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The Theory of Implementation Revisited:
A New Approach to the Implementation Problem

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This paper is Part 1 of my Ph.D. dissertation, entitled

"Three Essays on Decentralized Decision Making under Uncertainty."

This paper is almost identical with the paper forthcoming in Journal of Economic Theory, entitled,

"A New Approach to the Implementation Problem".

This version will be much readable, and will serve to make clearer an essense of the implementation problem.

Abstract

We consider the implementation problem with incomplete information, where for each agent, only knowledge about his and his neighbors' preferences is always sufficient to enforce his optimal behavior. It is shown that almost every social choice rule is implementable.

1. Introduction

In this paper, we consider the theory of implementation in which information is incomplete.[1] When a society is large, it is implausible to assume complete information which implies that each agent knows every agent's preference.[2] The decentralization of information follows as an inevitable consequence. This circumstance of incomplete information leads to the decentralization of decision making, which is described by a game form. The theory of implementation concerns the problem of designing a game form the equilibria of which have properties that are desirable according to a specified criterion of social welfare, i.e., a social choice rule.

Using dominant strategy equilibrium as a basic equilibrium concept, Gibbard [4] and Satterthwaite [13] have introduced the notion of strategy-proofness: A social choice rule is said to be strategy-proof if for each agent, truthful revelation about his own preference is always a dominant strategy. They showed that every social choice rule which is strategy-proof is dictatorial, and therefore, there is no reasonable rule which is strategy-proof.

The impossibility of strategy-proofness depends on the following asssumption about informational availability: That is, each agent knows nothing about other agents' preferences. This is too restrictive to represent the general case, on which, nevertheless, the concept of dominant strategy equilibrium will be based. One conjecture is that when this assumption is weakened in a reasonable way, the impossibility theorem collapses. Can this conjecture be true ?

The main purpose is to provide an answer to this question. Our answer is "yes", previded that we permit probabilistic choice, which is explored by Gibbard [5], Pattanaik and Peleg [11] and so on.

We reconsider the notions of weak implementation and strong implementation in Maskin's sense (see Maskin [6,7] or Moulin [8]). These employ Nash equilibrium as a basic solution concept. We require in addition to several requirements of implementation that for each agent, only knowledge about his own and his two neighbors' preferences is always sufficient to enforce his optimal behaviour: We introduce the class of game forms in which each agent anounces his own and his two neighbours' preferences only, and we will show that the truthful revelation is a Nash equilibrium that sustains the social optimum assigned by the social choice rule.

The following two possibility theorems will be presented: The first is concerned with weak implementation, which requires the existence of a game form in which social optima are sustained by Nash equilibria. It is shown that all social choice rules are weakly implementable under the above informational requirement.

The second is concerned with strong implementation, which requires that for every preference profile, the corresponding social optimum is sustained by all Nash equilibria. We shall confine attentions to the class of ordinary social choice rules, which depend on the ordinal property of preferences only. It is shown, as the main contribution of this paper, that almost every ordinary social choice rule with some weak decisiveness condition, i.e., the no veto power condition, is strongly implementable under the above informational requirement.

In Section 2, we present the basic model-setting, and, in Section 3, the possibility theorem of weak implementation will be proved. In Sections

4, 5 and 6, we will present a characterization of ordinary social choice rules which are strongly implementable: That is, under the no veto power condition, the extended monotonicy condition is a necessary and sufficient condition of strong implementation of ordinary social choice rules. We will argue that the extended monotonicity condition is very weak, and we can say that almost every ordinary social choice rule are strongly implementable. Moreover, without contradiction of strong implementation, we can require a game form to satisfy the uniqueness of, not only Nash-equilibrium outcome but also, Nash equilibrium itself; i.e., to satisfy that the truthful revelation is always a unique Nash equilibrium. This will strengthen the meaning of our informational requirement.

2. Basic Model-Setting

N is the set of all agents, which represents a society. We assume that N is finite, and $n := |N| \geq 3$. A is the set of all alternatives. We assume that A is finite and $m := |A| \geq 2$. U is the set of all von Neumann-Morgenstern utility functions on A, u, which satisfy that for all $a,b \in A$,

$$u(a) \neq u(b) \text{ whenever } a \neq b.$$

Agent i has a von Neumann-Morgenstern utility function $u_i \in U$. Ψ is the set of all probabilities over A. An element $\rho \in \Psi$ is called a lottery. For each $\rho \in \Psi$ and for each $u \in U$, denote

$$u(\rho) = \sum_{a \in A} u(a)\rho(a).$$

u_i is called a cardinal preference of agent i. $w := (u_i)_{i \in N} \in U^n$ is a cardinal preference profile.

A <u>game form</u> is a mapping $g:S \to \Psi$, where $S := \times_{i \in N} S_i$, and $s_i \in S_i$ denotes a strategy of agent i. A <u>game</u> is a combination of a game form and a cardinal preference profile (g,w). A joint strategy $s^* \in S$ is said to be a <u>Nash equilibrium</u> of (g,w) if for every $i \in N$,

$$u_i(g(s^*)) \geq u_i(g(s_i, s^*_{-i})) \text{ for all } s_i \in S_i.$$

$NE(g,w)$ is the set of all Nash equilibria of (g,w), and the set of all Nash-equilibrium outcomes is defined as

$$g(NE(g,w)) := \{\rho \in \Psi: \rho = g(s) \text{ for some } s \in NE(g,w)\}.$$

A <u>probabilistic social choice rule</u>, briefly a <u>social choice rule</u>, is defined as a function $\varsigma:U^n \to \Psi$. For any cardinal preference profile $w \in U^n$, one interprets $\varsigma(w) \in \Psi$ as the welfare optimum.

