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Approximated Formulas of Average Distances Associated with Regions and Their Applications to Location Problems

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1. Introduction

This study is concerned with a problem of measuring average distances between two coplanar regions. The objectives are : 1)to derive and check accuracies of the approximated average distances between two regions under conditions of uniform distributions ; and 2)to apply these approximated average distances to a formulation of an approximated objective functions of the Weber Problem, which is well known mathematical programming problem in a field of city planning.

In this study, we consider two coplanar geographical regions A and B, and let P and Q be two points on the respective regions and let $r(P,Q)$ be a distance between P and Q (Figure 1). In additions, let S_A and S_B be the areas of the respective regions. Throughout this paper, we assume that points P and Q are mutually independent and uniformlly distributed in each region. Then the average distance $\bar{r}(A,B)$ between two regions is defined by

$$\bar{r}(A,B) = \frac{1}{S_A S_B} \int_A \int_B r(P,Q)\, dP dQ. \tag{1}$$

Of course, the average distance Equation (1) depends on forms and locations of regions A and B. It is needless to say that it is desirable to reduce (1) into a simple functional form as posible, from the stand-point of engineers. Although we assume the conditions of uniform distributions mentioned above, these reductions cannot be made in general. If one of the regions has no area, the derivation of the average distance becomes easier. Without loss of generality, let B have no area and let it be a point Q (Figure 2). Then the average distance $\bar{r}(A,Q)$ between region A and point Q is as follows:

$$\bar{r}(A,Q) = \frac{1}{S_A}\int_A r(P,Q)dP. \qquad (2)$$

Schweitzer(1968) derived $\bar{r}(A,Q)$ for a circular disk A, when the point Q lies outside of the disk A. Koshizuka(1977) derived $\bar{r}(A,Q)$ for a circular disk A, when the point Q lies inside of the disk A. These two results relating to a circular disk are expressed by the complete elliptic integrals of the first and the second kind. Love(1972) calculated $\bar{r}(A,Q)$ for a rectangle A. This rectangle case is reduced to a complicated function including arc-hyperbolic functions. The cases of A above are the only three cases that $\bar{r}(A,Q)$ are derived. On the other hand, when A and B have areas, $\bar{r}(A,B)$ was obtained for two non-overlapping circular disks A and B [Bouwkamp,1978]. Bouwkamp shows that $\bar{r}(A,B)$ is reduced to Appell's hypergeometric function. Ghosh(1951) shows the derivation of probability density functions for two rectangle disks.

The results introduced above have complicated forms, and especially for the cases concerning circular disks, we always have to do a numerical calculation if we need a value of $\bar{r}(A,B)$ or $\bar{r}(A,Q)$, and for Love's rectangle case, Bennett and Mirakhor(1974) pointed out that calculating $\bar{r}(A,Q)$ is very time-consuming.

Vaughan(1984) obtained an approximated average distance between two circular disks which are not overlapping (Figure 3). Letting h be the inter-centroid distance between two circular disks A and B, which have radii of α and β respectively, Vaughan's approximated formula $\tilde{r}(A,B)$ of $\bar{r}(A,B)$ is expressed as

$$\tilde{r}(A,B) = h + \frac{\alpha^2+\beta^2}{8h} \qquad \text{if} \qquad h \geq \alpha+\beta. \qquad (3)$$

Equation (3) is to be referred later in this paper. Vaughan also obtained the approximated formulas for average disntance $\bar{r}(A,B)$ when A and B are disjoint polygons, by use of the generating function of Legendre polynomial. Vaughan did not check its accuracy for real regions in cities. We checked its accuracies for pairs from 23 special wards in Tokyo[Kurita and Koshizuka,1988]. Our results show that Vaughan's formulas are very accurate and useful. However, there is a weakness in Vaughan's method not to be overlooked. The weakness is that Vaughan's approximation is not accurate when A and B or A and Q are not disjoint. If we regard the point Q as a facility and a region A as its attracting region, we can use Vaughan's approximation only when the facility Q is outside of the region A. We claim that if the regions are circular disks, the accurate approximated formulas of $\bar{r}(A,B)$ can be obtained.

First of all we will show how to derive the approximation for circular disks A and B in the next section. Then we will use the results to approximate the objective function of the Weber Problem.

2. Circular Disks

Suppose a region A to be a circular disk of radius α and its centroid to be a point O. Q is a point which lies inside or outside of disk A. Let h be the distance between O and Q, and let ς be h/α (Figure 4). The average distance $\bar{r}(A,Q)$ between A and Q is derived as follows[Koshizuka and Kurita,1983]:

$$\bar{r}(A,Q) = \frac{4\alpha}{9\pi}[(7+\varsigma^2)E(\varsigma) - 4(1-\varsigma^2)K(\varsigma)] \qquad \text{if} \qquad \varsigma \leq 1; \tag{4}$$

$$\bar{r}(A,Q) = \frac{4h}{9\pi}[(7+\varsigma^2)E(\frac{1}{\varsigma}) - (2+\varsigma^2-\frac{3}{\varsigma^2})K(\frac{1}{\varsigma})] \qquad \text{if} \qquad \varsigma \geq 1. \tag{5}$$

In the equations (4) and (5), $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of the first and the second kind. (4) and (5) are for cases of $h \leq \alpha$ and $h \geq \alpha$ respectively. These complete elliptic integrals can be expressed by the hypergeometric functions as follows:

$$K(\varsigma) = \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2},1;\varsigma^2) = \frac{\pi}{2}{}_2F_1(\frac{1}{2},\frac{1}{2};1;\varsigma^2); \tag{6}$$

$$E(\varsigma) = \frac{\pi}{2}F(\frac{-1}{2},\frac{1}{2},1;\varsigma^2) = \frac{\pi}{2}{}_2F_1(\frac{-1}{2},\frac{1}{2};1;\varsigma^2). \tag{7}$$

In Equations (6) and (7), F is Gauss' hypergeometric function and ${}_2F_1$ is Pochhammer's generalized hypergeometric function:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{8}$$

$$F(a,b,c;z) = {}_2F_1(a,b;c;z),$$

where

$$(a)_0 = 1 \qquad \text{and}$$
$$(a)_n = a(a+1)(a+2)\ldots(a+n-1) \qquad \text{if} \qquad n \geq 2. \tag{9}$$

The right side of equation (8) converges if $|z| \leq 1$.

According to equations (8) and (9), (6) can be transformed into Equation (10).

$$K(\varsigma) = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{(2n-1)!!}{2^n n!}]\varsigma^{2n} = \frac{\pi}{2}[1+\frac{1}{4}\varsigma^2+\frac{9}{64}\varsigma^4+\frac{25}{256}\varsigma^6+\ldots]. \tag{10}$$

$E(\varsigma)$ can also be transformed into Equation (11).

$$E(\varsigma) = \frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{-1}{2n-1}\frac{(2n-1)!!}{2^n n!}]\varsigma^{2n} = \frac{\pi}{2}[1 - \frac{1}{4}\varsigma^2 - \frac{3}{64}\varsigma^4 - \frac{5}{256}\varsigma^6 - \ldots]. \qquad (11)$$

Substituting (10) and (11) into (4) and (5) yields expansion formulas of $\bar{r}(A,Q)$ by ς or $1/\varsigma$ as follows.

$$\bar{r}(A,Q) = \alpha[\frac{2}{3} + \frac{1}{2}\varsigma^2 + \frac{1}{32}\varsigma^4 + \ldots], \quad \text{if} \quad \varsigma \le 1, \qquad (12)$$

$$\bar{r}(A,Q) = h[1 + \frac{1}{8}\frac{1}{\varsigma^2} + \frac{1}{192}\frac{1}{\varsigma^4} + \ldots], \quad \text{if} \quad \varsigma \ge 1. \qquad (13)$$

We can obtain the approximation $\tilde{r}(A,Q)$ for $\bar{r}(A,Q)$ by neglecting high order terms of (12) and (13). We have chosen the first 2 terms, and made approximated formulas, Equation (14):

$$\tilde{r}(A,Q) = \begin{cases} \alpha[\frac{2}{3} + \frac{1}{2}\varsigma^2] = \frac{2}{3}\alpha + \frac{1}{2}\frac{h^2}{\alpha} & \text{if } h < \alpha, \\ h[1 + \frac{1}{8}\frac{1}{\varsigma^2}] = h + \frac{1}{8}\frac{\alpha^2}{h} & \text{if } h \ge \alpha. \end{cases} \qquad (14)$$

The formulas for $h \ge \alpha$ in Equation (14) has been derived in Vaughan(1984) too, but the formulas for $h < \alpha$ in Equation (14) cannot be derived by Vaughan's method. To check the relative bias of $\tilde{r}(A,Q)$ from $\bar{r}(A,Q)$, we can let the radius α of circular disk A be constant without loss of generality. With α set at the value of 1, we calculate the precise value of the average distance $\bar{r}(A,Q)$ for each h from $h = 0$ to $h = 2$ by increasing step 0.01. Let ϵ be the relative bias of $\tilde{r}(A,Q)$:

$$\epsilon(h) = \frac{\tilde{r}(A,Q) - \bar{r}(A,Q)}{\bar{r}(A,Q)}.$$

Figure 5 shows $\epsilon(h)$. Because $\tilde{r}(A,Q)$ of (14) has a discontinuous point at $h = \alpha$, $\epsilon(h)$ is discontinuous at $h = \alpha$. We can see that $\tilde{r}(A,Q)$ overapproximates $\bar{r}(A,Q)$ when $0 < h < \alpha$, and that $\tilde{r}(A,Q)$ underapproximates $\bar{r}(A,Q)$ when $\alpha \le h$. And the absolute value of the relative bias of $\tilde{r}(A,Q)$ has a maximum value of about 3% at $h = \alpha$. We conclude that the approximated formulas (14) fit very well.

