No. 367

AHP applied to Binary and Ternary Comparisons

by
Iwaro Takahashi

June 1988

AHP applied to Binary and Ternary Comparisons

1 Introduction

The intrinsic feature of AHP (Analitic Hierarchy Process) is to evaluate objects by a eigen vector of a matrix whose components are ratios taken by paired comparisons [1] [2] [3]. But often we have only binary information such as "good or bad", or "victory or defeat" in sport games or matches, by paired comparisons.

Let us call such a problem as "binary comparisons". AHP is a very useful tool to evaluate objects in binary comparisons, and in this special case we have some beautifull results in eigen value problems (§3).

Further in case of sport games "tie" often occurs, and also in general evaluation we have often situations "ofject i is as good as object j" in addition to binary informations. Let us call such a problem as "ternary comparison". Say, in sport games, it is often difficult to compair a team to another, each belonging to a different league. But our ternary comparison method gives an appropriate criterion for this problem.(§4)

2 Basic Property

Let $E = \{1, 2, \cdots, n\}$ be the set of objects $i = 1, 2, \cdots, n$. Let a_{ij} ($>$o) be the ratio of evaluation of object i to object j. Thus we have a matrix

$$(2.1) \qquad A = \begin{bmatrix} 1 & a_{12} & \cdots a_{1n} \\ a_{21} & 1 & \cdots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots 1 \end{bmatrix}$$

which is called comparison matrix.

We have the following well known theorem in linear algebra,

Theorem 1. (Perron & Frobenius)

The maximal eigen value of a matrix, whose components are all positve, is a simple positve root, and the components of the corresponding eigen vetor can be all positive. [1] [4]. ――――――

Thus a comparison matrix A has a maximal eigen value λ ($>$o) and the corresponding eigen vector

$$\boldsymbol{w} = [w_1\ w_2\ \cdots w_n],\ w_i > \text{o}\ (i = 1, \cdots, n)$$

The basic idea of AHP is to take w_i as evaluation of object i ($i = 1, \cdots, n$), [2] [3].

If we have

(2. 2) $a_{ij}\, a_{ji}=1\ (i \neq j)$

then $\{i, j\}$ is called consistent. If all $\{i, j\}$ $(i, j=1, \cdots, n;\ i \neq j)$ are consistent, A is a reciprocal matrix and the maximal eigenvalue λ alway satisfies

(2. 3) $\lambda \geqq n$

[1]. Hereafter we assume that all $\{i, j\}$ are consistent. Further if we have

(2. 4) $a_{ij}\, a_{jk}\, a_{ki}=1\ (i \neq j \neq k \neq i)$

then $\{i, j, k\}$ is called consistent. Then we have the following theorem.

Theorem 2. The necessary and suficient condition for a_{ij} (components of a comparison matrix A) to be represented as

(2. 5) $a_{ij}=w_i/w_j\ (i, j=1, \cdots, n)$

by some positive value w_i $(i=1, \cdots, n)$ is that all $\{i, j, k\}$ $(i, j, k=1, \cdots, n;\ i \neq j \neq k \neq i)$ are consistent.

Proof: If is clear that if (2. 5) is valid then we have (2. 4) for all $\{i, j, k\}$. So we have only to show the validity of (2. 5) from (2. 4). Let us take $w_1, \cdots, w_n$ as

(2. 6) $w_1=1,\ w_2=a_{21},\ w_3=a_{31}, \cdots, w_n=a_{n1}$

, then for any i, j we have

$$a_{ij}\, a_{j1}\, a_{1i}=1, \quad a_{1i}=1/a_{i1}$$

from (2. 4) and (2. 2). So we have

$$a_{ij}=\frac{1}{a_{j1}\cdot a_{1i}}=\frac{1}{w_j \cdot 1/w_i}=\frac{w_i}{w_j} \qquad \rule{4cm}{0.4pt}.$$

If components a_{ij} of a comparison matrix A satisfy (2. 5), then $E=\{1, 2, \cdots, n\}$ (or A it self) is called consistent. Thus theorem 2 states that the necessary and sufficient condition of consistency of $E=\{1, 2, \cdots, n\}$ is that any 3-set $\{i, j, k\} \subseteqq E$ is consistent.

The following theorem is fundamental and well known [1][3], but we will give our own elementary proof for it.