3. Weak Implementation

We introduce the notion of weak implementation according to Maskin [6,7]: A game form g is said to <u>weakly implement</u> a social choice rule ζ if for each $w \varepsilon U^n$, there exists a Nash equilibrium s of (g,w) such that

$$g(s) = \zeta(w).$$

Fix a social choice rule ζ arbitrarily. We construct a game form $\tilde{g}$ which satisfies the following property: $S_i = U^3$ for all $i \varepsilon N$, and

$$\tilde{g}(s) = \zeta(w) \text{ whenever there is } i_0 \varepsilon N \text{ such that } s_i = w^i \text{ for all } i \varepsilon N/\{i_0\},$$

where $w = (u_i)_{i\varepsilon N}$, $w^i = (u_{i-1}, u_i, u_{i+1})$, and we write u_0 for u_n and u_{n+1} for u_1. We can check easily the existence of such $\tilde{g}$ from the fact that $n \geq 3$. We interpret $\tilde{g}$ as follows: Each agent announces an opinion about his own and his two neighbours' cardinal preferences. If all agents other than agent i_0 agree with a certain preference profile w, then $\zeta(w)$ is realized irrespective of agent i_0's opinion. Notice that a joint strategy $s \varepsilon S$ is a Nash equilibrium of $(\tilde{g},w)$ whenever there is $w' \varepsilon U^n$ such that

$$w'^i = s_i \text{ for all } i \varepsilon N,$$

where $w' = (u'_i)_{i\varepsilon N}$, and $w'^i = (u'_{i-1}, u'_i, u'_{i+1})$. Hence, for each $w \varepsilon U^n$, the truthful revelation, $(w^i)_{i\varepsilon N}$, is a Nash equilibrium of $(\tilde{g},w)$, and satisfies

$$\tilde{g}((w^i)_{i \in N}) = \zeta(w).$$

Thus, the following theorem holds true.

Theorem 1. $\tilde{g}$ weakly implements ζ.

Theorem 1 means that every social choice rule is weakly implementable, i.e, every social choice rule is weakly implemented by a game form. In the game form $\tilde{g}$, the truthful revelaton about cardinal preferences of agents i-1, i and i+1 is an optimal strategy of agent i, as long as the others conform the truthful revelation. Since the truthful revelation sustains the corresponding social optimum, we can say that for each agent i, complete knowledge about his own and his two neighbors' cardinal preferences is sufficient to enforce his optimal strategies.

4. Strong Implementation and Ordinary Social Choice Rules

Weak implementation will be inadequete, because this requires only the sustainability of the social optimum by a certain Nash equilibrium. There may exist multiple equilibria which sustain lotteries different from the social optimum.

Strong implementation, which is explored by Maskin [6], requires that all Nash equilibria sustain the social optimum: To be precise, a game form g is said to <u>strongly implement</u> a social choice rule ζ if for every $w \varepsilon U^n$,

$$g(NE(g,w)) = \{\zeta(w)\}.$$

Notice that if g strongly implements ζ, then g weakly implements ζ.

Our main purpose is to present a characterization of strongly implementable social choice rules. We shall confine attentions to the class of social choice rules which depend on the ordinal property of preference profile only. For the precise description of this confinement, several additional notations must be introduced: For each cardinal preference $u \varepsilon U$, we define $R(u)$ as a binary relation over A such that $aR(u)a$ for all $a \varepsilon A$, and for each $a \varepsilon A$ and $b \varepsilon A/(a)$,

$$[aR(u)b] \leftrightarrow [u(a) > u(b)].$$

$R(u)$ represents the ordinal property of u. Notice that $R = R(u)$ is a strict ordering, i.e., a binary relation over A that satisfies the following four properties;

Reflexivity: aRa for all $a \varepsilon A$.

Connectedness: aRb or bRa for all $a \varepsilon A$ and all $b \varepsilon A/(a)$.

Asymmetry: $[aRb] \rightarrow not[bRa]$ for all $a \varepsilon A$ and all $b \varepsilon A/(a)$.

Transitivity: $[aRb$ and $bRc] \rightarrow [aRc]$.

Ω is the set of all strict orderings.[3] Notice that for each $R \in \Omega$, there exists $u \in U$ such that $R(u) = R$.

R_i denotes an ordinal preference of agent i. Notice that $R_i = R(u_i)$ whenever u_i is the cardinal preference of agent i. $e := (R_i)_{i \in N} \in \Omega^n$ denotes an ordinal preference profile, and $e(w) := (R(u_i))_{i \in N}$ is the ordinal preference profile corresponding to the cardinal preference profile $w \in U^n$.

Based on these notations, we shall confine attentions to the class of social choice rules, ζ, which satisfies that

$$\zeta(w) = \zeta(w') \text{ whenever } e(w) = e(w').$$

Thus, ζ is represented by the following function f from Ω^n to Ψ; i.e., for every $w \in U^n$,

$$\zeta(w) = f(e(w)).$$

A function $f : \Omega^n \rightarrow \Psi$ is called an <u>ordinary social choice rule</u>. Weak implementation and strong implementation, which have already been defined above, are redefined with respect to ordinary social choice rules in the following way: A game form g is said to weakly implement an ordinary social choice rules f if for every $w \in U^n$, there is a Nash equilibrium s of (g,w) such that $g(s) = f(e(w))$. A game form g is said to strongly implement an ordinary social choice rule f if for every $w \in U^n$, $g(NE(g,w)) = f(e(w))$.

