No. 371

The Folk Theorem in Repeated Partnership Games with Imperfect Monitoring and Discounting

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June 1988

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February 1988

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ABSTRACT

We consider repeated partnership games in which decisions of all players can not be monitored perfectly. We argue that under the additive separability of utilities and the independency of commonly observable signals, the Folk Theorem holds true.

1. INTRODUCTION

Many authors have already explored the idea of "the Folk Theorem", which is that, if the single-period game is repeated infinitely and previous decisions of all players are common knowledge in every period, then all individually rational outcomes can be attained with supergame equilibria when the discount factor is near to unity (see Aumann and Shapley (1976), Rubinstein (1979), Fudenberg and Maskin (1983), and so on). It, however, is an open question to answer whether the Folk Theorem holds on the assumption that each player can not monitor previous decisions of the other players perfectly in every period, that is, on the assumption of imperfect monitoring. Radner (1985) showed that all efficient outcomes which dominate the single-period Nash-equilibrium outcome will be attained with supergame equilibria in the repeated principal-agent game with imperfect monitoring. In the repetition of the game, the principal has an opportunity to observe the realizations of random signals which depend on the agent's decisions over a number of periods, and use some statistical test to infer whether the agent was choosing the appropriate actions. Thus, the principal could employ the analogue of a statistical quality control chart to deter "cheating" by the agent.

The study of Radner (1985) leaves something to be desired in the following two respects. The one is that he has not present the complete proof of the Folk Theorem; i.e., he did not clarify the sustainability of inefficient outcomes. The other is that the principal-agent game with imperfect monitoring is very special in the sense that decisions of only one player, i.e., the single agent, can not be monitored perfectly, whearas decisions of the other player, i.e., the principal, can be observed directly by the agent.

The latter will be more essential: Radner, Myerson and Maskin (1986) considered games that are called pertnership game, in which decisions of more-than-one players can not be monitored perfectly, and presented an example in which efficiency can not be attained with supergame equilibria. Recently, Matsushima (1987) showed that the argument of Radner et al. depends on a certain "separability" condition of random signal, which is too restrictive to be the representative of the general case.

Our puppose in this paper is to present the complete proof of the Folk Theorem in the case of a specialized partnership game in which decisions of every player can not be monitored perfectly. We assume that the utility function of each player is additively separable, and that for each player, there is a commonly observable random signal which depends on his decision only. Under these assumptions, we can apply the essential idea of Radner (1985) to the case in which decisions of every player can not be monitored perfectly; i.e., we can construct n independent statistical tests that deter cheating of their respective players. Moreover, we assume that for every player, the reservation value of his own is smaller than his utility when the reservation value of another player is enforced. This will be typical in the case that decisions of all players induce positive external effect. Under these three assumptions, we show that reservation values are attained with supergame equilibria, using the idea of Radner (1985), and we can present the complete proof of the Folk Theorem.

Throughout this paper, we shall confine attentions to the case in which the discount factor is less than unity. Discounting future utilities will exclude the drawback of the limit-of-mean criterion under the assumption of no discounting, which is that, it ignores any finite time interval into account (to be precise, see Radner (1981,1986), Rubinstein (1979) and Matsushima (1987)).[1]

2. MODEL SETTING

We shall consider the following single-period game G, which is called the partnership game (for the general definition, see Radner (1986)). N is the set of all players, which is finite and $n := |N| \geq 2$. Player i has the action set $A_i = A := [\underline{\alpha}, \bar{\alpha}]$, where $0 < \underline{\alpha} < \bar{\alpha} < 1$. The utility function for player i, $u_i : \times_{j \in N} A_j \to R$, is defined as the following additively separable function: for each $a = (a_j)_{j \in N} \in \times_{j \in N} A_j$,

