Institute of Socio-Economic Planning
Discussion Paper Series

No. 365

FIXED POINT ALGORITHMS FOR STATIONARY POINT PROBLEMS *

Yoshitsugu Yamamoto

June 1988

* to appear in the Proceedings of the 13th International Symposium on Mathematical Programming

Institute of Socio-Economic Planning
University of Tsukuba
Tsukuba, Ibaraki 305, Japan

FIXED POINT ALGORITHMS FOR STATIONARY POINT PROBLEMS

Yoshitsugu Yamamoto
Institute of Socio-Economic Planning
University of Tsukuba
Tsukuba, Ibaraki 305, Japan

ABSTRACT: Over several years, we have had a new class of fixed point algorithms called variable dimension algorithms. They were originally for finding a fixed point on the unit simplex and then have been extended to the solution of systems of nonlinear equations. Recently were developed several algorithms tailored for the stationary point problem, which could be viewed as a fixed point problem without any boundary condition. We first demonstrate that the stationary point problem affords a unifying view on problems arising from various fields such as mathematical economics, game theory and mathematical programming. Then we review the basic and common ideas of the variable dimension algorithms, path-following and the primal-dual pair of subdivided manifolds by taking two algorithms for the stationary point problem and the equilibrium problem.

1. INTRODUCTION

The aim of this paper is to provide an introductory review of variable dimension algorithms for stationary point problems. The algorithm can be traced back to the works of Kuhn [K3] and Shapley [S3] but was substantially improved by van der Laan and Talman [LT1,2,3,4]. It is distinguished by three features: no artificial dimension, arbitrary starting point and generating simplices of variable dimensions. The algorithm was originally developed for finding Brouwer's fixed point on the unit simplex and then extended to the solution of systems of nonlinear equations on R^n. In this process of extension several new algorithms joined in and now it is not a sole algorithm but a class of algorithms based on the same idea. See [T3,4],[K2],[KY2] and [B] for introduction and [F2,3],[KY1,3],[L],[LS],[LT5,6,7,8,9],[T1],[TV],[W] and [Y1,2,3,6] for algorithms and theory.

The stationary point problem, often referred to as the variational inequality problem, is the one embracing problems of economic equilibrium, nonlinear complementarity, nonlinear program and traffic assignment. The early studies were carried out mainly for infinite-dimensional problems in the area of control theory, for which the reader is referred to the books by Kinderlehrer and Stanpacchia [KS] and Glowinski et al.[GLT]. The problem can be indeed reformulated as a fixed point problem through the projection, but solving the original problem directly sounds more reasonable than solving the reformulated one. Mathematical equivalence does not always imply computational equivalence. van der Laan and Talman's adapting their variable dimension algorithm to the linear complementarity problem with upper and lower bounds led a series of works of developing a tailored algorithm for the stationary point problems. It is closely related with the unproper integer labelling [LTV2] and also with the generalization of Sperner's Lemma [F1],[F4,5] and [Y6].

In the following sections after citing several problems that are reduced to the stationary point problem, we will briefly review the primal-dual pair of subdivided manifolds in Section 3 and take two variable dimension algorithms to explain the basic and common idea. We confine ourselves to the variable dimension algorithm, so the reader is referred to the survey by Allgower and Georg [AG], the monograph by Todd [T2] and the book by Garcia and Zangwill [GZ] for other algorithms such as homotopy method.

2. EXISTENCE OF A STATIONARY POINT AND APPLICATIONS

Let K be a closed convex subset of R^n and let f be a continuous function from K into R^n. A point s of K is called a stationary point of the problem (f,K) if it satisfies the so-called variational inequality:

(2.1) $\langle x - s, f(s)\rangle \geq 0$ for any $x \in K$.

This inequality geometrically means that $f(s)$ is an inward normal to K when s is on the boundary of K and s is a zero point of f when s is in the interior of K. If we define the normal cone to K at point x of R^n as

$$N_K(x) = \emptyset \qquad \text{if } x \notin K;$$
$$= \{ y : \langle y, z-x\rangle \leq 0 \text{ for any } z \in K \} \qquad \text{if } x \in K,$$

then (2.1) will hold if and only if s satisfies the generalized equation (see [R])

$$0 \in f(s) + N_K(s).$$

The following theorem (2.2) states the equivalence between the existence of a stationary point and Brouwer's Fixed Point Theorem (see for example [I]).

(2.2) Theorem: Suppose K is a nonempty compact convex subset of R^n. Then the following two statements are equivalent:

(2.3a) any continuous function g from K into K has a fixed point;

b) for any continuous function f from K into R^n there is a stationary point of (f,K).

Proof: Let $f(x) = g(x) - x$ and let s be a stationary point of (f,K). Then since g maps K into K, it is readily seen that $f(s) = 0$, i.e., s is a fixed point of g. To show the converse let $r(z) = \text{argmin} \{ \|z-x\| : x \in K \}$ for each $z \in R^n$ and consider $g(x) = r(x-f(x))$, where $\|.\|$ denotes Euclidean norm. r is nonexpansive and hence continuous (see for example [KS, Corollary 2.4]). By construction g is also continuous and has a fixed point, which is seen to be a stationary point of (f,K). //

Now let us demonstrate how the stationary point problem serves as a unifying model of various problems arising from the areas of mathematical economics, game theory and mathematical programming.

Exchange Economy

As Sonnenschein [S1] showed the crucial point whether a given function f from the n-dimensional unit simplex $S^n = \{ x : x \in R^{n+1}_+ ; \langle e,x \rangle = 1 \}$ into R^{n+1} is an excess demand function generated by aggregating individual utility maximizing behavior in some market is Walras' Law:

(2.4) $\langle x,f(x) \rangle = 0$ for any $x \in S^n$,

where e is an (n+1)-dimensional vector of ones and R^{n+1}_+ is the nonnegative orthant of R^{n+1}. Therefore we assume here only (2.4) besides the continuity. Under this condition there is a price vector $s \in S^n$ satisfying $f(s) \leq 0$. This price is called an equilibrium price. The existence of s is usually shown by applying Brouwer's Fixed Point Theorem (2.3a) to the function

$$g(x) = (x+f^+(x))/(1+\langle e,f^+(x) \rangle)$$

from S^n into itself, where $f^+_i(x) = \max \{0,f_i(x)\}$ for $i=1,\ldots,n+1$. We show below that a stationary point of $(-f,S^n)$ is an equilibrium price and (2.3b) also guarantees its existence.

(2.5) Theorem: Let f be a continuous function from the n-dimensional unit simplex S^n into R^{n+1} satisfying Walras' Law (2.4). Then a stationary point of $(-f,S^n)$ is an equilibrium price and vice versa.

Proof: Suppose s is a stationary point of $(-f,S^n)$ and so $f(s) \in N_{S^n}(s)$. Then

$$f_i(s) = \mu - \lambda_i, \quad \lambda_i \geq 0, \ \lambda_i s_i = 0, \quad i=1,\ldots,n+1$$

for some μ and λ_i. By (2.4) and $s \in S^n$, $\mu = \mu\sum_i s_i = \langle s,f(s) \rangle = 0$.

