No.382

SAVAGE'S UTILITY WITH NON-ADDITIVE PROBABILITEIES ON FINITE STATE SPACE

by

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June, 1988

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ABSTRACT

Schmeidler-Gilboa's representation generalizes Savage's expected utility to cope with Ellsberg paradox, so that the probability measure over states of the world need not be additive. This paper examines a similar generalization under Savage's formulation when the set of states is finite, while Savage's states are continuously divisible. Our axiomatization requires that the set X of consequences is infinite in contrast to Savage's arbitrary X. Three representational forms are axiomatized to give non-additivity, complementary additivity, and additivity of probability measures, respectively.

Theories of decision making under uncertainty provide subjective expected utility models that represent numerically the personal beliefs (subjective probabilities) and preferences of a decision maker. Under various formulations numerous axiom systems and their numerical representations have been proposed (see a survey by Fishburn (1981)). One of the most elegant and well-known axiomatization of subjective probability and utility was given by Savage (1954). He developed an act-oriented theory which applies a preference relation over the set of acts. Acts are defined as functions from the states of the world into the consequences.

There is abundant evidence that people's carefully considered decisions often violate the assumptions of subjective expected utility theories (see Allais (1953), Davidson, Suppes and Siegel (1957), Ellsberg (1961), Slovic and Tversky (1974), Kahneman and Tversky (1979), Grether and Plott (1979), and others). Recently, Schmeidler (1984) and Gilboa (1987) generalized Savage's utility theory to accommodate Ellsberg-type violation of additive probability so that their theories do not require additivity of probability measures. Schmeidler adopted the idea of lottery acts (functions from the states of the world into probability distributions on the consequence space) introduced by Anscombe and Aumann (1963). Gilboa used Savage's basic formulation, so that the set of states requires to be continuously divisible.

This paper axiomatizes a generalization of Savage's expected utility to accommodate Ellsberg-type violation of additive probability under Savage's formulation when the set of states is finite. The axiomatization requires that the set of consequences is infinite, while it is arbitrary in the theories of Savage, Schmeidler, and Gilboa. The axioms are shown to give a utility function over consequence space, and non-additive prabability measures over states. The expectation of a utility function with respect to non-additive probability measures is given as Choquet integration as in Schmeidler-Gilboa's representation, which is discussed in the next section. Section 3 states the axioms and the representation theorems that include two special representations with complementary additivity and additivity of probability measures, respectively. Section 4 establishes a representation for all binary acts, then Section 5 completes the proofs of the representation theorems in Section 3.

2. REPRESENTATIONS WITHOUT ADDITIVITY

Let S be the set of states. Subsets of S are called events. For events in $2^S = \{A: A \subseteq S\}$, denote the complement $S \setminus A$ of A by A^c. A Savage's act is a function from S into the set X of consequences. F is the set of all Savage's acts. For all $f,g \in F$, all $x \in X$, and all $A \in 2^S$,

$f =_A g$ <u>means that</u> $f(s) = g(s)$ <u>for every</u> $s \in A$,

$f =_A x$ <u>means that</u> $f(s) = x$ <u>for every</u> $s \in A$.

Let $\prec$ on F be the basic binary preference relation with $\sim$ and $\precsim$ defined in the usual way: for $f,g \in F$, $f \sim g$ if not$(f \prec g)$ and not$(g \prec f)$; $f \precsim g$

if $f \prec g$ or $f \sim g$. Savage's qualitative probability relation $<^*$ on 2^S is defined as follows: for all $A,B \in 2^S$,

$$A <^* B \text{ \underline{iff} } f \prec g \text{ \underline{whenever} } x \prec y,\ f =_A y,\ f =_{A^c} x,\ g =_B y,\ \text{\underline{and} } g =_{B^c} x.$$

Schmeidler-Gilboa's representation has the form: for all $f,g \in F$,

$$f \prec g \text{ iff } \int_S u(f(s))d\pi(s) < \int_S u(g(s))d\pi(s),$$

where π is monotonic but not necessarily additive on 2^S, and u is a real valued function on X. Monotonicity of π means that if $A \subseteq B$ then $\pi(A) \leq \pi(B)$. The integration in the representation is defined in Choquet's sense (see Choquet (1955)) to account for non-additive probability measures as follows. Since we are concerned with a finite S, assume that $S = \{s_1,\ldots, s_n\}$, and $u_1 \geq \ldots \geq u_n$, where $u_i = u(f(s_i))$ for $s_i \in S$. Let π_f denote the decumulative distribution function induced by π through f so that

$$\pi_f(t) = \pi(\{s\colon u(f(s)) \geq t\})$$

with $\pi_f(t) = 0$ for $t > u_1$ and $\pi_f(t) = 1$ for $t \leq u_n$. Then Choquet integration is given by

$$\int_S u(f(s))d\pi(s) = \sum_{i=1}^{n-1} (u_i - u_{i+1})\pi_f(u_i) + u_n.$$

In Schmeidler-Gilboa representation, u is unique up to a positive linear transformation, and π is unique, so that u and $\pi(A)$ for $A \subseteq S$ can be interpreted as a utility function on X and a subjective probability for an event A, respectively. However, for two disjoint events, we may have $\pi(A + B) \neq \pi(A) + \pi(B)$. The interpretation of π follows from that $A <^{*} B$ iff $\pi(A) < \pi(B)$, and that by Choquet integration, expected utilities are calculated with respect to decumulative distributions.

On the other hand, it is possible to define the integration for non-additive probability measures using cumulative distributions. Let π^{*} denote the dual measure of π such that $\pi^{*}(A^{c}) = 1 - \pi(A)$ for $A \varepsilon 2^{S}$. Also let π_{f}^{*} denote the cumulative distribution function induced by π^{*} through f so that

$$\pi_{f}^{*}(t) = \pi^{*}(\{s: u(f(s)) \leq t\}).$$

Then the definition of Choquet integration is rearranged to give

$$\begin{aligned}\int_{S} u(f(s))d\pi(s) &= \sum_{i=1}^{n-1} (u_{i} - u_{i+1})(1 - \pi^{*}(\bigcup_{j=i+1}^{n} \{s_{j}\}) + u_{n} \\ &= \sum_{i=1}^{n-1} (u_{i+1} - u_{i})\pi^{*}(\bigcup_{j=i+1}^{n} \{s_{j}\}) + \sum_{i=1}^{n-1} (u_{i} - u_{i+1}) + u_{n} \\ &= \sum_{i=1}^{n-1} (u_{i+1} - u_{i})\pi^{*}(\bigcup_{j=i+1}^{n} \{s_{j}\}) + u_{1} \\ &= \sum_{i=1}^{n-1} (u_{i+1} - u_{i})\pi_{f}^{*}(u_{i+1}) + u_{1}.\end{aligned}$$

Thus an integration of u with respect to cumulative distributions is defined by

$$\int_S u(f(s))d\pi^*(s) = \sum_{i=1}^{n-1}(u_{i+1} - u_i)\pi_f^*(u_{i+1}) + u_1,$$

which is referred to as the dual Choquet integration. This means that expected utilities are calculated with respect to cumulative distributions.

Let $<^{**}$ be the dual of $<^*$ such that

$$A <^{**} B \text{ \underline{iff} } g \prec f \text{ \underline{whenever} } x \prec y,\ f =_A x,\ f =_{A^c} y,$$
$$g =_B x, \text{ \underline{and} } g =_{B^c} y,$$

so $A <^* B$ iff $B^c <^{**} A^c$. Since $A <^{**} B$ iff $\pi^*(A) < \pi^*(B)$, it is natural to interpret $\pi^*(A)$ as a subjective probability for an event $A\varepsilon 2^S$.

Although a utility function is common, the subjective probability measures may differ in the definitions of the integration. It easily follows that a necessary and sufficient condition for π and π^* to be identical is given by the following <u>complementary additivity</u> condition: for all $A\varepsilon 2^S$,

$$\pi(A) + \pi(A^c) = 1.$$

Throughout the paper, the integration always stands for the dual Choquet integration.