5. Extended Monotonicity and No veto Power

In Section 6, we shall present a characterization of ordinary social choice rules which are strongly implementable. Before presenting, we must introduce two conditions of social choice rules, i.e., the extended monotonicity condition and the no veto power condition.

Define

$$U(R) := \{u \varepsilon U: R(u) = R\}.$$

$U(R)$ is the set of all von Neumann-Morgenstern utility functions the ordinal properties of which are represented by R. R is said to <u>prefer</u> ρ <u>to</u> ρ' if either $\rho = \rho'$ or

$$u(\rho) > u(\rho') \text{ for all } u\varepsilon U(R).$$

We shall write $\rho R \rho'$ if R prefers ρ to ρ'. $\rho R \rho'$ implies that ρ is prefered to ρ' by all von Neumann-Morgenstern utility functions corresponding to R. Notice that R is a partial ordering over ψ by means of first-order stochastic dominance: That is,

Lemma 1. $\rho R \rho'$ <u>if and only if for every</u> $a \varepsilon A$,

$$\sum_{a'Ra} \rho(a') \geq \sum_{a'Ra} \rho'(a').$$

<u>Proof</u>. Suppose that for every $a \varepsilon A$,

$$\sum_{a'Ra} \rho(a') \geq \sum_{a'Ra} \rho'(a').$$

For each $r \varepsilon (1,\ldots,m)$, a_r is the (r)-th highest alternative of R. Let $u \varepsilon U(R)$.

Notice

$$u(\rho) = u(a_m) + \sum_{r=1}^{m-1} \{u(a_r) - u(a_{r+1})\} \sum_{aRa_r} \rho(a),$$

and

$$u(\rho') = u(a_m) + \sum_{r=1}^{m-1} \{u(a_r) - u(a_{r+1})\} \sum_{aRa_r} \rho'(a).$$

Since $u(a_r) - u(a_{r+1}) > 0$ for all $r \varepsilon \{1,\ldots,m-1\}$,

$$u(\rho) - u(\rho')$$

$$= \sum_{r=1}^{m-1} \{u(a_r) - u(a_{r+1})\}\{ \sum_{aRa_r} \rho(a) - \sum_{aRa_r} \rho'(a)\} \geq 0.$$

Hence, $\rho R \rho'$ holds.

Suppose that there exists $q \varepsilon \{1,\ldots,m\}$ such that

$$\sum_{aRa_q} \rho'(a) > \sum_{aRa_q} \rho(a).$$

We construct $u \varepsilon U(R)$ as follows; i.e.,

$$u(a_r) - u(a_{r+1}) = 1 \text{ whenever } r \neq q,$$

and

$$u(a_q) - u(a_{q+1}) = D.$$

Notice

$$u(\rho') - u(\rho)$$

$$= \sum_{r=1}^{m-1} \{u(a_r) - u(a_{r+1})\}\{ \sum_{aRa_r} \rho'(a) - \sum_{aRa_r} \rho(a)\}$$

$$\geq D\{ \sum_{aRa_q} \rho'(a) - \sum_{aRa_q} \rho(a)\} - m + 2.$$

Let D be a real number such that

$$D > \{ \sum_{aRa_q} \rho'(a) - \sum_{aRa_q} \rho(a)\}^{-1} (m - 2).$$

Then, $u(\rho') > u(\rho)$. Thus, not$[\rho R \rho']$.

Q.E.D.

For each lottery $\rho\varepsilon\Psi$, define a binary relation $>_\rho$ on Ω as follows;

$$R' >_\rho R \text{ if and only if } [\rho R\rho'] \rightarrow [\rho R'\rho'] \text{ for all } \rho'\varepsilon\Psi.$$

$R' >_\rho R$ implies that R' prefers ρ more than R.

Lemma 2. $R' >_\rho R$ <u>if and only if</u> $aRb \rightarrow aR'b$ <u>for all</u> $b\varepsilon A$ <u>and for all</u> $a\varepsilon A$ <u>such that</u> $\rho(a) > 0$.

<u>Proof</u>. The case of $\rho = \rho'$ is trivial. We shall consider the case of $\rho \neq \rho'$. Suppose that $aRb \rightarrow aR'b$ for all $b\varepsilon A$ and for all $a\varepsilon A$ such that $\rho(a) > 0$. Moreover, suppose $\rho R\rho'$. Fix $u'\varepsilon U(R')$ arbitrarily. There exists $u\varepsilon U(R)$ such that

$$u(a) = u'(a) \text{ if } \rho(a) > 0,$$

and

$$u(a) \geq u'(a) \text{ if } \rho(a) = 0.$$

Since $\rho R\rho'$, u satisfies $u(\rho) > u(\rho')$. Notice

$$u(\rho) = \sum_{a\varepsilon A} \rho(a)u(a) = \sum_{\rho(a)>0} \rho(a)u'(a) \leq \sum_{a\varepsilon A} \rho(a)u'(a) = u'(\rho),$$

and

$$u(\rho') = \sum_{a\varepsilon A} \rho'(a)u(a) \geq \sum_{a\varepsilon A} \rho'(a)u'(a) = u'(\rho').$$

Hence, $u'(\rho) > u'(\rho')$, and therefore, $\rho R'\rho'$.

