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Water Quality Control in River Systems

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2.2 Water Quality Control in River Systems

2.2.1 Introduction

Modern industrial, urban (households) and agricultural activities have produced a tremendous amount of wastes or by-products that are eventually discharged into water bodies of streams, lakes, and estuaries. In terms of the deterioration of water quality, there are two major types of wastes dissolved in the rivers streams:

i) organic matters of suspended particulates and sediments in relation to "oxygen deficits or low transparency",

ii) inorganic matters of nitrogen and phosphorus in relation to outbreak of algae bloom due to "water eutrophication".

The traditional objective of controlling water quality with respect to organic pollutants, is to decrease the level of BOD (Biological Oxygen Demand), or to increase the level of DO (Dissolved Oxygen) up to some predetermined standards values. The standard levels of BOD and DO are determined depending on the required water quality for particular water utilization. So far, the construction of sewage treatment facilities to cut organic pollutants has been a major control mean before they are discharged into streams. Direct aeration of injecting or mixing air into stream is also often used to increase the level of dissolved oxygen in waters (James 1978, Rinaldi et al 1979).

The second problem of controlling inorganic-pollutants has emerged as one of the most serious issues in most urban areas after a certain scale of sewage treatment systems is developed. The biological treatment of the organic wastes in sewage plants may not necessarily result in reducing the level of inorganic matters in spite of cutting organic matters in their effluent water. Abundant inorganic materials of nitrogen oxides (NO_3) and phosphates (PO_4) in effluent water may trigger a large scale of algae blooms in stream and estuaries. Then, the decaying process of bloomed algae or benthic flora contributes to consuming a large volume of dissolved oxygen at their natural bacterial degradation.

Lack of economically efficient and practical instruments to control inorganic wastes may add further problem of setting the balance between control costs of reducing the damages from deteriorated water quality, and that of increasing amenity of water quality. Given management objectives of controlling water quality in terms of the target or standards levels of pollutants, "real-time" observation, monitoring, and forecasting of water quality are the basic requirements for the water quality control to be practical in regulating both organic and inorganic pollutants. The systems control theory may work as one of the most powerful methodologies that can deal with those control tasks involved in "planning", "designing" and "operational" phases.

In this section, we describe some of the new applications of systems control theory to water quality problems above mentioned. The first part describes a general view of water quality models in river streams which incudes inputs from pollution sources and in-stream aeration as a control function. The second part deals with the identification of distributed source function by using the on-line monitoring data on the specific points. The numerical identification algorithms is given with the emphasis on "non-well posed nature" of the identification problem. The third part is concerned with the theoretical development to utilize optimal impulse control theory to the in-stream aeration problem. Final part is devoted to the multi-objective optimization problems in the river basin management.

2.2.2 Control Models of Water Quality Problems

Figure 2.2.1 displays a schematic illustration of pollution sources, pathways, sewage treatment, and transport or reaction process in a river stream system. There are four major pollution sources are involved in the urban river streams:

i) industrial use of water   ii) municipal use of water

iii) urban run-off   iv) agricultural use of water

Pollutants from sources i) and ii) are collected via drainage network into sewage plants that are distributed in various levels and scales. Some are centralized urban sewage systems and some are individually operated plants attached to industrial firms or private houses. Both outputs from sewage system form point-sources of pollution load into river streams. However, for example, in many Japanese municipals, considerable amount of discharges from household use of water are not necessarily connected to urban sewage systems. The untreated waste water from households has been one of the major contributors to the deterioration of water quality. The urban run-off of surface precipitation and irrigation water are another sources of untreated wastes water. They contain various pollutants, in particular, of inorganic and some toxic chemicals or heavy-metals. Those pollutant loads form typical distributed source $w(t,x)$ that flows directly into river streams or penetrates into the underground waters that eventually spring into river streams.

The balance of water quality variables $S(t,x)$ in terms of pollution related substances S can be described within a unit reach of river streams:

$$\partial S(t,x)/\partial t = D_e \partial^2 S(t,x)/\partial x^2 + v\partial S(t,x)/\partial x + f(t,x,S)$$

(rate of change) (dispersion) (advection) (reaction and decay)

$$+ \quad w(t,x) \quad + \quad u(t,x,S) \qquad (2.2.1)$$

(pollutant inputs) (control)

where notations of variables and parameters are:

$S(t,x)$: a water quality vector along river stream

$f(t,x,S)$: a reaction function of water quality S in stream

$w(t,x)$: an input vector of pollutants into stream at x.

$u(t,x,S)$: a control vector operated in stream at x.

D_e : a dispersion coefficient.

v : a advection coefficient (a mean flow rate).

The two transportation factors of material S are dispersion (molecular diffusion and mixing by turbulent eddies) and advection (transport due to the mean river velocity). The chemical reaction of the materials is the third factor (decomposition and deposition of the material). The pollutant input $w(t,x)$ represents a series of pollutant inflows along the river stream x, and the control $u(t,x,S)$ denotes an remedial control measure such as the in-stream aeration at $x = x_d$:

$$u(t,x,S) = q(t,S)\delta(x-x_d) \qquad (2.2.2)$$

The well established model for (2.2.1) is "Streeter Phelps" equation in which BOD and DO are used as state variables of water quality $S(t)$, (Gromiec et al 1983).

$$\partial S_1/\partial t = D_e \partial S_1/\partial x^2 - v\partial S_1/\partial x - k_1 S_1 + w(t,x) \qquad (2.2.3)$$

$$\partial S_2/\partial t = D_e \partial S_1/\partial x^2 - v\partial S_2/\partial x - k_2 S_1 + k_3(S_s - S_2) + u(t,x) \qquad (2.2.4)$$

where

S_1 : Biological Oxygen Demand (BOD)

S_2 : Dissolved Oxygen (DO)

k_1 : a decay coefficient of BOD

k_2 : a consumption coefficient of DO

k_3 : a coefficient of surface aeration

S_d : saturated level of DO

There are several simplified versions of the equation (2.2.1). One is associated with the assumption of no dispersive transport where advective flow is dominant in comparison to dispersion flow:

$$\partial S(t,x)/\partial t = v\partial S(t,x)/\partial x + f(t,x,S) + w(t,x) + u(t,x,S) \qquad (2.2.5)$$

For this plug-flow type model, we have further simplification into the ordinary differential equation. For example, applying the characteristic transformation method to the convection term of the equations (2.2.5):

$$dx/d\tau = v, \quad \text{and} \quad dt/d\tau = 1, \qquad (2.2.6)$$

we have

$$dS_1(\tau)/d\tau = f_1(v(t-\tau), S_1) + w(v(t-\tau), \tau) \qquad (2.2.7)$$

$$dS_2(\tau)/d\tau = f_2(v(t-\tau), S_2) + u(v(t-\tau), \tau). \qquad (2.2.8)$$

This is a set of ordinary differential equations along the characteristic curve $x = v(t - \tau)$. In the following discussion, we will use this set of BOD-DO equations as an example of mathematical description of control problem.

Another simplification is a static version of the equation (2.2.1) in which the term of time variation dS/dt is set to zero.

$$D_e dS_1{}^2/dx^2 - vdS_1(x)/dx + f_1(x, S) + w(x) = 0 \quad (2.2.9)$$

$$D_e dS_2{}^2/dx^2 - vdS_2(x)/dx + f_2(x, S) + u(x) = 0 \quad (2.2.10)$$

This is usually used for planning purpose to allocate a cascade of sewage plants w(x) for regulating total output or to design distribution of aeration devices u(x) along the river stream.

2.2.3 Optimal Control Scheme in Water Quality

Let us now formulate a general control problem at a unit reach of river system based on the ordinary differential equations (2.2.7) and (2.2.8). As to control variables, there are two major inputs, $w(\tau)$ and $u(\tau)$, that can be a control function to the water quality system. The one is associated with pollutant input $w(\tau)$ from sewage plant, and the other is the direct aeration input $u(\tau)$ in the river stream.