Therefore $f(s) \leq 0$, that is s is an equilibrium. When $f(s) \leq 0$, by (2.4) $f_i(s) = 0$ if $s_i > 0$ and $f_i(s) \leq 0$ if $s_i = 0$. It is readily seen that $f(s) \in N_{S^n}(s)$, namely s is a stationary point of $(-f, S^n)$. //

When the economy involves production, the equilibrium is defined as follows: Let $A(p) \in R^{(n+1)\times m}$ be the technology matrix of input-output coefficients at price p, y denote the activity level, $d(p)$ denote the demand function at price p and b denote the endowment. Then a pair $(p,y) \in S^n \times R^m_+$ is called an equilibrium of this economy if

(2.6a) $d(p) - b - A(p)y \leq 0$;

b) $A(p)^t p \leq 0$;

c) $\langle y, A(p)^t p\rangle = 0$;

d) $\langle p, A(p)y - d(p) + b\rangle = 0$.

(2.7) Theorem: Let $x = (p,y) \in R^{n+1+m}$, $f(x) = (d(p)-b-A(p)y, A(p)^t p)$ and $K = S^n \times R^m_+$. If $d(p) - b$ satisfies Walras' Law (2.4), an equilibrium of this economy is a stationary point of $(-f,K)$ and vice versa.

For this problem Mathiesen [M] proposed solving a sequence of linear complementarity problems obtained by taking a linear approximation of the problem and Dirven and Talman [DT4] proposed a simplicial fixed point algorithm for the problem with a constant technology matrix.

Noncooperative n-Person Game

Player j has m_j+1 pure strategies and his expected loss is given by $L_j(x^1,x^2,\ldots,x^n)$ when player i chooses mixed strategy x^i for $i=1,\ldots,n$ from his strategy space being an m_i-dimensional unit simplex S^{m_i}. Let the marginal loss of player j when he chooses his h^{th} pure strategy while others keep their mixed strategy be defined as $L_j(x^{-j},e^h)$, where (x^{-j},e^h) abbreviates $(x^1,\ldots,x^{j-1},e^h,x^{j+1},\ldots, x^n)$ and e^h denotes the h^{th} unit vector of R^{m_j+1}. An equilibrium strategy is a point $s \in \Pi_i S^{m_i}$ satisfying

(2.8) $L_i(s) \leqq L_i(s^{-i},e^h)$ for any $h=1,\dots,m_i+1$ and $i=1,\dots,n$.

If we define

$$f_{ih}(x) = L_i(x) - L_i(x^{-i},e^h);$$
$$f_i(x) = (f_{i1}(x),\dots,f_{im_i+1}(x));$$
$$f(x) = (f_1(x),\dots,f_n(x)),$$

the condition (2.8) is simply rewritten as

$$f(s) \leqq 0.$$

By the definition of L_i we readily see that $\langle x^i,f_i(x)\rangle = 0$ for $i=1,\dots,n$, and obtain the following theorem.

(2.9) Theorem: A stationary point of $(-f,\Pi_i S^{m_i})$ is an equilibrium strategy of the above game and vice versa.

In this example K is a cross product of several unit simplices. The same region appears in the competitive equilibrium problem of international trade market [L2] (see also [P1]). When the set K is a simplex or the cross product of several simplices as above, several path-following algorithms have been developed for (f,K) in [DET],[DLT],[DT1,2,3], [ELT],[ET],[LT5,9]. See also a survey [LT8]. Note that the normal cone $N_K(x)$ is known a priori for this kinds of K.

Nonlinear Program

Consider the following nonlinear program:

(2.10) minimize $g_0(x)$ subject to $g_i(x) \leqq 0$ for $i=1,\dots,m$.

Under some appropriate constraints qualification (see for example [BS]) we see that a local minimizer x of this problem satisfies

(2.11a) $\nabla g_0(x) + Dg(x)u = 0;$

b) $g_i(x) \leqq 0$ for $i=1,\dots,m;$

c) $u \geqq 0;\ \ \langle u,g(x)\rangle = 0$

for Karush-Kuhn-Tucker multiplier vector $u = (u_1,\ldots,u_m)$, where $g(x) = (g_1(x),\ldots, g_m(x))$ and $Dg(x)$ is its Jacobian matrix at x.

(2.12) Theorem: Suppose g_i is a convex function for $i=1,\ldots,m$. Then under some appropriate constraint qualification a point satisfying (2.11) is a stationary point of $(\nabla g_0,K)$ and vice versa, where K is the feasible region of (2.10).

Robinson [R] proposed another formulation.

(2.13) Theorem: Let $f(x,u) = (\nabla g_0(x) + Dg(x)u,\ g(x))$ and $K = R^n \times R^m_+$. Then (2.11) holds if and only if (x,u) is a stationary point of $(-f,K)$.

Phan-huy-Hao [P2] also proposed an interesting method for reducing nonliear programs to stationary point problems on the unit simplex of multipliers (see also [D2]). We consider the problem (2.10) with a constraint $x \in X$ added for some compact subset X of R^n. Given a multiplier vector $\bar{u} = (u_0,u) = (u_0,u_1,\ldots,u_m)$ of the m-dimensional unit simplex S^m of R^{1+m}, consider the problem with the aggregated objective function:

(2.14) minimize $\langle\bar{u},\bar{g}(x)\rangle$ subject to $x \in X$,

where $\bar{g}(x) = (g_0(x),g(x)) = (g_0(x),g_1(x),\ldots,g_m(x))$. Let us denote by $x(\bar{u})$ the optimal solution which we assume for simplicity is unique. Let

$$f_0(\bar{u}) = \langle u,g(x(\bar{u}))\rangle$$
$$f_i(\bar{u}) = -\ g_i(x(\bar{u})) \quad \text{for} \quad i=1,\ldots,m$$

and $f(\bar{u}) = (f_0(\bar{u}),f_1(\bar{u}),\ldots,f_m(\bar{u}))$.

(2.15) Theorem: Suppose there is a point z of X such that $g(z) < 0$. Let $\bar{s} = (s_0,s) = (s_0,s_1,\ldots,s_m)$ be a stationary point of (f,S^m). Then the optimal solution $x(\bar{s})$ of (2.14) with multiplier $\bar{s}$ is an optimal solution of the original nonlinear program.