Next, suppose that there exist $a\varepsilon A$ and $b\varepsilon A/\{a\}$ such that $\rho(a) > 0$, aRb and $bR'a$. We construct $\rho'\varepsilon\Psi$ as follows;

$$\rho'(c) = \rho(c) \text{ whenever } c \neq a \text{ and } c \neq b,$$

$$\rho'(a) = 0,$$

and

$$\rho'(b) = \rho(a) + \rho(b).$$

Notice that $\rho R \rho'$, whereas $\rho' R' \rho$, and therefore, not$[R' >_\rho R]$.

Q.E.D.

$R' >_\rho R$ implies that R' prefers every alternative in the support of ρ more than R. Notice that if the support of ρ is equal to A, then $R' >_\rho R$ implies $R' = R$: That is,

Lemma 3. Suppose that $\rho \varepsilon \Psi$ is totally mixed, i.e., $\rho(a) > 0$ for all $a \varepsilon A$. If $R' >_\rho R$, then $R' = R$.

Definition 1. An ordinary social choice rule f possesses the extended monotonicity condition, briefly EM, if and only if for all $e, e' \varepsilon \Omega^n$,

$$f(e) = f(e') \text{ whenever } R_i' >_{f(e)} R_i \text{ for all } i \varepsilon N,$$

where $e = (R_i)_{i \varepsilon N}$ and $e' = (R_i')_{i \varepsilon N}$.

EM implies that the social optimum is unchanged when the preference profile is changed from e to e', if for each agent i, R_i' prefers f(e) more than R_i.

EM is a necessary condition of strong implementation: That is,

Proposition 1. Suppose that an ordinary social choice rule f is strongly implementable, i.e., is strongly implementated by a game form g. Then, f possesses EM.

Proof. Fix $e = (R_i)_{i\varepsilon N} \varepsilon \Omega^n$ arbitrarily, and let $w = (u_i)_{i\varepsilon N}$ be a cardinal preference profile such that $e = e(w)$. Since g strongly implements f, there is a Nash equilibrium s of (g,w), which satisfies

$$g(s) = f(e).$$

Let $e' = (R_i')_{i\varepsilon N} \varepsilon \Omega^n$ be an ordinal preference profile such that

$$R_i' >_{f(e)} R_i \text{ for all } i\varepsilon N.$$

From Lemma 2, there exists $w' = (u_i')_{i\varepsilon N} \varepsilon U^n$ such that $e' = e(w')$, and for every $i\varepsilon N$,

$$u_i'(a) = u_i(a) \text{ if } f(e)(a) > 0,$$

$$u_i'(a) \leq u_i(a) \text{ if } f(e)(a) = 0.$$

We can show in the same way as the proof of Lemma 2 that for each $\rho\varepsilon\Psi$, if $u_i(f(e)) \geq u_i(\rho)$, then $u_i'(f(e)) \geq u_i'(\rho)$. Since $u_i(f(e)) \geq u_i(g(s_i', s_{-i}))$ for all $s_i'\varepsilon S_i$,

$$u_i'(f(e)) \geq u_i'(g(s_i', s_{-i})) \text{ for all } s_i'\varepsilon S_i.$$

This implies that s is also a Nash equilibrium of (g,w'). Since g strongly implements f, $g(s) = f(e(w')) = f(e')$. Therefore, $f(e) = f(e')$, that is, f possesses EM.

Q.E.D.

Moreover, we introduce the following condition of ordinary social choice rules, i.e., the no veto power condition: For each $a\varepsilon A$, define a subset $\phi(a)$ of Ω^n by

$$\phi(a) := \{e = (R_i)_{i\varepsilon N} \varepsilon \Omega^n : \text{ there is } i_0 \varepsilon N \text{ such that a is the top alternative of } R_i \text{ for all } i\varepsilon N/\{i_0\}\}.$$

$\phi(a)$ is a set of all preference profiles in which at least $(n - 1)$ agents agree with a.

Definition 2. An ordinary social choice rule f possesses the no veto power condition, briefly NVP, if and only if for every $a \in A$,

$$f(e)(a) = 1 \text{ for all } e \in \phi(a).$$

NVP requires that the majority is followed whenever at least $(n - 1)$ agents agree with a certain alternative.

6. Main Contribution

We will argue that under NVP, EM is a sufficient condition of strong implementation. Fix an ordinary social choice rule f arbitrarily. We construct a game form $\hat{g}:S \to \Psi$ in the following way: Let

$$S_i = \Omega^3 \text{ for all } i \varepsilon N.$$

The interpletation is that each agent announces an opinion about his own and his two neighbours' true ordinal preferences. Denote $s_i = (s_i^1, s_i^2, s_i^3) \varepsilon S_i$. s_i^1, s_i^2 and s_i^3 represent agent i's opinions about agent i-1's, agent i's and agent i+1's ordinal preferences respectively. Fix a joint strategy $s \varepsilon S$ arbitrarily. Three cases arise as follows.