Here, we will take the aeration input u as a control variable, and pollutant input $w(\tau)$ is a given function. In addition, for simplicity we use again t as auxiliary variable τ of the characteristic curve. Then, the control problem is to determine such an control $u(t)^{o}$ that minimizes the following performance functional J over the time period [0 - T]:

$$J = \text{minimize } \{\int G(s(t), u(t))dt\} \qquad (2.2.11)$$

subject to the system equation of water quality vector S:

$$dS/dt = F(t)S + w(t) + u(t) \qquad (2.2.12)$$

The linear quadratic function G is commonly used in the process control scheme which consists of squared deviation of the state variables S(t) from target values of $S^{*}(t)$, and control cost associated with the squared value of in-stream aeration u(t):

$$G = (1/2)(S-S^{*})'Q_1(S-S^{*}) + (1/2)u'Q_2u. \qquad (2.2.13)$$

where Q_1 and Q_2 are constant non-negative matrices, and (') denotes a transpose of the concerned matrices.

In this setting, the optimal control u(t) can be theoretically derived by the variation principle of Hamiltonian-Jacobi function (Bryson and Ho 1975).

Following the variation principle , we have

$$H(S, u, m) = G(S, u) + m\{F(t)S + w + u\} \qquad (2.2.14)$$

where m(t) is an adjoint variable that satisfies;

$$\partial m/\partial t = \partial H(S, u, m)/\partial S = Q'(S-S^*)+ F(t)'m \qquad (2.2.15)$$

The necessary condition for the minimum of the functional (2.2.14) is that the first derivative of H with respect to u is equal to zero as is the case of Lagrangian optimization scheme. The sufficient condition is the convexity of the Hamiltonian H with respect to S and u that guarantees the existence of the minimum of Hamiltonian H:

$$H_u = \partial H/\partial u = Q_2 u + m = 0 \qquad (2.2.16)$$

$$H_{ss} = \partial^2 H/\partial S^2 > 0, \quad H_{uu} = \partial^2 H/\partial u^2 > 0 \qquad (2.2.17)$$

Then, the optimal control $u^o(t)$ can be derived as a function of the adjoint state variable m(t) and S(t) by solving (2.2.15) and (2.2.16).

$$u^o(t) = - Q_2^{-1} m(t), \qquad (2.2.18)$$

where (-1) of the superscript denotes the inverse of the concerned matrix.

The linear control theory (the regulator problem) tell us that, if the system equation f(S, w, u) is linear in both variables S and u together with quadratic performance function G (as is our case of BOD-DO equation), then we can construct such a feedback control u^o with the specific gain matrix R (Riccati Coefficient Matrix):

$$u^o = -Q_2^{-1}R(t)S(t), \qquad (2.2.19)$$

where the gain matrix R(t) is the solution of the following Riccati equation, and (') denotes the transpose of the concerned matrices:

$$- dR/dt = F(t)'R + R'F(t)- R\, Q_2^{-1}R + Q_1 \qquad (2.2.20)$$

$$R(T)= 0 \qquad (2.2.21)$$

Figure 2.2.2 illustrates the simple scheme of feedback control in the river stream such as for in-stream aeration. There are, however, a number of computational and operational difficulties associated with solving the variational equations of control function. We will discuss later this problem in the case of optimal impulse aeration at a specific control point.

Needless to say, we have a great number of theoretical problems in terms of applying modern systems control theory in the river system. Among them we will pick up the following interesting theoretical problems:

(A) how to identify the distributed pollution input function $w(t,x)$ from the observed data of the water quality,

(B) how to construct an optimal control scheme of impulse type (on-off control) rather than continuous feedback control, and

(C) how to aid the decision makers (policy makers) to make trade-offs concerning "multiple objectives" of water quality control problems in the framework of river basin management.

2.2.4 Identification of Pollution Source

Let us consider a problem of identifying the unknown pollution source $w(t,x)$ in the simplified "Streeter Phelps" equations (2.2.7) and (2.2.8). In the DO equation (2.2.7), we assume that the unknown $w(t,x)$ is expressed by the following functional form:

$$w(t,x) = y(t)z(x) \tag{2.2.22}$$

where $f(t)$ is a given function of time t and $z(x)$ is a unknown function of location x along the river reach [a, b]. This assumption means that the known $f(t)$ represents a kind of time varying pattern of pollutant discharge (BOD) which is uniform at any point $x \in [a, b]$. But the $z(x)$ includes some unknown source information not only on location of pollutant sources but also on the intensity of the discharged pollutants into the river stream. The observation points are set to be at the both upper and lower points x=a and x=b of the unit reach as shown in Fig. 2.2.3.

Case of No Dispersion

For computational simplicity, we introduce new variable $x' = x/v$ and $S_2' = (S_d - S_2)/v$, and use again x and S_2, respectively, instead of x' and S_2'. Then the Streeter Phelps equations for the plug-flow case are reduced to:

$$dS_1/d\tau = -k_1S_1 + y(\tau)z(x) \tag{2.2.23}$$

$$dS_2/d\tau = k_2S_1 - k_3S_2 \tag{2.2.24}$$

along the characteristic curve:

$$x = t - \tau \tag{2.2.25}$$

The measurement of BOD are usually carried out by laboratory monitoring devices rather than by on-site field devices. The DO can be measured continuously and automatically at the monitoring station on a river side. Since both water quality measures have different values in terms of operational costs and efficiency of source monitoring, it would be beneficial to establish both identification schemes.

Now, we have two problems of identifying unknown source distribution z(x) by observing either S_1 (BOD) or S_2 (DO) based on the simplified "Streeter Phelps" equation (Ikeda, Miyamoto & Sawaragi 1974)

[A] Identification of z(x) by BOD data:

Since the BOD equation (2.2.23) hasn't any other external variable except the BOD, the solution $S_1(\tau+b,b)$ at x = b can be easily obtained by integrating equation (2.2.21) from $t=\tau+a$ to $t=\tau+b$ along the characteristic curve that is linear in this case:

$$S_1(\tau+b,b)\exp(k_1 b) = S_1(\tau+a,a)\exp(k_1 b) + \int_{\tau+a}^{\tau+b} z(t-\tau)y(t)\exp(k_1 t)dt \qquad (2.2.26)$$

This equation can be rearranged in the form of integral equation:

$$\widetilde{S_1}(\tau) = \int_a^b K(\tau, s)z(s)ds \equiv T\cdot z(s) \qquad (2.2.27)$$

where T is an integral operator of which kernel K is defined as

$$K(\tau,s) = y(\tau+s)\exp(ks) \qquad (2.2.28)$$

and the observed data S_1 is given by

$$\widetilde{S_1}(\tau) = S_1(\tau+b,b)\exp(k_1 b) - S_1(\tau+a,a)\exp(k_1 a). \qquad (2.2.29)$$

The identification of z(s), thus, is reduced to solving the integral equation (2.2.27), given the data $\widetilde{S_1}(\tau)$ observed at the point x=a and x=b, and the kernel $K(\tau,s)$. Theoretically, this equation (2.2.27) can be solved by getting "inverse operator" T^{-1}

$$z(s) = T^{-1}\widetilde{S_1}(\tau).$$

However, this integral equation called the Fredholm equation of the first kind, has a specific feature of "non-well posedness" in terms of stability associated with getting inverse operator T^{-1}.

[B] Identification of z(x) by DO data:

The DO equation (2.2.24) contains two states variables of S_1 and S_2. This coupling of the BOD and DO seems to make this identification problem more complex than the case [A]. However, the linearity of the coupling equations help for us to get the same type of integral equation as in the case A. For computational simplicity, let the upper point a=0, that is, $0 \leq x \leq b$.