Proof: In the same way as in the proof of (2.5) there are μ and $\lambda_0, \lambda_1, \ldots, \lambda_m$ such that

$$f_i(s) = \mu + \lambda_i, \quad \lambda_i \geq 0, \quad \lambda_i s_i = 0 \quad \text{for} \quad i=0,1,\ldots,m.$$

Therefore

$$\text{(2.16)} \qquad \mu + \lambda_0 = f_0(\bar{s}) = \langle s,g(x(\bar{s}))\rangle = -\textstyle\sum_{i=1}^{m} s_i(\mu+\lambda_i) = \mu(s_0-1).$$

Suppose $s_0 > 0$. Then $\lambda_0 = 0$ and $(2-s_0)\mu = 0$, which implies $\mu = 0$ and hence $f_i(\bar{s}) \geq 0$ for all i. Thus

$$g(x(\bar{s})) \leq 0 \quad \text{and} \quad \langle s,g(x(\bar{s}))\rangle = 0.$$

Now let x be an arbitrary feasible point of the nonliear program. Then

$$\begin{aligned} s_0 g_0(x(\bar{s})) &= \langle \bar{s},\bar{g}(x(\bar{s}))\rangle \leq \langle \bar{s},\bar{g}(x)\rangle \\ &= s_0 g_0(x) + \langle s,g(x)\rangle \leq s_0 g_0(x), \end{aligned}$$

which yields the optimality of $x(\bar{s})$. It remains to show that $s_0 > 0$. Suppose the contrary, then $\mu = -\lambda_0/2$ by (2.16) and hence $f_0(\bar{s}) = \mu + \lambda_0 = \lambda_0/2 \geq 0$. On the other hand for the point z in the condition

$$f_0(\bar{s}) = \langle s,g(x(\bar{s}))\rangle = \langle \bar{s},\bar{g}(x(\bar{s}))\rangle \leq \langle \bar{s},\bar{g}(z)\rangle = \langle s,g(z)\rangle < 0.$$

This is a contradiction. //

Generalized Complementarity Problem ([HP1],[K1],[S2])
By replacing the usual nonnegativity constraints of the complementarity problems with partial orderings generated by a pair of cones we obtain the following generalized complementarity problem of finding a point s such that

$$s \in K, \quad f(s) \in K^+ \quad \text{and} \quad \langle s,f(s)\rangle = 0,$$

where K is a cone of R^n and $K^+ = \{ y : \langle y,x\rangle \geq 0 \text{ for any } x \in K \}$.

(2.17) Theorem ([K1, Lemma 3.1]): A solution of the generalized complementarity problem is a stationary point of (f,K) and vice versa.

A typical approach for solving the stationary point problem (f,K) is an iterative method generating a sequence of points each of which is a stationary point of a linearized problem: x^{k+1} is given as a(n) (approximate) solution of

$$0 \in f^k(x) + N_K(x) \equiv f(x^k) + A(x^k)(x-x^k) + N_K(x).$$

There are various methods depending on the choice of the matrix $A(x^k)$. If $Df(x^k)$ or its approximation is chosen, we have Newton, quasi-Newton, successive overrelaxation, linearized Jacobi methods, and a fixed symmetric positive definite matrix yields the projection method. Newton-type methods have the same drawback as their correspondents for ordinary equations. Namely, their convergence is not global but local. See [PC] for convergence conditions.
The projection method is based on the fixed point formulation of the stationary point used in the proof of Theorem (2.2) and hence is not globally convergent unless the function f has some monotonicity. Though the function f is a gradient mapping of some scalar valued function only if its Jacobian matrix is symmetric at every point, there are several algorithms exploiting the idea of the nonlinear program. See for example [HM] and [IFI]. Instead of solving the linearized problem on the entire set K one may solve it on a subset K^k of K and update the subset so that the stationary point of (f^k,K^k) may converge to a solution. As K^k is usually taken a convex hull of several and possibly few vertices of K and one adds and deletes vertices to obtain a new subset K^{k+1}. This is called the simplicial decomposition method. How to update K^k is very crucial for the convergence. See for example [LH] and [PY]. For further information of algorithms of these kinds the reader is referred to the paper by Dafermos [D1] and the recent survey by Harker and Pang [HP2] and their references.

In the following section we show that the idea of path-following, which is common in the area of fixed point algorithms, yields an algorithm which converges under the assumption that f is continuous and K is compact convex. For simplicity of discussion we will assume that f is continuously differentiable.

3. PDM AND THE BASIC MODEL

In this section we will give a brief review of subdivided manifolds, a basic theorem for fixed point algorithms and a primal-dual pair of subdivided manifolds introduced by Kojima and Yamamoto [KY1] as a unifying framework for variable dimension algorithms.

We call an m-dimensional convex polyhedral set in R^k an m-cell or a cell when there is no confusion. If a cell B is a face of cell C, we write $B \prec C$. We call a finite or countable collection M of m-cells a subdivided manifold if

(3.1a) the intersection of two cells of M is either empty or their common face;

b) each (m-1)-cell of $\bar{M}$ lies in at most two m-cells of M;

c) M is locally finite: each point $x \in |M|$ has a neighborhood which intersects only finitely many cells of M,

where

$$\bar{M} = \{ B : B \text{ is a face of some m-cell of } M \}$$
$$|M| = \bigcup \{ C : C \in M \}.$$

We call the collection of (m-1)-cells of $\bar{M}$ each of which lies in exactly one m-cell of M the boundary of M and denote it by ∂M. A continuous mapping h from $|M|$ into R^n is piecewise continuously differentiable (abbreviated by PC^1) on M if the restriction of h to each cell of M has a continuously differentiable extension. We denote the Jacobian matrix of h at point x of cell C by $Dh(x;C)$. A point $c \in R^n$ is a regular value of the PC^1 mapping $h : |M| \to R^n$ if

$$x \in B \prec C \in M \text{ and } h(x) = c \text{ imply } \dim\{Dh(x;C)y : y \in B\} = n.$$

We see by Sard's theorem that almost every vector of R^n is a regular value. In case that $m = n+1$, we obtain (3.2) (see [A],[K2] and [KY2]).

(3.2) Theorem: Let M be a subdivided $(n+1)$-manifold in R^k and $h : |M| \to R^n$ be a PC^1 mapping on M. Suppose that $c \in R^n$ is a regular value of h. Then $h^{-1}(c)$ is a disjoint union of paths and loops, where by path we mean a connected one-dimensional manifold homeomorphic to one of the unit intervals $(0,1)$, $(0,1]$ and $[0,1]$, and by loop a connected one-dimensional manifold homeomorphic to the one-dimensional sphere. Furthermore

(3.3a) $h^{-1}(c) \cap C$ is either empty or a disjoint union of smooth one-dimensional manifolds for each cell $C \in M$;

b) loops do not intersect $|\partial M|$;

c) $x \in h^{-1}(c)$ is an endpoint of a path if and only if $x \in |\partial M|$;

d) if $|M|$ is a closed subset of R^k, every path homeomorphic to $(0,1)$ or $(0,1]$ is unbounded.

We say that a pair of subdivided manifolds P and D of R^k is a primal-dual pair of subdivided manifolds (PDM in short) if there is an operator d from $\bar{P} \cup \bar{D}$ to $\bar{P} \cup \bar{D} \cup \{\emptyset\}$ and a positive integer ℓ such that

(3.4a) $X^d \in \bar{D} \cup \{\emptyset\}$ for each $X \in \bar{P}$, and $Y^d \in \bar{P} \cup \{\emptyset\}$ for each $Y \in \bar{D}$;

b) if $Z^d \neq \emptyset$, $(Z^d)^d = Z$ and $\dim Z + \dim Z^d = \ell$;

c) if $Z_1, Z_2 \in \bar{P}$ (or $\bar{D}$), $Z_1 \prec Z_2$ and $Z_1{}^d \neq \emptyset$, $Z_2{}^d \neq \emptyset$, then $Z_2{}^d \prec Z_1{}^d$.