$$u_i(a) = \sum_{j \in N} a_j - c(a_i),$$

where $c : A \to R$ is continuous, increasing and strictly convex. We interpret that player i's decision, a_i, induces positive external effect at the expense of $c(a_i)$. A joint action $a \in \times_{i \in N} A_i$ is said to be a Nash equilibrium of G if for every $i \in N$,

$$u_i(a) \geq u_i(a_i', a_{-i}) \text{ for all } a_i' \in A_i .$$

We define a joint action $a^* \in \times_{i \in N} A_i$ as follows: for each $i \in N$,

$$a_i^* - c(a_i^*) \geq a_i - c(a_i) \text{ for all } a_i \in A_i .$$

Notice that $a_i^* = a_j^* := \alpha^*$ for all $i, j \in N$, and $a^* = (\alpha^*, \ldots, \alpha^*)$ is the unique Nash equilibrium of G. Let

$$V := \{v \in R^n : v = u(a) \text{ for some } a \in \times_{i \in N} A_i \}.$$

V is the set of feasible outcomes. We will call $\hat{v}_i = \max_{a_i} \min_{a_{-i}} u_i(a)$ player i's reservation value, and refer to $\hat{v} = (\hat{v}_i)_{i\in N}$ as the maximin point. Denote $\underline{a} := (\underline{\alpha},\ldots,\underline{\alpha})$. Notice that $\hat{v}_i = \hat{v}_j$ for all $i,j\in N$, and $\hat{v}_i = u_i(a_i^*,\underline{a}_{-i})$. We can prove easily the following lemma from the increasingness of c.

Lemma 1.

$\hat{v}_i \leq u_i(a_j^*,\underline{a}_{-j})$ for all $i,j\in N$.

Lemma 1 implies that for every player, the reservation value of his own is smaller than the utility when the reservation value of another player is enforced. An outcome $v\in V$ is said to be individually rational if $v \geq \hat{v}$. Notice that both $u(a^*)$ and $u(a_i^*,\underline{a}_{-i})$ are individually rational.

The infinitely repeated game with the single-period game G and a discount factor $\delta\in(0,1)$ is denoted by $G^\infty(\delta)$. The players can not observe realizations of the other players' past actions directly. The players, however, can observe, and therefore condition on, realizations of n+1 independent signals, $\omega_0,\omega_1,\ldots,\omega_n$. ω_0 does not depend on $a\in A$: Let Ω_0 := [0,1], which is the set of feasible ω_0. ω_0 is uniformly distributed on Ω. The dependence of strategies on ω_0 enlarges the possibility of correlation of strategies, and plays an important role in employing n independent statistical test to deter cheating of the players, as will be noted later.

For each $i \varepsilon N$, ω_i depends on $a_i \varepsilon A_i$ only: Let $\Omega_i := \{0,1\}$, which is the set of feasible ω_i. We assume that the probability that $\omega_i = 1$ occurs when a_i is chosen by player i is equal to a_i. Notice from the assumption that $0 < \underline{\alpha} < \bar{\alpha} < 1$ that both the probability that $\omega_i = 0$ and the one that $\omega_i = 1$ are always larger than zero, which are equal to $1 - a_i$ and a_i respectively. We denote $\omega := (\omega_0, \ldots, \omega_n)$ and $\Omega := \Omega_0 \times \ldots \times \Omega_n$.

The interpretation is that $u_i(a)$ is the expected value of instantaneous payoff: When a is played and ω occurs in period t, the realized instantaneous payoff of player i recieved at the end of period t is

$$\pi_i(t) = \sum_{i \varepsilon N} \omega_i - c(a_i).$$

Notice that $u_i(a)$ is equal to the expected value of $\pi_i(t)$.