Integration of the DO equation (2.2.24) in the same manner as for the former case results in:

$$S_2(\tau+b,b)\exp(k_2 b)-S_2(\tau,0) = k_3 \int \exp(k_3 s)S_1(\tau+s,s)ds \qquad (2.2.30)$$

For the S_1 in the integrand, it follows from (2.2.26) that

$$S_1(\tau+s,s) = S_1(\tau,0)\exp(-k_1 s)+ \int \exp\{k(p-s)\}f(\tau+p)z(p)dp. \quad (2.2.31)$$

Substituting (2.2.30) into the right hand side of (2.2.30) and arranging the term, we have

$$\widetilde{S_2}(\tau) = \int \exp\{-(k_1-k_2)s\}ds \int \exp(k_1 p)y(\tau+p)z(p)dp. \qquad (2.2.32)$$

where $S_2(\tau)$ of this equation is a known time function, given the observed DO and BOD data at x=o or x=b as follows:

$$\widetilde{S_2}(\tau) = (1/k_3)\{S_2(\tau+b,b)$$

$$-S_2(\tau,0)-S_1(\tau)\int \exp\{(k_2-k_1)p\}dp\}. \qquad (2.2.33)$$

Now, interchanging the integration order of s and p in (2.2.32), and performing the integration with respect to s, we obtain the following integral equation of the Fredholm of the first kind:

$$\widetilde{S_2}(\tau) = \int K(\tau,p)z(p)dp \equiv T\cdot z(p) \qquad (2.2.34)$$

where integral kernel $K(\tau,p)$ has a rather complicated form in comparison to the BOD case:

$$K(\tau,p) = \{1/(k_2-k_1)\}[\exp\{(k_1-k_2)b\}-\exp\{(k_1-k_2)p\}] \cdot$$
$$\cdot \exp(k_1 p)y(\tau+p). \qquad (2.2.35)$$

Case of Dispersive Flow

The solution of the dispersion model (2.2.3) can be obtained in the form of the integral equation under the proper assumptions for the initial and boundary conditions:

$$S_1(0, x) = 0, \quad t = 0, \ x \in [0, b].$$

$$S_1(t, 0) = 0, \quad t \in [0, T], \ x = 0.$$

Then, we have the following solution at the observation point x=b by using fundamental solution U of the parabolic partial differential equation (Ito 1966):

$$S_1(t,b) = \int_0^t\int_0^b U(t-\tau,b,s)f(\tau)z(s)d\tau ds, \quad 0 \leq t \leq T \qquad (2.2.36)$$

$$U(t,x,s) = \{1/(2\sqrt{D_e t})\}\exp\{-\frac{(x-s)^2}{4D_e t} + \frac{v}{2D_e}x - (\frac{v^2}{4D_e} + k_1)t\} \qquad (2.2.37)$$

Let the kernel K be

$$K(t,s) = \int_0^t U(t-\tau,b,s)f(\tau)d\tau \ , \qquad (2.2.38)$$

then we obtain the same type of integral equation as the former case.

$$\widetilde{S_1}(t) = \int_0^b K(t,\tau)z(\tau)d\tau, \qquad (2.2.39)$$

where $\widetilde{S_1}(t)$ is the known function defined by the observation data at x=0 and x=b. In this case, the kernel K has a strong smoothing property of exponential decay function, which contrasts with the high frequency noise involved in the observed data $\widetilde{S_1}$. We will see later this point in the numerical examples.

Identification Algorithm by Regularization Method

It is well known in the applied mathematics that so-called "inverse problem" of finding z(s) on the basis of a given function

$S(\tau)$ in (2.2.24),(2.2.32) and (2.2.39) is "non-well posed" in the sense of Hadamard. The concept of "well-posed" means that:

(1) the solution $z(s)$ exists,

(2) it is unique, and

(3) it depends continuously on the observed data $S(\tau)$.

Failure of the above conditions is called that "the problem is non-well posed or improperly posed". Even when we may have an unique solution in physical sense as in most of the engineering identification problems, the condition of "continuous dependence" on the data does not necessarily hold. This leads to important problem that a small change in the data $S(\tau)$ may trigger a drastic change of the solution $z(s)$. As the measurement error in the observed data $S(\tau)$ always exists in practical cases, we cannot obtain the solution $z(s)$ even in approximation. The integral equation of the first kind is the very one that has "numerical instability" in relation to identification scheme for numerical solution (Lavrentiev 1967).

There has been a number of studies to relax this difficulty of "incorrectly posed problem": methods of generalized inverses (Rao and Mittra 1971), penalty function methods (Phillips 1962), etc. Here, we will use Tihonov's regularization (1963, 1977). The regularization method is to construct a family of approximate solutions of "well-posed equation" z^{α} that belong to such a compact subset of the original space of the solution $z(s)$. The original equation is transformed into an equivalent quadratic cost functional M^{α} to be minimized with the regularization term $\alpha N(z(s))$ of a small positive parameter α:

$$M^{\alpha}(z;S) = \int_{\rho}^{t}\{\int_{0}^{b}K(\tau,s)z(s)ds - S(\tau)\}^{2}d\tau + \alpha N\{z(s)\}^{2} \qquad (2.2.40)$$

The second term is crucial in numerical method for solving the equation. Roughly speaking, this term prevents the solution $z^{\alpha}(s)$ from its diverging out of the admissible bound.

The solution $z^{\alpha}(s)$ is obtained by solving the following Euler equation of minimize the functional $M^{\alpha}(z)$ (2.2.40):

$$\int_0^T\int_0^b K(p,s)K(p,\tau)z(\tau)dpd\tau + \alpha\, z(s) = \int_0^T K(\tau,s)\widetilde{S}_1(\tau)d\tau \ . \qquad (2.2.41)$$

The finite-difference scheme of the above integral equation is:

$$(K^tK + \alpha I)Z = K^t\widetilde{S}_1 \qquad (2.2.42)$$

where discrete matrices (kernel K, source Z, and observed data S) are defined in the interval $0 < s < b$ and $0 < t < T$ as follows:

$K = \{K^{ij}\} = K[(i-1)\Delta t,\ (j-1)\Delta s]$

$Z = \{z_j\} = z[(j-1)/2\cdot\Delta s]$

$S = \{S_i\} = S[(i-1)/2\cdot\Delta t]$

$\Delta s = (b-a)/n, \quad \Delta t = T/m, \quad 1\leq i\leq m$ and $1\leq j\leq n$

I : unit matrix.

The accuracy of the solution does, of course, depend on the magnitude of regularization factor . The larger we take the parameter α , the more stabilized but smoothed solution we get.

<u>Example of Source Identification</u>

In order to examine the validity of the identification scheme (2.2.42) in finding pollution source $z(s)$, experimental trials were undertaken at the part of the model river as shown in Figure 2.2.4 (Sawaragi and Ikeda 1975, Matsuoka et al 1977). The model river is designed in the scale of 1/250 times length at some urban river system in Osaka, Japan. The chlorine solution is injected as pollutant source in the following impulsive way:

$$f(t) = \begin{cases} 1.0, & 0 \leq t \leq \Delta t, \\ 0.0 & \Delta t \leq t \leq T. \end{cases} \qquad z(s) = \begin{cases} k, & s_i \leq s \leq s_{(i+1)} \\ 0, & \text{otherwise.} \end{cases}$$

along the several source points z(s).

Figure 2.2.5 displays the observed data S(t) at the measurement point D. It is assumed that there is no chemical reaction along the model stream, i.e. $k_1 = 0$. Other dimensions of the model river are:

Total length: 2,500 cm, Depth : 16.0 cm,

Width: 16.0 cm (partly 32.0 cm)

Velocity: v = 9.95 cm/sec,

Estimated dispersion coefficient: $D_e = 54.4\ cm^2/sec$.

Numerical results of identifying z(s) from the observation data S(t) are illustrated in the Figure 2.2.6 with different values of regularization parameter α ($10^{-19} \sim 0.003$). The figures (a), (b) and (c) show the cases where the model river has two sources A and B with the different assumption of the values for dispersion parameter. The figure (d) is the case that has three input souces A, B and C. All cases show that the smaller the regularization paramerter is, the more accurate peak points we obtain, but we have negative values in some parts for sources. On the otherhand, with larger value of α, we have a positive, smoothed, but less peak values of source distribution z(s).