We call Z^d the dual of Z and ℓ the degree of the PDM. (3.4b) means that the operator d pairs off the cells and (3.4c) is a kind of complementarity. Since the dimensions of a cell and its dual sum up to the constant ℓ, (3.4c) means that when the dimension of a primal cell increases, that of dual one decreases. Given a PDM with degree ℓ let

$$\langle P,D;d\rangle = \{\ X \times X^d : X \in \bar{P};\ \ X^d \neq \emptyset\ \},$$

which could be written as

$$\langle P,D;d\rangle = \{\ Y^d \times Y : Y \in \bar{D};\ \ Y^d \neq \emptyset\ \}.$$

(3.5) Theorem ([KY1], Theorems 3.2 and 3.3): Let (P,D) form a PDM with degree ℓ together with the dual operator d. Then $M = \langle P,D;d\rangle$ is a subdivided ℓ-manifold and has the boundary

$$\partial M = \{\ X \times Y : X \in \bar{P};\ Y \in \bar{D};\ X \times Y \text{ is an } (\ell-1)\text{-cell of } \bar{M};\ \text{either } X^d \text{ or } Y^d \text{ is empty}\ \}.$$

The basic model in [KY1] for the variable dimension fixed point algorithms is the system

(3.6) $$h(x,y) \equiv f(x) + y = 0;\quad (x,y) \in |M|.$$

Suppose f is a continuously differentiable mapping from $|P|$ into R^n. If (P,D) is a PDM with degree $n+1$ and $0 \in R^n$ is a regular value of the PC^1 mapping $h(x,y)$, then we can apply Theorem (3.2) to system (3.6) and obtain paths and loops of solutions. The keystone of the algorithms is now the PDM, namely how we furnish the subdivided manifold M with the boundary such that we may easily find a starting point there and obtain a stationary point after toiling along a path.

4. VARIABLE DIMENSION ALGORITHMS FOR STATIONARY POINT PROBLEMS

As the first example we take the algorithm by Talman and Yamamoto [TY] for the stationary point problem (f,K) on a compact convex polyhedral set

$$K = \{\ x \in R^n : \langle a^i,x\rangle - b_i = 0 \ \text{ for } i=1,\dots,k;\ \ \langle c^i,x\rangle - d_i \leq 0 \ \text{ for } i=1,\dots,m\ \}$$

and show how the PDM and the basic model (3.6) yield an algorithm.

The normal cone $N_K(x)$ is

$$N_K(x) = \{ y \in R^n : y = \sum_i \mu_i a^i + \sum_i \nu_i c^i; \quad \nu_i \geq 0, \ (\langle c^i, x\rangle - d_i)\nu_i = 0, \ i=1,\ldots,m \}.$$

This is identical for all points x on a face of K, and so we denote by $N_K(F)$ the normal cone corresponding to face F. Then a face F contains a stationary point if and only if

$$0 \in f(F) + N_K(F).$$

Let w be an arbitrary starting point of K and wF be the convex hull of w and face F not containing w. Replacing F by wF in the above system we obtain

$$0 \in f(wF) + N_K(F)$$

or

$$0 = f(x) + y; \quad x \in wF; \quad y \in N_K(F).$$

The dimensions of wF and $N_K(F)$ are $\dim F+1$ and $n-\dim F$, respectively and sum up to $n+1$. Then it is quite natural to define a PDM as follows:

$$P = \{ wF : F \text{ is a facet of } K \text{ not containing } w \};$$
$$D = \{ N_K(v) : v \text{ is a vertex of } K \};$$
$$wF \leftarrow d \rightarrow N_K(F) \text{ for each face } F \text{ of } K \text{ not containing } w.$$

Then $M = \langle P,D;d\rangle$ is a subdivided $(n+1)$-manifold and the boundary consists of

(4.1a) $F \times N_K(F)$ for face F not containing w;

b) $wG \times N_K(F)$ for face F and its facet G such that $w \notin G$, $w \in F$;

c) $\{w\} \times N_K(v)$ for vertex v of K other than w.

According to the basic model (3.6) we consider the system

(4.2) $$h(x,y) \equiv f(x) + y = 0; \quad (x,y) \in |M|.$$

(4.3) Theorem: Suppose that $0 \in R^n$ is a regular value of $h : |M| \to R^n$ and the starting point $w \in K$ is not a stationary point of (f,K). Then there is a path of solutions to (4.2) leading from $(w,-f(w))$ to a point $(s,-f(s))$ such that s is a stationary point of (f,K).

Proof: Clearly $(w,-f(w))$ is a solution of (4.2). Since K is compact, $N_K(v)$ covers R^n as v ranges over all vertices of K. By the assumption that w is not a stationary point $-f(w)$ does not lie in $N_K(w)$. Therefore

$$(w,-f(w)) \in \{w\} \times N_K(v) \subset |\partial M|$$

for some vertex v of K. Then by Theorem (3.2) we obtain a path of solutions to (4.2) which is homeomorphic to either $(0,1]$ or $[0,1]$. Since $|M|$ is the finite union of closed subsets $wF \times N_K(F)$, it is still closed. Moreover, by the compactness of K and the continuity of f the entire set of solutions to (4.2) is contained in the compact set $K \times f(K)$. Therefore by (3.3d) the path is homeomorphic to $[0,1]$ and has two distinct end points in $|\partial M|$. Let (s,y) be the end point other than $(w,-f(w))$. Then $y = -f(s)$, and $(s,-f(s))$ lies in a cell of (4.1a) or b). Since otherwise $(s,-f(s)) \in \{w\} \times N_K(v)$ for some vertex v and hence $s = w$ and $f(s) = f(w)$. This contradicts that (s,y) is different from $(w,-f(w))$. In either case it is contained in $F \times N_K(F)$ for some face F of K. This completes the proof. //

Thus tracing the path of solutions of (4.2) from $(w,-f(w))$ one will reach a stationary point. When projected on the x-space one leaves the starting point w along one of the line segments connecting w to vertices of K. In this sense it has as many directions as the vertices of K. This algorithm reduces to the (n+1)-ray algorithm of when applied to the equilibrium problem on the unit simplex (see [LT9]). An example of the path projected on the x-space is shown in Figure 1 below. The function is

(4.4a) $\quad f_1(x_1,x_2) = -x_1 \qquad$ if $\eta \leq 0$;

b) $\qquad\qquad = -(\cos \eta\pi)x_1 + (\sin \eta\pi)x_2 \qquad$ if $0 < \eta < 1$;

c) $\qquad\qquad = x_1 \qquad$ if $\eta \geq 1$;

d) $\quad f_2(x_1,x_2) = - x_2 \qquad$ if $\eta \leq 0$;

e) $\qquad\qquad = -(\sin \eta\pi)x_1 - (\cos \eta\pi)x_2 \qquad$ if $0 < \eta < 1$;

f) $\qquad\qquad = x_2 \qquad$ if $\eta \geq 1$,

a continuous function which is the identity within the radius of 1/2 and minus the identity outside the unit circle and $K = \{ x \in R^2 : -1 \leq x_1, x_2 \leq 1 \}$. The origin is the unique stationary point of this problem. When one moves conducted by the vector field illustrated by small arrows, one will coil round the circle with a radius of $1/\sqrt{2}$ and will never reach the stationary point.