A strategy for player i is defined as a sequence of functions $\sigma_i := (\sigma_i(t))_{t=1}^{\infty}$, where $\sigma_i(1) \varepsilon A_i$, and for each $t \geq 2$, $\sigma_i(t): \Omega^{t-1} \rightarrow A_i$.[2] $\sigma := (\sigma_i)_{i \varepsilon N}$ is a strategy profile. A t-period history of ω is denoted by $\omega^t = (\omega(1), \ldots, \omega(t))$. $\sigma|_{\omega^t} := (\sigma_i|_{\omega^t})_{i \varepsilon N}$ is the strategy profile induced by σ after the t-period history ω^t. Associated with any strategy profile σ are stochastic streams of instantaneous payoffs $(\pi_i(1), \ldots, \pi_i(t), \ldots)$ and actions $(a_i(1), \ldots, a_i(t), \ldots)$. The normalized expected present value of this stream is

$$v_i(\delta, \sigma) := (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} E[\pi_i(t) | \sigma],$$

where

$$E[\pi_i(t)|\sigma] := \sum_{j\varepsilon N} E[a_j(t)|\sigma] - E[c(a_i(t))|\sigma].$$

A strategy profile σ is said to be a supergame equilibrium of $G^{\infty}(\delta)$ if for each $i\varepsilon N$,

$$\sigma_i \varepsilon \underset{\sigma_i'}{\text{argmax}}\ v_i(\delta, \sigma_i', \sigma_{-i}).$$

3. SUSTAINABILITY OF RESERVATION VALUES

A feasible outcome $v \varepsilon V$ is said to be sustained approximately by supergame equilibria if for each $\varepsilon > 0$, there exists a discount factor $\bar{\delta} \varepsilon (0,1)$ such that, for every $\delta \varepsilon [\bar{\delta},1)$, there is a supergame equilibrium σ of $G^{\infty}(\delta)$ that satisfies

$$|v_i - v_i(\sigma,\delta)| \leq \varepsilon \text{ for all } i \varepsilon N.$$

Theorem 1. $u(a_i^*, \underline{a}_{-i})$ is sustained approximately by supergame equilibria.

Theorem 1 implies that reservation values are attained with supergame equilibria. In the next section, we will present the proof of Theorem 1.

4. Proof of Theorem 1

Notice that

$$u_i(a_i^*, \underline{a}_{-i}) = \hat{v}_i = \alpha^* + (n-1)\underline{\alpha} - c(\alpha^*),$$

and for $j \neq i$,

$$u_j(a_i^*, \underline{a}_{-i}) = \alpha^* + (n-1)\underline{\alpha} - c(\underline{\alpha}).$$

Suppose that $\alpha^* = \underline{\alpha}$. Then, $u(a_i^*, \underline{a}_{-i}) = u(a^*)$. Notice that for every $\delta \varepsilon (0,1)$, $u(a^*)$ is a supergame-equilibrium outcome of $G^\infty(\delta)$. Therefore, $u(a_i^*, \underline{a}_{-i})$ is approximately sustained by supergame equilibria.

Suppose that $\alpha^* > \underline{\alpha}$. Then, for every $i \varepsilon N$ and every $j \varepsilon N/\{i\}$,

$$\hat{v}_j < u_j(a_i^*, \underline{a}_{-i}).$$

T is a positive integer, and $\varepsilon > 0$ is a positive real number. Fix a player $i \varepsilon N$ arbitrarily. Let $H(i,T,\varepsilon)$ be the event that

$$\left|\frac{1}{T}\sum_{t=1}^{T} \omega_i(t) - \underline{\alpha}\right| > \varepsilon.$$

Suppose that a strategy for player i, σ_i, satisfies that for $t = 1,\dots,T$, $\sigma_i(t)$ depends on ω_i^{t-1} only. If player i plays such σ_i, the probability of $H(i,T,\varepsilon)$ does not depend on the other players' strategies. Thus, the probability is denoted by $P\{H(i,T,\varepsilon)|\sigma_i\}$. Moreover, if player i plays such σ_i, the expected values of $a_i(t)$ and $c(a_i(t))$ do not depend on the other players' strategies for $t = 1,\dots,T$. Hence the amounts are denoted by $E[a_i(t) \mid \sigma_i]$ and $E[c(a_i(t)) \mid \sigma_i]$.