2.2.5 Optimal Impulse Control in River Stream

Let us consider a problem of designing an optimal aerator system based on the simplified "Streeter Phelps" equation (2.2.12) with a minor modification:

$$dS/dt = F(t)S + w(t) + u. \qquad (2.2.43)$$

where w(t) is a known pollution source function, and the control function u(t) is assumed to consist of a series of aerators which has the aeration capacity q^i of on-off control type working on the sequential time $t = T^i$. In another words, a series of aerators with maximum power q^i, (i= 1,2,..,N) locates at t^i along the river stream:

$$u(t) = -\Sigma q^i \delta(t - t^i), \quad 0 < i < N. \qquad (2.2.44)$$

$$0 \leq t^1 \leq t^2 \leq \ \ldots \leq t^{-1} \leq t^N \leq T$$

The performance function J is the combination of the system's final state h(T,S(T)),that is, the state at the most down stream, the cost of impulse aeration operations, and integration of transient state of water quality $S(t;\ 0,S_0)$ over the interval [0, T]:

$$J = h(T, S(T)) + c(T^i, S(t^i;0,S_0), q^i) + \int_0^T g(t, S(t;0,S_0)dt \qquad (2.2.45)$$

where $S(t;0,S_0)$ is a trajectory of the solution of (2.2.43) starting from the initial state $(t, S(t))=(0, S_0)$. The functions h, c and g are assumed to have enough smooth natures as to guarantee an optimal solution (such as twice continuously differentiable and monotonously increasing functions, etc.). The simplest form of g(t,S) would be a quadratic function of the deviation from the fixed DO standard values as defined in (2.2.13).

Then the optimal control problem is to find such a series of aerators located sequentially along the stream that minimizes performance function J. According to the theory of "Quasi Variational Inequality" by Bensoussan and Lions (1975), this control problem is reduced to finding the auxiliary functional V(t,x) which satisfies the following "Quasi Variational Inequality (Q.V.I.) that is compatible conditions for the Hamilton-Jacobi Variational Principle:

$$\partial V(t,S)/\partial t + F(t)S\, \partial V(t,S)/\partial S + g(t,S) > 0 \qquad (2.2.46)$$

$$V(t,S) < \min [c(t,S,q) + V(t, S-q)] \qquad (2.2.47)$$

$$[V(t,S)/ t + F(t)S\, V(t,S)/ S + g(t,S)]$$
$$[V(t,S) - \min \{c(t,S,q)+g(t,S-q)\}] = 0 \qquad (2.2.48)$$

$$V(T,S) = h(T,S) \qquad (2.2.49)$$

Roughly speaking, the first and second inequalities describe the necessary conditions of the optimality. The third one is sufficient condition, and the last one is the terminal condition.

Then we can construct an optimal impulse control u that takes q^i (i=1,2, ..,N) as an aeration power, based on the following function V(t,S) called "Free Boundary":

$$V(t,S) = \min [c(t,S,q) + V(t, S-q)], \quad 0 < t < T. \qquad (2.2.50)$$

The surface of the function V becomes a switching curve on which the impulse control u must operate with the aeration power q^i. For example, starting the initial state $(0,S_0)$ without control, the trajectory $S(t;0,S_0)$ first comes across the surface V, and then takes control u as its aeration power of q^i toward an optimal final state.

Here, we have also same computational problems of solving variational inequality as we had in the Hamiltonian approach to continuous aeration case (Section 2.2.3). However, for a case of linear system with a quadratic performance function J, we can obtain the switching curve by some iterative analytical procedure (Shima 1978, Koivo and Phillips 1975).

Iterative Procedure for Free Boundary V

Let $M(t,x)$ be the performance function J without control u:

$$M(t,S) = h(T,S(T;0,S_0)) + \int g(t,S(t;0,S_0)dt. \qquad (2.2.51)$$

Differentiation of the function $M(t,S)$ along the trajectory $S(t;0,S_0)$ demonstrates that M satisfies the Q.V.I. conditions (2.2.46) and (2.2.49):

$$\partial M(t,S)/\partial t + F(t,S)\partial M(t,S)/\partial S + g(t,S) = 0 \qquad (2.2.52)$$

$$M(T,S)=h(T,S) \qquad (2.2.53)$$

Then, we can obtain the free boundary $V(t,S)$ by the following iterative procedure (Shima 1978):

(1) obtain such an impulse function q that minimizes:

$$N(t,S,q)= \min [g(t,S,q) + M(t,S-q)], \qquad (2.2.54)$$

where the first term corresponds to the cost function which use the impulse control q, and the second term is the performance function without impulse control.

(2) define the curve G of the function :

$$G(t,S)=\{M(t,S)- N(t,S,q)\}, \qquad (2.2.55)$$

and divide it into two positive and negative domains:

$$G^+ =\{(t,S):G(t,S)>0, 0<t<T\},$$
$$G^- =\{(t,S):G(t,S)<0, 0<t<T\}. \qquad (2.2.56)$$

Apparently, it seems that we had better use impulse control in the domain G^+, but not in the domain G^- when the trajectory starts without control. However, we need to make sure the other switching possibility of impulse control over the entire domain.

(3) let J_t or $J_{t+\delta t}$ be the performance functions of using impulse control q at $p=t$ or $p=t+\delta t$, respectively. Then, the difference of two performance functions $J_t-J_{t+\delta t}$ can be expanded as the

following variational equation of the order o(t):

$$\partial c(t,S,q)/\partial t + F(t)\partial c(t,S,q)/\partial S + \partial M(t,S-q)/\partial t$$
$$+ F(t)\partial M(t,S-q)/\partial S + g(t,S) + o(\delta t) = Q(t,S) \qquad (2.2.57)$$

The resulting function Q(t,S) is divided into two domain:

$$Q^+ = \{(t,S):Q(t,S)>0\},$$
$$Q^- = \{(t,S):Q(t,S)<0\} \qquad (2.2.58)$$

The surface Q called "Free Boundary" can actually guarantee whether the impulse control is optimal or not be switched or not.

(4) Repeat the procedure (2)-(3) insofar as the trajectory hits again the surface G and Q for the next possible switching.

Thus, we have the iterative procedure of determining optimal control strategy about switching the impulse control q (on-off control) taking account of both surfaces of G(t,S) and Q(t,S) based on the procedure (1) -(3).

<u>Simple Example</u>

Let the DO equation (2.2.8), impulse control function u(t), and the performance function J be (Shima 1978):

$$dS/dt = -S + 1 + u, \quad (k_1=1,\ W(t)=1) \qquad (2.2.59)$$

$$u = -\Sigma\, q^i\delta(t-t^i), \quad q^i> 0,\ i=1,\ldots,N \qquad (2.2.60)$$

$$J = \alpha N + (1/2)\int_o^T\{S(p)\}^2dp + (1/2)\{S(T)\}^2 \qquad (2.2.61)$$

respectively, where α is an unit operation cost of the impulse aeration. From the system equation (2.2.58), we have the trajectory S without any control (u=0) passing a point (t,S):

$$S(p;t,S) = \exp\{-(p-t)\}(S-1) + 1 \qquad (2.2.62)$$

Then, M(t,S) of the performance function without control becomes:

$$M(t,S)= (1/4)[(1+\exp\{-2(T-t)\}(S-1)^2+4(S-1)+(T-t+1)] \qquad (2.2.63)$$

From the necessary condition for optimality (2.2.53), we get

$$q = S + [1-\exp\{2(T-t)\}]/[1+\exp\{2(T-t)\}] > 0 \qquad (2.2.64)$$

as an optimal impulse control.

By using this impulse control q, any trajectory S(t;(t,S)) can be taken from the point (t,S) to the point (t, S-q) of the curve G^*:

$$(t,S-q) = (t,\ S-[1-\exp\{2(T-t)\}]/[1+\exp\{2(T-t)\}]\ ) \qquad (2.2.65)$$

Figure 2.2.7 illustrates the optimal trajectories in the simple case (T=1, α =0.5) starting various initial states (E_1), and curves G, Q. The trajectory AP_1B is the solution S(t) of (2.2.61) without control which is tangent to the curve G at the point P_1. The trajectory which starts at the initial states below the point A, does not need to take any impulse control as shown $E'_1E_3E_4$, since the trajectories are belong to the domain G^{-1}. The trajectories starting from the points between A and C will hit the "free boundary" Q at the point E_2, and then they need to take impulse control q until arriving at the curve G^*. Hence, the optimal trajectory becomes $E_1E_2E_3E_4$. Other trajectory starting from the points belong to the domain $G^+ \cup Q^+$ need to take the impulse control up to the curve G^*, then no use of control as indicated $E''_1E_2E_3E_4$.

All trajectories of this simple numerical example have less than one-time switching: either no control or one impulse control. If the trajectories stating on G^* after taking an impulse control, hit again the section CP_1, then we need to repeat whole procedure to determine the free boundary.

