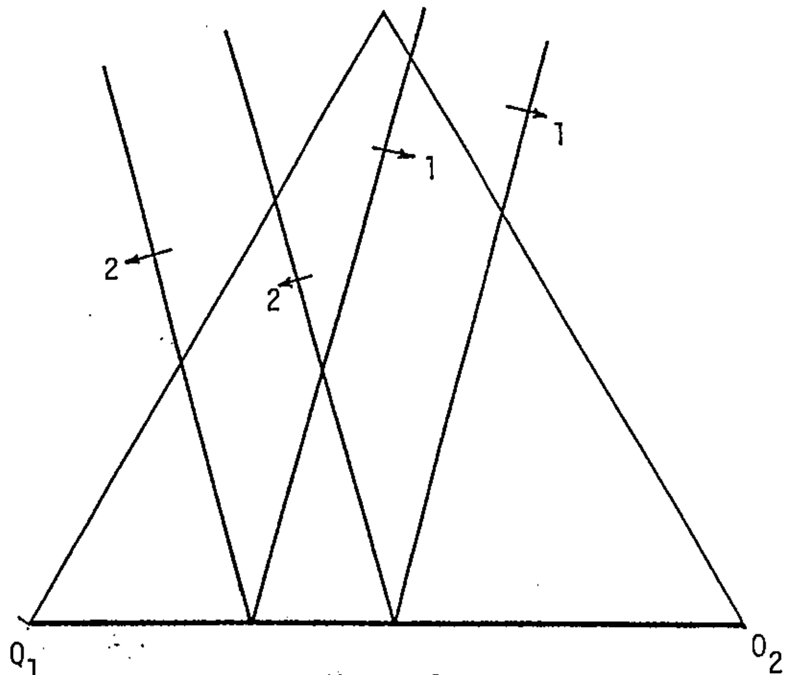

Figure 0

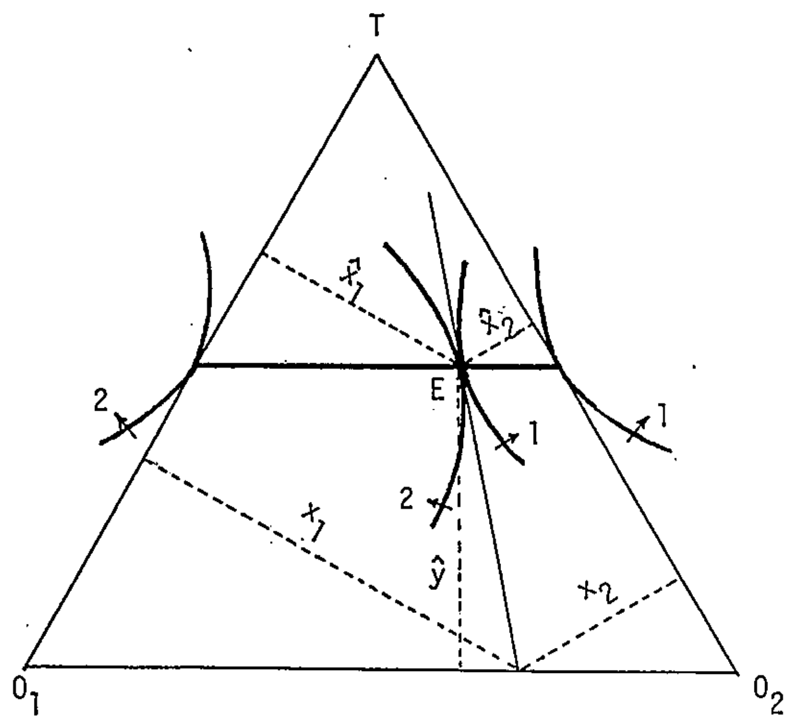