3. AXIOMS AND THEOREMS

Several additional notations and definitions will be useful in stating axioms. When F_1 and F_2 are subsets of F, $F_1 \prec F_2$ means $f \prec g$ for all $f \in F_1$ and all $g \in F_2$; $F_1 \precsim F_2$ means $f \precsim g$ for all $f \in F_1$ and all $g \in F_2$. Let xAf denote the act f' with $f' =_A x$ and $f' =_{A^c} f$. A binary act is denoted by xAy, and a constant act is $f =_S x$ for some $x \in X$, so every $x \in X$ is identified as a constant act.

Let m be a mapping from F into X that assigns a constant act m(f) for each $f \in F$ such that $f \sim m(f)$ if it exists. A partition of S is a set of nonempty events that are mutually disjoint and whose union equals S. For an n-partition $P = \{A_1, \ldots, A_n\}$, if $f =_{A_i} x_i$ for all i, then m(f) is denoted by $m_P(x_1, \ldots, x_n)$, and sometimes we write $f_P(x_1, \ldots, x_n)$ instead of f. When $P = \{A, A^c\}$ and f = xAy, denote m(f) $= m_A(x, y)$. Also, xAy is sometimes written by $f_A(x, y)$. Suppose that $\{x_1, \ldots, x_n\} = \{x_{i_1}, \ldots, x_{i_n}\}$, $P = \{A_1, \ldots, A_n\}$, $P' = \{A_{i_1}, \ldots, A_{i_n}\}$, and $P = P'$. Then by the definition of Savage's acts, $f_P(x_1, \ldots, x_n) = f_{P'}(x_{i_1}, \ldots, x_{i_n})$. If $P'' = \{A_1+A_2, A_3, \ldots, A_n\}$ and $x_1 = x_2$, then $f_P(x_1, \ldots, x_n) = f_{P''}(x_2, \ldots, x_n)$. In particular, xAx = x and xAy = yA^cx.

A null event $A \in 2^S$ means that for all $f, g \in F$, $f \sim g$ whenever $f =_{A^c} g$. Two acts $f, g \in F$ are said to be comonotonic if and only if there exist no $s, t \in S$ such that $f(s) \prec f(t)$ and $g(t) \prec g(s)$. Comonotonicity was introduced in Schmeidler (1984), and crucial in Schmeidler-Gilboa's representation.

The following axioms apply to all $f, g \in F$, all partitions, all

$A, B \in 2^S$, and all $x, y, z, w, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$:

A1: $\prec$ *on* F *is an asymmetric weak order*.

A2: *If* $f \prec xAy$ *then* $f \prec aAy$ *for some* $a \in X$ *with* $a \prec x$;

if $xAy \prec f$ *then* $aAy \prec f$ *for some* $a \in X$ *with* $x \prec a$.

A3: *If* A *is not null*, *and* xAf *and* yAf *are comonotonic*, *then*

$x \prec y$ *iff* $xAf \prec yAf$.

A4: *If* $x_1 \precsim \cdots \precsim x_n$ *and* $y_1 \precsim \cdots \precsim y_n$ *with* $x_i \precsim y_i$, *then*

$$f_A(m_P(x_1, \ldots, x_n), m_P(y_1, \ldots, y_n)) \sim f_P(m_A(x_1, y_1), \ldots, m_A(x_n, y_n)).$$

A1 is a Savage's basic ordering axiom, which says that $\prec$ is asymmetric, and that $\prec$ and $\sim$ are transitive. A2 is a density axiom, and requires that X is an infinite set. A3 is an independence axiom that weakens one of Savage's sure-thing principle. A4 is a version under uncertainty of the condition under risk proposed in Fishburn (1988, Chapter 3) that modified the third axiom in Chew (1984). In particular, when $P = \{B, B^c\}$, A4 is similar to the isometry conditoion if $A \neq B$, and the bisymmetry condition if $A = B$ in Pfanzagl (1968). It will be shown in Lemma 1 that m(f) exists for every $f \in F$ when S is finite.

The following lemma shows implications of A1, A2, and A3.

LEMMA 1: *Suppose that* A1-A3 *hold*. *Then the following conditions hold*.

(1) <u>If</u> $x_1 \precsim \cdots \precsim x_n$, <u>then</u> $x_1 \precsim f_P(x_1, \ldots, x_n) \precsim x_n$.

(2) <u>If</u> $x_i \sim y_i$ <u>for</u> $i = 1, \ldots, n$, <u>then</u> $f_P(x_1, \ldots, x_n) \sim f_P(y_1, \ldots, y_n)$.

(3) <u>If</u> $f \prec x$, <u>then</u> $f \prec a \prec x$ <u>for some</u> $a \varepsilon X$;

<u>if</u> $x \prec f$, <u>then</u> $x \prec a \prec f$ <u>for some</u> $a \varepsilon X$.

(4) $f_P(x_1, \ldots, x_n) \sim a$ <u>for some</u> $a \varepsilon X$.

(5) <u>If</u> A <u>is not null</u>, <u>then</u> $x \prec y$ <u>iff</u> $xAz \prec yAz$.

<u>Proof</u>: (1) Suppose that $x_1 \precsim \cdots \precsim x_n$. Let $P = \{A_1, \ldots, A_n\}$. Define $f_i, g_i \varepsilon F$ for $i = 1, \ldots, n$ by

$$f_i =_{A_j} x_1 \text{ for } j \leq i \qquad g_i =_{A_j} x_j \text{ for } j \leq n-i$$
$$\quad =_{A_j} x_j \text{ for } j > i \qquad\quad =_{A_j} x_n \text{ for } j > n-i.$$

Then by A3, $f_n \precsim \cdots \precsim f_1$ and $g_1 \precsim \cdots \precsim g_n$. Since $f_n = x_1$, $g_n = x_n$, and $f_1 = g_1 = f_P(x_1, \ldots, x_n)$, A1 implies that $x_1 \precsim f_P(x_1, \ldots, x_n) \precsim x_n$.

(2) Suppose that $x_i \sim y_i$ for $i = 1, \ldots, n$. Let $P = \{A_1, \ldots, A_n\}$. Define $f_i \varepsilon F$ for $i = 1, \ldots, n+1$ by

$$f_i =_{A_j} x_j \quad \text{for } j \geq i$$
$$\quad =_{A_j} y_j \quad \text{for } j < i.$$

Then f_i and f_{i+1} are comonotonic. Applying A3 successively, we obtain that $f_1 \sim f_2 \sim \ldots \sim f_{n+1}$. Thus by A1, $f_1 \sim f_{n+1}$, so $f_P(x_1, \ldots, x_n) \sim f_P(y_1, \ldots, y_n)$.

(3) Suppose that $f \prec x$. When $x \prec f$, the proof is similar. Then $f \prec xAx$, so by A2, $f \prec bAx$ for some b with $b \prec x$. Since $bAx = xA^c b$, $f \prec xA^c b$. Thus by A2, $f \prec dA^c b$ for some d with $d \prec x$. If $d \precsim b$, then by (1), $dA^c b \precsim b$, so letting $a = b$, $f \prec a \prec x$. If $b \precsim d$, then $dA^c b = bAd \precsim d$, so letting $a = d$, $f \prec a \prec x$.

(4) With no loss of generality, assume that $x_1 \precsim \ldots \precsim x_n$. If $x_1 \sim x_n$, then by A1, $x_1 \sim x_i$ for $i = 1, \ldots, n$. Thus by (2), $f_P(x_1, \ldots, x_n) \sim f_P(x_1, \ldots, x_1) = x_1$, so $f \sim x_1$. Suppose $x_1 \prec x_n$. Then by (1), $x_1 \precsim f \precsim x_n$. If either $x_1 \sim f$ or $f \sim x_n$, then the conclusion of (4) holds, so assume that $x_1 \prec f \prec x_n$. Let $Y = \{x \in X: f \prec x \prec x_n\}$, so by (3), Y is nonempty and bounded. Let a be the greatest lower bound of Y. We are to show that $f \sim a$. If $f \prec a$, then by (3), $f \prec b \prec a$ for some $b \in X$, contradicting the definition of a. If $a \prec f$, then a similar contradiction easily obtains. Hence $f \sim a$.