Define a strategy for player i, $\underline{\sigma}_i$, in the following way: $\underline{\sigma}_i(1) = \underline{\alpha}$, and for $t \geq 2$,

$$\underline{\sigma}_i(t)(\omega^{t-1}) = \underline{\alpha} \text{ for all } \omega^{t-1}.$$

Using Chebychev's inequality, we can show in the same way as Radner [8] that

$$\lim_{T\to\infty} P\{H(i,T,\varepsilon) \mid \underline{\sigma}_i\} = 0.$$

We can find, for each T, a pair of positive real numbers $(\varepsilon(T), W(T))$ such that

$$\lim_{T\to\infty} \varepsilon(T) = 0, \ \lim_{T\to\infty} W(T) = \infty,$$

and

$$\lim_{T\to\infty} W(T)P\{H(i,T,\varepsilon(T)) \mid \underline{\sigma}_i\} = 0.$$

For every positive integer T and every strategy for player i, σ_i, such that $\sigma_i(t)$ depends on ω_i^{t-1} only, define

$$z_i(T,\sigma_i) := \frac{1}{T}\sum_{t=1}^{T} E[a_i(t) - c(a_i(t)) \mid \sigma_i] - W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i\}.$$

Notice

$$\lim_{T\to\infty} z_i(T,\underline{\sigma}_i) = \underline{\alpha} - c(\underline{\alpha}).$$

Let $\sigma_i^{[T]}$ be a strategy for player i, such that $\sigma_i^{[T]}(t)$ depends on ω_i^{t-1} only, and $\sigma_i^{[T]}$ maximizes $z_i(T,\sigma_i)$ amongst all strategies for player i, σ_i, such that $\sigma_i(t)$ depends on ω_i^{t-1} only. Notice

$$z_i(T,\sigma_i^{[T]}) \geqq z_i(T,\underline{\sigma}_i). \tag{4.1}$$

Lemma 2.

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[a_i(t) \mid \sigma_i^{[T]}] = \underline{\alpha},\ \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[c(a_i(t)) \mid \sigma_i^{[T]}] = c(\underline{\alpha}),$$

and

$$\lim_{T\to\infty} W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T]}\} = 0.$$

Proof of Lemma 2. Suppose that $P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T]}\}$ does not converge to zero as T approaches to unity. Then, there is a positive real number ξ such that for every T', there exists $T \geqq T'$ that satisfies

$$P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T]}\} \geqq \xi.$$

When T is sufficiently large, $W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T]}\}$ is near to infinity. This implies that we can find a positive integer T such that

$$z(T,\sigma_i^{[T]}) < z(T,\underline{\sigma}_i),$$

which is a contradiction of (4.1). Therefore,

$$\lim_{T\to\infty} P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T]}\} = 0. \tag{4.2}$$

(4.2) implies

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[a_i(t) \mid \sigma_i^{[T]}] = \underline{\alpha},$$

and, together with the convexity of c, that

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} E[c(a_i(t)) \mid \sigma_i^{[T]}] = c(\underline{\alpha}).$$

Thus, from (4.1), $\lim_{T\to\infty} z_i(T,\sigma_i^{[T]}) = \underline{\alpha} - c(\underline{\alpha})$ must hold. This implies

$$\lim_{T\to\infty} W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T]}\} = 0.$$

Q.E.D.