2.2.6 Water Quality Control Problems in River Basin Management

Let us consider the water quality control problem in the river basin management. In the concepts of water basin management, there has been a number of theoretical frameworks to deal with conflicting social goals involved in the water utilization and preservation. One of the general approaches recently developed is the "trade-off" analysis for weighing the economic costs of abating or controlling water quality versus economic benefits of utilizing cleaner water both for production and aesthetic purposes.

In terms of control theory, this approach is called "multi-objective optimization" in decision support framework. A multi-objective optimization can be described in the following way:

$$\max J(s)= \max[J_1(s),\ J_2(s), \ldots,\ J_p(s)] \tag{2.2.66}$$

$$\text{subject to} \quad s \in S\ ,\ S=\{s: f_j(s) \leq 0,\ j=1,2,\ldots,m\} \tag{2.2.67}$$

where J_i are the i-th competing objectives (or attributes), $(i=1,2,\ldots,p)$, and S is the feasible set of the solution described by equality or inequality constraints with respect to control variables related to water quantity and quality (Haimes et al 1975).

The set of optimal solutions satisfying (2.2.66) and (2.2.67) is called "Parato optimal" (or non-inferior solutions), and it defines a hypersurface on which there is no improvement of the objectives without lowering at least one of others. The final optimal solution must be singled out of this "Parato" set on the basis of the decision makers' preference to the combination of each objective. In order to single the final solution out, therefore, we need a preference structure of the decision maker.

Theoretically, the preference structure of the decision maker can be given by the marginal rate of utility function in terms of the substitution in the objective function space (see Figure 2.2.8). The utility function which represents a global preference structure of the decision maker, might select an optimal solution which maximizes the expected utility. In practical cases, however, the information on the decision maker's preference is not necessarily given in explicit way as the utility function. For assessing the preference structure of the decision makers, the multi-attribute utility function method is one of the most advantageous methods (Keeny and Raiffa 1976). But it has based on some rigid assumptions of preferential and utility independences that are difficult to be identified in many cases.

Possibly, we may extract it from the partial information about the preference attitude of the decision maker based on the his behavior on each alternative solution near a "Parato" optimal solution. This approach is called "interactive method" to the multi-objective optimization (Geoffrion et al 1972, Wierzbicki 1980, Nakayama et al 1980).

In this section, we describe an application of "interactive method" of multi-objective optimization to the water quality control problem in river basin management. The example is the Yodo River basin located in the western center of the Japanese main island (Sawaragi et al 1978). It has three major tributary rivers (Katsura River, and Uji River which flows out of the Lake Biwa, and Kizu River) as upper streams, and one main stream of Yodo River that flows into Osaka Bay as shown schematically in Figure 2.2.9.

We consider the inter-regional conflicts between "upper region" of two major tributary rivers (Katsura River and Uji River) and "lower region" of main river concerning to the water quality improvement versus the treatment costs. For simplicity of the multiobjective problem, we assume a constant flow rate and a constant amount of intake water, and constant BOD load to be generated in each river basin as shown in Figure 2.2.9. In addition, it is also assumed that there is no explicit natural decaying process involved in this river basin system.

Then the optimization problem is reduced to how to allocate the BOD input to be discharged into each river region, taking account of operation costs of the waste treatment plants, and resulting water quality at the outlet of the Yodo river. We formulate the optimization problem in the following way. (Sawaragi et al 1978)

Let s_1, s_2 and s_3 be control variables defined by

s_1: treatment ratio of the wasted water (BOD) at the treatment plant in the Katsura River of the upper region,

s_2: same in the Uji River of the upper region,

s_3: same in the Yodo River of the lower region.

within the domain set as a physical constraint:

$$0.45 < s_1,\ s_2,\ s_3 < 1.0 \qquad (2.2.68)$$

Then, we define the following three objective functions as components of the multiobjective function $J(J_1,J_2,J_3)$:

(1) treatment cost at the upper region;

$$J_1 = f_1(s_1,s_2)$$
$$= 287.58 + 2295.59(s_1-0.45)^2 + 404.46(s_2-0.45)^2 \qquad (2.2.69)$$

(2) treatment cost in the lower region;

$$J_2 = f_2(s_3)$$
$$= 1050.73 + 10035.34(s_3-0.45)^2 \qquad (2.2.70)$$

(3) BOD concentration at the outlet of the Yodo River to the Osaka Bay;

$J_3 = f_3(s_1,s_2,s_3)$

$$= 36.03 - (8.05s_1 + 1.04s_2 + 24.0s_3) \qquad (2.2.71)$$

A kind of "interactive procedure" used in this example is based on relaxation of the constraint, that is, transforming the minimization of the utility $U(J_1,J_2,J_3)$ on the "Parato" surface into that of the $U(J_1,J_2, f_3(J_1, J_2)$ with some appropriate assumption on the smoothness of the objective functions and the utility function.

<u>Formulation of the Interactive Relaxation</u>

Instead of having three objective functions, we transformed the original problem into the following single objective one:

$$\text{Minimize } J_1 = f_1(s_1,s_2) \qquad (2.2.72)$$

$$\text{subject to } J_2 = f_2(s_3) \leq e_2$$

$$J_3 = f_3(s_1,s_2,s_3) \leq e_3.$$

The multiplier method of -constraint is used to solve the problem (2.2.72) (Nakayama et al 1980), which provides not only the "Parato" solution but also the values of Lagrange multiplier. According to the Lagrangian formula, we have the following relation at the optimal solution S^o on the "Parato" curve and indifference curve of the utility function U,

$$M_i = U_i/U_1, \quad T_i = T_i/T_1, \quad (i=2,3)$$

where, M_i is the marginal rate of substitution between J_i and J_1, and T_i is the trade-off ratio which is the direction ratio of the normal of the "Parato" surface between J_i and J_1.

First, we calculate the upper and lower bounds for J_2 and J_3 within the interval $0.45 < s_1, s_2, s_3 < 1.0$, respectively:

$$1050.73 < J_2 < 4086.42 \; , \quad 2.94 < J_3 < 21.14$$

We choose $(J_2,J_3) = (3000, \; 7.0)$ as an initial point, and solve the relaxed problem (2.2.72), setting the constraints e_2=3000 and e_3=7.0.

$(J_1, J_2, J_3) = (695.84, 3000, 7.0)$

on the "Parato" surface. The decision maker are asked to answer which direction of J_2 he or she might take considering both the values of J_2 and the trade-off ratio T_2. Then, J_2 being fixed at 3000, we search the preferred value of J_3 with interaction to the decision maker. Next, J_3 being fixed at 6.0 as the preferred one, we proceed further up to obtain a satisfactory and converged solution in terms of the difference of the marginal substitute M and the trade-off ratio T ($\{M-T\} \to 0$). Table 2.2.1 displays the whole process of the IRM for water quality control problem. This decision maker has put the primary weight on water quality (BOD) with the stress on a relatively high ratio of water treatment (s_1=0.945, s_2=0.8814, and s_3=0.973) in all river streams.

2.2.7 Conclusion

We have described some of the application of the modern control theory to the water quality problems. Those are:(1) identification of pollution source function from the point monitoring data, (2) optimal impulse control of the in-stream aeration, and (3) multiobjective optimization of water quality management in river basin system.

The theoretical frameworks described in the above problems differ in scale and complexity. In recent years, a variety of the optimal control algorithms are available, and several analytical tools of systems engineering could provide a basic environment for a fairly large area of the water quality control problems (Beck 1980).

However, lack of effective engineering measures to control pollutants discharges in the sense of real-time regulation, has limited the practical applicability of the modern control theory to a rather narrow area such as optimal operations at sewage treatment plants. The problem of instrumentation adds another difficult issue of controlling and monitoring a variety of variables and conditions in a spatial and temporal dimensions of river stream system.

As concern has grown over the ecological (eutrophication of river water), public health (hazardous and toxic risks of drinking water) and economic issues (social costs of siting and constructing sewage plants), we need to develop an integrated framework of control theory both in dynamic and static contexts.

At the same time, to have better applications of the control theory to a wider area of water quality problems, we need a further elaboration of inputs, outputs and internal dynamic characteristics of water quality system toward adequate matching in theoretical frameworks of the modern control theory.