Theorem 3. If an nxn comparison matrix $A=[a_{ij}]$ is consistent then the maximal eigenvalue of A is equal to n, and its eigenvector is

(2. 7) $w=[1, a_{21}, \cdots, a_{n1}]$

Proof: The proper equation of A is

$$\begin{vmatrix} 1-\lambda & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & 1-\lambda & \cdots & w_2/w_n \\ \vdots & \vdots & & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & 1-\lambda \end{vmatrix} = 0 .$$

Multipling i-th column by w_i and dividing j-th column by w_j, we have

$$\begin{vmatrix} 1-\lambda & 1 & \cdots & 1 \\ 1 & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1-\lambda \end{vmatrix} = 0$$

Adding i-th column ($i=2, 3, \cdots, n$) to the first column we have

$$\begin{vmatrix} n-\lambda & 1 & \cdots & 1 \\ n-\lambda & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ n-\lambda & 1 & \cdots & 1-\lambda \end{vmatrix} = (n-\lambda) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\lambda & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1-\lambda \end{vmatrix} = 0$$

Subtracting the first row from all other rows we have

$$(n-\lambda) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -\lambda \end{vmatrix} = \pm(n-\lambda)\lambda^{n-1} = 0$$

which states that eigenvalues of A are n and zero, so the maximal eigenvalue is n.

Further multipling w (represented by (2. 7)) by A we have $Aw=nw$. So w is the eigen vector for $\lambda=n$.

3 Binary Comparisons

We consider, say, sport games or matches among n teams $\{1, 2, \cdots, n\}$. In such case the information taken from a match between team i and j is only "vicfory of defeat". If team i wins and j loses then let a_{ij} and a_{ji} be

(3. 1) $\qquad a_{ij}=\theta,\ a_{ji}=1/\theta$

with a paraneter θ (>1). Then from a league tourmament in $\{1, 2, \cdots, n\}$ we have a comparison matrix $A=\{a_{ij}\}$, whose non diagonal elements are θ or $1/\theta$. Let us call such a problem a "binary comparison (with parameter θ)".

In a general evaluation problem if we require only "good or bad" on paired comparison this becomes also a binary comparison.

In a binary comparison on $E=\{1, 2, \cdots, n\}$ if the condition

(3. 2) $\qquad a_{ij}>1, a_{jk}>1$ implies $a_{ik}>1$ $(i\neq j\neq k\neq i)$

holds then let $\{i, j, k\}$ ($\subseteq$E) be called "logically consistent". Note that if $\{i, j, k\}$ is consistent then this is logically consistent, but the inverse is not necessarily true. If any 3-set $\{i, j, k\}$ $\subseteq E$ is logically consistent then let "the binary comparison on E be called logically consistent."

We construct a directed graph corresponding to a comparison matrix $A = [a_{ij}]$ of a binary comparison on $E = \{1, 2, \cdots, n\}$ by the following way ; we have a directed arc connecting point i to j iff $a_{ij} > 1$ (that is $a_{ij} = \theta$), being E the set of points of the graph. (see Ex 1, Ex 2 …).

Then the graph of a logically consistent binary comparison on E is acyclic and any two points have a directed arc, so E is a totally ordered set, where i precedes j iff $a_{ij} > 1$. Let us renumber points in E along this order, then the comparison matrix A is written as

$$(3.\ 3)\qquad A = \begin{bmatrix} 1 & \theta & \theta & \cdots & \theta \\ 1/\theta & 1 & \theta & \cdots & \theta \\ 1/\theta & 1/\theta & 1 & \cdots & \theta \\ \vdots & \vdots & \vdots & & \vdots \\ 1/\theta & 1/\theta & 1/\theta & \cdots & 1 \end{bmatrix}$$

Theorem 4. A comparison matrix A (represented in the from (3. 3)) of a logically consistent binary comparison has the maximal eigen valne

$$(3.\ 4)\qquad \lambda = 1 + \theta\ (w + w^2 + \cdots + w^{n-1})$$

and the corresponding eigen vector

$$(3.\ 5)\qquad \boldsymbol{w} = [1, w, w^2, \cdots, w^{n-1}]$$

where

$$(3.\ 6)\qquad \theta^2 w^n = 1$$

Proof: Firstly we show that

$$(3.\ 7)\qquad A\boldsymbol{w} = \lambda\,\boldsymbol{w}$$

, that is, $\boldsymbol{w}$ is the eigen vector corresponding to λ. Let $(A\boldsymbol{w})_i$ be the i-th component of $A\boldsymbol{w}$, then we have

$$(3.\ 8)\qquad \begin{aligned} &(A\boldsymbol{w})_1 = 1 + \theta\ (w + w^2 + \cdots + w^{n-1}) = \lambda \\ &(A\boldsymbol{w})_i = (1 + w + \cdots + w^{i-2})/\theta + w^{i-1} + \theta\ (w^i + \cdots + w^{n-1}) \\ &\qquad (i = 2, \cdots, n, \text{ the third term vanishes when } i = n) \end{aligned}$$

On the other hand we have

$$\begin{aligned} \lambda w^{i-1} &= w^{i-1} + \theta\ (w^i + \cdots + w^{n-1} + w^n + \cdots + w^{n+i-2}) \\ &= w^{i-1} + \theta\ (w^i + \cdots + w^{n-1}) + \theta\, w^n\ (1 + \cdots + w^{i-2}) \\ &= w^{i-1} + \theta\ (w^i + \cdots + w^{n-1}) + 1/\theta\,(1 + \cdots + w^{i-2}), \end{aligned}$$

which is equal to (3. 8), so we have (3. 7).