(5) Suppose that A is not null. If either $\{x, y\} \precsim z$ or $z \precsim \{x, y\}$, then xAz and yAz are comonotonic, so by A3, the conclusion of (5) holds. Thus assume that either $x \prec z \prec y$ or $y \prec z \prec x$. If $x \prec y$, then $x \prec z \prec y$, so by (1) and A1, $xAz \prec yAz$. Suppose that $xAz \prec yAz$. If $y \prec z \prec x$, then by (1) and A1, $yAz \prec xAz$, a contradiction. Therefore, $x \prec z \prec y$, so $x \prec y$. [Q.E.D.]

The main purpose of the paper is to prove the following:

THEOREM 1: Suppose that S is finite, $\prec$ is not empty, and A1-A4 hold. Then there are monotonic measure π on 2^S, and real-valued function u on X such that for all $f, g \in F$,

$$f \prec g \text{ iff } \int_S u(f(s))d\pi(s) < \int_S u(g(s))d\pi(s).$$

Moreover, π is unique, and u is unique up to a positive linear transformation.

The proof of the theorem will appear in Section 5. As is discussed in the previous section, the uniqueness of π depends upon the definition of the integration, so that if we apply Choquet integration with respect to the dual measure π^* of π, then the representation of the theorem is tantamount to Schmeidler-Gilboa's representation when S is finite.

The following axiom implies that π is complementarily additive, so π is identical to π^*.

A5: For all $x,y,z,w \in X$ and all $A,B \in 2^S$,

$$m_A(m_B(x, y), m_B(z, w)) \sim m_B(m_A(x, z), m_A(y, w)).$$

This axiom is a version of the isometry condition. As a special case of complementary additivity, the following stronger version of A4 implies that π is additive.

$A4^*$: For all partitions of S, all $x_1,\ldots,x_n,y_1,\ldots,y_n \varepsilon X$, and all $A \varepsilon 2^S$,

$$f_A(m_P(x_1,\ldots, X_n), m_P(y_1,\ldots, y_n)) \sim f_P(m_A(x_1, y_1),\ldots, m_A(x_n, y_n)).$$

Implications of A5 and $A4^*$ are given by

THEOREM 2: Suppose that S is finite, $\prec$ is not empty, and A1-A3 hold. Then the representation of Theorem 1 holds with the following properties:

(1) π is complementarily additive if A4 and A5 hold.

(2) π is additive if $A4^*$ holds.

The proof of the theorem will appear in Section 5. Theorem 2(2) gives an expected utility representation with an additive subjective probability measure under Savage's formulation when S is finite.

4. A REPRESENTATION FOR BINARY ACTS

We assume throughout the rest of the paper that A1-A4 hold. This section proves the following proposition, which implies that the representation of Theorem 1 holds for all binary acts.

PROPOSITION 1: Suppose that $\prec$ is not empty. Then there are two real valued functions π on 2^S, and u on X such that for all $x,y,z,w \varepsilon X$, and all $A,B \varepsilon 2^S$, if $x \precsim y$ and $z \precsim w$ then

$xAy \precsim zBw$ *iff* $\pi(A)u(x) + (1-\pi(A))u(y) \leq \pi(B)u(z) + (1-\pi(B))u(w)$;

$A \subseteq B$ *implies* $\pi(A) \leq \pi(B)$.

Moreover, π *is unique*, *and* u *is unique up to a positive linear transformation*.

First we show two lemmas, and then give the proof of Proposition 1. Let

$$X^* = \{x \in X:\ a \prec x \prec b \text{ for some } a,b \in X\},$$
$$X^a = \{x \in X:\ x \precsim a \text{ and } a \in X\},$$
$$X_a = \{x \in X:\ a \precsim x \text{ and } a \in X\},$$
$$X_{max} = \{x \in X:\ x \prec a \text{ for no } a \in X\},$$
$$X_{min} = \{x \in X:\ a \prec x \text{ for no } a \in X\}.$$

Let N be any set of consecutive integers. Given a nonnull $A \subset S$, we define a *standard sequence* with respect to $a \in X^*$ as a set $\{a_i: a_i \in X, i \in N\}$ that satisfies that there exist $b,c \in X$ such that not$(b \sim c)$, either $\{b, c\} \precsim a \precsim \{a_i\}$ and $bAa_i \sim cAa_{i+1}$ for all $i,i+1 \in N$, or $\{a_i\} \precsim a \precsim \{b, c\}$ and $a_iAb \sim a_{i+1}Ac$ for all $i,i+1 \in N$. Let $\precsim^A$ be a binary relation on $X^a \times X_a$ induced by $\precsim$ as follows: for ordered pairs $xy, zw \in X^a \times X_a$, $xy \precsim^A zw$ iff $xAy \precsim zAw$.

Lemma 2 shows that the triple $\langle X^a, X_a, \precsim^A \rangle$ is an *additive conjoint structure* (Krantz, et al. (1971, Chapter 6)). Then Lemma 3 shows that an additive representation holds for a subset of all binary acts with a fixed nonnull $A \subset S$.

LEMMA 2: Suppose that $a \varepsilon X^*$, and $A \subset S$ is not null. Then for all $x,y,z \varepsilon X^a$ and all $x',y',z' \varepsilon X_a$, the following five conditions hold:

(1) If $xAb' \precsim yAb'$ for some $b' \varepsilon X_a$, then $xAx' \precsim yAx'$;
if $bAx' \precsim bAy'$ for some $b \varepsilon X^a$, then $xAx' \precsim xAy'$.

(2) If $xAz' \sim zAy'$ and $zAx' \sim yAz'$, then $xAx' \sim yAy'$.

(3) If $xAx' \precsim yAy' \precsim zAx'$, then $bAx' \sim yAy'$ for some $b \varepsilon X^a$;
if $xAx' \precsim yAy' \precsim xAz'$, then $xAb' \sim yAy'$ for some $b' \varepsilon X_a$.

(4) Every strictly bounded standard sequence with respect to a is finite.

(5) not$(bAb' \sim cAb')$ for some $b,c \varepsilon X^a$ and some $b' \varepsilon X_a$;
not$(bAb' \sim bAc')$ for some $b \varepsilon X^a$ and some $b',c' \varepsilon X_a$.

LEMMA 3: If $A \subset S$ is not null, then there is a real valued function u on X such that for some $0 < \alpha < 1$, and for all $x,y,z,w \varepsilon X$ with $x \precsim y$ and $z \precsim w$,

$$xAy \precsim zAw \text{ iff } \alpha u(x) + (1-\alpha)u(y) \leq \alpha u(z) + (1-\alpha)u(w).$$

Moreover, α is unique, and u is unique up to a positive linear transformation.

Proof of Lemma 2: (1) This easily follows from A3.

(2) Suppose that $xAz' \sim zAy'$ and $zAx' \sim yAz'$. Then by A1, $m_A(x, z') \sim m_A(z, y')$ and $m_A(z, x') \sim m_A(y, z')$. We are to show that $m_A(x, x') \sim$

$m_A(y, y')$, so $xAx' \sim yAy'$. The following three cases cover all possibilities: $\{x', y'\} \precsim z'$; $\{x', z'\} \precsim y'$; $\{y', z'\} \precsim x'$.

<u>CASE</u> 1($\{x', y'\} \precsim z'$): By A4 and Lemma 1(2),

$$\begin{aligned} f_A(m_A(x, x'), m_A(z', z')) &\sim f_A(m_A(x, z'), m_A(x', z')) \\ &\sim f_A(m_A(z, y'), m_A(x', z')) \\ &\sim f_A(m_A(z, x'), m_A(y', z')) \\ &\sim f_A(m_A(y, z'), m_A(y', z')) \\ &\sim f_A(m_A(y, y'), m_A(z', z')). \end{aligned}$$

Hence by A1 and Lemma 1(5), $m_A(x, x') \sim m_A(y, y')$.