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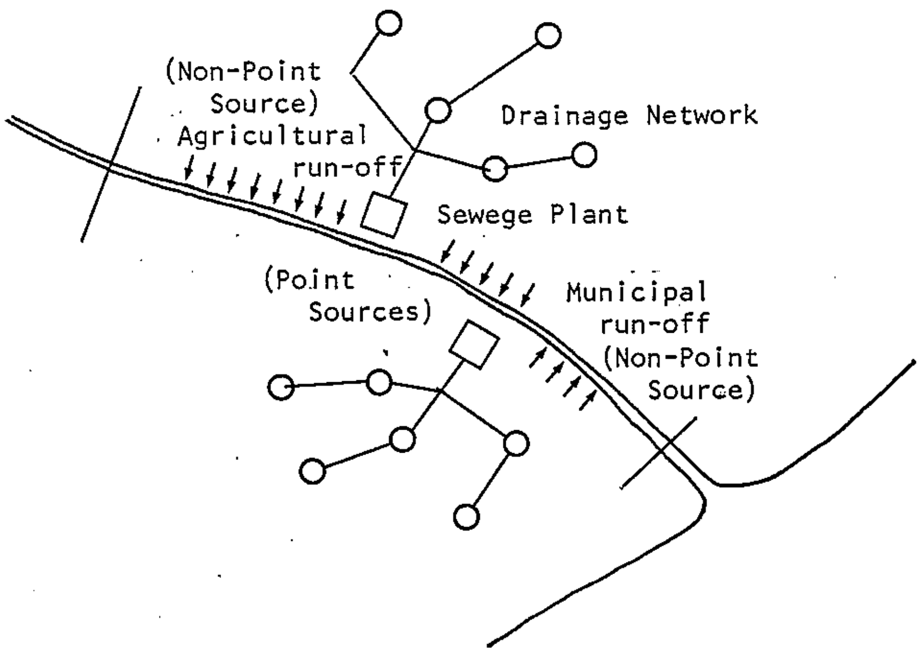
Captions of Figures

Fig. 2.2.1



(Non-Point
Source)
Agricultural
run-off
Drainage Network
Sewege Plant
(Point
Sources)
Municipal
run-off
(Non-Point
Source)

Fig. 2.2.2

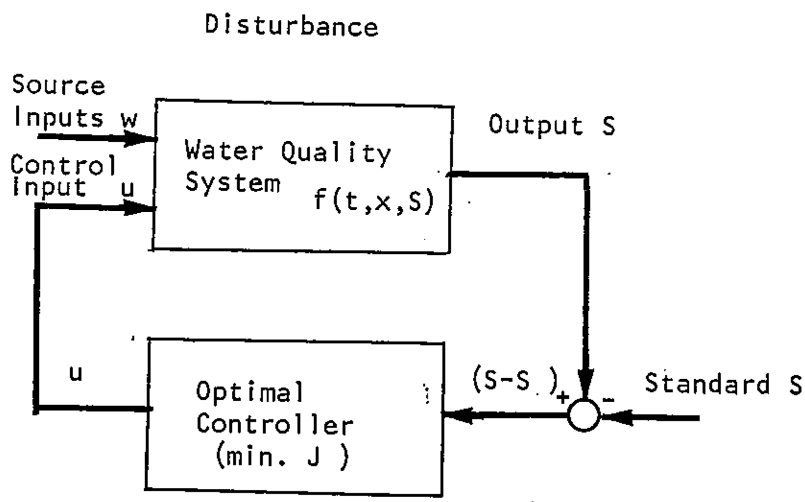

Disturbance
Source
Inputs w
Control
Input u
Water Quality
System
f(t,x,S)
Output S
u
Optimal
Controller
(min. J)
(S-S)
+
-
Standard S

Fig. 2, 2, 3

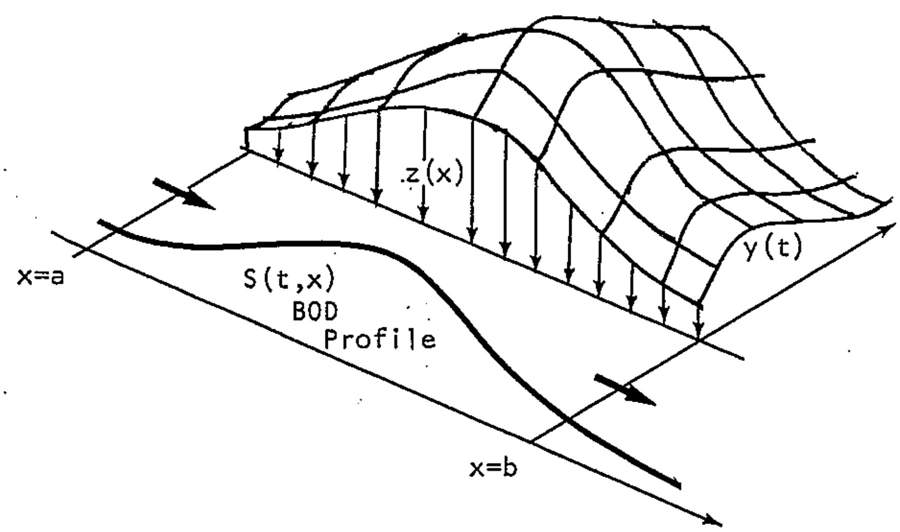


Fig. 2.2.4

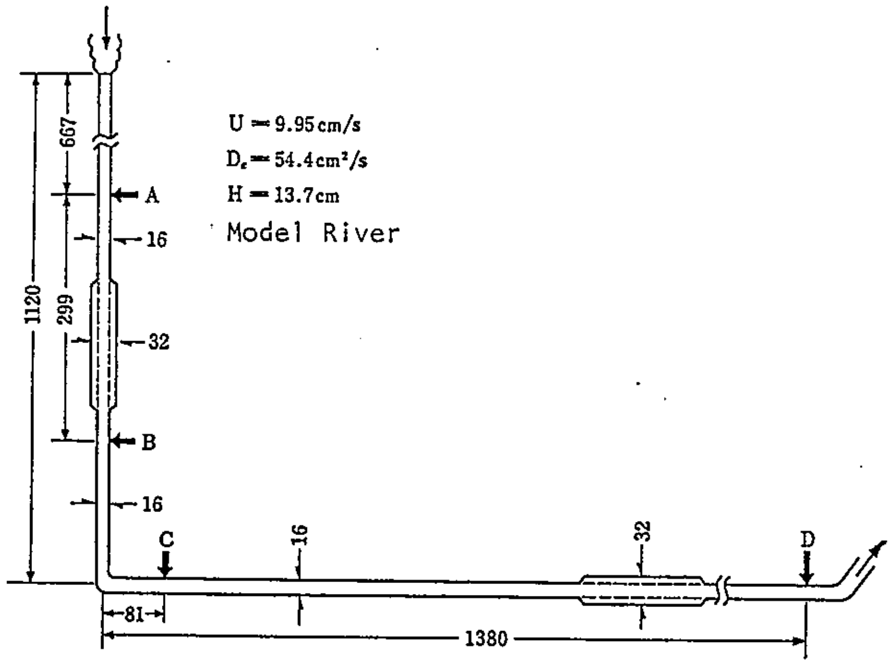

U = 9.95 cm/s
D_c = 54.4 cm²/s
H = 13.7 cm
Model River
667
A
16
1120
299
32
B
16
C
16
32
D
81
1380