For every positive integer T, every $\delta\varepsilon(0,1)$, and every strategy for player i, σ_i, such that $\sigma_i(t)$ depends on ω_i^{t-1} only, define

$$z_i(T,\delta,\sigma_i) := \left(\sum_{t=1}^{T}\delta^{t-1}\right)^{-1}\sum_{t=1}^{T}\delta^{t-1}E[a_i(t) - c(a_i(t) \mid \sigma_i]$$

$$- W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i\}.$$

Let $\sigma_i^{[T,\delta]}$ be a strategy for player i that maximizes $z_i(T,\delta,\sigma_i)$ amongst all strategies for player i, σ_i, such that $\sigma_i(t)$ depends on ω_i^{t-1} only. Notice

$$z_i(T,1,\sigma_i) = z_i(T,\sigma_i).$$

Let $\eta > 0$ be a positive real number. From the continuity of $z_i(T,\delta,\sigma_i)$ in δ, we can apply the argument in Lemma 2 to the case that δ is near to unity: For each $\xi > 0$, we can find T such that for every δ near to unity,

$$\left|\left(\sum_{t=1}^{T}\delta^{t-1}\right)^{-1}\sum_{t=1}^{T}\delta^{t-1}E[a_i(t) \mid \sigma_i^{[T,\delta]}] - \underline{\alpha}\right| < \eta,$$

$$\left|\left(\sum_{t=1}^{T}\delta^{t-1}\right)^{-1}\sum_{t=1}^{T}\delta^{t-1}E[c(a_i(t)) \mid \sigma_i^{[T,\delta]}] - c(\underline{\alpha})\right| < \eta,$$

and

$$W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T,\delta]}\} < \eta.$$

Fix such η, T and δ arbitrarily. Define

$$K := \left(\sum_{t=1}^{T}\delta^{t-1}\right)^{-1}\sum_{t=1}^{T}\delta^{t-1}E[(n-1)a_i(t) - c(a_i(t)) \mid \sigma_i^{[T,\delta]}] + \alpha^*$$

$$- W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T,\delta]}\},$$

and

$$M := \left(\sum_{t=1}^{T}\delta^{t-1}\right)^{-1}\sum_{t=1}^{T}\delta^{t-1}E[(n-1)a_i(t) \mid \sigma_i^{[T,\delta]}] + \alpha^* - c(\alpha^*)$$

$$+ (n-1)W(T)P\{H(i,T,\varepsilon(T)) \mid \sigma_i^{[T,\delta]}\}.$$

Notice that for $i \neq j$,

$$|u_i(a_j^*, \underline{a}_{-j}) - K| < (n+1)\eta,$$

and

$$|\hat{v}_i - M| < 2(n-1)\eta.$$

Let $D := K - M$. When η is sufficiently small,

$$|\{u_i(a_j^*, \underline{a}_{-j}) - \hat{v}_i\} - D| < (3n-1)\eta \text{ whenever } i \neq j.$$

Since δ is near to unity, we can find a real number $x(D)$ such that

$$0 < x(D) < \frac{1}{n}, \text{ and } W(T) = \frac{\delta^T}{1-\delta^T}x(D)D.$$

Based on the above notations, we define n strategy profiles, $(\sigma^{(j)})_{j\varepsilon N}$, in the follwing way: For each $t = 1,\ldots,T$ and each ω^{t-1},

$$\sigma_j^{(j)}(t)(\omega^{t-1}) = \alpha^*.$$

For every $t = 1,\ldots,T$,

$$\sigma_i^{(j)}(t) = \sigma_i^{[T,\delta]}(t) \text{ whenever } i \neq j.$$

If $k \neq j$, $\omega^T \varepsilon H(k,T,\varepsilon(T))$ and $\frac{k-1}{n} \leq \omega_0(T) \leq \frac{k-1}{n} + x(D)$, then,

$$\sigma^{(j)}|_{\omega^T} = \sigma^{(k)},$$

and, otherwise,

$$\sigma^{(j)}|_{\omega^T} = \sigma^{(j)}.$$

When $\sigma^{(j)}$ is played in $G^\infty(\delta)$, player $i \neq j$ obtains

$$v_i(\delta,\sigma^{(j)})$$

$$= (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\{\sum_{k\neq j}E[a_k(t)|q_k^{[T,\delta]}] + \alpha^* - E[c(a_i(t))|q_i^{[T,\delta]}]\}$$

$$+ \delta^T x(D)\sum_{k\neq j}P\{H(k,T,\varepsilon(T))|q_k^{[T,\delta]}\}v_i(\delta,\sigma^{(k)})$$

$$+ \delta^T[1 - x(D)\sum_{k\neq j}P\{H(k,T,\varepsilon(T))|q_k^{[T,\delta]}\}]v_i(\delta,\sigma^{(j)}).$$