At the next stage we consider the case of two circular disks. If two circular disks are not overlapping(which means disjoint), an approximated average distance between them was derived as Equation (3) by Vaughan(1984). However, we cannot derive the approximated formula by Vaughan's method if two disks are not disjoint. Below we will show how to derive the average distance when one circular disk completely includes the other. When two circular disks are partially overlapping the derivation of the approximate formula remains to be an open problem.

First, we consider a case where one circular disk A completely includes the other circular disk B in it (Figure 6). We can calculate the approximated average distance between them on the basis of the expansion (12) of $\bar{r}(A,Q)$. Consider a point Q on the included disk B and let h' be the distance between the point Q and the centroid of the larger disk A. The average distance $\bar{r}(A,Q)$ is then expressed as Equation (15) by use of the expansion (12).

$$\bar{r}(A,Q) = \frac{2}{3}\alpha + \frac{h'^2}{2\alpha} + \frac{h'^4}{32\alpha^3} + \dots \quad . \tag{15}$$

We express the point Q by a polar coordinate (r,θ) which has the origin of the centroid of the disk B shown in Figure 6. h' is then expressed as (16) by the cosine theorem.

$$h' = \sqrt{h^2 + r^2 + 2hr\cos\theta} \tag{16}$$

Because the point Q is uniformly distributed on B, integrating $\bar{r}(A,Q)$ over B according to (15) and (16) yields the average distance between A and B:

$$\begin{aligned}
\bar{r}(A,B) &= \frac{1}{S_B}\int_B \bar{r}(A,Q)dQ = \frac{1}{\pi\beta^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{\beta}[\frac{2}{3}\alpha + \frac{h'^2}{2\alpha} + \frac{h'^4}{32\alpha^3} + \dots]rdrd\theta \\
&= \frac{2\alpha}{3\pi\beta^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{\beta} rdrd\theta \\
&\quad + \frac{1}{2\pi\alpha\beta^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{\beta}(h^2 + r^2 + 2hr\cos\theta)rdrd\theta \\
&\quad + \frac{1}{32\pi\alpha^3\beta^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{\beta}(h^2 + r^2 + 2hr\cos\theta)^2 rdrd\theta + \dots \\
&= \alpha[\frac{2}{3} + \frac{1}{4}(\frac{\beta}{\alpha})^2 + \frac{1}{2}(\frac{h}{\alpha})^2 - \frac{1}{96}(\frac{\beta}{\alpha})^4 - \frac{3}{64}(\frac{\beta}{\alpha})^2(\frac{h}{\alpha})^2 - \frac{1}{32}(\frac{h}{\alpha})^4 + \dots].
\end{aligned} \tag{17}$$

We chose the first three terms of Equation (17) to develop our approximate formula as Equation (18):

$$\tilde{r}(A,B) = \frac{2}{3}\alpha + \frac{1}{4}\frac{\beta^2}{\alpha} + \frac{1}{2}\frac{h^2}{\alpha}, \qquad \text{if} \quad \alpha \geq \beta \quad \text{and} \quad h + \beta \leq \alpha. \tag{18}$$

Because the bias of the approximated average distance $\tilde{r}(A,Q)$ between a point Q and the circular disk A is less than 3%, which has been shown above, the bias of $\tilde{r}(A,B)$ in the Equation (18) has the bias of less than 3% in general.

3. Circle Circumferences

In this section we will formulate the average distance between two circle circumferences on the basis of Equations (3) and (18). The result will be applicable to some mathematical analyses in city planning.

Let us suppose a circle A' which has the radius of $\alpha + \Delta\alpha(\Delta\alpha > 0)$ with the same centroid as that of A (Figure 7). This time, we consider the average distance $\bar{r}(A', B)$ between A' and B. Let $S_{A'}$ be the area of A' and let S_R be the area of the ring made by the radius' increase $\Delta\alpha$(hatched area in Figure 7). Then

$$S_A = \pi\alpha^2, \qquad S_{A'} = \pi(\alpha + \Delta\alpha)^2 \qquad \text{and}$$
$$S_R = S_{A'} - S_A = \pi\{(\alpha + \Delta\alpha)^2 - \alpha^2\} = 2\alpha\Delta\alpha + (\Delta\alpha)^2$$

hold. And we define $\bar{r}(R, B)$ as the average distance between the ring R and the circular disk A. According to the definitions above, we can express the average distance $\bar{r}(A', B)$ between disks A' and B as follows:

$$\begin{aligned} \bar{r}(A', B) &= \frac{S_A\,\bar{r}(A, B) + S_R\,\bar{r}(R, B)}{S_{A'}} \\ &= \frac{\alpha^2\,\bar{r}(A, B) + \{2\alpha\Delta\alpha + (\Delta\alpha)^2\}\,\bar{r}(R, B)}{(\alpha + \Delta\alpha)^2}. \end{aligned} \tag{19}$$

Solving (19) for $\bar{r}(R, B)$, we have

$$\begin{aligned} \bar{r}(R, B) &= \frac{(\alpha + \Delta\alpha)^2\,\bar{r}(A', B) - \alpha^2\,\bar{r}(A, B)}{2\alpha\Delta\alpha + (\Delta\alpha)^2} \\ &= \frac{\alpha}{2}\,\frac{\bar{r}(A', B) - \alpha^2\,\bar{r}(A, B)}{\Delta\alpha + \frac{(\Delta\alpha)^2}{2\alpha}} + \bar{r}(A', B). \end{aligned} \tag{20}$$

If we let $\Delta\alpha$ become 0, we have

$$\bar{r}(A', B) \to \bar{r}(A, B) \qquad \text{as} \qquad \Delta\alpha \to 0$$

and the first term of equation (20) is the derivative $\partial\bar{r}(A, B)/\partial\alpha$. So if we define

$$\bar{r}(C_A, B) = \lim_{\Delta\alpha \to +0} \bar{r}(R, B),$$

we obtain

$$\bar{r}(C_A, B) = \frac{\alpha}{2}\frac{\partial}{\partial\alpha}\bar{r}(A, B) + \bar{r}(A, B). \tag{21}$$

Here, $\bar{r}(C_A, B)$ means the average distance between circumference C_A of radius α and the disk B.

In the reduction of Equation (21) above, we have considered an increase $\Delta\alpha$ of the radius of A. Let us now suppose an increase $\Delta\beta$ of the radius of B. At this time, we can obtain the average distance $\bar{r}(C_A, C_B)$ between two circumferences C_A and C_B which have the radii of α and β respectively. To reduce $\bar{r}(C_A, C_B)$, we only substitute β for α, $\bar{r}(C_A, B)$ for $\bar{r}(A, B)$ and $\bar{r}(C_A, C_B)$ for $\bar{r}(C_A, B)$ in Equation (21). The reason for this manipulation will be obvious from Figure 8. From this substitutions and Equation (21) itself, Equation (22) is derived:

$$\begin{aligned}\bar{r}(C_A,C_B) &= \frac{\beta}{2}\frac{\partial}{\partial\beta}\bar{r}(C_A,B)+\bar{r}(C_A,B)\\ &= \frac{\beta}{2}\frac{\partial}{\partial\beta}\left\{\frac{\alpha}{2}\frac{\partial}{\partial\alpha}\bar{r}(A,B)+\bar{r}(A,B)\right\}+\frac{\alpha}{2}\frac{\partial}{\partial\alpha}\bar{r}(A,B)+\bar{r}(A,B)\\ &= \frac{\alpha\beta}{4}\frac{\partial^2}{\partial\alpha\partial\beta}\bar{r}(A,B)+\frac{\alpha}{2}\frac{\partial}{\partial\alpha}\bar{r}(A,B)+\frac{\beta}{2}\frac{\partial}{\partial\beta}\bar{r}(A,B)+\bar{r}(A,B). \quad (22)\end{aligned}$$

We have obtained $\bar{r}(C_A, C_B)$, Equation (22) perfectly.