Case 1. Opinions are consistent each other: That is, there is $e = (R_i)_{i\varepsilon N} \varepsilon \Omega^n$ such that

$$s_i = (R_{i-1}, R_i, R_{i+1}) \text{ for all } i\varepsilon N.$$

In this case, the unanimous opinion is followed, i.e.,

$$\hat{g}(s) = f(e).$$

Case 2. Opinions are consistent amongst only (n - 1) agents, except agent i_0: That is, there is $e = (R_i)_{i\varepsilon N} \varepsilon \Omega^n$ such that

$$s_i = (R_{i-1}, R_i, R_{i+1}) \text{ for all } i\varepsilon N/\{i_0\},$$

whereas

$$s_{i_0} \neq (R_{i_0-1}, R_{i_0}, R_{i_0+1}).$$

Notice from $n \geq 3$ that in Case 2, such e exists uniquely. In Case 2, $\hat{g}(s)$ is described by i_0, e and s_{i_0}. We shall require that in Case 2, $\hat{g}(s)$ is completely described by i_0, e and agent i_0's opinion about his own ordinal preference $s^2_{i_0}$ only. Hence, we write

$$\hat{g}(s) = \mu(i_0, e, s^2_{i_0}).$$

$\mu(i, e, R)$ will be specified later.

Case 3. Otherwise: That is, opinions are consistent amongst at most $(n - 2)$ agents.

Without loss of generality, let $A := \{1,\dots,m\}$. We define a product $\otimes$ on Ω as follows: Let $r\varepsilon\{1,\dots,m\}$. If a is the (r)-th highest alternative of R', then the (r)-th highest alternative of $R \otimes R'$ is equal to the (a)-th highest alternative of R. In Case 3, $\hat{g}(s)$ assigns probability 1 to the top alternative of

$$s^2_1 \otimes s^2_2 \otimes \dots \otimes s^2_n.$$

Notice that for every $i\varepsilon N$ and every $a\varepsilon A$, there is $R\varepsilon\Omega$ such that a is the top alternative of

$$s^2_1 \otimes \dots \otimes s^2_{i-1} \otimes R \otimes s^2_{i+1} \otimes \dots \otimes s^2_n.$$

We will specify $\mu(i,e,R)$ in the following way, where $e = (R_i)_{i\varepsilon N}\varepsilon\Omega^n$. Let $a_r\varepsilon A$ be the (r)-th highest alternative of R. Define

$$A_1 := \{a\varepsilon A\colon aR_i a_1\}.$$

A_1 is the set of all alternatives that are prefered to a_1 by R_i. For each $r\varepsilon\{2,\ldots m\}$, define

$$A_r := \{a\varepsilon A\colon aR_i a_r \text{ and } a\notin A_q \text{ for all } q\varepsilon\{1,\ldots r-1\}\}.$$

A_r is the set of all alternatives that are prefered to a_r by R_i but are not prefered to a_q by R_i for all $q\varepsilon\{1,\ldots,r-1\}$. $\mu(i,e,R)$ is a lottery such that for every $r\varepsilon\{1,\ldots,m\}$,

$$\mu(i,e,R)(a_r) = \sum_{a\varepsilon A_r} f(e)(a).$$

$\mu(i,e,R)$ is well-defined, because $\{A_r\}_{r=1}^m$ is a partition of A.

Lemma 4. $\mu(i,e,R)R\mu(i,e,R')$ <u>for all</u> $R'\varepsilon\Omega$.

<u>Proof</u>. Fix $r\varepsilon\{1,\ldots,m\}$ arbitrarily. By definition,

$$\sum_{aRa_r} \mu(i,e,R)(a) = \sum_{a\varepsilon \bigcup_{q=1}^{r} A_q} f(e)(a),$$

where $\{a_q\}_{q=1}^m$ and $\{A_q\}_{q=1}^m$ are defined according to R. By definition,

$$\bigcup_{q=1}^{r} A_q := \{a\varepsilon A\colon aR_i a_q \text{ for some } q\varepsilon\{1,\ldots,r\}\}.$$

Let a'_r be the (r)-th alternative of R', and define $\{A'_r\}_{r=1}^{m}$ according to R' in the same way as above. We define a permutation ν on $\{1,\ldots,m\}$ such that for every $q \varepsilon \{1,\ldots,m\}$,

$$a_q = a'_{\nu(q)}.$$

By definition,

$$\sum_{aRa_r} \mu(i,e,R')(a) = \sum_{a \varepsilon \bigcup_{q=1}^{r} A'_{\nu(q)}} f(e)(a).$$

Notice that if a is included in $\bigcup_{q=1}^{r} A'_{\nu(q)}$, then there is $q \varepsilon \{1,\ldots,r\}$ such that $aR_i a'_{\nu(q)}$, i.e., $aR_i a_q$. This implies that $\bigcup_{q=1}^{r} A'_{\nu(q)}$ is a subset of $\bigcup_{q=1}^{r} A_q$.

Hence, for every $r \varepsilon \{1,\ldots,m\}$,

$$\sum_{aRa_r} \mu(i,e,R)(a) \geq \sum_{aRa_r} \mu(i,e,R')(a).$$

From Lemma 1, this means $\mu(i,e,R) R \mu(i,e,R')$.

Q.E.D.

Lemma 5. $\mu(i,e,R) = f(e)$ <u>if and only if</u> $R >_{f(e)} R_i$.