Next we show that λ is the maximal eigen value of A. Let the maximal eigen value of A be ρ, then the transposed matrix A' of A has the same maximal eigen value ρ. From Theorem 1, ρ is positive and the corresponding eigen vector $\boldsymbol{x}$ of A' is positive (that is, its components are all positive).

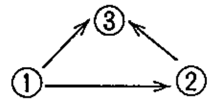
Now we have

$$\rho\ (x'w) = (\rho x)'w = (A'x)'\ w = x'A\,w = \lambda\ (x'w) \tag{3. 9}$$

, and both x and w are positive vectors so $x'w$ cannot vanish, therefore we have $\lambda = \rho$.

Example 1 For a comparison matrix and its graph shown below

$$A = \begin{bmatrix} 1 & \theta & \theta \\ 1/\theta & 1 & \theta \\ 1/\theta & 1/\theta & 1 \end{bmatrix}$$



we have

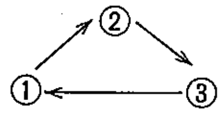
$\lambda = 3.0536$ $w = [1,\ 0.63,\ 0.3968]$ ($\theta = 2$). from Theorem 4. Instead w we often use standardized vector $\overline{w}$, the sum of whose elements is equal to unity. Thus we have

$\overline{w} = [0.4934,\ 0.3108,\ 0.1958]$.

Of course we often encounter non consistent binary comparisons in the real world. Though A is not consistent the evalnation based on the eigen vector gives an apropriate criterion.

.**Example 2.**

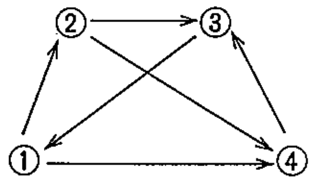
$$A = \begin{bmatrix} 1 & \theta & 1/\theta \\ 1/\theta & 1 & \theta \\ \theta & 1/\theta & 1 \end{bmatrix}$$



$\lambda = 3.5$ $w = [1,\ 1,\ 1]$ ($\theta = 2$)

Example 3.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & \theta & 1/\theta & \theta \\ 1/\theta & 1 & \theta & \theta \\ \theta & 1/\theta & 1 & 1/\theta \\ 1/\theta & 1/\theta & \theta & 1 \end{bmatrix} \end{array}$$



$\lambda = 4.644$ ($\theta = 2$)

$w = [1,\ 0.9396,\ 0.7724,\ 0.6899]$

$\overline{w} = [0.2940,\ 0.2762,\ 0.2133,\ 0.2028]$.

Let A represent scores of a league tournament on $E = \{1,\ 2,\ 3,\ 4\}$. Both teams 1 and 2 win two matches and lose one, but team 1 gets higher evaluation than team 2. The reason is rather complicated but mainly is that team 1 defeats team 2.

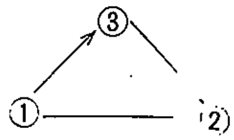
4 Ternary Comparisons

In case of sport games "tie" often occurs. We assume that if team i ties team j then $a_{ij}=1$, then nondiagonal elements of the comparison matrix $A=[a_{ij}]$ are θ, $1/\theta$ or 1. Let such a problem be called a "ternary comparison". In general evaluation problem if object i is as good as object j, or i is equivalent to j then a_{ij} is to be unity, which leads to a ternary comparison. In the graph representation of a ternary comparison, if $a_{ij}=1$ then point i is to be connected to point j with an undirectde arc.

In this section we will give an evaluation criterion $n\bar{w}$ and show how this is appropriate for evalution amang different groups of objects through various examples.