Substituting the expansions of the average distances $\bar{r}(A, B)$'s of Equation (3) or (18) yields expansion formulas of $\bar{r}(C_A, C_B)$. The results are as follow:

$$\bar{r}(C_A, C_B) = h[1+\frac{1}{4}(\frac{\alpha}{h})^2+\frac{1}{4}(\frac{\beta}{h})^2+\frac{1}{64}(\frac{\alpha}{h})^4+\frac{1}{16}(\frac{\alpha}{h})^2(\frac{\beta}{h})^2+\frac{1}{64}(\frac{\beta}{h})^4+\ldots] \quad \text{if} \quad \alpha+\beta<h; \quad (23)$$

$$\bar{r}(C_A, C_B) = \alpha[1+\frac{1}{4}(\frac{\beta}{\alpha})^2+\frac{1}{4}(\frac{h}{\alpha})^2+\frac{1}{64}(\frac{\beta}{\alpha})^4+\frac{3}{64}(\frac{\beta}{\alpha})^2(\frac{h}{\alpha})^2+\frac{1}{64}(\frac{h}{\alpha})^4+\ldots] \quad \text{if} \quad \alpha\geq\beta \quad \text{and} \quad h+\beta\leq\alpha. \quad (24)$$

Neglecting the higher order terms of Equations (23) and (24) yields approximated formulas for $\bar{r}(C_A, C_B)$.

4. Application to one Type of Location Problems

In this section the approximated formulas derived in section 2 are applied to a facility location model. Among many variations of facility location models, we consider a case involving a single facility and multiple uniformly populated regions. Suppose that there are n regions $A_1, A_2, \ldots, A_n$ in a plane, and that demand population is uniformly distributed in each region. The objective of our location model is to minimize total travel distance from demand point to the facility F where F is represented as

$$F = (x, y).$$

Figure 9 shows a situation with five population regions. The unknown position of the demand is given by

$$P = (u, v).$$

The total cost function is then given by

$$\phi(F) = \sum_{j=1}^{n} \rho_j \int_{A_j} r(P, F)\, dP, \tag{25}$$

where $r(P, F)$ is the distance between F and P and ρ_j embodies the appropriate population density of A_j. Total population of each region is

$$\kappa_j = \rho_j S_j \qquad (j = 1, 2, \ldots, n)$$

where S_j is the area of region A_j. The integration of Equation (25) then can be regarded as

$$\begin{aligned} \rho_j \int_{A_j} r(P, F)\, dP &= \rho_j S_j \times \frac{1}{S_j} \int_{A_j} r(P, F)\, dP \\ &= \kappa_j \bar{r}(A_j, F) \qquad\qquad (j = 1, 2, \ldots, n) \end{aligned}$$

where $\bar{r}(A_j, F)$ is the average distance from region A_j to the facility F. Then we can rewrite Equation (25) as Equation (26):

$$\phi(F) = \sum_{j=1}^{n} \kappa_j \bar{r}(A_j, F). \tag{26}$$

The location problem (also called the Weber problem) is then represented as (27).

$$\text{minimize} \qquad \phi(F) = \sum_{j=1}^{n} \kappa_j \bar{r}(A_j, F). \tag{27}$$

It will be trivial that $\phi(F)$ is convex for any A_j's and κ_j's [Cooper,1974]. The precise way to optimize Problem (27) in general will be argued in the next section and in Appendix A.

As we have mentioned in the section 1, it is tiresome to calculate $\phi(F)$ and its derivatives (naturally we have to calculate not only $\phi(F)$ but its first or the second partial derivatives for the optimization of (27)). A primitive approximation for the objective function $\phi(F)$ has thus usually been used. This approximation is as follows: letting P_j be the centroid of each region A_j; and substituting $r(P_j, F)$ for the average distance $\bar{r}(A_j, F)$ in Equation (27). An approximate problem is then formulated as

$$\text{minimize} \qquad \phi_1(F) = \sum_{j=1}^{n} \kappa_j \, r(P_j, F). \tag{28}$$

$\phi_1(F)$ is a convex function too. Let F^* be the precise solution of Problem (27) and F_1^* be the solution of Problem (28). If $\phi(F^*) \simeq \phi(F_1^*)$, it must be worthwhile to solve Problem (28). However there exists a doubt if the solutions F_1^* and $\phi(F_1^*)$ are biased or have errors. The error problem of Problem (28) is called *data aggregation problem* [Casillas,1987].

We propose a new method for approximation of $\phi(F)$ in terms of the approximate average distance $\tilde{r}(A, Q)$ of Equation (14). In our method proposed, the circular disk A_j is substituted for each region A_j, where the area of the disk A'_j equals the area of A_j and the centroid of A'_j is the same as that of A_j. And we use the approximated average distances between these circles A_j's and the point F. Letting the radius of the circle A_j be α_j, then we have

$$S_j = \pi \, {\alpha_j}^2 \qquad (j = 1, 2, \ldots, n)$$

Of course $\alpha_j = \sqrt{S_j/\pi}$. In this case the inter centroid distance h in Equation (14) is

$$h_j(F) = r(P_j, F) \qquad (j = 1, 2, \ldots, n)$$

The approximate average distance from the region A_j to the point F is then

$$\tilde{r}(A'_j, F) = \begin{cases} \frac{2}{3}\alpha_j + \frac{1}{2}\frac{\{h_j(F)\}^2}{\alpha_j} & \text{if } h_j(F) < \alpha_j, \\ h_j(F) + \frac{1}{8}\frac{{\alpha_j}^2}{h_j(F)} & \text{if } h_j(F) \geq \alpha_j. \end{cases} \tag{29}$$

According to the preparations above, we can formulate the new approximate problem as

$$\text{minimize} \qquad \phi_2(F) = \sum_{j=1}^{n} \kappa_j \, \tilde{r}(A'_j, F). \tag{30}$$

The approximated objective function $\phi_2(F)$ is discontinuous when F is on any circumferences of substituted disks. This is because the approximated function

$\tilde{r}(A_j, F)$ is discontinuous at $h_j(F) = \alpha_j$. However, $\tilde{r}(A_j, F)$ is convex when $F \notin$ (the circumference of A'_j). This convexity can easily be proved by evaluating the first partial derivatives and the Hessian of $\tilde{r}(A'_j, F)$ where the point F and P_j are represented by the Cartesian coordinate system (the proof is not discussed in this paper). We can expect Problem (30) to be solved by use of Newton's method.

4.1 Two Circular Disks

Before discussing the optimization issue for a real regional data, let us consider on Problem (30) comparing to Problem (28). Suppose that the number of the regions is restricted to two. Let A_1 and A_2 be such regions. In this situation, it is well known that : if the constant weight κ_1 equals κ_2, any point on the line $\overline{P_1P_2}$ can be the solution F_1^* of (28) and ; if $\kappa_1 > \kappa_2$, then $F_1^* = P_1$. However, this dose not hold for the case of Problem (30). Let us discuss the behaviour of F_2^* below.

Figure 10 shows two disjoint circular disks which have the centroids P_1 and P_2 and have the radii of α_1 and α_2 respectively. First, it is obvious that the solution F_2^* for Problem (30) is on the line segment $\overline{P_1P_2}$ because of the convexity of $\tilde{r}(A'_j, F)$ $(j = 1, 2)$. Let the line length be $2d$ and the middle point of the line be a point O. Suppose a coordinate x along the line $\overline{P_1P_2}$ has the origin O as Figure 10 shows. It means that x represents the point F. In this case, if F is outside of the disks A_1 and A_2 (this condition is the same as $-d + \alpha_1 \leq x \leq d - \alpha_2$), then the approximated average distances are as follows:

$$\tilde{r}(A'_1, F) = d + x + \frac{\alpha_1^2}{8(d+x)};$$

$$\tilde{r}(A'_2, F) = d + x + \frac{\alpha_2^2}{8(d-x)}.$$

The function ϕ_2 is naturally, a function of $x, d, \alpha_1, \alpha_2, \kappa_1$ and κ_2. So, the objective function ϕ_2 is formulated as

$$\phi_2(x; d; \alpha_1, \alpha_2; \kappa_1, \kappa_2) = \kappa_1\left\{d + x + \frac{\alpha_1^2}{8(d+x)}\right\} + \kappa_2\left\{d - x + \frac{\alpha_2^2}{8(d-x)}\right\}, \quad \text{if} \quad -d + \alpha_1 \leq x \leq d - \alpha_2. \tag{31}$$

And the first and the second derivatives are obtained as follows:

$$\frac{\partial \phi_2}{\partial x} = \kappa_1\left\{1 - \frac{\alpha_1^2}{8(d+x)^2}\right\} + \kappa_2\left\{-1 + \frac{\alpha_2^2}{8(d-x)^2}\right\}; \tag{32}$$

$$\frac{\partial^2 \phi_2}{\partial x^2} = \frac{1}{4}\left\{\frac{\kappa_1\alpha_1^2}{(d+x)^3} + \frac{\kappa_2\alpha_2^2}{(d-x)^3}\right\} \qquad > 0. \tag{33}$$

Equation (33) shows the convexity of ϕ_2 about x.

