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Abstract Recently A. V. Goldberg and R. E. Tarjan have proposed a negative cycle method for finding minimum-cost flows by selecting negative cycles of minimum-mean length and have shown that the complexity of their method is strongly polynomial. In the present paper we examine whether Goldberg and Tarjan's approach to ordinary minimum-cost flows can be applied to submodular flows. We prove two key theorems, which are significant in their own right, and show that the negative cycle methods for submodular flows of S. Fujishige and U. Zimmermann which may not terminate in finitely many steps can be made terminate by selecting negative cycles of minimum-mean length of certain special type.

1. Introduction

Recently A. V. Goldberg and R. E. Tarjan [7] have proposed a negative cycle method for finding minimum-cost flows by selecting negative cycles of minimum-mean length, and have shown that the complexity of their method is strongly polynomial. Goldberg and Tarjan's method is a version Klein's

For a distributive lattice D with $\emptyset$, $V \in D$ and a submodular function f on D with $f(\emptyset)=0$, (D,f) is called a submodular system on V [4] and we define $B \subset R^V$ by

(2.2) $$B=\{x \in R \mid x(X) \leq f(X) \text{ for all } X \in D,\ x(V)=f(V)\},$$

where for each $X \in D$ $x(X)=\sum_{e \in X} x(e)$. We call $B \subset R^V$ the <u>base polyhedron</u> associated with (D,f) [4].

Given a directed graph G, base polyhedron B, lower and upper capacity functions $\underline{c}$, $\bar{c}$: $A \to R$ (for each $a \in A$, $\underline{c}(a) \leq \bar{c}(a)$), and a cost function γ: $A \to R$, a <u>submodular flow problem</u> in the network $N=(G=(V,A), \underline{c}, \bar{c}, B, \gamma)$ is described as follows [1]:

$$\text{Minimize} \quad \sum_{a \in A} \gamma(a)\Phi(a)$$

(2.3) $$\text{subject to} \quad \underline{c}(a) \leq \Phi(a) \leq \bar{c}(a) \quad (a \in A),$$

(2.4) $$\partial\Phi \in B,$$

where Φ: $A \to R$ is a flow and $\partial\Phi$: $V \to R$ is the boundary of the flow Φ defined by

(2.5) $$\partial\Phi(v)=\sum_{\partial^+ a=v} \Phi(a) - \sum_{\partial^- a=v} \Phi(a) \quad (v \in V).$$

If Φ satisfies (2.3) and (2.4), Φ is called a <u>submodular flow</u> in N. An optimal solution of the above problem is called an <u>optimal submodular flow</u> in N. For $b \in B$ and $v \in V$ we define $dep(b,v) \subseteq V$ by

(2.6) $$dep(b,v)=\{u \in V \mid \exists\, d>0: b+d(X_v-X_u) \in B\}.$$

Here, for any $u \in V$ X_u is a unit vector in R^V such that

(2.7) $$X(w)_u=\begin{cases} 1 & (w=u), \\ 0 & (w \in V-\{u\}). \end{cases}$$

The function dep: $B \times V \to 2^V$ is called a <u>dependence function</u> [2]. Moreover, for any $u \in V$ we define

(2.8) $$\tilde{c}(b,u,v)=\max\{d \mid d \geq 0,\ b+d(X_v-X_u) \in B\},$$

where if $b+d(X_v-X_u) \in B$ for all $d \geq 0$, then we define $\tilde{c}(b,u,v)=+\infty$. We call

$\tilde{c}(b,u,v)$ the exchange capacity from v to u associated with b (see [2]). We can easily show that $\tilde{c}(b,u,v)\neq 0$ if and only if $u \in dep(b,v)$. For $b \in B$ we define

(2.9) $$\tilde{A}_b:=\{(u,v) \in V^2 \mid \tilde{c}(b,u,v)>0,\ u\neq v\}.$$

$\tilde{G}_b=(V,\tilde{A}_b)$ is called the exchangeability graph associated with b. For a submodular flow Φ in N let

(2.10) $$A^*_\Phi:=\{a\in A \mid \Phi(a)<\bar{c}(a)\},$$

(2.11) $$B^*_\Phi:=\{\bar{a}=(\partial^- a,\partial^+ a) \mid a \in A,\ \underline{c}(a)<\Phi(a)\},$$