<u>CASE</u> 2($\{x', z'\} \precsim y'$): By A4 and Lemma 1(2),

$$\begin{aligned} f_A(m_A(x, x'), m_A(z', y')) &\sim f_A(m_A(x, z'), m_A(x', y')) \\ &\sim f_A(m_A(z, y'), m_A(x', y')) \\ &\sim f_A(m_A(z, x'), m_A(y', y')) \\ &\sim f_A(m_A(y, z'), m_A(y', y')) \\ &\sim f_A(m_A(y, y'), m_A(z', y')). \end{aligned}$$

Hence by A1 and Lemma 1(5), $m_A(x, x') \sim m_A(y, y')$.

<u>CASE</u> 3($\{y', z'\} \precsim x'$): By A4 and Lemma 1(2),

$$\begin{aligned} f_A(m_A(x, x'), m_A(z', x')) &\sim f_A(m_A(x, z'), m_A(x', x')) \\ &\sim f_A(m_A(z, y'), m_A(x', x')) \\ &\sim f_A(m_A(z, x'), m_A(y', x')) \\ &\sim f_A(m_A(y, z'), m_A(y', x')) \\ &\sim f_A(m_A(y, y'), m_A(z', x')). \end{aligned}$$

Hence by A1 and Lemma 1(5), $m_A(x, x') \sim m_A(y, y')$.

(3) Suppose that $xAx' \precsim yAy' \precsim zAx'$. If $xAx' \sim yAy'$ or $yAy' \sim zAx'$, then take $b = x$ or z, respectively, so $bAx' \sim yAy'$. Thus assume that $xAx' \prec yAy' \prec zAx'$. Then by A1 and A3, $x \prec z$. Let $Y = \{w \varepsilon X^a\text{: } yAy' \prec wAx' \text{ and } w \precsim z\}$, so by A2, Y is nonempty and bounded. Let b be the greatest lower bound of Y, so by A2, $x \prec b \prec z$. We are to show that $yAy' \sim bAx'$. If $yAy' \prec bAx'$, then by A2, $yAy' \prec cAx'$ for some $c \prec b$, contradicting the definition of b. If $bAx' \prec yAy'$, then by A2, $cAx' \prec yAy'$ for some $c \varepsilon X^a$ with $b \prec c$. By Lemma 1(3), $b \prec d \prec c$ for some $d \varepsilon X^a$; by A3, $dAx' \prec cAx'$, so $dAx' \prec yAy'$. This contradicts the definition of b. Hence we must have $yAy' \sim bAx'$. When $xAx' \precsim yAy' \precsim xAz'$, it similarly follows that $xAb' \sim yAy'$ for some $b' \varepsilon X_a$.

(4) Suppose that $c \prec d$ for $c, d \varepsilon X^a$, and that $\{a_i\} \subset X_a$ is a standard sequence with $dAa_i \sim cAa_{i+1}$. When $\{a_i\} \subset X^a$ and $c \prec d$ for $c, d \varepsilon X_a$, the proof is similar. Then by A3, $a_i \prec a_{i+1}$. Let $Y^b = \{a_i\text{: } a_i \precsim b\}$ for some $b \varepsilon X_a$. Suppose that $a_{i+1} \varepsilon Y^b$ for every $a_i \varepsilon Y^b$. Then $Y^b \prec b$. Let b' be the least upper bound of Y^b, so $Y^b \prec b'$ and $dAa_k \prec cAb'$ for all

$a_k \varepsilon Y^b$. Thus by A3, $dAa_k \prec cAb' \prec dAb'$ for $a_k \varepsilon Y^b$. By (3), $cAb' \sim dAb''$ for some $b'' \varepsilon X_a$, so $a_k \prec b'' \prec b'$. By the assumption and the definition of b', there exists $a' \varepsilon Y^b$ such that $b'' \prec a' \prec b'$, so by A3, $dAb'' \prec dAa'$. Since $dAa' \prec cAb'$, $dAb'' \prec cAb'$. This is a contradiction. Therefore, $a_{i+1} \notin Y^b$ for some $a_i \varepsilon Y^b$.

Let $Y_b = \{a_i : b \precsim a_i\}$ for some $b \varepsilon X_a$. Then a similar analysis of the preceding paragraph gives that $a_{i-1} \notin Y_b$ for some $a_i \varepsilon Y_b$. Hence every strictly bounded standard sequence with respect to a is finite.

(5) This easily follows since A is not null. [Q.E.D.]

Proof of Lemma 3: Suppose that $A \subset S$ is not null. First we construct two real valued functions, ϕ and ψ, on X such that for all $x,y,z,w \varepsilon X$ with $x \precsim y$ and $z \precsim w$,

(i) $xAy \precsim zAw$ iff $\phi(x) + \psi(y) \leq \phi(z) + \psi(w)$.

(ii) if ϕ' and ψ' satisfy (i) instead of ϕ and ψ, respectively, then there exist constants, $\alpha > 0$, β_1, and β_2 such that

$$\phi'(x) = \alpha\phi(x) + \beta_1 \text{ for all } x \varepsilon X \backslash X_{max},$$
$$\psi'(x) = \alpha\psi(x) + \beta_2 \text{ for all } x \varepsilon X \backslash X_{min}.$$

Lemma 2 implies that for $a \varepsilon X^*$, the triple $\langle X^a, X_a, \precsim^A \rangle$ is an additive conjoint structure. Thus by Theorem 2 of Chapter 6 in Krantz, et al. (1971), there exist two functions, ϕ_a and ψ_a, on X^a and X_a, respectively such that for all $x,y \in X^a$ and all $z,w \in X_a$,

$$xAz \precsim yAw \text{ iff } \phi_a(x) + \psi_a(z) \leq \phi_a(y) + \psi_a(w).$$

If $a \precsim b$ for $a,b \in X^*$, then $X^a \times X_b \subseteq X^a \times X_a$ and $X^a \times X_b \subseteq X^b \times X_b$. Thus for all $x,y \in X^a$ and all $z,w \in X_b$,

$$\begin{aligned} xAz \precsim yAw &\text{ iff } \phi_a(x) + \psi_a(z) \leq \phi_a(y) + \psi_a(w) \\ &\text{ iff } \phi_b(x) + \psi_b(z) \leq \phi_b(y) + \psi_b(w). \end{aligned}$$

By the uniqueness of additive representations, there are real valued functions $k(a, b) > 0$, $k_1(a, b)$, and $k_2(a, b)$ for all $a,b \in X^*$ with $a \precsim b$ such that

$$\begin{aligned} \phi_a(x) &= k(a, b)\phi_b(x) + k_1(a, b) \text{ for all } x \in X^a, \\ \psi_a(x) &= k(a, b)\psi_b(x) + k_2(a, b) \text{ for all } x \in X_b. \end{aligned}$$

Given ϕ_a and ψ_a for a fixed $a \in X^*$, scale ϕ_b and ψ_b for $b \in X_a \setminus X_{max}$, and ϕ_c and ψ_c for $c \in X^a \setminus X_{min}$ such that $k(a, b) = k(c, a) = 1$ and $k_i(a, b) = k_i(c, a) = 0$ for $i = 1, 2$. Then it easily follows that if $b \precsim c$ and $b,c \in X^*$ then $\phi_b = \phi_c$ on X^b, and $\psi_b = \psi_c$ on X_c. Therefore, for all $x \in X$, define

$$\begin{aligned} \phi(x) &= \phi_a(x) \text{ if } x \precsim a \text{ for some } a \in X^*, \\ \psi(x) &= \psi_a(x) \text{ if } a \precsim x \text{ for some } a \in X^*, \end{aligned}$$

so ϕ on $X\backslash X_{max}$, and ψ on $X\backslash X_{min}$ are uniquely specified. With ϕ and ψ thus defined, let

$$\phi(y) = \sup\{\phi(x): x\varepsilon X\backslash X_{max}\} \quad \text{for } y\varepsilon X_{max},$$
$$\psi(y) = \inf\{\psi(x): x\varepsilon X\backslash X_{min}\} \quad \text{for } y\varepsilon X_{min}.$$