Let us discuss the first order condition

$$\frac{\partial \phi_2}{\partial x} = 0. \tag{34}$$

Equation (34) is an implicit function of $x, d, \alpha_1, \alpha_2, \kappa_1$ and κ_2. Let x^* satisfy the condition (34) and also satisfy

$$-d + \alpha_1 \leq x^* \leq d - \alpha_2.$$

We will find the relation between x^* and α_1 or κ_1. First, fixing d, α_2, κ_1 and κ_2, suppose x^* to be a function of α_1, which is the radius of the disk A_1' (i.e. $x^* = x^*(\alpha_1)$). Then the next equation is provided by the implicit function theorem:

$$\begin{aligned} \frac{dx^*}{d\alpha_1} &= -\frac{\frac{\partial}{\partial \alpha_1}\left\{\frac{\partial \phi_2}{\partial x}(x^*)\right\}}{\frac{\partial^2 \phi_2}{\partial x^2}(x^*)} \\ &= \frac{\alpha_1}{(d+x^*)^2} \times \frac{1}{\frac{\kappa_1 \alpha_1^2}{(d+x^*)^3} + \frac{\kappa_2 \alpha_2^2}{(d-x^*)^3}} > 0. \end{aligned} \tag{35}$$

Inequality (35) means that if $-d + \alpha_1 \leq x^* \leq d - \alpha_2$ then an increase of the radius α_1 yields an increase of x^* (Figure 11). This interesting characteristics of the solution x^* (i.e. F^*) has not been argued before.

Next, fixing d, α_1, α_2 and κ_2, suppose x^* to be a function of κ_1, the constant population weight of the disk A_1' (i.e. $x^* = x^*(\kappa_1)$). Then, as well as Equation (35), Equation (36) follows by the implicit function theorem:

$$\begin{aligned} \frac{dx^*}{d\kappa_1} &= -\frac{\frac{\partial}{\partial \kappa_1}\left\{\frac{\partial \phi_2}{\partial x}(x^*)\right\}}{\frac{\partial^2 \phi_2}{\partial x^2}(x^*)} \\ &= -\frac{1 - \frac{\alpha_1^2}{8(d+x^*)^2}}{\frac{1}{4}\frac{\kappa_1 \alpha_1^2}{(d+x^*)^3} + \frac{\kappa_2 \alpha_2^2}{(d-x^*)^3}} < 0. \end{aligned} \tag{36}$$

Because we suppose that $-d + \alpha_1 \leq x^* \leq d - \alpha_2$, we have $\alpha_1 \leq d + x^*$. From this condition it is obvious that the numerator of (36) is positive,

$$\text{i.e.} \qquad 1 - \frac{\alpha_1^2}{8(d+x^*)^2} > 0,$$

and this yields the inequality of (36). Inequality (36) means that if $-d + \alpha_1 \leq x^* \leq d - \alpha_2$ then an increase of κ_1 yields a decrease of x^* (i.e. being closer to the circular disk A_1). This sounds quite natural.

In the above mentioned remarks, the uniqueness of the solution x^* on the line $\overline{P_1P_2}$ and the behaviour of x^* when κ_1 increases can be easily obtained in fact without the approximated average distances (i.e. it can be obtained from the convexity of $\bar{r}(A_j, F)$ directly). However, the relation between the solution x^* and the radius α_1 (or α_2) cannot be derived directly from the characteristics of $\bar{r}(A_j, F)$. We have shown that the approximated average distances provide us with an insight on the nature of the solution for two circular disks.

4.2 Algorithms for the Problems

We will solve Problems (27),(28) and (30) by use of Newton's method (a descent method, see Strang(1986)). If $f(F) = f(x, y)$ is the objective function, the Newton's method is described as follows,

$$F^{k+1} = F^k + \gamma^k d^k, \qquad k = 0, 1, 2, \ldots$$

where

$$\begin{aligned} F^0 &= (\text{the initial value of the variable } F), \\ \nabla f(F) &= (\text{the gradient of } f(F)), \\ \nabla^2 f(F) &= (\text{the Hessian matrix of } f(F)), \\ d^k &= (\text{the descent direction vector}) \\ &= -\{\nabla^2 f(F^k)\}^{-1} \nabla^t f(F^k), \\ \gamma^k &= (\text{the step size decided by a line search algorithm}). \end{aligned}$$

In this study we use Goldstein's rule to decide the step size γ^k. As shown above, we must calculate the objective function value, its gradient and its Hessian. This calculation can be done for our 3 problems as shown in Appendix A, B and C.

4.3 An Example of 4 Circular Disks

We now give an example of 4 regions as shown in Figure 12. In this example we suppose that $A_1, \ldots, A_4$ are circular disks (i.e. $A_j = A'_j$) with radii $\alpha_1, \ldots, \alpha_4$ respectively. The weight for the circular disk A_j is κ_j. Each variables are described in the Figure 12. We then compare the solution of Problem (28) with that of Problem (30). In Figure 12, the frame of outside is 10×10. The solution F_1^* of Problem (28) is represented by $\circ$, and the solution F_2^* of Problem (30) is represented by $\bullet$. Let the lower left corner of the frame be the origin O of the orthogonal coordinates as shown in the figure. The solutions are then provided as

$$F_1^* = (x_1^*, y_1^*) = (5.282, 4.450)$$

and

$$F_2^* = (x_2^*, y_2^*) = (5.779, 5.799).$$

The distance between them is $r(F_1^*, F_2^*) = 1.438$. We now check the bias of the objective function. First, from section 2, it is obvious that $\phi(F) \simeq \phi_2(F)$ because A_j is a circular disk in this 4-region case. We have

$$\phi_2(F_2^*) = 34.017.$$

Second, we have

$$\phi_2(F_1^*) = 36.400.$$

The relative bias of $\phi_2(F_1^*)$ from $\phi_2(F_2^*)$ is then around 7%. The bias on the objective function of the solution of the approximated problem (28) is very large. The above is an example of cases where the familiar approximation (28) is of no use.

4.4 A Real Regional Data

In this section, Problems (27),(28) and (30) are solved for real regional data. We consider 23 special wards in Tokyo, Japan as our study regions. We suppose each special ward to be a polygon with a finite number of vertices, digitizing their coordinates (Figure 13). The numbers of the vertices of a polygon range from 27 (the Chiyoda special ward) to 68 (the Nerima special ward) in our digitized data. Table 1 shows the night populations in 1982 of the special wards. These populations are κ_j's. We suppose that the population is uniformly distributed in each region.