Fig. 2.2.5

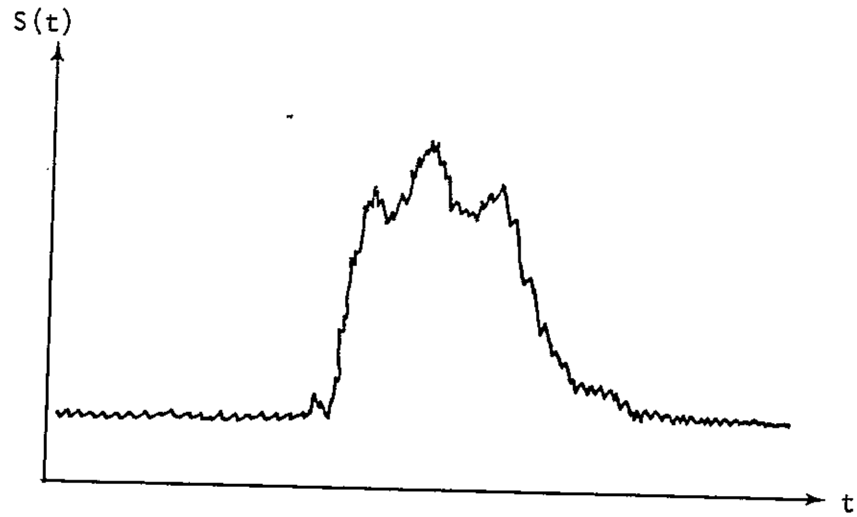
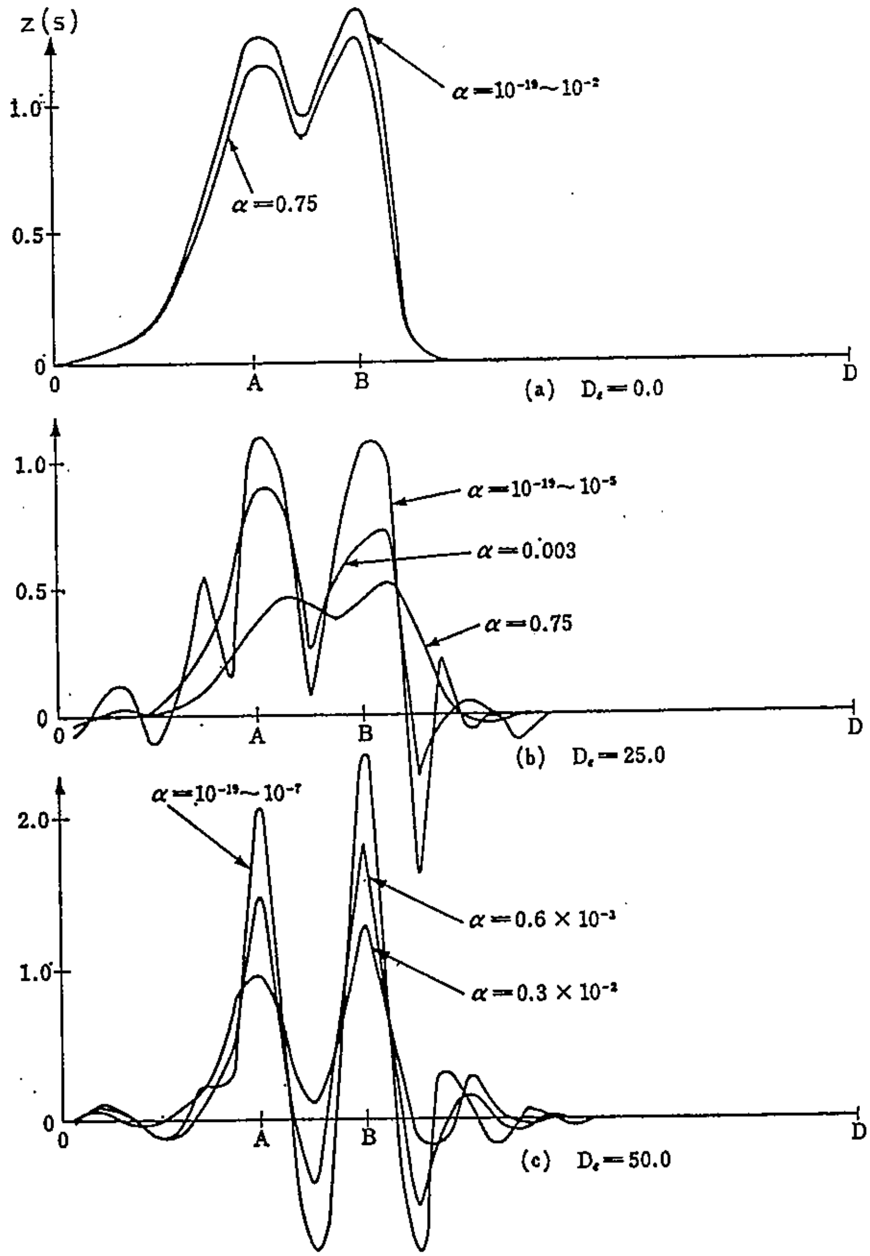


Fig. 2.2.6
(1)



z(s)
1.0
0.5
0
$\alpha=10^{-19}\sim10^{-2}$
$\alpha=0.75$
0
A
B
D
(a) $D_\varepsilon=0.0$
1.0
0.5
0
$\alpha=10^{-19}\sim10^{-5}$
$\alpha=0.003$
$\alpha=0.75$
A
B
D
(b) $D_\varepsilon=25.0$
2.0
1.0
0
$\alpha=10^{-19}\sim10^{-7}$
$\alpha=0.6\times10^{-3}$
$\alpha=0.3\times10^{-2}$
A
B
D
(c) $D_\varepsilon=50.0$

Fig. 2.2.6
(2)

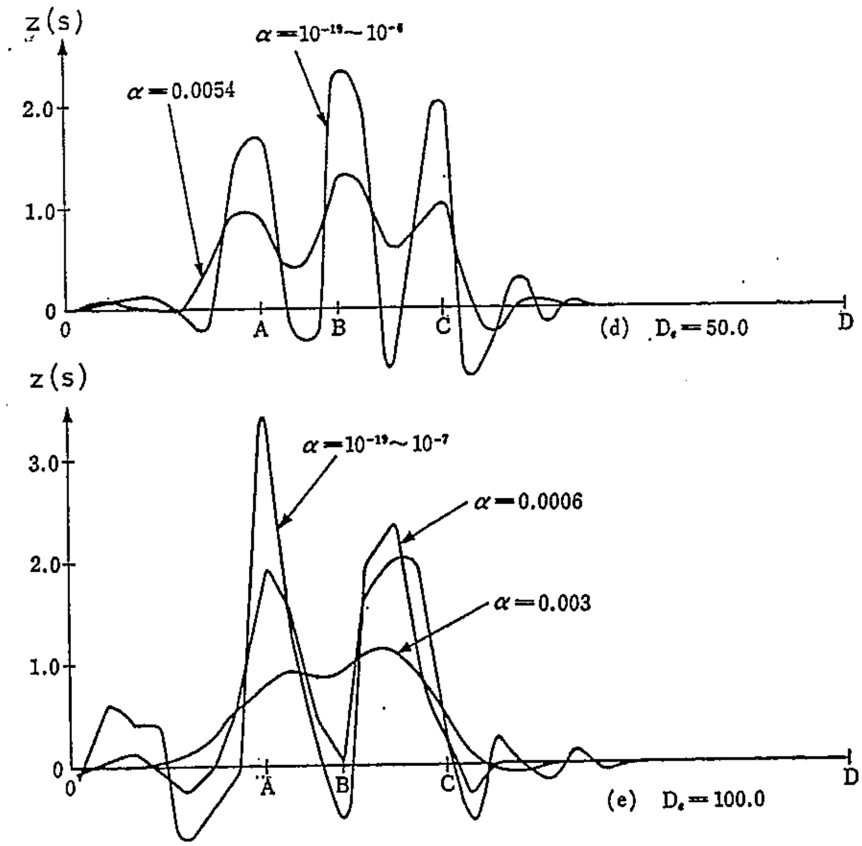

z(s)
α = 10^-19 ~ 10^-6
α = 0.0054
2.0
1.0
0
0
A
B
C
(d) D_ε = 50.0
D
z(s)
3.0
α = 10^-19 ~ 10^-7
α = 0.0006
2.0
α = 0.003
1.0
0
0
A
B
C
D
(e) D_ε = 100.0