(2.12) $$A_\Phi:=\tilde{A}_{\partial\Phi} \cup A^*_\Phi \cup B^*_\Phi.$$

Here, $\bar{a}=(\partial^- a,\partial^+ a)$ is the reorientation of a. $G_\Phi=(V,A_\Phi)$ is called the auxiliary graph associated with submodular flow Φ. For each $a \in A_\Phi$ we define a capacity function c_Φ by

(2.13) $$c_\Phi(a)=\begin{cases} \bar{c}(a)-\Phi(a) & \text{if } a \in A^*_\Phi, \\ \Phi(\bar{a})-\underline{c}(\bar{a}) & \text{if } \bar{a} \in A \text{ and } a \in B^*_\Phi, \\ \tilde{c}(\partial\Phi,\partial^+ a,\partial^- a) & \text{if } a \in \tilde{A}_{\partial\Phi}. \end{cases}$$

A cost function γ_Φ by

(2.14) $$\gamma_\Phi(a)=\begin{cases} \gamma(a) & \text{if } a \in A^*_\Phi, \\ -\gamma(\bar{a}) & \text{if } \bar{a} \in A \text{ and } a \in B^*_\Phi, \\ 0 & \text{if } a \in \tilde{A}_{\partial\Phi}. \end{cases}$$

Then, $N_\Phi=(G_\Phi=(V,A_\Phi), c_\Phi, \gamma_\Phi)$ is called the auxiliary network associated with Φ.

In network N_Φ the capacity of a cycle is the minimum of the capacities of its arcs, where a cycle is a directed closed path. The cost of a cycle Q is the sum of the costs of its arcs, denoted by $\gamma_\Phi(Q)$, relative to cost function γ_Φ and a negative cycle is a cycle of negative cost. The following theorem characterizes optimal submodular flows.

Theorem 2.1 [2, 14]. A submodular flow Φ is optimal if and only if there are no negative cycles in N_Φ.

The following lemma will be used in the next section.

Lemma 2.1 [2]. Suppose $b \in B$ and let u_i, $v_i (i=1,2,\ldots,q)$ be 2q distinct vertices in V such that

$$v_i \in \mathrm{dep}(b,u_i) \qquad (i=1,2,\ldots,q),$$

$$v_j \notin \mathrm{dep}(b,u_i) \qquad (1 \leq i < j \leq q).$$

For arbitrary $d_i (i=1,2,\ldots,q)$ satisfying

$$0 < d_i \leq \tilde{c}(b,u_i,v_i) \qquad (i=1,2,\ldots q),$$

let b^* be a vector in R^V defined by

$$b^* = b + \sum_{i=1}^{q} d_i (\chi_{u_i} - \chi_{v_i}).$$

Then, we have $b^* \in B$.

3. Minimum-mean Cycles in the Auxiliary Network

Based on Theorem 2.1 and Lemma 2.1 in the preceding section, the following primal algorithm for the independent flow problem was proposed by Fujishige [2]. We rewrite it for the submodular flow problem (cf. also [14])

(PA) Begin with any submodular flow Φ.

Do the following (†) while there is a negative cycle in N_Φ:

(†) Find a negative cycle Q of the fewest arcs in N_Φ and change the flow Φ along the cycle Q by

$$\Phi(a) := \begin{cases} \Phi(a)+d & \text{if } a \in Q \cap A_\Phi^*, \\ \Phi(a)-d & \text{if } \bar{a} \in Q \cap B_\Phi^*, \\ \Phi(a) & \text{otherwise}, \end{cases} \tag{3.1}$$

where d is the capacity of the cycle Q.

Because we select negative cycles of the fewest arcs, the succesively obtained flows Φ are submodular flows in N and have smaller costs than the previous ones. However, this algorithm may not find an optimal submodular flow in a finite number of steps (see [2], [14]).

Adopting Goldberg and Tarjan's approach [7], we give a new cycle selection rule for submodular flows which guarantees that the primal algorithm (PA) always finds an optimal submodular flow in a finite number of steps. We need a few further definitions to describe the cycle selection rule.

The <u>mean cost of a cycle</u> in a directed graph with arc costs is its cost divided by the number of arcs it contains. A <u>minimum-mean cycle</u> is a cycle whose mean cost is as small as possible. The <u>minimum cycle mean</u> is the mean cost of a minimum-mean cycle.