<u>Proof</u>. Suppose $R >_{f(e)} R_i$. Let the (r)-th highest alternative of R, a_r, be in the support of f(e), i.e., $f(e)(a_r) > 0$. Suppose that a_r is included in A_q. Then $a_r R_i a_q$, and therefore, $a_r R a_q$, as is checked from Lemma 2. This implies that $r \leq q$. Since a_r is included in $\bigcup_{k=1}^{r} A_k$, it must hold that $r = q$, that is, $a_r \varepsilon A_r$. This means that for every $r \varepsilon \{1,\ldots,m\}$, there is no

alternative $a \varepsilon A$ such that $a \neq a_r$ and $f(e)(a) > 0$. Thus, $\mu(i,e,R) = f(e)$.

Next, suppose not$[R >_{f(e)} R_i]$. Then, there exist $a_r \varepsilon A$ and $a_k \varepsilon A/\{a_r\}$ such that $f(e)(a_r) > 0$, $a_r R_i a_k$, whereas $a_k R a_r$, i.e., $r > k$. Since

$$\bigcup_{q=1}^{k} A_k := \{a \varepsilon A\colon a R_i a_q \text{ for some } q \varepsilon \{1,\ldots,k\}\},$$

a_r must be included in $\bigcup_{q=1}^{k} A_k$, that is, there is an integer $h < r$ such that $a_r \varepsilon A_h$. Notice by definition that $a_q \varepsilon A_q$ whenever A_q is nonempty. Therefore,

$$\mu(i,e,R)(a_h) \geqq f(e)(a_h) + f(e)(a_r) > f(e)(a_h),$$

that is, $\mu(i,e,R) \neq f(e)$.

Q.E.D.

From Lemmas 4 and 5, it is shown that if $R >_{f(e)} R_i$, then

$$f(e) R \mu(i,e,R') \text{ for all } R' \varepsilon \Omega.$$

In the game form $\hat{g}$, the truthful revelation is a Nash equilibrium which sustains the social optimum assigned by the ordinary social choice rule f: That is,

Proposition 2. $\hat{g}$ <u>weakly implements</u> f.

<u>Proof</u>. Fix $e = (R_i) \varepsilon \Omega^n$ arbitrarily. By definition,

$$\hat{g}((e^i)_{i \varepsilon N}) = f(e),$$

where $e^i = (R_{i-1}, R_i, R_{i+1})$. We will prove that $(e^i)_{i\varepsilon N} \varepsilon S$ is a Nash equilibrium of $(\hat{g}, w)$ for all $w \varepsilon U^n$ such that $e(w) = e$. By definition of $\hat{g}$, notice that if $s_i \neq e^i$, then $(s_i, (e^j)_{j \neq i})$ is included in Case 2, and

$$\hat{g}(s_i, (e^j)_{j \neq i}) = \mu(i, e, s_i^2).$$

From Lemmas 4 and 5,

$$f(e) R_i \mu(i, e, R) \text{ for all } R \varepsilon \Omega.$$

Since $f(e) = \hat{g}((e^j)_{j \varepsilon N})$,

$$\hat{g}((e^j)_{j \varepsilon N}) R_i \hat{g}(s_i, (e^j)_{j \neq i}) \text{ for all } s_i \varepsilon S_i.$$

This implies that for every $u_i \varepsilon U(R_i)$,

$$u_i(\hat{g}((e^j)_{j \varepsilon N})) \geq u_i(\hat{g}(s_i, (e^j)_{j \neq i})) \text{ for all } s_i \varepsilon S_i,$$

that is, $(e^j)_{j \varepsilon N}$ is a Nash equilibrium of $(\hat{g}, w)$ for all $w \varepsilon U^n$ such that $e(w) = e$.

Q.E.D.

In the game form $\hat{g}$, we can show that for each agent $i \varepsilon N$, the best responses are described by the ordinal property of u_i, i.e., his ordinal preference $R(u_i)$ only: That is,

Proposition 3. <u>For each</u> $R \varepsilon \Omega$, <u>each</u> $i \varepsilon N$ <u>and each</u> $s \varepsilon S$, <u>there exists</u> $\tilde{s}_i \varepsilon S_i$ <u>such that</u>

$$\hat{g}(\tilde{s}_i, s_{-i}) R \hat{g}(s_i', s_{-i}) \text{ for all } s_i' \varepsilon S_i.$$

Proof. Fix $R\varepsilon\Omega$, $i\varepsilon N$ and $s\varepsilon S$ arbitrarily. Suppose that there is $e = (R_j)_{j\varepsilon N}\varepsilon\Omega^n$ such that $s_j = e^j$ for all $j\varepsilon N/\{i\}$. From the definition of $\hat{g}$, together with the fact that $\hat{g}((e^j)_{j\varepsilon N}) = \mu(i,e,R_i)$, we can easily check that for every $s_i' = (s_i'^1, s_i'^2, s_i'^3)\varepsilon S_i$,

$$\hat{g}(s_i', s_{-i}) = \mu(i,e,s_i'^2).$$

Let $\tilde{s}_i = (\tilde{s}_i^1, \tilde{s}_i^2, \tilde{s}_i^3)$ be a strategy of agent i such that

$$\tilde{s}^2 = R.$$

From Lemma 4,

$$\hat{g}(\tilde{s}_i, s_{-i}) = \mu(i,e,R) R \mu(i,e,s_i'^2) = \hat{g}(s_i', s_{-i})$$

for all $s_i'\varepsilon S_i$.