Figure 1

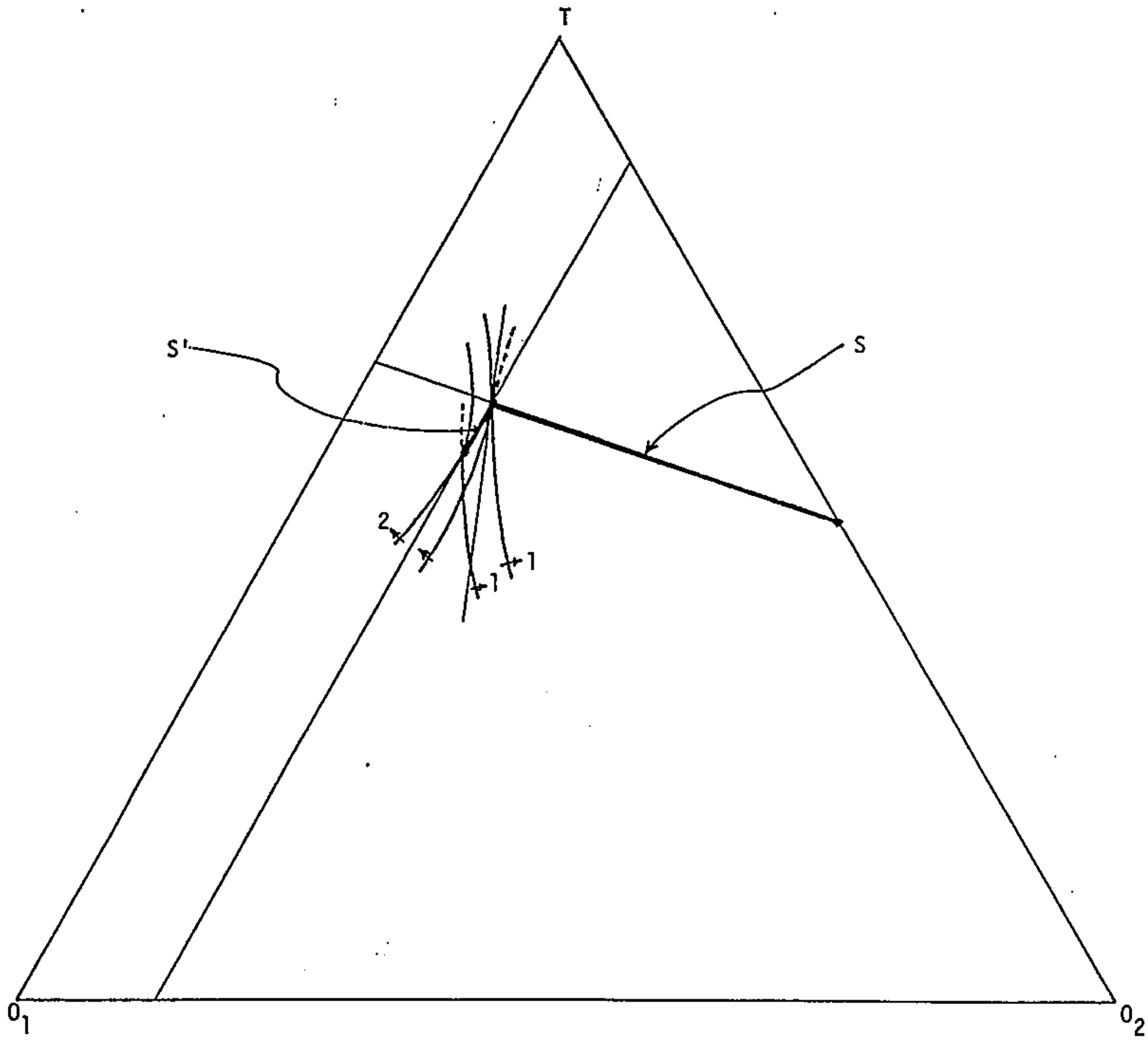

Figure 3