A cycle Q in N_Φ is a <u>feasible cycle</u> if we change flow Φ along it by (3.1) the resultant flow is also submodular flow in N.

Suppose we are given a one-to-one mapping π: $V \to \{1,2,\ldots,|V|\}$ and let

(3.2) $\quad q_\Phi(v)=\min\ \{\pi(w) \mid \text{arc } (v,w) \text{ lies on a minimum-mean cycle in } N_\Phi\}.$

If there is no minimum-mean cycle containing v in N_Φ, define $q_\Phi(v)=|V|+1$. Then, our cycle selection rule can be described as follows:

(*) Select a minimum-mean cycle Q in N_Φ such that for each arc a on Q we have $q_\Phi(\partial^+ a)=\pi(\partial^- a)$

Such a minimum-mean cycle can be found in $O(|V||A|)$ time using an algorithm of Karp [9]. The primal algorithm (PA) with minimum-mean cycle selection rule (*) instead of selecting negative cycles of the fewest arcs is valid due to the following theorem.

Theorem 3.1. The cycle Q selected by the above rule (*) in the primal algorithm is a feasible cycle.

Proof: Let μ be the minimum cycle mean of N_ϕ, denote by $A_\phi(Q)$ the set of arcs in Q, and define

(3.3) $$C^+=A_\phi(Q) \cap \tilde{A}_{\partial\phi}$$

Suppose $C^+=\{(u_i,v_i) \mid i \in I\}$. Then, u_i, v_i $(i \in I)$ are distinct vertices because Q is a minimum-mean cycle. We claim that there exists a permutation (u_{i_1},v_{i_1}), (u_{i_2},v_{i_2}),..., (u_{i_p},v_{i_p}) $(p=|I|)$ of arcs in C^+ such that for any k, l with $1 \leq k < l \leq p$

(3.4) $$(u_{i_l},v_{i_k}) \notin \tilde{A}_{\partial\phi}$$

Then, the theorem immediately follows from Lemma 2.1. To prove the claim, we suppose on the contrary that there were no permutation of C^+ satisfying (3.4), which will lead us to a contradiction.

It is easy to see that if the claim is not true, then there exist some arcs (u_{j_1},v_{j_1}), (u_{j_2},v_{j_2}),..., (u_{j_q},v_{j_q}) $(2 \leq q \leq p)$ of C^+ such that for each $r=1,2,\ldots,q$ $(u_{j_r},v_{j_{r+1}}) \in \tilde{A}_{\partial\phi}$ $(j_{q+1}=j_1)$. For any two vertices x, y on Q denote the directed path from x to y on Q by $P(x,y)$. For each $r=1,2,\ldots,q$ adding arc $(u_{j_r},v_{j_{r+1}})$ to path $P(v_{j_{r+1}}, u_{j_r})$ we get a cycle, which we denote by $Q(v_{j_{r+1}}, u_{j_r})$. We show that $Q(v_{j_{r+1}}, u_{j_r})$ is also a minimum-mean cycle.

Let $\gamma_\phi(Q)$ denote the cost of Q relative to γ_ϕ, and ℓ be the number of arcs of Q, and ℓ_r be the number of arcs of $Q(v_{j_{r+1}}, u_{j_r})$ $(r=1,2,\ldots q)$. Since $\gamma_\phi(a)=0$ for every arc $a \in \tilde{A}_{\partial\phi}$, we have

(3.5) $$\gamma_\phi(Q(v_{j_{r+1}}, u_{j_r}))=\gamma_\phi(Q)-\gamma(P(u_{j_r},v_{j_{r+1}})).$$

Hence we have

(3.6) $$\textstyle\sum_{r=1}^{q} \gamma_\phi(Q(v_{j_{r+1}}, u_{j_r}))=q\gamma_\phi(Q)-q^*\gamma_\phi(Q)=(q-q^*)\gamma_\phi(Q)$$