Next one is the $(2^{n+1}-2)$-ray algorithm by Doup, van der Laan and Talman [DLT] for the equilibrium problem on the unit simplex S^n. The problem would be reduced to the stationary point problem $(-f,S^n)$,

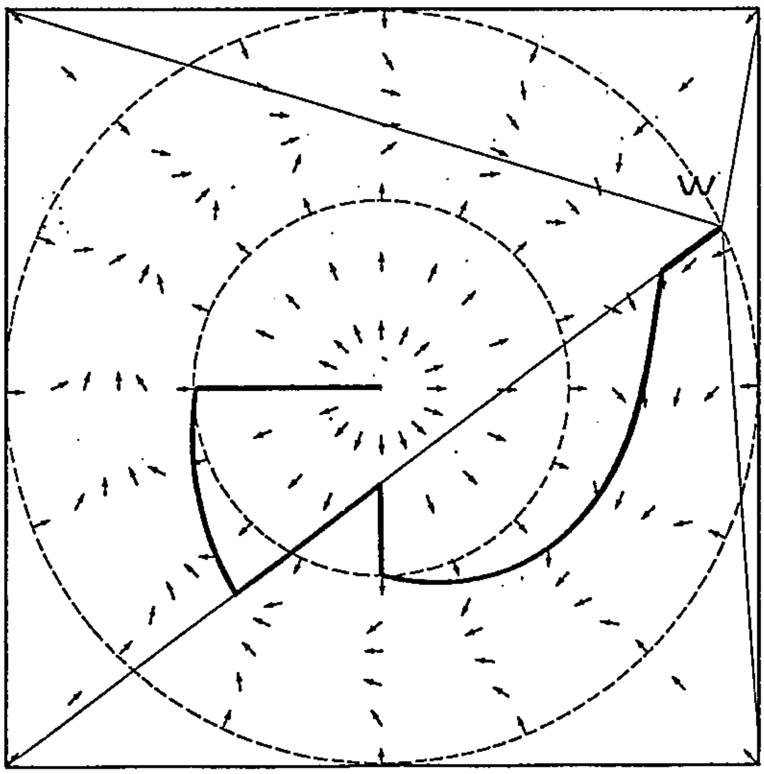

Fig.1. Path projected on the x-space for the function of (4.4)

however the algorithm admits of economic interpretation of the trajectory it follows as a tatonnement process (see also [LT9]). Starting with an arbitrary point, say w, of S^n it increases the prices of the commodities in excess demand and decreases those of the commodities in excess supply proportionally to the starting point w. To make the complementarity between variables and function values more conspicuous we assume that $w > 0$. In general steps the algorithm traces a path of points x satisfying

$$\begin{aligned} x_i/w_i &= \max\{x_j/w_j : j=1,\dots,n+1\} \quad \text{if} \quad f_i(x) > 0; \\ &= \min\{x_j/w_j : j=1,\dots,n+1\} \quad \text{if} \quad f_i(x) < 0. \end{aligned}$$

We call a vector of -1, 0 and 1 a sign vector. To construct a PDM for this algorithm let $I^+(t)$, $I^0(t)$ and $I^-(t)$ be the index set of positive, zero and negative components of a sign vector t, respectively. For an $(n+1)$-dimensional sign vector t with nonempty $I^+(t)$ and $I^-(t)$ let

$$\begin{aligned} X(t) = \{\, x \in S^n : \; x_i/w_i &= \max\{x_j/w_j : j=1,\dots,n+1\} \quad \text{if} \quad i \in I^+(t) \\ &= \min\{x_j/w_j : j=1,\dots,n+1\} \quad \text{if} \quad i \in I^-(t) \,\}. \end{aligned}$$

$X(t)$ is the convex hull of the starting point w and a subset

$$\begin{aligned} F(t) = \{\, x \in S^n : \; x_i/w_i &= \max\{x_j/w_j : j=1,\dots,n+1\} \quad \text{if} \quad i \in I^+(t) \\ &= 0 \quad \text{if} \quad i \in I^-(t) \,\} \end{aligned}$$

of face $F = \{\, x \in S^n : x_i = 0 \text{ if } i \in I^-(t) \,\}$ and is of $|I^0(t)|+1$ dimension. Figure 2 illustrates $X(t)$'s of S^2. The dual of $X(t)$ is given by

$$Y(t) = \{\, y \in R^{n+1} : \; \begin{aligned} y_i &\geq 0 \quad \text{if} \quad i \in I^+(t) \\ y_i &= 0 \quad \text{if} \quad i \in I^0(t) \\ y_i &\leq 0 \quad \text{if} \quad i \in I^-(t) \,\}, \end{aligned}$$

which is of $|I^+(t)|+|I^-(t)|$ dimension. Then

$$\begin{aligned} P &= \{\, X(t) : t \in R^{n+1} \text{ is a sign vector with nonempty } I^+(t) \text{ and } I^-(t) \,\} \text{ and} \\ D &= \{\, Y(t) : t \in R^{n+1} \text{ is a sign vector} \,\} \end{aligned}$$

form a PDM with degree n+2. Then by Theorem (3.5) we see that $M = \langle P, D; d\rangle$ is an (n+2)-subdivided manifold and the boundary consists of the following three kinds of (n+1)-cells:

(4.5a) $F(t) \times Y(t)$ for t with nonempty $I^+(t)$ and $I^-(t)$;

b) $X(t') \times Y(t)$ for $t \geq 0$ (or $t \leq 0$) and t' obtained by replacing one of zeros of t by -1 (or $+1$);

c) $\{w\} \times Y(t)$ for t with empty $I^0(t)$ and nonempty $I^+(t)$ and $I^-(t)$.