In this 23-ward case, the solutions are derived as follows (the coordinates are on km unit):

$$F^* = (x^*, y^*) = (17.457, 20.397);$$
$$F_1^* = (x_1^*, y_1^*) = (16.981, 20.460);$$
$$F_2^* = (x_2^*, y_2^*) = (17.413, 20.439).$$

F^*, F_1^* and F_2^* are represented by '□','•' and '○' respectively in Figure 14. The errors on the locations of F_1^* and F_2^* are then

$$r(F^*, F_1^*) = 0.480 \qquad \text{km}$$

and

$$r(F^*, F_2^*) = 0.061 \qquad \text{km}.$$

These results show that F_2^* is nearer to the real optimal solution F^* than F_1^*. The function values of ϕ at F^*, F_1^* and F_2^* are

$$\phi(F^*) = 7.7184 \times 10^7 \qquad \text{person} \times \text{km},$$
$$\phi(F_1^*) = 7.7258 \times 10^7 \qquad \text{person} \times \text{km},$$
$$\phi(F_1^*) = 7.7185 \times 10^7 \qquad \text{person} \times \text{km}.$$

The biases of $\phi(F_1^*)$ and $\phi(F_2^*)$ from $\phi(F^*)$ are then

$$\phi(F_1^*) - \phi(F^*) = 74,523 \qquad \text{person} \times \text{km}$$

and

$$\phi(F_2^*) - \phi(F^*) = 1,253 \qquad \text{person} \times \text{km}.$$

The difference between two biases expresses the superiority of the solution F_2^* (the inferiority of F_1^*). The relative differences are as follows,

$$\frac{\phi(F_1^*) - \phi(F^*)}{\phi(F^*)} \times 100 = 0.10\%,$$
$$\frac{\phi(F_2^*) - \phi(F^*)}{\phi(F^*)} \times 100 = 0.0016\%.$$

4.5 Discussion

As shown in the optimizing issues in the section 4.2 and Appendix A, it is complicated and time consuming to solve the original problem : Problem (27). Though it is quite easy to solve Problem (28), the solution F_1^* is biased and has errors, as shown in the sections 4.3 and 4.4. Our approximated problem (30) is easily solved and has little error. As the information required for our circular disks approximation is only the area and the centroid of each region, it is recommended that Problem (30) always be used in the location model.

5. Other Applications

In this paper, we argue the application of the approximated formula between a point and a circular disk (see Section 4). However, there are many other subjects to which the approximate formulas in this paper are applicable.

The average distance between two circle-circumferences are applied to an approximation of the average distance between two circular disks, when the points P and Q are not uniformuly distributed. Let P and Q be points on the circular disks A and B respectively, and let w be the distance between P and A's centroid, and z be the distance between Q and B's centroid. If the densities of the points P and Q are proportionaly reduced to e^{-az} and e^{-bw}, then we can obtain a good approximated average distance between A and B [Kurita and Koshizuka,1988].

The approximated average distances between two circular disks could be easily extended to a formulation of location models where the facility is not only a point but has some area. This model can be applied to the alalyses of public facilities such as parks.

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Appendix A. On Problem (27)

We first consider the derivation of the value of $\phi(F)$ of Problem (27), and second, the derivation of the values of its derivatives.

Because the objective function is provided as Equation (25), if we can calculate $\int_{A_j} r(P,F)dP$ and its derivatives, we can evaluate $\phi(F)$, its gradient and the Hessian. We then define

$$\begin{aligned}\varphi(F) = \varphi(x,y) &= \int_A r(P,F)dP \\ &= \int_A \sqrt{(u-x)^2+(v-y)^2}\,du\,dv. \end{aligned} \tag{37}$$

Let $\psi(u,v)$ be a function satisfying the condition

$$\frac{\partial\psi}{\partial u} = \sqrt{(u-x)^2+(v-y)^2}.$$

φ of Equation (37) is then reduced into Equation (38) by Green's theorem:

$$\varphi = \int_A \frac{\partial\psi}{\partial u}\,du\,dv = \oint_C \psi(u,v)\,dv \tag{38}$$

where C is the counterclockwise circumference of the region A. Let us evaluate the line integral. We define

$$t = v - y \qquad \text{and} \qquad s = u - x. \tag{39}$$

ψ is then yielded as

$$\begin{aligned}\psi(u,v) &= \int \sqrt{(u-x)^2+t^2}\,du \\ &= \int \sqrt{s^2+t^2}\,ds \qquad\qquad (s = u - x \rightarrow ds = du) \\ &= \frac{1}{2}\left\{s\sqrt{s^2+t^2} + t^2 \ln\left(s + \sqrt{s^2+t^2}\right)\right\}. \end{aligned} \tag{40}$$

It can be proved that ψ has no singular point (the proof is not presented here). Because $(u,v) \in C$ in the above line integration, let u be a function of v as

$$u = u(v), \qquad \text{where} \qquad (u(v),v) \in C.$$

Let us replace the u in the above definition of s in (39) as

$$s = u(v) - x.$$

Substituting $\varphi(x,y)$ in Equation (40) into (38) now yields a line integration,

$$\varphi(x,y) = \frac{1}{2}\oint_C s\sqrt{s^2+t^2}\,dv + \frac{1}{2}\oint_C t^2 \ln\left(s+\sqrt{s^2+t^2}\right)dv. \qquad (41)$$

This line integration yields elliptic integrals when the A is a circular disk.

In our real regional data, each region is a polygon with finite vertices (See Section 4.4). Let the vertices of the region A be counterclockwise

$$(u_0,v_0),(u_1,v_1),\ldots,(u_m,v_m)=(u_0,v_0).$$

A has m vertices. Let us assume that $v_k - v_{k-1} \neq 0$ for any k. This assumption is not strong : if there exists k such that $v_k = v_{k-1}$, we can make a rotation of the orthogonal axes (u,v) which yields $v_k \neq v_{k-1}$. The $u = u(v)$ above is then written as

$$u(v) = a_k v + u_{k-1} \quad \text{if} \quad v \in [v_{k-1}, v_k) \quad (k = 1,2,\ldots,m),$$

where

$$a_k = \frac{u_k - u_{k-1}}{v_k - v_{k-1}} \qquad (k = 1,2,\ldots,m).$$

Substituting $u(v)$ for u into Equation (41) of $\varphi(x,y)$ above yields E quation (42):

$$\begin{aligned}
\varphi(x,y) &= \oint_C \psi(u(v),v)\,dv \\
&= \sum_{k=1}^{m}\int_{(u_{k-1},v_{k-1})}^{(u_k,v_k)} \psi(a_k v + u_{k-1}, v)\,dv \\
&= \frac{1}{2}\sum_{k=1}^{m}\left[\int_{v_{k-1}}^{v_k} s\sqrt{s^2+t^2}\,dv + \int_{v_{k-1}}^{v_k} t^2\ln\left(s+\sqrt{s^2+t^2}\right)dv\right], \qquad (42)
\end{aligned}$$

where

$$s = a_k v + u_{k-1} - x \quad \text{and} \quad t = v - y \qquad (k = 0,1,\ldots,m).$$

We now define the following for any k:

$$\begin{aligned}
\lambda_k &= 1 + a_k^2, \\
\mu_k &= 2\{a_k(u_{k-1} - x) - y\}, \\
\nu_k &= (u_{k-1} - x)^2 + y^2.
\end{aligned}$$

The first integral in Equation (42) of $\varphi(x,y)$ can be then calculated as follows (we do not discuss the manipulation in detail):

$$
\begin{aligned}
&\int s\sqrt{s^2+t^2}\,dv \\
&= \int (a_k v + u_{k-1} - x)\sqrt{(a_k v + u_{k-1} - x)^2 + (v-y)^2}\,dv \\
&= a_k \int v\sqrt{\lambda_k v^2 + \mu_k v + \nu_k}\,dv \ + (u_k - x)\int \sqrt{\lambda_k v^2 + \mu_k v + \nu_k}\,dv \\
&= a_k \Bigg[\frac{1}{3\lambda_k}(\lambda_k v^2 + \mu_k v + \nu_k)^{\frac{3}{2}} + \frac{\mu_k(2\alpha_k v + \mu_k)}{8\lambda_k}\sqrt{\lambda_k v^2 + \mu_k v + \nu_k} \\
&\quad + \frac{\mu_k(\mu_k^2 - 4\lambda_k\nu_k)}{16{\lambda_k}^2\sqrt{\lambda_k}} \ln\left|2\lambda_k v + \mu_k + 2\sqrt{\lambda_k(\lambda_k v^2 + \mu_k v + \nu_k)}\right|\Bigg] \\
&\quad + (u_k - x)\Bigg[\frac{2\lambda_k v + \mu_k}{4\lambda_k}\sqrt{\lambda_k v^2 + \mu_k v + \nu_k} \\
&\quad - \frac{{\mu_k}^2 - 4\lambda_k\nu_k}{\lambda_k\sqrt{\lambda_k}} \ln\left|2\lambda_k v + \mu_k + 2\sqrt{\lambda_k(\lambda_k v^2 + \mu_k v + \nu_k)}\right|\Bigg]. \qquad (43)
\end{aligned}
$$

For the second integral in Equation (42), which is

$$\int_{v_{k-1}}^{v_k} t^2 \ln\left(s + \sqrt{s^2+t^2}\right) dv,$$

we have to do numerical integrations. Here we use the alorithm of Romberg's integration in practice. We have thus calculated the value of $\varphi(x,y)$ for a region A_j. Calculations for $A_1, A_2, \ldots, A_n$ naturally yield the value of our objective function,

$$
\begin{aligned}
\phi(x,y) &= \sum_{j=1}^{n} \rho_j \int_{A_j} r(P,F)\,dP \\
&= \sum_{j=1}^{n} \rho_j \int_{A_j} \sqrt{(u-x)^2 + (v-y)^2}\,du\,dv.
\end{aligned}
$$

Let us consider the derivatives of $\phi(F)$ now. The first and the second partial derivatives are obtained directly from the line-integral form of $\varphi(x,y)$ in Equation (41) as follows:

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial x} = -\oint_C \sqrt{s^2+t^2}\,dv, && \frac{\partial\varphi}{\partial y} = \oint_C \sqrt{s^2+t^2}\,dv; \\
&\frac{\partial^2\varphi}{\partial x^2} = \oint_C \frac{s}{\sqrt{s^2+t^2}}\,dv, && \frac{\partial^2\varphi}{\partial y^2} = -\oint_C \frac{t}{\sqrt{s^2+t^2}}\,du, \\
&\frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial^2\varphi}{\partial y\partial x} = \oint_C \frac{t}{\sqrt{s^2+t^2}}\,dv.
\end{aligned}
$$

These derivatives can be obtained without use of a numerical calculation in the same manner as Equation (43) can be. The weighted sums (the weights are ρ_j's) of the derivatives above yield the the first and the second derivatives of our objective function ϕ.