Suppose that there is no $e = (R_j)_{j\varepsilon N}\varepsilon\Omega^n$ such that $s_j = e^j$ for all $j\varepsilon N/\{i\}$. $\tilde{s}_i = (\tilde{s}_i^1, \tilde{s}_i^2, \tilde{s}_i^3)\varepsilon S_i$ is a strategy of agent i such that

$$\tilde{s}_i^1 \neq s_{i-1}^2 \text{ and } \tilde{s}_i^3 \neq s_{i+1}^2.$$

Notice that $(\tilde{s}_i, s_{-i})$ is included in Case 3. Let $\tilde{R}$ be an ordinal preference such that the top alternative of R is equal to the top alternative of

$$s_1^2 \boxtimes \dots \boxtimes s_{i-1}^2 \boxtimes \tilde{R} \boxtimes s_{i+1}^2 \boxtimes \dots \boxtimes s_n^2.$$

Let $\tilde{s}_i^2 = \tilde{R}$. By definition of $\hat{g}$, $\hat{g}(\tilde{s}_i, s_{-i})$ assigns propability 1 to the top alternative of R. Therefore,

$$\hat{g}(\tilde{s}_i, s_{-i}) R \hat{g}(s_i', s_{-i}) \text{ for all } s_i' \varepsilon S_i.$$

Q.E.D.

Proposition 3 quarrantees that the set of Nash equilibrium depends on the ordinal property of preference profile only, that is,

$$NE(g,w) = NE(g,w') \text{ whenever } e(w) = e(w').$$

Based on these argument, we shall show that EM, together with NVP, is a sufficient condition of strong implementation: That is,

Theorem 2. <u>Suppose that</u> f <u>possesses</u> NVP <u>and</u> EM. <u>Then</u> $\hat{g}$ <u>strongly implements</u> f.

<u>Proof</u>. We have already shown that $\hat{g}$ weakly implements f. All we have to do is to show that if s is a Nash equilibrium of $(\hat{g},w)$, then $\hat{g}(s) = f(e(w))$.

Suppose that s is included in Case 1, i.e, there exists $e = (R_i)_{i \varepsilon N} \varepsilon \Omega^n$ such that $s_i = e^i = (R_{i-1}, R_i, R_{i+1})$ for all $i \varepsilon N$. Suppose that s is a Nash equilibrium of $(\hat{g},w)$. From the argument of the proof of Proposition 3, notice that for every $i \varepsilon N$,

$$\mu(i,e,R(u_i)) = \hat{g}(s) = f(e)$$

must hold, where $w = (u_i)_{i\varepsilon N}$. This implies, from Lemma 5, that $R(u_i) >_{f(e)} \hat{R}_i$ for all $i\varepsilon N$. From EM, $\hat{g}(s) = f(e(w))$ must hold.

Suppose that s is included in Case 2, i.e, there exist $i_0\varepsilon N$ and $e\varepsilon\Omega^n$ such that $s_i = e^i$ for all $i\varepsilon N/\{i_0\}$, whereas $s_{i_0} \neq e^{i_0}$. Notice that for every $i\varepsilon N/\{i_0\}$, there is no $e'\varepsilon\Omega$ such that $s_j = e'^j$ for all $j\varepsilon N/\{i\}$. Suppose that s is a Nash equilibrium of (g,w). From the argument in the proof of Proposition 3, $\hat{g}(s)$ must assign probability 1 to the top alternative of $R(u_i)$ for all $i\varepsilon N/\{i_0\}$. From NVP, $\hat{g}(s) = f(e(w))$ must hold.

Suppose that s is included in Case 3. Notice that for every $i\varepsilon N$, there is $e\varepsilon\Omega$ such that $s_j = e^j$ for all $j\varepsilon N/\{i\}$. Suppose that s is a Nash equilibrium of (g,w). From the argument in the proof of Proposition 3, $\hat{g}(s)$ must assign probability 1 to the top alternative of $R(u_i)$ for all $i\varepsilon N$. From NVP, $\hat{g}(s) = f(e(w))$ must hold.

Q.E.D.

7. Discussion

7.1. Weakness of Extended Monotonicity

The extended monotonicity condition is amazingly weak: Suppose that an ordinary social choice rule f possesses NVP and assigns positive probability to every alternative whenever the preferences are conflicting, i.e,

(1) f(e) is totally mixed whenever there is no $a\varepsilon A$ such that $e\varepsilon\phi(a)$.

Notice from Lemma 3 and Theorem 2 that such a rule f satisfies EM, and therefore, is strongly implementable. We can say that almost every ordinary social choice rule with NVP satisfies (1), and therefore, almost every ordinary social choice rule with NVP is strongly implementable.

7.2. Uniqueness of Nash Equilibrium

Strong Implementation does not require the uniqueness of Nash equilibrium. Under the multiple existence of Nash equilibria, it is difficult to interplet the notion of Nash equilibrium conceptually. This will make obscure the meaning of informational availability in our argument. From the proof of Theorem 2, however, we can find that the uniqueness of Nash equilibrium will be compatible with strong implementation in general: That is,

Proposition 4. Suppose that f satisfies NVP and (1). Then $(e^i)_{i\varepsilon N}$ is a unique Nash equilibrium of $(\hat{g},w)$ whenever $e = e(w)$ and there is no $a\varepsilon A$ such that $e\varepsilon\phi(a)$.

For almost every ordinary social choice rule, the truthful revelation in the game form $\hat{g}$ is a unique Nash equilibrium whenever the preferences are conflicting.