According to the basic model (3.6) we consider the set of solutions to the system

$$-f(x) + y = 0; \quad (x,y) \in |M|.$$

For a given starting point w of S^{n+1} $(x,y) = (w, f(w))$ is a

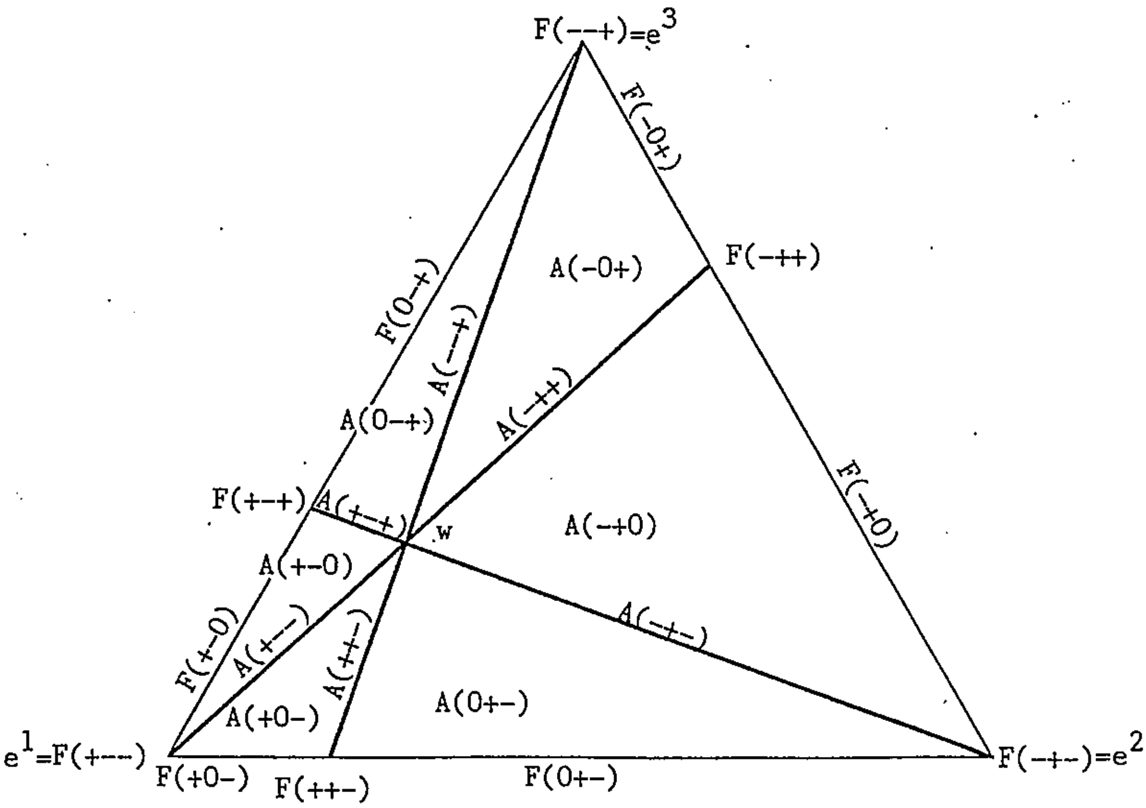

Fig. 2. Two-dimensional unit simplex and $X(t)$'s

trivial solution of this system and it lies in a cell of (4.5c) and hence on the boundary of M. In fact it does not occur that $f(w) \geqq 0$ and $\neq 0$ since f satisfies Walras' Law, and w is already an equilibrium when $f(w) \leqq 0$. Therefore there is a path from $(w,f(w))$ to another end point, say $(s,f(s))$, which is in a cell of either (4.5a) or b). When it is in a cell of (4.5a),

$$\begin{aligned} s_i &> 0, & f_i(s) &\geqq 0 \quad \text{for} \quad i \in I^+(t); \\ &\geqq 0, & &= 0 \quad \text{for} \quad i \in I^0(t); \\ &= 0, & &\leqq 0 \quad \text{for} \quad i \in I^-(t). \end{aligned}$$

Therefore $f(s) \leqq 0$ by Walras' Law and hence s is an equilibrium. We see $f(s) = 0$ in the case of (4.5b).

A possibility of applying the well-known predictor-corrector method for tracing the path of solutions to (4.2) is shown in [Y5]. Since the cell wF is the convex hull of w and all the vertices of F, instead of (4.2) the following system is considered:

$$\begin{aligned} &f(\lambda_w w + \textstyle\sum_v \lambda_v v) + \sum_i \mu_i a^i + \sum_I \nu_i c^i = 0; \\ &\lambda_w + \textstyle\sum_v \lambda_v = 1; \\ &\lambda_w \geqq 0; \ \lambda_v \geqq 0 \quad \text{for all} \quad v; \\ &\nu_i \geqq 0 \quad \text{for} \quad i \in I, \end{aligned}$$

where I is the index set of binding inequality constraints of F. In general neither wF nor the normal cone $N_K(F)$ is simplicial. That makes a difficulty of nonuniqueness of combination coefficients λ, μ and ν. An idea of simplicial decomposition is presented in [Y5] and also in [Y4] for the case of an affine function f. The algorithm of [TY] traces the path of solutions to the system (4.2) with f replaced by its piecewise linear approximation $\bar{f}$ on a triangulation T of K, namely

$$(4.6) \qquad \bar{h}(x,y) \equiv \bar{f}(x) + y = 0; \quad (x,y) \in |M|.$$

They generalized the V-triangulation by Doup and Talman [DT1] to triangulate K so that for each face F of K the set wF is also triangulated by the lower dimensional simplices. Collecting the families $\{ \sigma \times N_K(F) : \sigma \in \bar{T};\ \sigma \subset wF;\ \dim \sigma = \dim wF \}$ for all faces F of K not containing w, one obtain a refinement of M. This is not necessarily simplicial but $\bar{h}$ is affine on each of its cells. Therefore the regular value assumption for piecewise linear functions (see for example [E]) will suffice to ensure the existence of the path of solutions to (4.6) leading from $(w,\bar{f}(w)) = (w,f(w))$ to a stationary point of $(\bar{f},K)$, that is an approximate stationary point of (f,K). Let σ be a simplex of $\bar{T}$ restricted to wF which contains x, $v^1,\dots,v^{k+1}$ be its vertices and I be the index set of binding inequality constraints at face F. Then the system is

(4.7a) $\sum_j \lambda_j f(v^j) + \sum_i \mu_i a^i + \sum_I \nu_i c^i = 0;$

b) $\sum_j \lambda_j = 1;\quad \lambda_j \geq 0$ for all j;

c) $\nu_i \geq 0$ for $i \in I$.

At the start we solve the linear program

$$\begin{aligned} &\min.\ \textstyle\sum_i b_i\mu_i + \sum_i d_i\nu_i; \\ &\text{s.t. } \lambda_1 f(w) + \textstyle\sum_i \mu_i a^i + \sum_i \nu_i c^i = 0; \\ &\qquad \nu_i \geq 0 \text{ for all } i;\quad \lambda_1 = 1, \end{aligned}$$

and obtain an initial basic solution corresponding to the zero-dimensional starting simplex $\{w\}$. The optimal basic variables determine the normal cone $N_K(v)$ containing $-f(w)$ and further give the cell $w\{v\} \times N_K(v)$ of M which the path of solutions of (4.6) passes through. Since this problem is the dual of

$$\min.\ \langle f(w), x\rangle \quad \text{s.t. } x \in K,$$

it follows that the algorithm begins to move toward the first candidate for the statinary point. In general steps when some λ_j vanishes we obtain a facet τ of σ, then check if τ lies in the boundary of wF. If not, there must be a unique simplex σ' in wF sharing τ with σ. So we make an ordinary replacement. If τ is in the boundary of wF, which is wG for some face G of F, we introduce into the system (4.7) the coefficient vectors c^i's of the inequality constraints which become newly active at G. If ν_i vanishes, we first see if $y = \sum_i \mu_i a^i + \sum_I \nu_i c^i$ hits the boundary of $N_K(F)$. This will be done by checking if there remains a coefficient vector c^k, $k \in I$ on the opposite side of the subspace spanned by a^i's and c^i's corresponding to basic variables to the dropped one. If there is such a c^k, we introduce it to the system (4.7). If not, we delete the inequality constraints corresponding to the nonbasic variables to obtain a new face E of K containing F as a facet.