Appendix B. On Problem (28)

The objective function $\phi_1(F)$ of Problem (28) is as follows:

$$\begin{aligned}\phi_1(F) &= \sum_{j=1}^{n} \kappa_j r(P_j, F) \\ &= \sum_{j=1}^{n} \kappa_j \sqrt{(x-u_j)^2 + (y-v_j)^2},\end{aligned} \tag{44}$$

where $F = (x, y)$ is the facility point and $P_j = (u_j, v_j)$ is the centroid of the region A_j. As (44) has a very simple functional form, it is easy to derive its derivatives. They are written as follows:

$$\begin{aligned}
\frac{\partial \phi_2}{\partial x} &= \sum_{j=1}^{n} \kappa_j \frac{x-u_j}{\sqrt{(x-u_j)^2+(y-v_j)^2}} = \sum_{j=1}^{n} \kappa_j \frac{x-u_j}{r(F,P_j)}, \\
\frac{\partial \phi_2}{\partial y} &= \sum_{j=1}^{n} \kappa_j \frac{y-v_j}{\sqrt{(x-u_j)^2+(y-v_j)^2}} = \sum_{j=1}^{n} \kappa_j \frac{y-v_j}{r(F,P_j)}, \\
\frac{\partial^2 \phi_2}{\partial x^2} &= \sum_{j=1}^{n} \kappa_j \frac{(y-v_j)^2}{\{(x-u_j)^2+(y-v_j)^2\}^{\frac{3}{2}}} = \sum_{j=1}^{n} \kappa_j \frac{(y-v_j)^2}{\{r(F,P_j)\}^3}, \\
\frac{\partial^2 \phi_2}{\partial y^2} &= \sum_{j=1}^{n} \kappa_j \frac{(x-u_j)^2}{\{(x-u_j)^2+(y-v_j)^2\}^{\frac{3}{2}}} = \sum_{j=1}^{n} \kappa_j \frac{(x-u_j)^2}{\{r(F,P_j)\}^3}, \\
\frac{\partial^2 \phi_2}{\partial x \partial y} &= \frac{\partial^2 \phi_2}{\partial y \partial x} \\
&= \sum_{j=1}^{n} \kappa_j \frac{(x-u_j)(y-v_j)}{\{(x-u_j)^2+(y-v_j)^2\}^{\frac{3}{2}}} = \sum_{j=1}^{n} \kappa_j \frac{(x-u_j)(y-v_j)}{\{r(F,P_j)\}^3}.
\end{aligned}$$

The results above are first obtained in Cooper(1961).

Appendix C. On Problem (30)

The objective function of Problem (30) is

$$\phi_2(F) = \phi_2(x,y) = \sum_{j=1}^{n} \kappa_j \tilde{r}(A'_j, F),$$

where

$$\tilde{r}(A'_j, F) = \begin{cases} \frac{2}{3}\alpha_j + \frac{1}{2}\frac{\{h_j(F)\}^2}{\alpha_j}_2 & \text{if } h_j(F) < \alpha_j, \\ h_j(F) + \frac{1}{8}\frac{\alpha_j{}^2}{h_j(F)} & \text{if } h_j(F) \geq \alpha_j, \end{cases}$$

and

$$h_j(F) = r(P_j, F) = \sqrt{(x-u_j)^2 + (y-v_j)^2}.$$

As we can see in the equations above, if we calculate the derivatives of the aproximated average distance $\tilde{r}(A_j, F)$ for each j, we can reasonably evaluate the objective function ϕ_2. The derivatives are shown below, where $\tilde{r}_j$ is an abbreviated form of the $\tilde{r}(A_j, F)$ (i.e. $\phi_2(x,y) = \sum \tilde{r}_j$).

First, if $h_j(x,y) < \alpha_j$ (i.e. the point $F = (x,y)$ is inside of the circular disk A_j'), the derivatives are derived as follows:

$$\frac{\partial \tilde{r}_j}{\partial x} = \frac{x-u_j}{\alpha_j}, \qquad \frac{\partial \tilde{r}_j}{\partial y} = \frac{y-v_j}{\alpha_j};$$

$$\frac{\partial^2 \tilde{r}_j}{\partial x^2} = \frac{1}{\alpha_j}, \qquad \frac{\partial^2 \tilde{r}_j}{\partial y^2} = \frac{1}{\alpha_j},$$

$$\frac{\partial^2 \tilde{r}_j}{\partial x \partial y} = \frac{\partial^2 \tilde{r}_j}{\partial y \partial x} = 0.$$

Second, if $h_j(x,y) < \alpha_j$ (i.e. the point $F = (x,y)$ is outside of the circular disk A_j'), the derivatives are derived as follows:

$$\frac{\partial \tilde{r}_j}{\partial x} = \frac{x-u_j}{h_j(x,y)}\left[1 - \frac{\alpha_j^2}{8\{h_j(x,y)\}^2}\right],$$

$$\frac{\partial \tilde{r}_j}{\partial y} = \frac{y-v_j}{h_j(x,y)}\left[1 - \frac{\alpha_j^2}{8\{h_j(x,y)\}^2}\right];$$

$$\frac{\partial^2 \tilde{r}_j}{\partial x^2} = \frac{(y-v_j)^2}{\{h_j(x,y)\}^3}\left[1 - \frac{\alpha_j^2}{8\{h_j(x,y)\}^2}\right] + \frac{\alpha_j^2(x-u_j)^2}{4\{h_j(x,y)\}^5},$$

$$\frac{\partial^2 \tilde{r}_j}{\partial y^2} = \frac{(x-u_j)^2}{\{h_j(x,y)\}^3}\left[1 - \frac{\alpha_j^2}{8\{h_j(x,y)\}^2}\right] + \frac{\alpha_j^2(y-v_j)^2}{4\{h_j(x,y)\}^5},$$

$$\frac{\partial^2 \tilde{r}_j}{\partial x \partial y} = \frac{\partial^2 \tilde{r}_j}{\partial y \partial x} = \frac{(x-u_j)(y-v_j)}{\{h_j(x,y)\}^3}\left[\frac{3\alpha_j^2}{8\{h_j(x,y)\}^2} - 1\right].$$

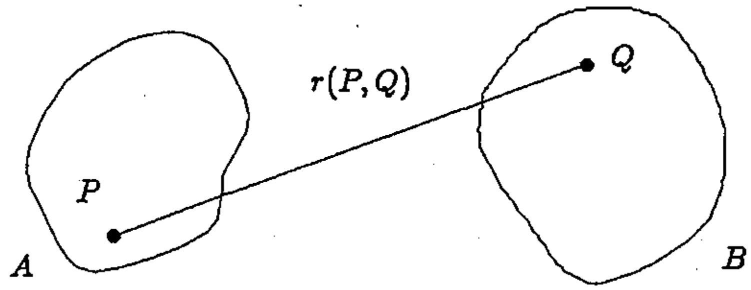


Figure 1 Two Coplanar Regions.

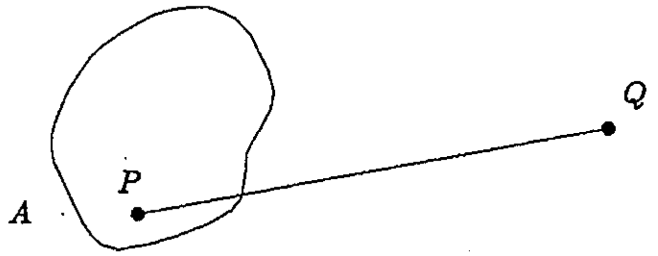


Figure 2 Region A and Point Q.