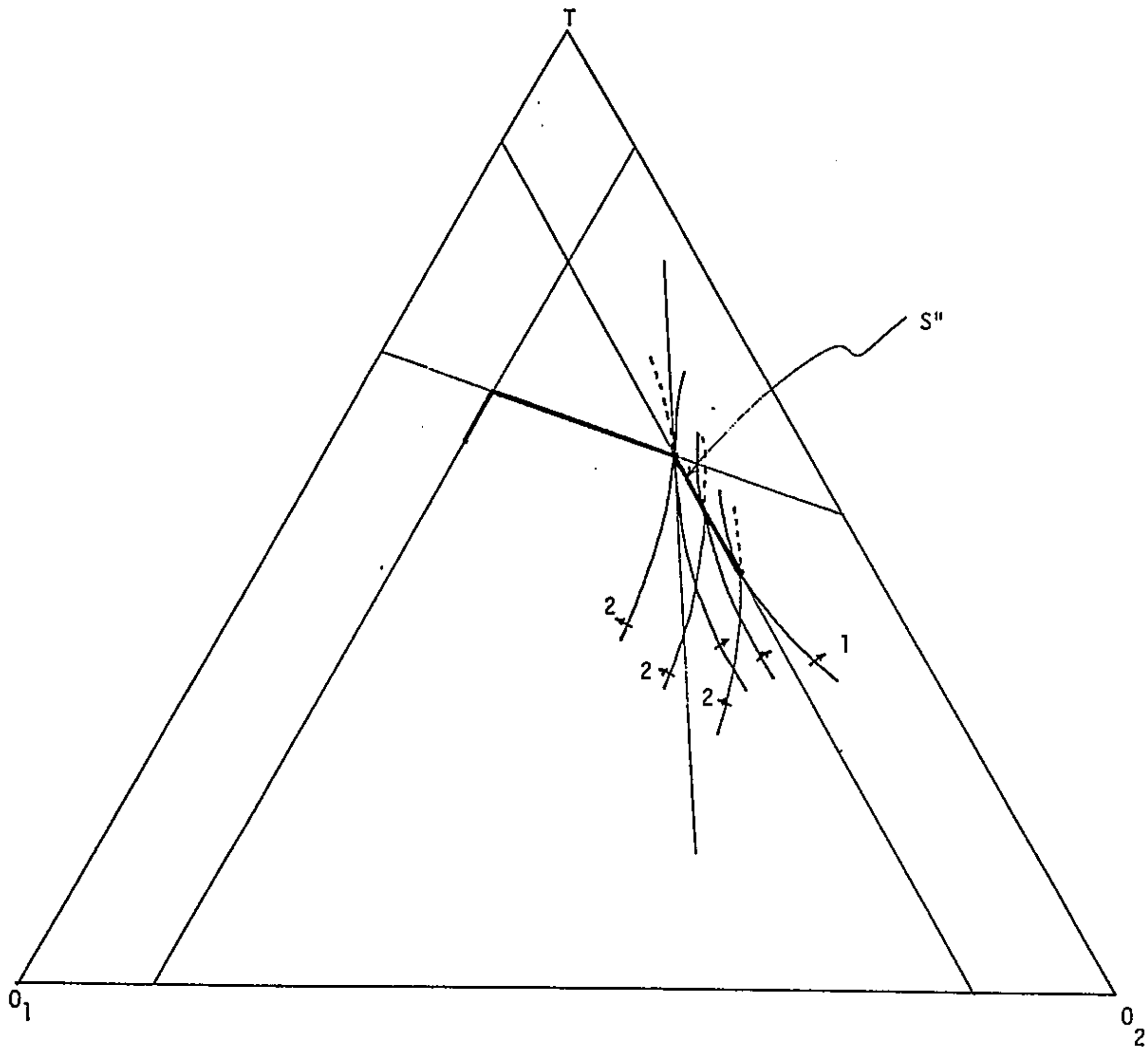

Figure 4

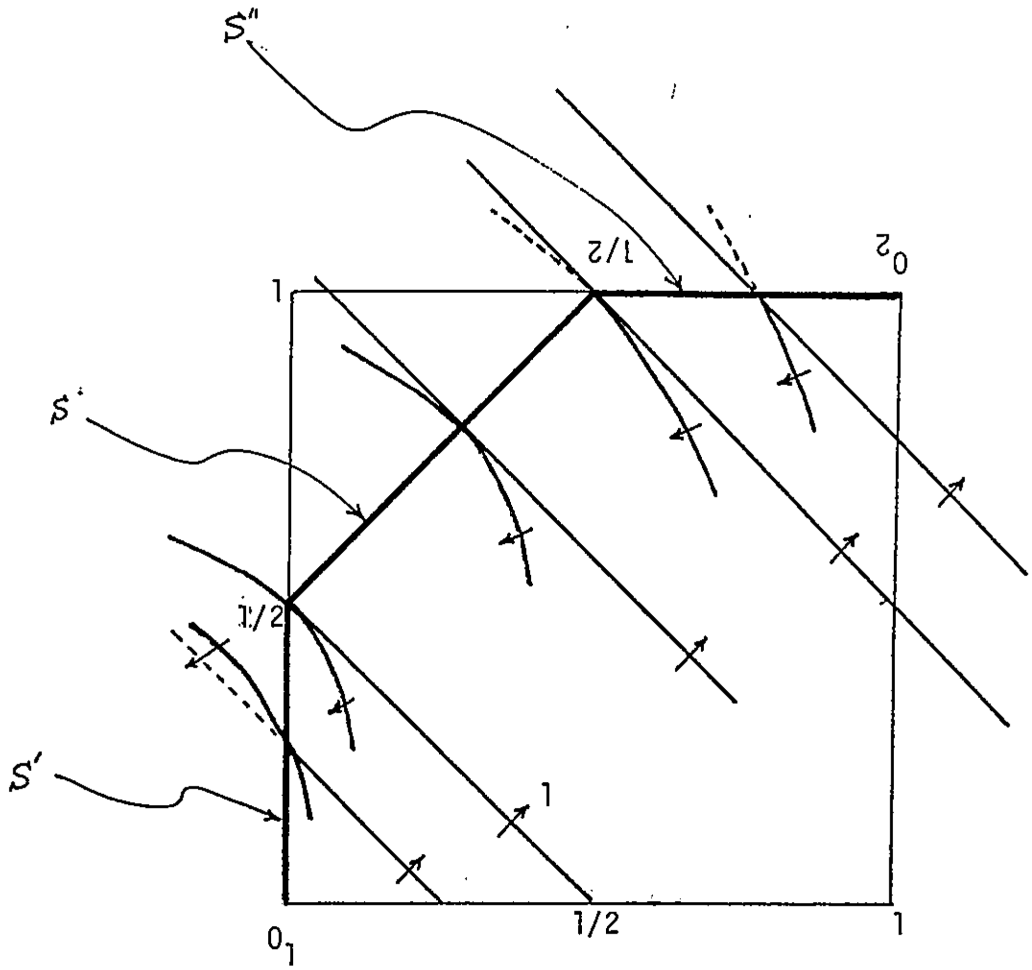

Figure 5