This is rewriten to be

$$v_i(\delta,\sigma^{(j)})$$

$$= (\sum_{t=1}^{T}\delta^{t-1})^{-1}\sum_{t=1}^{T}\delta^{t-1}E[(n-1)a_i(t) - c(a_i(t))|q_i^{[T,\delta]}] + \alpha^*$$

$$- \frac{\delta^T}{1-\delta^T}x(D)P\{H(i,T,\varepsilon(T))|\sigma_i^{[T,\delta]}\}\{v_i(\delta,\sigma^{(j)}) - v_i(\delta,\sigma^{(i)})\}.$$

Moreover, when $\sigma^{(j)}$ is played in $G^\infty(\delta)$, player j obtains

$$v_j(\delta,\sigma^{(j)})$$

$$= (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\{\sum_{k\neq j}E[a_k(t)|q_k^{[T,\delta]}] + \alpha^* - c(\alpha^*)\}$$

$$+ \delta^T x(D)\sum_{k\neq j}P\{H(k,T,\varepsilon(T))|q_k^{[T,\delta]}\}v_j(\delta,\sigma^{(k)})$$

$$+ \delta^{T}[1 - x(D) \sum_{k \neq j} P\{H(k,T,\varepsilon(T)) | \sigma_k^{[T,\delta]}\}] v_j(\delta, \sigma^{(j)}).$$

Thus is rewritten to be

$$v_j(\delta, \sigma^{(j)})$$

$$= (\sum_{t=1}^{T} \delta^{t-1})^{-1} \sum_{t=1}^{T} \delta^{t-1} E[(n-1)a_i(t) - c(a_i(t)) | \sigma_i^{[T,\delta]}] + \alpha^*$$

$$+ (n-1) \frac{\delta^T}{1-\delta^T} x(D) P\{H(i,T,\varepsilon(T)) | \sigma[T,\delta\}_i\} \{v_j(\delta, \sigma^{(i)})$$

$$- v_j(\delta, \sigma^{(j)})\},$$

where $i \neq j$. From the definitions of K, M and x(D),

$$v_j(\delta, \sigma^{(i)}) = K, \text{ and } v_j(\delta, \sigma^{(j)}) = M.$$

Hence, $u_j(a_i^*, a_{-i})$ and $\underline{v}_j$ approximate $v_j(\delta, \sigma^{(i)})$ and $v_j(\delta, \sigma^{(j)})$ respectively.

Notice from the definition of $\sigma^{(j)}$ that $\sigma^{(j)}$ is a supergame equilibrium of $G^{\infty}(\delta)$, if for $i \neq j$, $\sigma_i^{(j)}$ maximizes $z_i(T, \delta, \sigma_i)$ amongst all σ_i such that $\sigma_i(t)$ depends on ω_i^{t-1} only. We can easily check from the definition of $\sigma^{(j)}$ that $\sigma^{(j)}$ possesses this condition. Hence, $u(a_i^*, a_{-i})$ is sustained approximately by supergame equilibria.

5. FOLK THEOREM

Theorem 1 implies that reservation values are credible threats. Hence, we can prove the Folk Theorem in the following way.

Theorem 2. Suppose that v is individually rational. Then, v is sustained approximately by supergame equilibria.