REFERENCES

[A] J.C. Allexander, "The topological theory of an embedding method", in: H. Wacker ed., Continuation Method (Academic Press, New York, 1978) pp.36-67.

[AG] E.L. Allgower and K. Georg, "Predictor-corrector and simplicial methods for approximating fixed points and zero points of nonlinear mappings", in: A.Bachem, M.Grötschel and B.Korte eds., Mathematical Programming, The State of the Art, Bonn 1982 (Springer-Verlag, Berlin, 1983) pp.15-56.

[B] M. Broadie, "An introduction to the octahedral algorithm for the computation of economic equilibria", Mathematical Programming Study 23 (1985) 121-143.

[BS] M.S. Bazaraa and C.M. Shetty, Foundation of Optimization, Lecture Notes in Economics and Mathematical Systems 122 (Springer-Verlag, Berlin, 1976).

[D1] S. Dafermos, "An iterative scheme for variational inequalities", Mathematical Programming 26 (1983) 40-47.

[D2] T. Doup, Simplicial algorithm on the simplotope, Ph.D.Dissertation, Tilburg University (Tilburg, The Netherlands, Oct. 1987).

[DET] T. Doup, A.H. van den Elzen and A.J.J. Talman, "Simplicial algorithms for solving the nonlinear complementarity problem on the simplotope", in: A.J.J.Talman and G.van der Laan eds., The Computation and Modelling of Economic Equilibria (North-Holland, Amsterdam, 1987) pp.125-154.

[DLT] T. Doup, G. van der Laan and A.J.J. Talman, "The $(2^{n+1}-2)$-ray algorithm: A new simplicial algorithm to compute economic equilibria", Mathematical Programming 39 (1987) 241-252.

[DT1] T. Doup and A.J.J. Talman, "A new simplicial variable dimension algorithm to find equilibria on the product space of unit simplices", Mathematical Programming 37 (1987) 319-355.

[DT2] T. Doup and A.J.J. Talman, "A continuous deformation algorithm on the product space of unit simplices", Mathematics of Operation Research 12 (1987) 485-521.

[DT3] T. Doup and A.J.J. Talman, "The 2-ray algorithm for solving equilibrium problems on the unit simplex", Methods of Operations Research 57 (1987) 269-285.

[DT4] C.A.J.M. Dirven and A.J.J. Talman, "A simplicial algorithm for finding equilibria in economies with linear production technologies", FEW 217, Department of Economics, Tilburg University (Tilburg, The Netherlands, 1988).

[E] B.C. Eaves, "A short course in solving equations with PL-homotopies", SIAM-AMS Proceeding 9 (1976) 73-143.

[ELT] A.H. van den Elzen, G. van der Laan and A.J.J. Talman, "Adjustment processes for finding equilibria on the simplotope", FEW 196, Tilburg University (Tilburg, The Netherlands, 1985).

[ET] A.H. van den Elzen and A.J.J. Talman, "A new strategy-adjustment process for computing a Nash equilibrium in a noncooperative more-person game", REEKS TER DISCUSSIE No.85.06, Tilburg University (Tilburg, The Netherlands, 1985).

[F1] K. Fan, "Fixed-point and related theorems for non-compact convex sets", in: O.Moeschlin and D.Pallaschke (eds.), Game Theory and Related Topics (North-Holland, Amsterdam, 1979) 151-156.

[F2] R.M. Freund, "Variable dimension complexes, Part I: basic theory", Mathematics of Operations Research 9 (1984) 479-497.

[F3] R.M. Freund, "Variable dimension complexes, Part II: a unified approach to some combinatorial lemmas in topology", Mathematics of Operations Research 9 (1984) 498-509.

[F4] R.M. Freund, "Combinatorial theorems on the simplotope that generalize results on the simplex and cube", Mathematics of Operations Research 11 (1986) 169-179.

[F5] R.M. Freund, "Combinatorial analogs of Brouwer's fixed point theorem on a bounded polyhedron", Sloan W.P. No.1720-85, Massachusetts Institute of Technology (Cambridge, U.S.A., 1985, Revised 1986).

[GLT] R. Glowinski, J.L. Lions and R. Tremolieres, Numerical Analysis of Variational Inequalities (North-Holland, Amsterdam, 1981).

[GZ] C.B. Garcia and W.I. Zangwill, Pathways to Solutions, Fixed Points and Equilibria (Prentice-Hall, Inglewood Cliffs, U.S.A., 1981).

[HM] J.H. Hammond and T.L. Magnanti, "Generalized descent methods for asymmetric systems of equations", Mathematics of Operations Research 12 (1987) 678-699.

[HP1] G.J. Habetler and A.L. Price, "Existence theory for generalized nonlinear complementarity problem", Journal of Optimization Theory and Applications 7 (1971) 223-239.

[HP2] P.T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications", 87-12-06, Department of Decision Sciences, University of Pennsylvania (Philadelphia, U.S.A. 1987).

[I] V.I. Istrăţescu, Fixed Point Theory (D.Reidel, Dordrecht, The Netherlands, 1981).

[IFI] T. Ito, M. Fukushima and T. Ibaraki, "An iterative method for variational inequalities with application to traffic equilibrium problems", Journal of the Operations Research Society of Japan 31 (1988) 82-104.

[K1] S. Karamadian, "Generalized complementarity problem", Journal of Optimization Theory and Applications 8 (1971) 161-168.

[K2] M. Kojima, "An introduction to variable dimension algorithms for solving systems of equations", in: E.L.Allgower, K.Glashoff and H.-O.Peitgen eds., Numerical Solution of Nonlinear Equations (Springer-Verlag, Berlin, 1981) pp.199-237.

[K3] H.W. Kuhn, "Approximate search for fixed points", in: Computing Methods in Optimization Problems 2 (Academic Press, New York, 1969).

[KS] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications (Academic Press, New York, 1980).

[KY1] M. Kojima and Y. Yamamoto, "Variable dimension algorithms: basic theory, interpretations and extensions of some existing methods", Mathematical Programming 24 (1982) 177-215.

[KY2] M. Kojima and Y. Yamamoto, "Variable dimension algorithms and a class of primal-dual subdivided manifolds", in: Proceedings of the 3rd Mathematical Programming Symposium, Japan (Tokyo, Japan, 1982) pp.45-61.

[KY3] M. Kojima and Y. Yamamoto, "A unified approach to the implementation of several restart fixed point algorithms and a new variable dimension algorithm", Mathematical Programming 28 (1984) 288-328.