Fig. 2.2.7

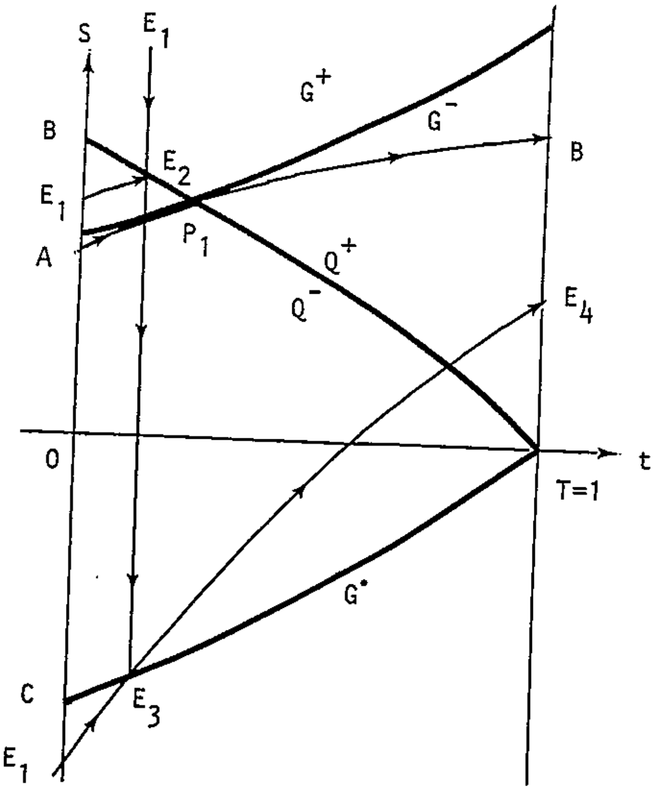


Fig. 2.2.8

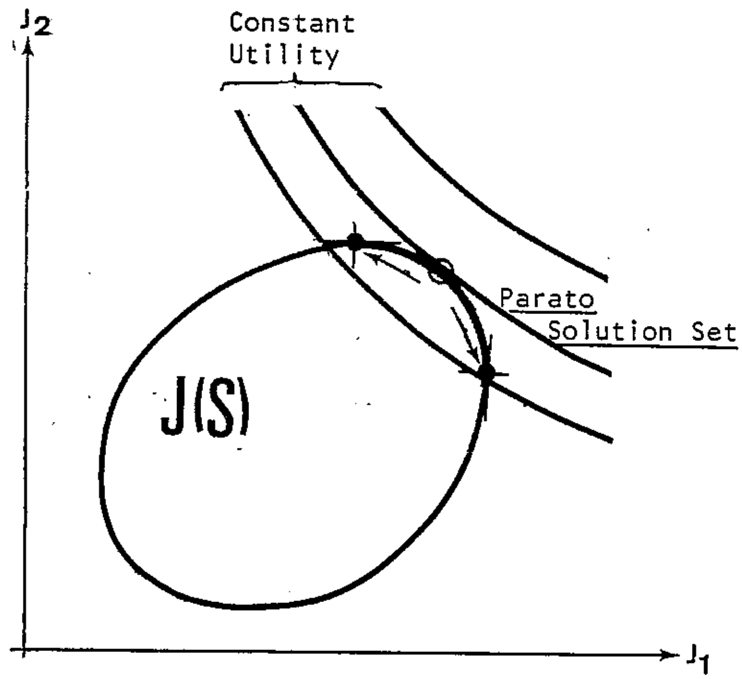
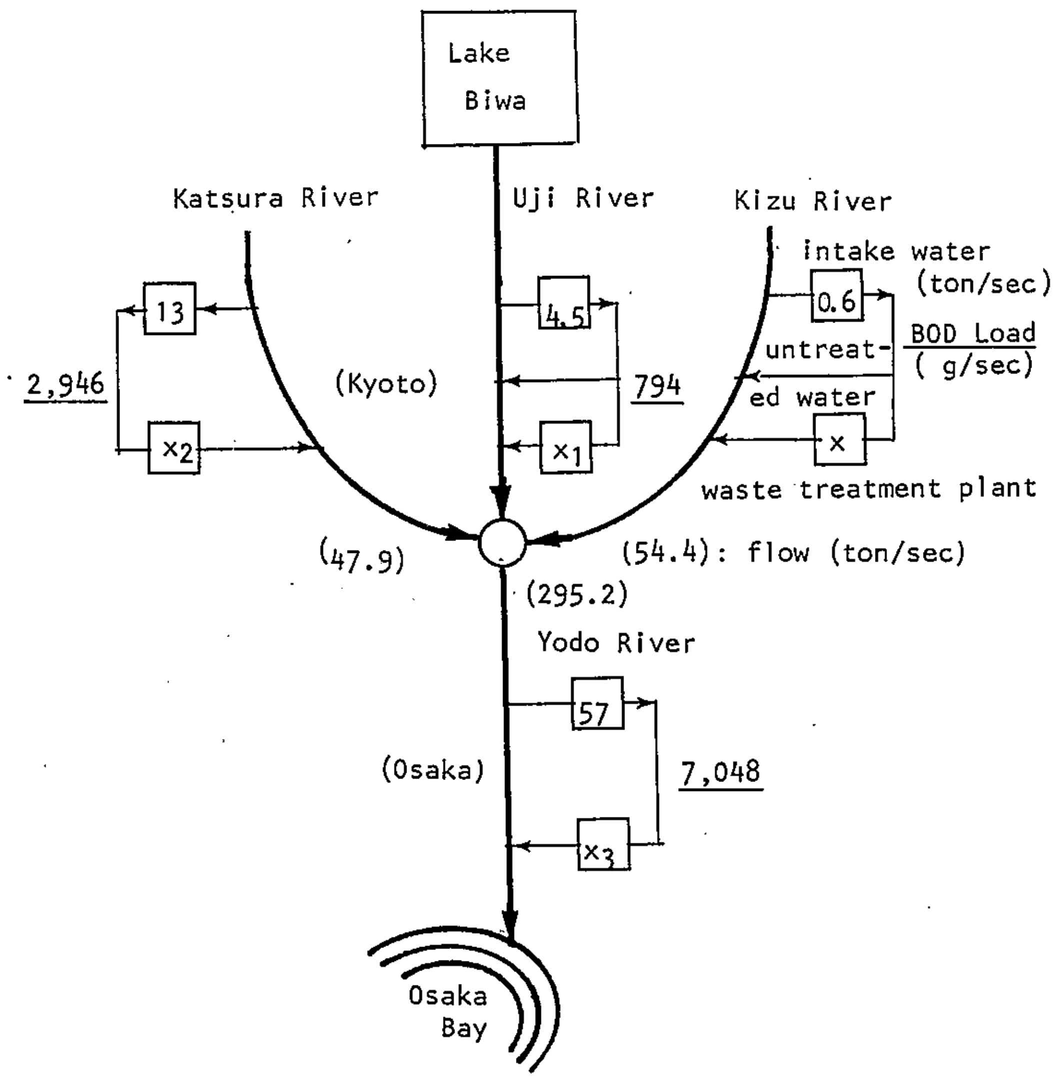


Fig. 2.2.9



Lake Biwa
Katsura River
Uji River
Kizu River
intake water (ton/sec)
13
4.5
0.6
2,946
(Kyoto)
794
untreated water
BOD Load (g/sec)
x2
x1
x
waste treatment plant
(47.9)
(54.4): flow (ton/sec)
(295.2)
Yodo River
57
(Osaka)
7,048
x3
Osaka Bay

TABLE.
DECISIONMAKING PROCESS BY IRM

f_2	f_3	f_1	x_1	x_2	x_3	T_2	T_3	M_2-T_2	M_3-T_3
3000	7.0	695.84	0.853	0.746	0.891	0.624	230.10		+
3000	5.0			infeasible					
3000	6.0	959.26	0.967	0.829	0.891	0.799	294.82	–	0
3500	6.0	637.11	0.823	0.723	0.944	0.514	212.50	+	
3200	6.0	814.77	0.908	0.786	0.913	0.674	260.99	–	
3300	6.0	749.94	0.879	0.764	0.923	0.617	244.45	0	+
3300	5.0	1025.69	0.992	0.847	0.923	0.781	309.17	–	0
3500	5.0	881.05	0.936	0.806	0.944	0.671	277.22	–	
3700	5.0	760.87	0.884	0.768	0.963	0.577	247.71	0	+
3700	4.0	1042.32	0.998	0.852	0.963	0.728	312.43		–
3700	4.5	893.42	0.941	0.810	0.963	0.653	280.07		+
3700	4.2	980.25	0.975	0.835	0.963	0.694	298.23	–	0
3800	4.2	907.70	0.947	0.814	0.973	0.647	283.31	0	0