Proof. Let $a \in A$ be a joint atrategy such that $u(a) = v$. T is a positive integer, and $\varepsilon > 0$ is a positive real number near to zero. Let $G(i,T,\varepsilon)$ be the event that

$$\left|\frac{1}{T}\sum_{t=1}^{T} \omega_i(t) - a_i\right| > \varepsilon.$$

Define a strategy for player i, $\phi_i^{[T,\delta]} = (\phi_i^{[T,\delta]}(t))_{t=1}^{\infty}$, that maximizes

$$\left(\sum_{t=1}^{T} \delta^{t-1}\right)^{-1} \sum_{t=1}^{T} \delta^{t-1} E[a_i(t) - c(a_i(t) | \sigma_i] - WP\{G(i,T,\varepsilon) | \sigma_i\}$$

amongst all σ_i such that $\sigma_i(t)$ depends on ω_i^{t-1} only. We can prove in the same way as the proof of Theorem 1 that for every $\eta > 0$, there exist T, ε and W such that when δ is near to unity,

$$\left|\left(\sum_{t=1}^{T} \delta^{t-1}\right)^{-1} \sum_{t=1}^{T} \delta^{t-1} E[a_i(t) | \phi_i^{[T,\delta]}] - a_i\right| < \eta, \tag{5.1}$$

$$\left|\left(\sum_{t=1}^{T} \delta^{t-1}\right)^{-1} \sum_{t=1}^{T} \delta^{t-1} E[c(a_i(t)) | \phi_i^{[T,\delta]}] - c(a_i)\right| < \eta, \tag{5.2}$$

and

$$WP\{G(i,T,\varepsilon) | \phi_i^{[T,\delta]}\} < \eta \tag{5.3}$$

for all $i \in N$. Fix such ε, W, η, T and δ arbitrarily, where the following inequalities are satisfied for the n strategy profiles $(\sigma^{(i)})_{i\in N}$, which is defined in the previous section; i.e.,

$$|v_i(\delta,\sigma^{(j)}) - u_i(a_j^*, \underline{a}_{-j})| < \eta \text{ for all } i,j \in N. \tag{5.4}$$

By choosing η near to zero, we can find a positive real number $\mu > 0$ such that

$$\mu + \eta < v_i - \hat{v}_i \text{ for all } i \in N. \tag{5.5}$$

Let D_i be a positive real number, such that

$$|D_i - v_i| \leq \mu. \tag{5.6}$$

From (5.4), (5.5) and (5.6),

$$D_i - v_i(\delta,\sigma^{(i)}) > 0 \text{ for all } i \in N.$$

When δ is near to unity, we can find a real number $x_i(D_i)$ such that

$$0 < x_i(D_i) < \frac{1}{n}, \text{ and } W = \frac{\delta^T}{1-\delta^T} x_i(D_i)\{D_i - v_i(\delta,\sigma^{(i)})\}. \tag{5.7}$$

Define

$$K_i((D_j)_{j\in N})$$

$$:= (\sum_{t=1}^{T} \delta^{t-1})^{-1} \sum_{t=1}^{T} \delta^{t-1} \{ \sum_{j\in N} E[a_j(t)|\phi_j^{[T,\delta]}] - E[c(a_i(t))|\phi_i^{[T,\delta]}]\}$$

$$- \frac{\delta^T}{1-\delta^T} \sum_{j\in N} x_j(D_j) P\{G(j,T,\varepsilon)|\phi_j^{[T,\delta]}\}\{D_i - v_i(\delta,\sigma^{(j)})\}.$$

From (5.3) and (5.7),

$$\frac{\delta^T}{1-\delta^T} x_j(D_j) P\{G(j,T,\varepsilon)|\phi_j^{[T,\delta]}\}|D_i - v_i(\delta,\sigma^{(j)})|$$

$$< \frac{|D_i - v_i(\delta,\sigma^{(j)})|}{D_j - v_j(\delta,\sigma^{(j)})}\eta,$$

which is smaller than

$$(v_j - \hat{v}_j - \mu - \eta)^{-1}\{|v_i - u_i(a_j^*,a_{-j})| + \mu + \eta\}\eta.$$