[L1] G. van der Laan, Simplicial Fixed Point Algorithms, Mathematical Centre Tracts 129 (Mathematisch Centrum, Amsterdam, 1980).

[L2] G. van der Laan, "The computation of general equilibrium in economies with a block diagonal pattern", Econometrica 53 (1985) 659-665.

[LH] S. Lawphongpanich and D.W. Hearn, "Simplicial decomposition of the asymmetric traffic assignment problem", Transportation Research 18B (1984) 123-133.

[LS] G. van der Laan and L.P. Seelen, "Efficiency and implementation of simplicial zero point algorithms", Mathematical Programming 30 (1984) 196-217.

[LT1] G. van der Laan and A.J.J. Talman, "A restart algorithm for computing fixed points without an extra dimension", Mathematical Programming 17 (1979) 74-84.

[LT2] G. van der Laan and A.J.J. Talman, "A restart algorithm without an artificial level for computing fixed points on unbounded regions", in: H.-O. Peitgen and H.-O. Walther eds., Functional Differential Equations and Approximation of Fixed Points (Springer-Verlag, Berlin, 1979) pp.247-256.

[LT3] G. van der Laan and A.J.J. Talman, "Convergence and properties of recent variable dimension algorithms", in: W. Forster ed., Numerical Solution of Highly Nonlinear Problems (North-Holland, Amsterdam, 1980) pp.3-36.

[LT4] G. van der Laan and A.J.J. Talman, "A class of simplicial restart fixed point algorithms without an extra dimension", Mathematical Programming 20 (1981) 33-48.

[LT5] G. van der Laan and A.J.J. Talman, "On the computation of fixed points in the product space of unit simplices and an application to noncooperative N-person games", Mathematics of Operations Research 7 (1982) 1-13.

[LT6] G. van der Laan and A.J.J. Talman, "Note on the path following approach of equilibrium programming", Mathematical Programming 25 (1983) 363-367.

[LT7] G. van der Laan and A.J.J. Talman, "Simplicial algorithms for finding stationary points, a unifying description", Journal of Optimization Theory and Applications 50 (1986) 262-281.

[LT8] G. van der Laan and A.J.J. Talman, "Computing economic equilibria by variable dimension algorithms: State of the art",FEW 270, Tilburg University (Tilburg, The Netherlands, 1987).

[LT9] G. van der Laan and A.J.J. Talman, "Adjustment process for finding economic equilibria", in: A.J.J.Talman and G.van der Laan eds., The Computation and Modelling of Economic Equilibria (North-Holland, Amsterdam, 1987) pp.85-124.

[LTV1] G. van der Laan, A.J.J. Talman and L. Van der Heyden, "Simplicial variable dimension algorithms for solving the nonlinear complementarity problem on a product of unit simplices using a general labelling", Mathematics of Operations Research 12 (1987) 377-397.

[LTV2] G. van der Laan, A.J.J. Talman and L. Van der Heyden, "Variable dimension algorithms for unproper labellings", Research Memorandum FEW 147, Tilburg University (Tilburg, The Netherlands, April 1984).

[M] L. Mathiesen, "Computation of economic equilibria by a sequence of linear complementarity problems", Mathematical Programming Study 23 (1985) 144-162.

[P1] J.-S. Pang, "Asymmetric variational inequality problems over product sets: Applications and iterative methods", Mathematical Programming 31 (1985) 206-219.

[P2] E. Phan-huy-Hao, "Quadratically constrained quadratic programming", Zeitschrift für Operations Research 26 (1982) 105-119.

[PC] J.-S. Pang and D. Chan, "Iterative methods for variational and complementarity problems", Mathematical Programming 24 (1982) 284-313.

[PY] J.-S. Pang and C.-S. Yu, "Linearized simplicial decomposition method for computing traffic equilibria on networks", Networks 14 (1984) 427-438.

[R] S.M. Robinson, "Generalized Equations", in: A.Bachem, M.Grotschel and B.Korte eds., Mathematical Programming, The Stete of the Art, Bonn 1982 (Springer-Verlag, Berlin, 1983) pp.346-367.

[S1] H. Sonnenschein, "Market excess demand functions", Econometrica 40 (1972) 549-563.

[S2] R. Saigal, "Extension of the generalized complementarity problem", Mathematics of Operations Research 1 (1976) 260-266.

[S3] L.S. Shapley, "On balanced games without side payments", in: T.C.Hu and S.M.Robinson eds., Mathematical Progarmming (Academic Press, New York, 1973) pp.261-290.

[T1] A.J.J. Talman, Variable Dimension Fixed Point Algorithms and Triangulations, Mathematical Centre Tracts 128 (Mathematisch Centrum, Amsterdam, 1980).

[T2] M.J. Todd, The Computation of Fixed Points and Applications, (Springer-Verlag, Berlin, 1976).

[T3] M.J. Todd, "Global and local convergence and monotonicity results for a recent variable dimension simplicial algorithm", in: W.Forster ed., Numerical Solution of Highly Nonlinear Problems (North-Holland, Amsterdam, 1980) pp.43-69.

[T4] M.J. Todd, "An introduction to piecewise-linear homotopy algorithms for solving systems of equations", in: P.R. Turner ed., Topics in Numerical Analysis (Springer-Verlag, Berlin, 1982) pp.149-202.

[TV] A.J.J. Talman and L. Van der Heyden, "Algorithms for the linear complementarity problem which allow an arbitrary starting point", in: B.C. Eaves et.al. eds., Homotopy Methods and Global Convergence (Plenum Press, New York, 1983) pp.267-286.

[TY] A.J.J. Talman and Y. Yamamoto, "A globally convergent simplicial algorithm for stationary point problems", Report No.86422-OR, Institut für Ökonometrie und Operations Research, Universität Bonn (Bonn, West Germany, June 1986) (to appear in Mathematics of Operations Research).

[W] A.H. Wright, "The octahedral algorithm, a new simplicial fixed point algorithm", Mathematical Programming 21 (1981) 47-69.

[Y1] Y. Yamamoto, "A new variable dimension algorithm for the fixed point problem", Mathematical Programming 25 (1983) 329-342.

[Y2] Y. Yamamoto, "A unifying model based on retraction for fixed point algorithms", Mathematical Programming 28 (1984) 192-197.

[Y3] Y. Yamamoto,"A variable dimension fixed point algorithm and the orientation of simplices", Mathematical Programming 30 (1984) 301-312.

[Y4] Y. Yamamoto, "A path following algorithm for stationary point problems", Journal of Operations Research Society of Japan 30 (1987) 181-198.

[Y5] Y. Yamamoto, "Stationary point problems and a path-following algorithm", in: Proceedings of the 8th Mathematical Programming Symposium, Japan (Hiroshima, Japan, 1987) pp.153-170.

[Y6] Y. Yamamoto, "Orientability of a pseudomanifold and generalization of Sperner's lemma", Journal of Operations Research Society of Japan 31 (1988) 19-43.