We are to show that ϕ and ψ satisfy (i) and (ii). Suppose that $x \precsim y$ and $z \precsim w$. First, assume that $z \precsim y$ and $x \precsim w$. Then $\{x, z\} \precsim a \precsim \{y, w\}$ for some $a\varepsilon X$. If $a\varepsilon X^*$, then by Lemma 2, and the constructions of ϕ and ψ,

$$xAy \precsim zAw \text{ iff } \phi_a(x) + \psi_a(y) \leq \phi_a(z) + \psi_a(w)$$
$$\text{iff } \phi(x) + \psi(y) \leq \phi(z) + \psi(w).$$

It easily follows from A3, and the constructions of ϕ and ψ that for all $b,c\varepsilon X$, $b \precsim c$ iff $\phi(b) \leq \phi(c)$, and $b \precsim c$ iff $\psi(b) \leq \psi(c)$. Thus if $a\varepsilon X_{min}$, then $a \sim \{x, z\}$, so we obtain

$$
\begin{aligned}
xAy \precsim zAw &\text{ iff } aAy \precsim aAw && \text{(by A1 and Lemma 1(2))}\\
&\text{ iff } y \precsim w && \text{(by A3)}\\
&\text{ iff } \psi(y) \leq \psi(w)\\
&\text{ iff } \phi(a) + \psi(y) \leq \phi(a) + \psi(w)\\
&\text{ iff } \phi(x) + \psi(y) \leq \phi(z) + \psi(w).
\end{aligned}
$$

If $a\varepsilon X_{max}$, then (i) similarly follows. Next assume that either $y \prec z$

or $w \prec x$. Then (i) easily follows from A1, A3, and the preceding paragraph. The constructions of ϕ and ψ immediately give (ii).

Suppose that ϕ and ψ satisfy (i) with $\phi(y) = \sup \{\phi(x): x \epsilon X^*\}$ for $y \epsilon X_{max}$, and $\psi(y) = \inf \{\psi(x): x \epsilon X^*\}$ for $y \epsilon X_{min}$. Then we show that there are constants, $\beta > 0$ and γ such that $\psi(x) = \beta\phi(x) + \gamma$ for all $x \epsilon X$. Define

$$\phi_{1a}(x) = \phi(m_A(a, x)), \quad \psi_{1a}(x) = \psi(m_A(a, x)) \text{ for } x \epsilon X_a \text{ if } a \notin X_{max};$$
$$\phi_{2a}(x) = \phi(m_A(x, a)), \quad \psi_{2a}(x) = \psi(m_A(x, a)) \text{ for } x \epsilon X^a \text{ if } a \notin X_{min}.$$

For $x,y,z,w \epsilon X_a$ and $a \notin X_{max}$, if $x \precsim y$ and $z \precsim w$, then by A3, $m_A(a, x) \precsim m_A(a, y)$ and $m_A(a, z) \precsim m_A(a, w)$, so

$$\begin{aligned}
&\phi_{1a}(x) + \psi_{1a}(y) \leq \phi_{1a}(z) + \psi_{1a}(w) \\
&\quad \text{iff } \phi(m_A(a, x)) + \psi(m_A(a, y)) \leq \phi(m_A(a, z)) + \psi(m_A(a, w)) \\
&\quad \text{iff } f_A(m_A(a, x), m_A(a, y)) \precsim f_A((a, z), m_A(a, w)) \qquad \text{(by (i))} \\
&\quad \text{iff } f_A(m_A(a, a), m_A(x, y)) \precsim f_A(m_A(a, a), m_A(z, w)) \quad \text{(by A1 and A4)} \\
&\quad \text{iff } m_A(x, y) \precsim m_A(z, w) \qquad \text{(by A3)} \\
&\quad \text{iff } xAy \precsim zAw.
\end{aligned}$$

If $x \precsim y$ and $z \precsim w$ for $x,y,z,w \epsilon X^a$ and $a \notin X_{min}$, then similarly we get

$$xAy \precsim zAw \text{ iff } \phi_{2a}(x) + \psi_{2a}(y) \leq \phi_{2a}(z) + \psi_{2a}(w).$$

Thus ϕ_{ia} and ψ_{ia} satisfy (i) on X_a for $i = 1$, and on X^a for $i = 2$.

Therefore, there are real valued functions, $k_i > 0$ and k_{1j} for $i = 1, 2$ such that for $a \notin X_{max}$ and $b \notin X_{min}$,

$$\phi(m_A(a, x)) = k_1(a)\phi(x) + k_{11}(a) \quad \text{for } x \in X_a \setminus X_{max},$$
$$\psi(m_A(x, b)) = k_2(b)\psi(x) + k_{12}(b) \quad \text{for } x \in X^b \setminus X_{min}.$$

Suppose that $x \notin X_{max}$, $w \notin X_{min}$, $y, z \in X^*$, and $x \precsim \{y, z\} \precsim w$. Then by A1 and A3, $m_A(x, y) \precsim m_A(z, w)$ and $m_A(x, z) \precsim m_A(y, w)$. It follows from A4, (i), and the preceding paragraph that

$$\begin{aligned}
&f_A(m_A(x, y), m_A(z, w)) \sim f_A(m_A(x, z), m_A(y, w)) \\
&\quad \text{iff } \phi(m_A(x, y)) + \psi(m_A(z, w)) = \phi(m_A(x, z)) + \psi(m_A(y, w)) \\
&\quad \text{iff } k_1(x)\phi(y) + k_{11}(x) + k_2(w)\psi(z) + k_{22}(w) \\
&\qquad\qquad = k_1(x)\phi(z) + k_{11}(x) + k_2(w)\psi(y) + k_{22}(w) \\
&\quad \text{iff } k_1(x)(\phi(y) - \phi(z)) = k_2(w)(\psi(y) - \psi(z)),
\end{aligned}$$

so there are constants, λ and δ, such that for all $x \notin X_{max}$ and all $w \notin X_{min}$,

$$k_1(x) = \lambda > 0 \text{ and } k_2(w) = \delta > 0.$$

Thus $\lambda(\phi(y) - \phi(z)) = \delta(\psi(y) - \psi(z))$, so $\delta\psi(y) - \lambda\phi(y) = \delta\psi(z) - \lambda\phi(z)$ for all $y, z \in X^*$. Therefore, there are constants, $\beta > 0$ and γ such that $\psi(x) = \beta\phi(x) + \gamma$ for all $x \in X^*$.

Suppose that $xAy \sim z_x$ for $x \in X^*$ and $y \in X_{max}$. Then by (i), $\phi(x) +$

$\psi(y) = \phi(z_x) + \psi(z_x)$. Since $z_x \varepsilon X^*$, $\sup_x \phi(x) = \sup_x \phi(z_x)$, and $\sup_x \psi(x) = \sup_x \psi(z_x)$, we have $\psi(y) = \sup_x \psi(x)$. Thus

$$\begin{aligned}\psi(y) &= \sup_x \psi(x) \\ &= \beta \sup_x \phi(x) + \gamma \\ &= \beta\phi(y) + \gamma,\end{aligned}$$

since $\phi(y) = \sup_x \phi(x)$. Similarly, $\psi(y) = \beta\phi(y) + \gamma$ for $y\varepsilon X_{min}$. Hence $\psi(x) = \beta\phi(x) + \gamma$ for all $x\varepsilon X$.

With $\beta > 0$ thus obtained, let $\alpha = 1/(1+\beta)$. Define $u(x) = \phi(x)/\alpha$ for all $x\varepsilon X$. Then the representation of the lemma easily follows. To show the uniqueness of u, suppose that u' satisfies the representation also. By the uniqueness of ϕ and ψ, it easily follows that there are constants, $\beta > 0$ and γ such that $u'(x) = \beta u(x) + \gamma$ for all $x\varepsilon X^*$. Suppose that $xAy \sim z$ for $x\varepsilon X_{min}$ and $y\varepsilon X^*$. Then $z\varepsilon X^*$. we obtain

$$\alpha u(x) + (1-\alpha)u(y) = u(z),$$

$$\alpha u'(x) + (1-\alpha)u'(y) = u'(z),$$

so $u'(x) = \beta u(x) + \gamma$. Similarly, $u'(x) = \beta u(x) + \gamma$ for $x\varepsilon X_{max}$. Hence the uniqueness part of the lemma obtains. [Q.E.D.]