Example 4.

$$A=\begin{bmatrix}1 & 1 & \theta \\ 1 & 1 & 1 \\ 1/\theta & 1 & 1\end{bmatrix}$$



$\lambda=3.054$ ($\theta=2$)

$w=[1, 0.7937, 0.63]$

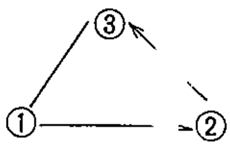
$\bar{w}=[0.4126, 0.3275, 0.26]$

(Incidentally we have $\lambda=1+w+\theta w^2$, $\theta w^3=1$,

$w=[1, w, w^2]$)

Example 5.

$$A=\begin{bmatrix}1 & \theta & 1 \\ 1/\theta & 1 & \theta \\ 1 & 1/\theta & 1\end{bmatrix}$$



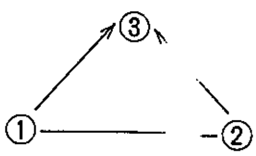
$\lambda=3.217$ ($\theta=2$)

$w=[1, 0.7937, 0.63]$

($\lambda=1+\theta w+w^2$, $\theta w^3=1$, $w=[1, w, w^2]$

Example 6. (consistent)

$$A=\begin{bmatrix}1 & 1 & \theta \\ 1 & 1 & \theta \\ 1/\theta & 1/\theta & \theta\end{bmatrix}$$

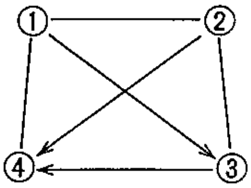


$\lambda=3$, $w=[1, 1, 1/\theta]$

$\bar{w}=[0.4\ 0.4\ 0.2]$ ($\theta=2$)

Example 7.

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & \theta & 1 \\ 1 & 1 & 1 & \theta \\ 1/\theta & 1 & 1 & \theta \\ 1 & 1/\theta & 1/\theta & 1 \end{bmatrix} \end{array}$$



$\lambda = 4.186$ ($\theta = 2$)

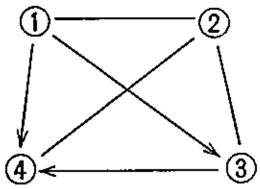
$w = [1, 0.9443, 0.8248, 0.5916]$

$\bar{w} = [0.2976, 0.2810, 0.2454, 0.1760]$.

Both teams 1 and 2 win one match and tie two matches, but team 1 gets higher evaluation.

Example 8

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & \theta & \theta \\ 1 & 1 & 1 & 1 \\ 1/\theta & 1 & 1 & \theta \\ 1/\theta & 1 & 1/\theta & 1 \end{bmatrix} \end{array}$$



$\lambda = 4.121$, ($\theta = 2$)

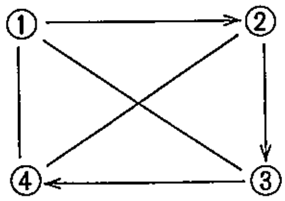
$w = [1, 0.7071, 0.7071, 0.5]$

$\bar{w} = [0.3431, 0.2426, 0.2426, 0.1716]$

Team 2 ties all matches and team 3 wins one and loses one, but they have the same evaluation. (Let $\theta = 4$ then we have $\lambda = 4.5$, $w = [1, 0.5, 0.5, 0.25]$)

Example 9

$$A = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & \theta & 1 & 1 \\ 1/\theta & 1 & \theta & 1 \\ 1 & 1/\theta & 1 & \theta \\ 1 & 1 & 1/\theta & 1 \end{bmatrix}$$



$\lambda = 4.3101$ ($\theta = 2$)

$w = [1, 0.8787, 0.8559, 0.6988]$

$\bar{w} = [0.2913, 0.2559, 0.2493, 0.2035]$ ————.

Generally each component of $n\,\bar{w}$ is appropriate criterion on the evaluation among different league tournaments. For example, we have the values of $n\,\bar{w}$ for Ex. 4, Ex. 6, Ex. 7 and Ex. 8,

Ex. 4 : $3\bar{w} = [1.238, 0.9825, 0.78]$

Ex. 6 : $3\bar{w} = [1.2, 1.2, 0.6]$

Ex. 7 : $4\bar{w} = [1.1902, 1.124, 0.9817, 0.7041]$

Ex. 8 : $4\bar{w} = [1.375, 0.972, 0.972, 0.688]$

In Ex. 6 team 1 wins one, ties one and gets score 1.2, while in Ex. 7 team 1 wins one, ties two and gets score 1.1902 slightly smaller than the former. Further team 3 in Ex. 7 and team 3 in Ex. 8 win one, lose one and tie one, but the former gets 0.9817 and the latter 0.972 slightly smaller than the latter.

It can be said that these evaluation scores are very close to our intuitive evaluation for abilities of these teams.

References

[1] Saaty, T.L.: A Scaling Method for Priorities in Hierarchical Structures, J. of Mathematical Psychology, Vol. 15, (1977), 234−281.

[2] Saaty, T.L.: The Analytic Hierarchy Process, Mc Graw-Hill (1980)

[3] Manabe, R.: AHP (Analytic Hierarchy Process) (in Japanese), Communications of the O.R. Society of Japan, Vol. 31, (1986), 475−478.

[4] Ito, N & others: Matrices and their Applicalins (in Japanese), Kinokuniya Publishing Co. (1987)