7.3. Epsilon Implementation

Introducing the following weaker version of strong implementation will make clearer the possibility of implementation; which is ε-implementation. Let ε be a nonnegative real number. An ordinary social choice rule f is said to be ε-equivalent to an ordinary social choice rule f' if for every $e \in \Omega^n$,

$$\max_{a \in A} |f(e)(a)-f'(e)(a)| \leq \varepsilon.$$

A game form g is said to ε-implement an ordinary social choice rule f if there is another rule f' such that

f' is ε-equivalent to f,

and

g strongly implements f'.

f is said to be ε-implementable if there is a game form g that ε-implements f. Notice that when ε = 0, ε-implementation is equal to strong implementation.

Proposition 5. Suppose ε > 0. Then, every ordinary social choice rule f with NVP is ε-implementable.

Proof. We can find another rule f' with NVP and (1) that is ε-equivalent to f. Since f' is strongly implementable, f is ε-implementable.

Q.E.D.

Notice from Proposition 4 and the proof of Proposition 5 that every ordinary social choice rule is ε-implemented by a game form that satisfies the uniqueness of Nash equilibrium whenever the preferences are conflicting in the above sense.

7.4. Minimal Efficiency

An ordinary social choice rule with (1) will assign positive probability to the alternative that is a low rank of all agents' preferences. This observation leads one to the study of implementation under some efficiency condition. We shall confine attentions to the class of ordinary social choice rules that possess the minimal efficiency condition, which is defined as follows: For every $e \in \Omega^n$, A(e) is defined as a minimal set among subsets of A, $\hat{A}$, such that for all $a \in \hat{A}$ and for all $b \notin \hat{A}$,

$$aR_i b \text{ for all } i \in N.$$

Any alternative that is not included in A(e) is disliked by all agents, and therefore, should not be realized: That is,

Definition 3. An ordinary social choice rule f possesses <u>the minimal efficiency condition</u> if

(i) the support of f(e) is a subset of A(e) for all $e \in \Omega^n$,

and

(ii) $f(e) = f(e')$ whenever $A(e) = A(e')$ and $[aR_i b] \leftrightarrow [aR_i' b]$ for all $a, b \in A(e)$.

The latter requirement (ii) implies that choice of social optima is independent of irrelevant alternatives that are disliked by all agents.

Suppose that an ordinary social choice rule f possesses NVP and the minimal efficiency condition, and satisfies that the support of f(e) is equal to A(e) whenever there is no $a\varepsilon A$ such that $e\varepsilon\Phi(a)$. Then, f possesses EM, and therefore, is strongly implementable. Hence, the generic posibility of strong implementation holds even though we restricts probabilistic choice in the reasonable way.

7.5. Role of Probabilistic Choice

However, our result will depend essentially on the permission of probabilistic choice: Probabilistic rules have received a considerable amount of attention in the theory of social choice, especially because the probabilistic framework provides many plausible rules for aggregating individual preferences which provide scope for incooperating certain notions of fairness and reasonable compromise (see Pattanaik and Peleg [11]).

An ordinary social choice rule f is said to be deterministic if for all $e\varepsilon\Omega^n$, there is an alternative $a\varepsilon A$ such that

$$f(e)(a) = 1.$$

For convenience, a deterministic ordinary social choice rule is denoted by a function $v:\Omega^n \to A$. If f possesses EM and is deterministic, i.e., is denoted by $v:\Omega^n \to A$, then for every $e,e'\varepsilon\Omega^n$,

$$v(e') = v(e) \text{ whenever } [v(e)R_i b] \to [v(e)R'_i b] \text{ for all } b\varepsilon A \text{ and all } i\varepsilon N.$$

This is the "well-known" monotonicity condition, which has been explored by Gibbard [4], Muller and Satterthwaite [9], Maskin [6], and so on. In Muller and Satterthwaite [9], the following result has been proved.

Proposition 6. Suppose that v possesses the monotonicity condition and the unanimity condition, i.e., satisfies that

$$v(e) = a \text{ whenever } aR_i b \text{ for all } b \varepsilon A \text{ and for all } i \varepsilon N.$$

Then, v is dictatorial, i.e., there is an agent $i\varepsilon N$ such that

$$v(e) = a \text{ whenever } aR_i b \text{ for all } b \varepsilon A.$$

NVP implies the unanimity condition, whereas the existence of dictator contradicts NVP. Thus, we have the following negative conclusion:

Proposition 7. If a deterministic ordinary social choice rule satisfies the unanimity condition and is strongly implementable, then, it is dictatorial. Moreover, there is no deterministic ordinary social choice rule which satisfies NVP and is strongly implementable.

Proof. Straightforward from Theorem 2 and Propositions 1 and 6.

Q.E.D.

7.6. Maskin's Nash Implementation

Maskin [6] considered the Nash implementation problem of deterministic social choice correspondences. He introduced revelation mechanisms in which each agent announces an opinions about all agents' preferences and recommends an alternative. Our work depends on Maskin [6] in the technical respect. However, Nash implementation makes sense on the assumption of complete information about all agents' preferences, and the extension to correspondences must accompany the multiple existence of Nash equilibria. From these criticisms, Maskin's notion is inappropriate for the study of implementation.

Footnotes

[1] The incompleteness of information in this paper is regarded as the special case of the Bayesian sense.

[2] Precisely, all agents' preferences are assumed to be common knowledge in Aumann's sense. See Aumann [1].

[3] Throughout this paper, we shall use "R" as both a function and variable.

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