<u>Proof of Proposition</u> 1: For nonnull $A \subset S$, Lemma 3 implies that there is a real valued function u_A on X such that for some $0 < \alpha_A < 1$ and for all $x,y,z,w\varepsilon X$ with $x \precsim y$ and $z \precsim w$,

$$xAy \precsim zAw \text{ iff } \alpha_A u_A(x) + (1-\alpha_A)u_A(y) \leq \alpha_A u_A(z) + (1-\alpha_A)u_A(w).$$

Define $\pi(A) = \alpha_A$ for nonnull $A \subset S$, $\pi(S) = 1$, and $\pi(A) = 0$ for null $A \varepsilon 2^S$.

Suppose that $A, B \subset S$ are not null. Then we show that $u_A = \alpha u_B + \beta$ for some constants, $\alpha > 0$ and β. Given $a \varepsilon X^*$, a binary relation $\precsim^a$ on $X^a \times X_a$ is defined as follows: for $xy, zw \varepsilon X^a \times X_a$,

$$xy \precsim^a zw \text{ iff } f_A(m_B(x, a), m_B(a, y)) \precsim f_A(m_B(z, a), m_B(a, w)).$$

By A1 and A4,

$$xy \precsim^a zw \text{ iff } f_B(m_A(x, a), m_A(a, y)) \precsim f_B(m_A(z, a), m_A(a, w)).$$

Since by A1 and Lemma 1(1), $m_C(x, a) \precsim m_C(a, y)$ and $m_C(z, a) \precsim m_C(a, w)$ for $C = A$ or B, Lemma 3 implies that

$$\begin{aligned} xy \precsim^a zw \text{ iff } & \pi(A)u_A(m_B(x, a)) + (1-\pi(A))u_A(m_B(a, y)) \\ & \leq \pi(A)u_A(m_B(z, a)) + (1-\pi(A))u_A(m_B(a, w)) \\ \text{iff } & \pi(B)u_B(m_A(x, a)) + (1-\pi(B))u_B(m_A(a, y)) \\ & \leq \pi(B)u_B(m_A(z, a)) + (1-\pi(B))u_B(m_A(a, w)). \end{aligned}$$

By the uniqueness of additive representations, there are real valued functions, $\omega > 0$ and σ, on X^* such that for all $a \varepsilon X^*$ and all $x \varepsilon X^a$,

$$\pi(A)u_A(m_B(x, a)) = \omega(a)\pi(B)u_B(m_A(x, a)) + \sigma(a).$$

Since $m_A(x, y) \varepsilon X^a$ for $x, y \varepsilon X^a$, it follows from A4, Lemma 3, and the above equation that for $x, y \varepsilon X^a$ with $x \precsim y$,

$$\begin{aligned}
&\pi(A)u_A(m_B(m_A(x, y), m_A(a, a))) \\
&\quad = \pi(A)u_A(m_A(m_B(x, a), m_B(y, a))) \\
&\quad = \pi(A)^2 u_A(m_B(x, a)) + (1-\pi(A))\pi(A)u_A(m_B(y, a)) \\
&\quad = \omega(a)\pi(B)\{\pi(A)u_B(m_A(x, a)) + (1-\pi(A))u_B(m_A(y, a))\} + \sigma(a).
\end{aligned}$$

On the other hand, by the preceding paragraph, Lemma 1(2), and A4,

$$\begin{aligned}
&\pi(A)u_A(m_B(m_A(x, y), m_A(a, a))) \\
&\quad = \pi(A)u_A(m_B(m_A(x, y), a)) \\
&\quad = \omega(a)\pi(B)u_B(m_A(m_A(x, y), a)) + \sigma(a) \\
&\quad = \omega(a)\pi(B)u_B(m_A(m_A(x, y), m_A(a, a)) + \sigma(a) \\
&\quad = \omega(a)\pi(B)u_B(m_A(m_A(x, a), m_A(y, a))) + \sigma(a).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&u_B(m_A(m_A(x, a), m_A(y, a))) \\
&\qquad = \pi(A)u_B(m_A(x, a)) + (1-\pi(A))u_B(m_A(y, a)).
\end{aligned}$$

Thus for all $x, y \varepsilon X^*$ with $x \precsim y$,

$$u_B(m_A(x, y)) = \pi(A)u_B(x) + (1-\pi(A))u_B(y).$$

If either $x, y \varepsilon X_{min}$ or $x, y \varepsilon X_{max}$, then the above equation easily follows, so we show that it also holds when $x \in X_{min}$ or $y \in X_{max}$. Suppose that $x \in X_{min}$ and $y \in X^*$. When $x \in X^*$ and $y \in X_{max}$, the proof is similar. Then $y \precsim z$ for some $z \in X^*$. Since $m_C(x, y)$ and $m_C(y, z)$ are in X^* for C = A or B, Lemma 3 and the preceding paragraphs imply that

$$\begin{aligned}
&u_B(m_B(m_A(x, y), m_A(y, z)) \\
&\quad = \pi(B)u_B(m_A(x, y)) + (1-\pi(B))u_B(m_A(y, z)) \\
&\quad = \pi(B)u_B(m_A(x, y)) + (1-\pi(B))(\pi(A)u_B(y) + (1-\pi(A))u_B(z)), \\
&u_B(m_A(m_B(x, y), m_B(y, z)) \\
&\quad = \pi(A)u_B(m_B(x, y)) + (1-\pi(A))u_B(m_B(y, z)) \\
&\quad = \pi(A)(\pi(B)u_B(x) + (1-\pi(B))u_B(y)) \\
&\qquad\quad + (1-\pi(A))(\pi(B)u_B(y) + (1-\pi(B))u_B(z)).
\end{aligned}$$

By A1 and A4, $m_B(m_A(x, y), m_A(y, z)) \sim m_A(m_B((x, y), m_B(y, z))$, so by Lemma 3, $u_B(m_B(m_A(x, y), m_A(y, z))) = u_B(m_A(m_B(x, y), m_B(y, z)))$. Thus $u_B(m_A(x, y)) = \pi(A)u_B(x) + (1-\pi(A))u_B(y)$.

Suppose next that $x \in X_{min}$ and $y \in X_{max}$. Applying the results in the preceding paragraph, the desired result similarly obtains. Hence by the uniqueness of u_A and u_B, we must have $u_A = \alpha u_B + \beta$ for some constants $\alpha > 0$ and β.

Under appropriate positive linear transformations, there is a real valued function u on X such that $u = u_A$ for all nonnull $A \subset S$. Noting that $xAy \sim y$ if A is null, and that $xSy \sim x$, it follows from the preceding paragraphs that for all $A \in 2^S$,

$$u(m_A(x, y)) = \pi(A)u(x) + (1-\pi(A))u(y).$$

Thus for all $x,y,z,w \in X$ with $x \precsim y$ and $z \precsim w$,

$$xAy \precsim zBw \text{ iff } m_A(x, y) \precsim m_B(z, w)$$
$$\text{iff } u(m_A(x, y)) \leq u(m_B(z, w))$$
$$\text{iff } \pi(A)u(x) + (1-\pi(A))u(y) \leq \pi(B)u(z) + (1-\pi(B))u(w),$$

so the representation of the proposition holds.

Next we show that $A \subseteq B$ implies $\pi(A) \leq \pi(B)$. Suppose $A \subseteq B$. Let $C = B \setminus A$, $f =_A x$, and $f =_{B^c} y$ with $x \prec y$. Since xCf and yCf are comonotonic, A3 implies that $xCf \prec yCf$ if C is not null; $xCf \sim yCf$ if C is null. Thus we have $xBy \precsim xAy$, so

$$\pi(B)u(x) + (1-\pi(B))u(y) \leq \pi(A)u(x) + (1-\pi(A))u(y).$$

Hence $\pi(A) \leq \pi(B)$.