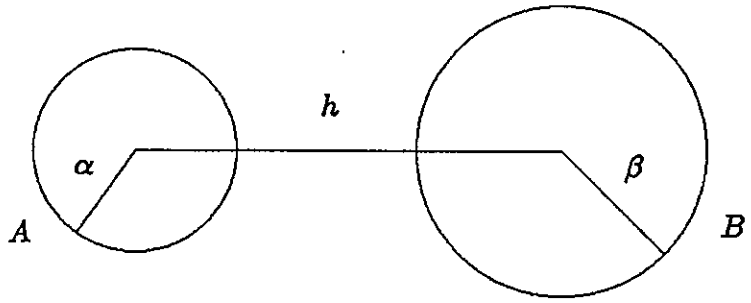
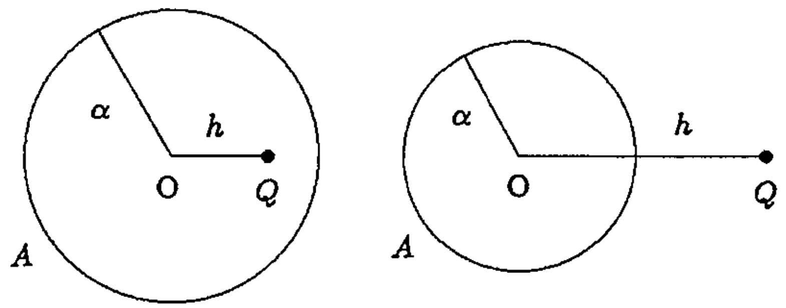


Figure 3 Two Non-Overlapping Circular Disks.



i) $\varsigma = \frac{h}{\alpha} \leq 1$ ii) $\varsigma = \frac{h}{\alpha} > 1$

Figure 4 Circular Disk A and Point Q.

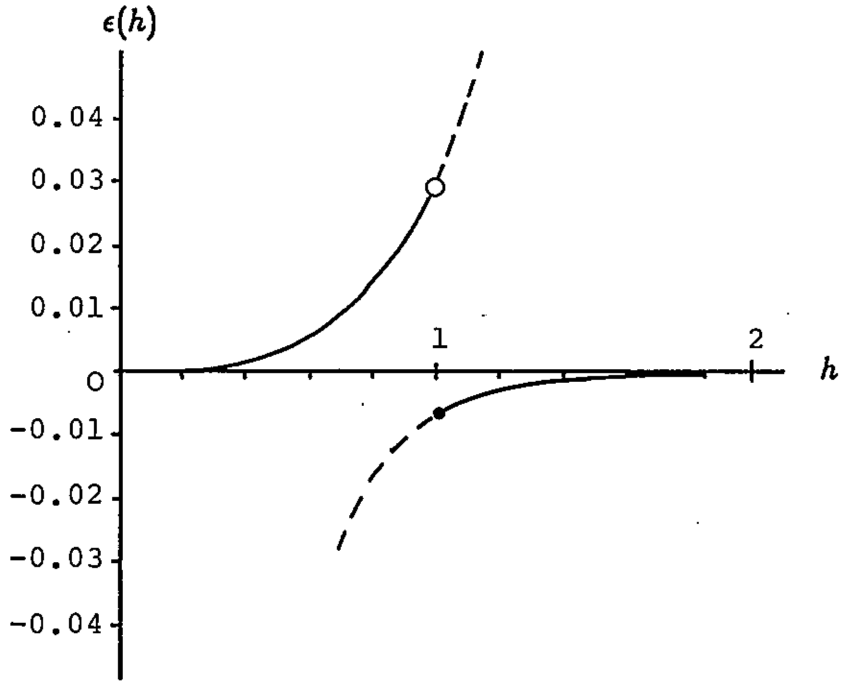


Figure 5 Relative Bias $\epsilon(h)$ of $\tilde{r}(A, Q)$, when $\alpha = 1$.

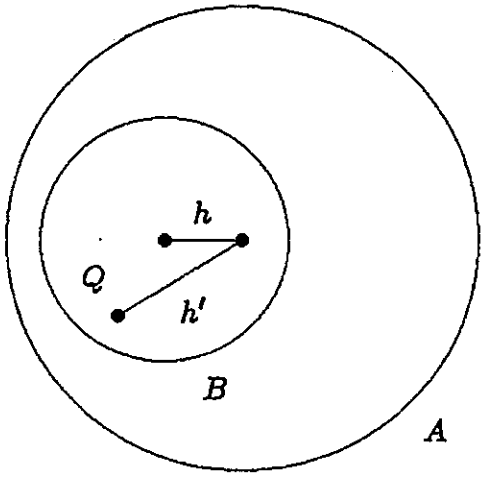


Figure 6 Circular Disks A and B.

(B is completely included in A).

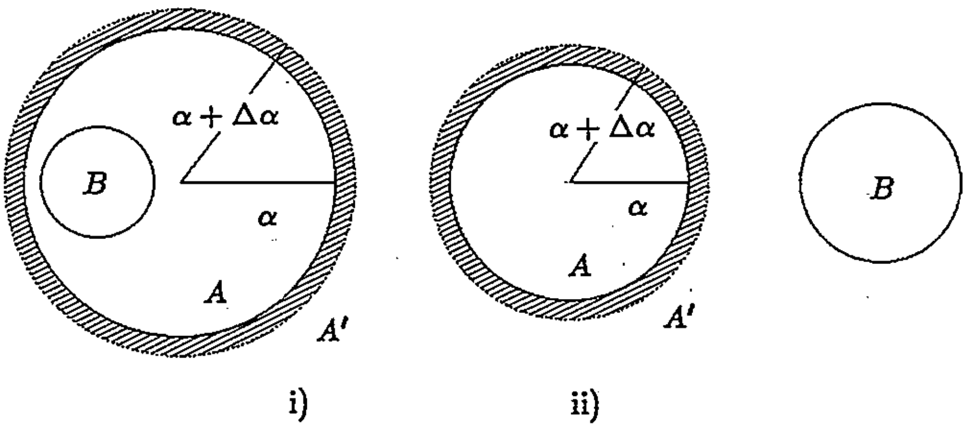


Figure 7 Circular Disk A' with Radius $\alpha + \Delta\alpha$.

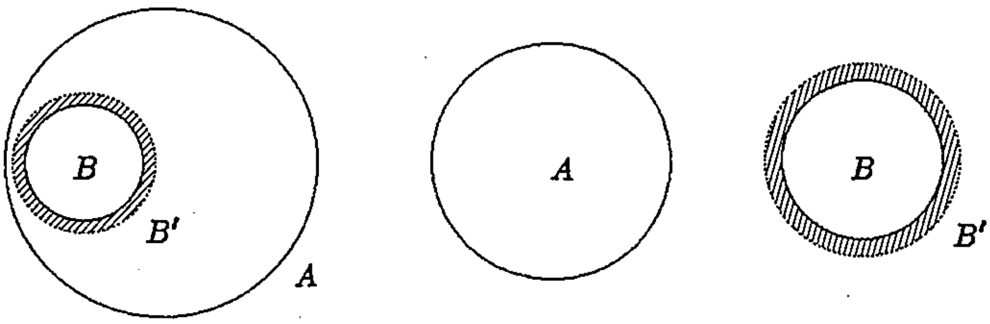


Figure 8 Circular Disk B' with Radius $\beta + \Delta\beta$.

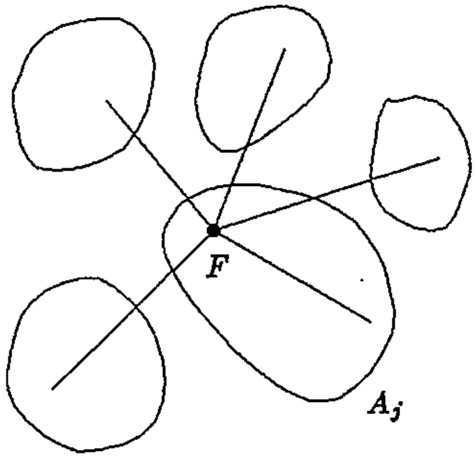


Figure 9 Populated Regions $A_1, \ldots, A_5$.