This implies, together with (5.1), (5.2) and (5.3), that

$$|K_i((D_j)_{j\varepsilon N}) - v_i|$$

$$< [n + 2 + \sum_{j\neq i}(v_j - \hat{v}_j - \mu - \eta)^{-1}\{|v_i - u_i(a_j^*,a_{-j})| + \mu + \eta\}]\eta,$$

which is smaller than μ when we choose η near to unity. From the fix point theorem, we can find $(D_j^*)_{j\varepsilon N}$ such that for every $i\varepsilon N$,

$$|D_i^* - v_i| < \mu, \text{ and } K_i((D_j)_{j\varepsilon N}) = D_i^*.$$

Based on the above argument, we define a strategy profile $\tilde{\sigma}$ in the follwing way: For $t = 1,\ldots,T$,

$$\tilde{\sigma}_i(t) = \phi_i^{[T,\delta]}(t) \text{ for all } i\varepsilon N.$$

If $\omega^T\varepsilon G(i,T,\varepsilon)$ and $\frac{i-1}{n} \leq \omega_0(T) \leq \frac{i-1}{n} + x_i(D_i^*)$, then

$$\tilde{\sigma}|_{\omega^T} = \sigma^{(i)},$$

and, otherwise,

$$\tilde{\sigma}|_{\omega^T} = \tilde{\sigma}.$$

Notice that when $\tilde{\sigma}$ is played in $G^{\infty}(\delta)$, player i obtains $v_i(\delta,\tilde{\sigma}) = \tilde{D}_i^*$. This implies that $v(\delta,\tilde{\sigma})$ approximates v. Notice that $\tilde{\sigma}$ is a supergame equilibrium of $G^{\infty}(\delta)$, if for every $i\varepsilon N$, $\tilde{\sigma}_i$ maximizes

$$(\sum_{t=1}^{T}\delta^{t-1})^{-1}\sum_{t=1}^{T}\delta^{t-1}E[a_i(t) - c(a_i(t)|\sigma_i] - WP\{G(i,T,\varepsilon)|\sigma_i\}$$

amongst all σ_i such that $\sigma_i(t)$ depends on ω_i^{t-1} only for t = 1,...,T. We can check easily from the definition of $\tilde{\sigma}$ that $\tilde{\sigma}$ satisfies this condition.

Q.E.D.

6. DISCUSSION

The sustainability of collusive behaviours, shown in Theorem 2, depends crucially on the additive separability of utilities and the independency of signals. If these assumptions do not hold, then we may not apply the idea of Radner (1985) to the case that decisions of more-than-one players can not be monitored perfectly. Moreover, the sustainability of reservation values shown in Theorem 1 will depend crucially on Lemma 1, together with the above assumptions; which is that, for every player, the reservation value of his own is smaller than the utility when the reservation value of another player is enforced. However, the lack of these assumptions does not imply necessarily the collapse of the Folk Theorem. The idea of a statistical quality control chart to deter cheating would be too specific to describe all supergame-equilibrium outcomes in general.[3] Matsushima (1987) tried to reduce the study of supergame equilibria to the solution of a class of static problem with utility-transfer, and conjectured that efficiency will be attained with supergame equilibria in general.

NOTES

1. Radner (1986) showed that when the discount factor is equal to unity, efficiency will be typically attained with supergame equilibria on the limit-of-mean criterion.

2. We assume that strategies of players are independent of their own previous decisions. This assumption plays no role in the sequel. We assume it only to avoid complicating the model with irrelevant generality.

3. In the case of perfect monitoring, Abreu (1983) presented a characterization of sequential equilibria that sustain all equilibrium outcomes. Moreover, Abreu, Pearce and Stacchetti (1986a,1986b) tried the same problem in the case of imperfect monitoring.

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