The uniqueness of π and u follows from the construction of u, and Lemma 3. [Q.E.D.]

5. PROOFS OF THE THEOREMS

Throughout the section let u on X, and π on 2^S be real valued functions obtained in Proposition 1. We extend u on X to on F as follows:

$$u(f) = u(x) \quad \text{when } f =_S x;$$
$$u(f) = u(m(f)) \quad \text{for all } f \in F.$$

Thus by A1, for all $f, g \varepsilon F$, $f \prec g$ iff $u(f) < u(g)$.

Proof of Theorem 1: We are to show that $u(f) = \int_S u(f(s))d\pi(s)$, so the conclusion of the Theorem obtains. Let $P = \{A_1, \ldots, A_n\}$ be an n-partition, and

$$f_n =_{A_i} x_i \text{ for } i = 1, \ldots, n,$$

where $x_1 \precsim \ldots \precsim x_n$ for $x_i \varepsilon X$. Also let B_0 be empty, and $B_i = \sum_{j=1}^{i} A_j$, so $B_n = S$. For all $f_2 \varepsilon F$, Proposition 1 gives the desired result. Thus we assume $n > 2$. Suppose that the conclusion is true for all n-partitions with $n < k$. Then it suffices to show that for $n = k$,

$$u(f_k) = \sum_{i=1}^{k} (\pi(B_i) - \pi(B_{i-1}))u(x_i),$$

since this easily leads to the definition of the dual Choquet integration.

We have the following three cases to examine:

CASE 1: $xAx_2 \sim x_1$ and $x_{k-1}Ay \sim x_k$ for some $x, y \varepsilon X$ and a nonnull $A \subset S$.

CASE 2: either not($xAx_2 \sim x_1$) for all $x \varepsilon X$ and all $A \subset S$
or not($x_{k-1}Ay \sim x_k$) for all $y \varepsilon X$ and all $A \subset S$.

CASE 3: not($xAx_2 \sim x_1$) and not($x_{k-1}Ay \sim x_k$) for all $x, y \varepsilon X$ and all $A \subset S$.

Let $P' = \{A_1, \ldots, A_{k-2}, A_{k-1}+A_k\}$ and $P'' = \{A_1+A_2, A_3, \ldots, A_k\}$, so they are (k-1)-partitions.

<u>CASE</u> 1: Since $xAx_2 \sim x_1$ and $x_{k-1}Ay \sim x_k$ for a nonnull $A \subset S$, Lemma 1(1) implies that $x \precsim x_1$ and $x_k \precsim y$. It follows from A1, A4, and Lemma 1(2) that

$$\begin{aligned}
f_k &= f_P(x_1, \ldots, x_k) \\
&\sim f_P(m_A(x, x_2), m_A(x_2, x_2), \ldots, m_A(x_{k-1}, x_{k-1}), m_A(x_{k-1}, y)) \\
&\sim f_A(m_P(x, x_2, \ldots, x_{k-2}, x_{k-1}, x_{k-1}), m_P(x_2, x_2, x_3, \ldots, x_{k-1}, y)) \\
&\sim f_A(m_{P'}(x, x_2, \ldots, x_{k-1}), m_{P''}(x_2, \ldots, x_{k-1}, y)),
\end{aligned}$$

so $f_k \sim m_A(m_{P'}(x, x_2, \ldots, x_{k-1}), m_{P''}(x_2, \ldots, x_{k-1}, y))$. Since $xAx_2 \sim x_1$ and $x_{k-1}Ay \sim x_k$, Proposition 1 implies that $u(x_1) = \pi(A)u(x) + (1-\pi(A))u(x_2)$ and $u(x_k) = \pi(A)u(x_{k-1}) + (1-\pi(A))u(y)$. Thus by the definition of $u(f_k)$, and the hypothesis of the induction, we get

$$\begin{aligned}
u(f_k) &= u(m_A(m_{P'}(x, x_2, \ldots, x_{k-1}), m_{P''}(x_2, \ldots, x_{k-1}, y)) \\
&= \pi(A)u(m_{P'}(x, x_2, \ldots, x_{k-1})) \\
&\qquad + (1-\pi(A))u(m_{P''}(x_2, \ldots, x_{k-1}, y)) \\
&= \pi(A)\{\pi(B_1)u(x) + \textstyle\sum_{i=2}^{k-2}(\pi(B_i) - \pi(B_{i-1}))u(x_i) \\
&\qquad + (1-\pi(B_{k-2}))u(x_{k-1})\} + (1-\pi(A))\{\pi(B_2)u(x_2) \\
&\qquad + \textstyle\sum_{i=3}^{k-1}(\pi(B_i) - \pi(B_{i-1}))u(x_i) + (1-\pi(B_{k-1}))u(y)\} \\
&= \textstyle\sum_{i=1}^{k}(\pi(B_i) - \pi(B_{i-1}))u(x_i).
\end{aligned}$$

<u>CASE</u> 2: Suppose that not$(xAx_2 \sim x_1)$ for all $x \in X$ and all $A \subset S$, and $x_{k-1}By \sim x_k$ for some $y \in X$ and a nonnull $B \subset S$. A similar proof applies when not$(x_{k-1}Ay \sim x_k)$ for all $y \in X$ and all $A \subset S$, and $xBx_2 \sim x_1$ for some $x \in X$ and a nonnull $B \subset S$. Let $z \sim x_1Ax_2$ for some $A \subset S$, so $x_1 \precsim z \precsim x_2$.

Then by Lemma 1(2) amd A4,

$$f_P(z, x_2,\ldots, x_k) \sim f_P(m_A(x_1, x_2), m_A(x_2, x_2),\ldots, m_A(x_k, x_k))$$
$$\sim f_A(m_P(x_1,\ldots, x_k), m_P(x_2, x_2, x_3,\ldots, x_k))$$
$$\sim f_A(m_P(x_1,\ldots, x_k), m_{P''}(x_2,\ldots, x_k)).$$

Since $f_k \sim m_P(x_1,\ldots, x_k)$, Proposition 1 implies

$$u(m_P(z, x_2,\ldots, x_k)) = \pi(A)u(f_k) + (1-\pi(A))u(m_{P''}(x_2,\ldots, x_k)).$$

Noting that $u(m_P(z, x_2,\ldots, x_k)) = \pi(B_1)u(z) + \sum_{i=2}^{k}(\pi(B_i)-\pi(B_{i-1}))u(x_i)$ by Case 1, and $u(z) = \pi(A)u(x_1) + (1-\pi(A))u(x_2)$ by Proposition 1, the above equation is rearranged to give

$$\pi(A)u(f_k) = \pi(B_1)u(z) + \textstyle\sum_{i=2}^{k}(\pi(B_i) - \pi(B_{i-1}))u(x_i)$$
$$- (1-\pi(A))\{\pi(B_2)u(x_2) + \textstyle\sum_{i=3}^{k}(\pi(B_i) - \pi(B_{i-1}))u(x_i)\}$$
$$= \pi(A)\{\pi(B_1)u(x_1) + \textstyle\sum_{i=2}^{k}(\pi(B_i) - \pi(B_{i-1}))u(x_i)\}.$$

Hence the desired result obtains.

<u>CASE</u> 3: Let $z \sim x_1Ax_2$ for some $A \subset S$, so $x_1 \precsim z \precsim x_2$. Then applying the result of Case 2, a similar analysis of Case 2 gives the desired result. [Q.E.D.]

<u>Proof of Theorem</u> 2: Let u and π be real valued functions that satisfy Theorem 1.

(1) Suppose that A5 holds. First we show that for nonnull $A \subset S$, the triple $\langle X, m_A, \precsim \rangle$ is a bisymmetric structure (Krantz, et al. (1970, Chapet 6)) which satisfies the following five conditions: for all $x,y,z,w \in X$,

(i) $\precsim$ on X is an asymmetric weak order.