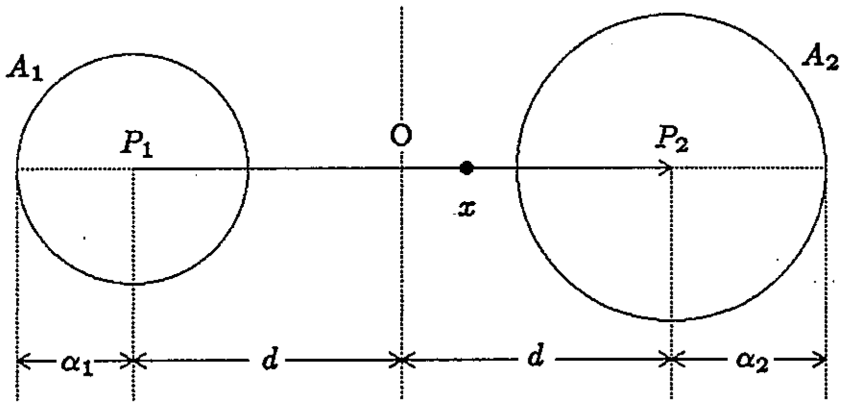


Figure 10 Two Circular Disks.

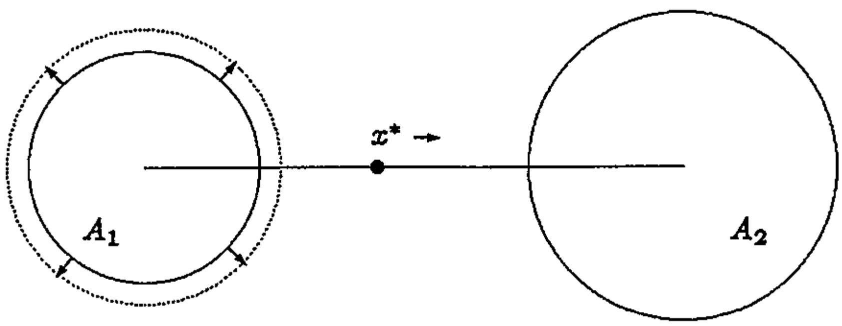


Figure 11 Increase of α Keeping.

the Optimal Solution, x^* away from A_1 .

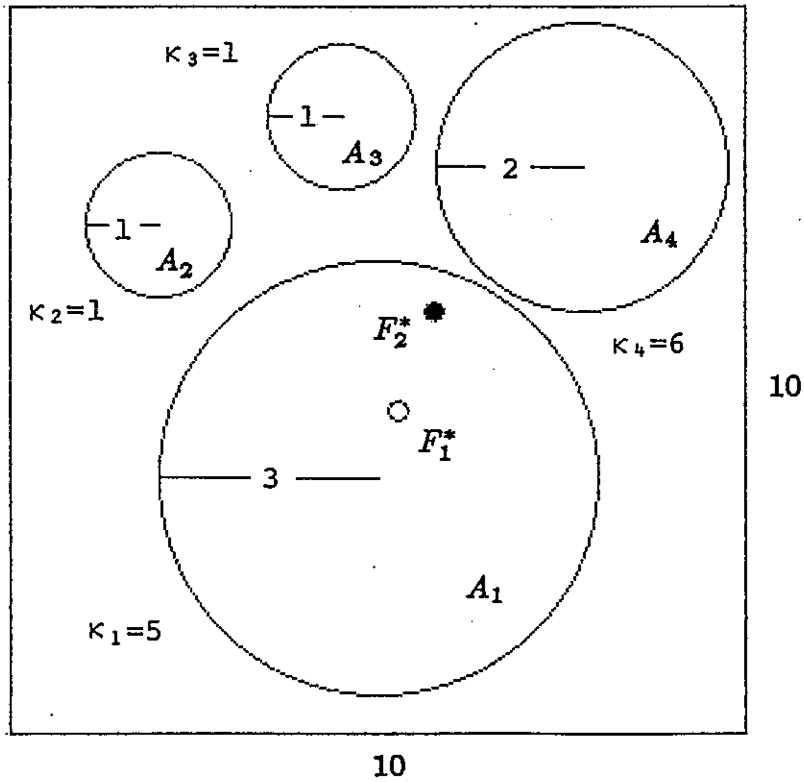


Figure 12 Four Circular Disks.

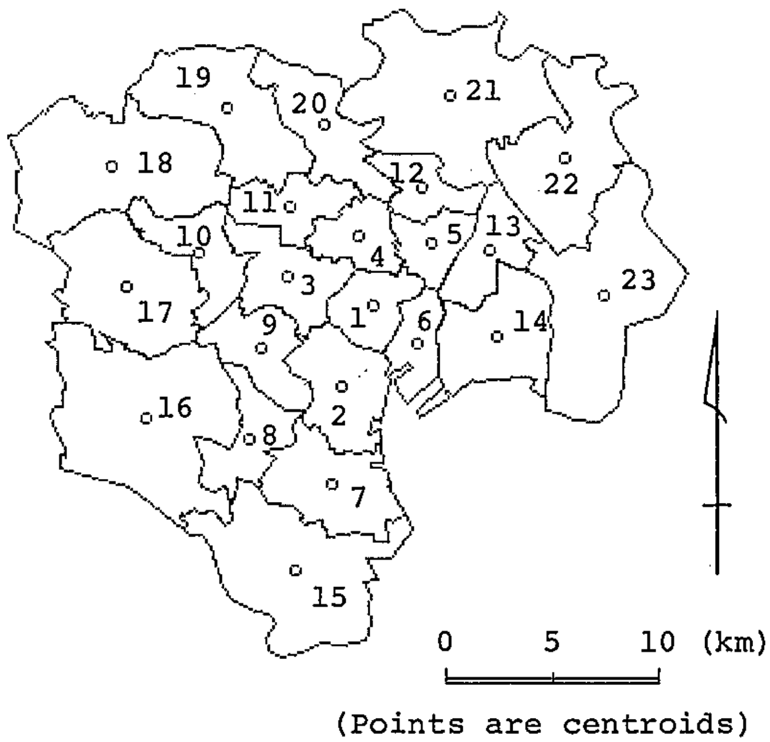


Figure 13 23 Special Wards in Tokyo.

Table 1 Populations of the Wards.

Special Ward	Population (person)	Special Ward	Population (person)
1. Chiyoda	50,493	13. Sumida	229,986
2. Minato	194,591	14. Koto	388,927
3. Shinjuku	332,722	15. Ota	662,814
4. Bunkyo	195,876	16. Setagaya	811,304
5. Taito	176,804	17. Suginami	539,842
6. Chuo	79,973	18. Nerima	587,887
7. Shinagawa	357,732	19. Itabashi	505,556
8. Meguro	569,166	20. Kita	367,579
9. Shibuya	242,442	21. Adachi	622,640
10. Nakano	335,936	22. Katsushika	419,017
11. Toshima	278,455	23. Edogawa	514,812
12. Arakawa	190,061		

(From the Census Data in 1980)

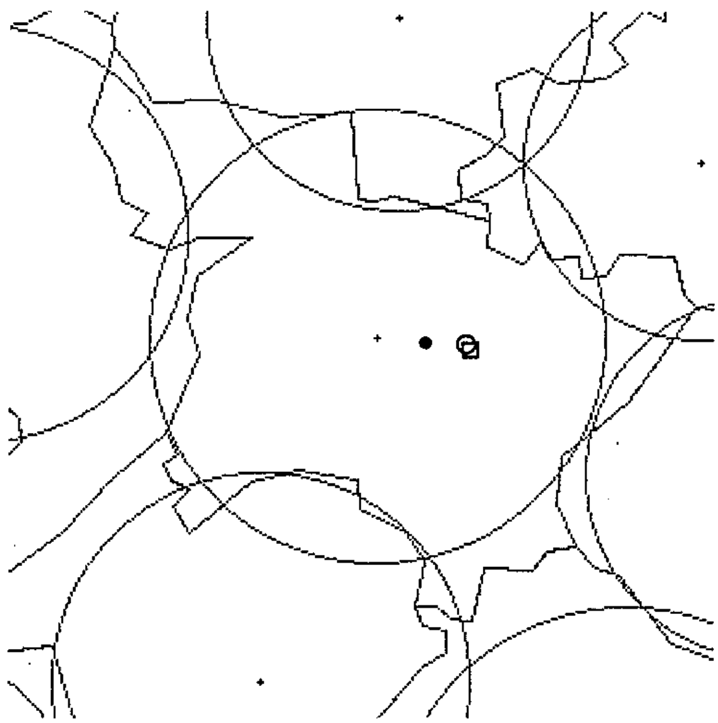


Figure 14 The Optimal Solutions F^*, F_1^* and F_2^*.

(F^* : □ , F_1^* : ● , F_2^* : ○)