(ii) $x \precsim y$ iff $m_A(x, z) \precsim m_A(y, z)$;

$x \precsim y$ iff $m_A(z, x) \precsim m_A(z, y)$.

(iii) $m_A(m_A(x, y), m_A(z, w)) \sim m_A(m_A(x, z), m_A(y, w))$.

(iv) If $m_A(x, z) \precsim w \precsim m_A(y, z)$ then $m_A(a, z) \sim w$ for some $a \in X$;

if $m_A(z, x) \precsim w \precsim m_A(z, y)$ then $m_A(z, a) \sim a$ for some $a \in X$.

(v) Every strictly standard sequence is finite, where $\{a_i: a_i \in X, i \in N\}$ is a standard sequence iff there exist $a,b \in X$ such that not$(a \sim b)$ and either $m_A(a_i, a) \sim m_A(a_{i+1}, b)$ for all $i,i+1 \in N$ or $m_A(a, a_i) \sim m_A(b, a_{i+1})$ for all $i,i+1 \in N$.

(i), (ii), and (iii) easily follow from A1, A5, and Lemma 1(5). Noting Lemma 1(5), the proofs of (iv) and (v) are similar to the proofs of Lemmas 2(3) and 2(4), respectively.

Since $xAx \sim x$, and $\langle X, m_A, \precsim \rangle$ is the bisymmetric structure, it follows from Theorem 10 of Chapter 6 in Krantz, et al. (1970) that there exists a real valued function v on X such that for all $x,y \in X$ and some constant $0 < \alpha < 1$,

$$x \precsim y \text{ iff } v(x) \leq v(y);$$

$$v(m_A(x, y)) = \alpha v(x) + (1-\alpha)v(y),$$

where α is unique, and v is unique up to a positive linear transformation. Then by the uniqueness of u and π in Theorem 1, we must have $\pi(A) = \alpha$ and $u = \beta v + \gamma$ for some constants, $\beta > 0$ and γ. Hence, for all $x,y \in X$ and all $A \in 2^S$,

$$u(xAy) = \pi(A)u(x) + (1-\pi(A))u(y),$$
$$u(yA^c x) = \pi(A^c)u(y) + (1-\pi(A^c))u(x).$$

Since $xAy \sim yA^c x$, we get $u(xAy) = u(yA^c x)$, so $\pi(A) + \pi(A^c) = 1$. Thus π is complementarily additive.

(2) Suppose that $A4^*$ holds. Note that $A4^*$ implies A4. To show additivity of π, it suffices to prove that if $A \subset B$ then $\pi(B) = \pi(A) + \pi(B \backslash A)$. For $x,y \in X$ with $x \prec y$, let $z \sim xCy$ for some nonnull $C \subset S$, and also let $w \sim zCx$, so $x \prec w \prec z \prec y$. Let $P = \{A, B\backslash A, B^c\}$ be 3-partition of S. Then by $A4^*$ and Lemma 1(2),

$$\begin{aligned} f_P(x, w, z) &\sim f_P(m_C(x, x), m_C(z, x), m_C(x, y)) \\ &\sim f_C(m_P(x, z, x), m_C(x, x, y)) \\ &\sim f_C(m_{B\backslash A}(z, x), m_B(x, y)). \end{aligned}$$

By Theorem 1,

$$u(f_P(x, w, z)) = \pi(A)u(x) + (\pi(B) - \pi(B\backslash A))u(w) + (1-\pi(B))u(z).$$

On the other hand, (1) implies

$$u(f_C(m_{B\setminus A}(z, x), m_B(x, y))$$
$$= \pi(C)u(m_{B\setminus A}(z, x)) + (1-\pi(C))u(m_B(x, y))$$
$$= \pi(C)\{\pi(B\setminus A)u(z) + (1-\pi(B\setminus A))u(x)\}$$
$$+ (1-\pi(C))\{\pi(B)u(x) + (1-\pi(B))u(y)\}$$
$$= \pi(B\setminus A)\{\pi(C)u(z) + (1-\pi(C))u(x)\} + (\pi(B) - \pi(B\setminus A))u(x)$$
$$+ (1-\pi(B))\{\pi(C)u(x) + (1-\pi(C))u(y)\}$$
$$= \pi(B\setminus A)u(w) + (\pi(B) - \pi(B\setminus A))u(x) + (1-\pi(B))u(z),$$

since by (1), $u(z) = \pi(C)u(x) + (1-\pi(C))u(y)$ and $u(w) = \pi(C)u(z) + (1-\pi(C))u(x)$.

Since $u(f_P(x, w, z)) = u(f_C(m_{B\setminus A}(z, x), m_B(x, y))$, we obtain

$$\pi(A)u(x) + (\pi(B) - \pi(B\setminus A))u(w) + (1-\pi(B))u(z)$$
$$= \pi(B\setminus A)u(w) + (\pi(B) - \pi(B\setminus A))u(x) + (1-\pi(B))u(z),$$

so $(\pi(A) - \pi(B) + \pi(B\setminus A))(u(x) - u(w)) = 0$. Hence, $\pi(B) = \pi(A) + \pi(B\setminus A)$. [Q.E.D.]

6. CONCLUSIONS

The purpose of this paper has been to axiomatize a generalization of the subjective expected utility under Savage's formulation when the set of states is finite. Our axiomatization required that the set X of consequences was infinite in contrast to Savage's arbitrary X. A genelerized representational form yielded a utility function on X, and a monotonic probability measure over states, and adopted Choquet integration or its dual as in Schmeidler-Gilboa's representation to account for non-additive probability measures.

The four axioms were shown to be sufficient for the representation, where the forth axiom was crucial in the finite state formulation. Also, we examined two additional axioms that gave comlementary additive or additive probability measures over states, respectively.

REFERENCES

M. Allais, Le comportement de l'homme rationel devant le risque: Critique des postulats at axioms de l'ecole americaine, Econometrica, 21 (1953), 503-546.

F.J. Anscombe and R.J. Aumann, A definition of subjective probability, Annals of Math. Statist., 34 (1963), 199-205.

S.H. Chew, An axiomatization of the rank dependent quasilinear mean generalizing the Gini mean and the quasilinear mean, Preprint, Dept. of Political Econ., Johns Hopkins Univ., Baltimore, 1984.

G. Choquet, Theory of capacities, Ann. Inst. Fourier, 5 (1955), 131-295.

D. Davidson, P. Suppes, and S. Siegel, "Decision Making: An Experimental Approach," Stanford Univ. Press, Stanford, CA, 1957.

D. Ellsberg, Risk, ambiguity and the Savage axioms. Quart. J. Econ., 75 (1961), 643-669.

P.C. Fishburn, Subjevtive expected utility: a review of normative theories, Theory and Decision, 13 (1981), 139-199.

P.C. Fishburn, "Nonlinear Preference and Utility Theory," Johns Hoppkins University Press, 1988.

I. Gilboa, Expected utility with purely subjective non-additive probabilities, J. of Math. Econ., 16 (1987), 65-88.

D.M. Grether and C.R. Plott, Economic theory of choice and the preference reversal phenomenon, American Econ. Rev., 69 (1979), 623-638.

D. Kahneman and A. Tversky, Prospect theory: an analysis of decision under risk, Econometrica, 47 (1979), 263-291.

D.H. Krantz, R.D. Luce, P. Suppes, and A. Tversky, " Foundations of Measurement," vol. 1, Academic Press, New York, 1970.

J. Pfanzagl, "Theory of Measurement," Wiley, New York, 1968.

L.J. Savage, "The Foundations of Statistics," Wiley, New York, 1954. Second revised edition, Dover Publications, New York, 1972.

D. Schmeidler, Subjective probability and expected utility without additivity, Preprint 84, Institute for Mathematics and Its Applications, University of Minnesota, Minneapolis, 1984.

P. Slovic and A. Tversky, Who accepts Savage's axiom? Behavioral Sci., 19 (1974), 368-373.