(3.7) $$\textstyle\sum_{r=1}^{q} \ell_r=(q-q^*)\ell$$

for some positive integer $q^* < q$. It follows from (3.6) and (3.7) that

(3.8) $$\sum_{r=1}^{q} [\gamma_\phi(Q(v_{j_{r+1}}, u_{j_r})) - \ell_r \mu] = (q-q^*)(\gamma_\phi(Q) - \ell\mu) = 0,$$

whereas

(3.9) $$\gamma_\phi(Q(v_{j_{r+1}}, u_{j_r})) - \ell_r \mu \geq 0 \qquad (r=1,2,\ldots,q),$$

due to the definition of μ. From (3.8) and (3.9) we see that each inequality in (3.9) holds with equality, that is, each $Q(v_{j_{r+1}}, u_{j_r})$ $(r=1,2,\ldots q)$ is also a minimum-mean cycle. The definition of Q implies that $\pi(v_{j_r}) < \pi(v_{j_{r+1}})$ $(r=1,2,\ldots,q)$. Therefore, we have $\sum_{r=1}^{q} \pi(v_{j_r}) < \sum_{r=1}^{q} \pi(v_{j_r})$, a contradiction. □

Theorem 3.2. A minimum-mean cycle of the fewest arcs is a feasible cycle.

Proof: The proof of this theorem is similar to that of Theorem 3.1. If the claim in the above proof is not true, then each $Q(v_{j_{r+1}}, u_{j_r})$ is also a minimum-mean cycle. (The proof up to this point is the same.) Here, we have $\ell_r < \ell$ which contradicts to the definition of Q. □

The cycle selection rule (*) can be regarded as a generalization of the rule of selecting augmenting paths in the maximum independent flow algorithm proposed by Fujishige [2] and later refined by P. Schönsleben [12] and E. L. Lawler and C. V. Martel [11]. Their rule is to always select a lexicographically minimum augmenting path of the fewest arcs.

In the next section, we show that the primal algorithm with minimum-mean cycle selection rule (*) terminates after a finite number of steps.

4. Analysis of the Primal Algorithm with Minimum-mean Cycle Selection

A function $p: V \to R$ is called a potential in N. A base $b^* \in B$ is called a maximum-weight base with respect to p if

(4.1) $$\sum_{v \in V} b^*(v)p(v) = \max_{b \in B} \sum_{v \in V} b(v)p(v)$$

It is well konwn that a base $b \in B$ is a maximum-weight base with respect to p if and only if $p(u)<p(v)$ implies $u \notin dep(b,v)$ $(u,v \in V)$. Optimal submodular flows are characterized in the following theorem (see , e.g., [5] [6]).

Theorem 4.1. A submodular flow Φ is optimal if and only if there exists a potential p such that

(i) for all $a \in A$

(4.2) $$\gamma(a)+p(\partial^+ a)-p(\partial^- a)>0 \Rightarrow \Phi(a)=\underline{c}(a)$$

(4.3) $$\gamma(a)+p(\partial^+ a)-p(\partial^- a)<0 \Rightarrow \Phi(a)=\bar{c}(a)$$

(ii) $\partial\Phi$ is a maximum-weight base of B with respect to p.

Furthermore, if p satisfies (i) and (ii) for some optimal submodular Φ, then p satisfies (i) and (ii) for any optimal submodular flow Φ.

Define $\gamma_{\Phi p}: A_\Phi \to R$ by

(4.4) $$\gamma_{\Phi p}(a)=\gamma_\Phi(a)+p(\partial^+ a)-p(\partial^- a) \qquad (a \in A_\Phi).$$

Then, the above (i) and (ii) hold if and only if for all $a \in A_\Phi$

(4.5) $$\gamma_{\Phi p}(a) \geqq 0.$$

We introduce a notion of approximate optimality, called ε-optimality, obtained by relaxing the above optimality condition (4.5), which plays a crucial role in our analysis. The notion of approximate optimality was first introduced by É. Tardos [13] for the ordinary minimum-cost flow problem (also see [6] for submodular flows). It was also used in the analysis of a primal algorithm for finding ordinary minimum-cost flows by Goldberg and Tarjan [7]. For any $\varepsilon \geqq 0$ a submodular flow Φ is called ε-optimal if for all $a \in A_\Phi$

(4.6) $\gamma_{\Phi p}(a) \geq -\varepsilon$

As is pointed out in [7] for ordinary minimum-cost flows, we can easily show that if all arc costs are integers and $\varepsilon < 1/|V|$, then an ε-optimal submodular flow is optimal. For a submodular flow Φ we denote by $\varepsilon(\Phi)$ the minimum ε such that Φ is ε-optimal, and by $\mu(\Phi)$ the mean cost of a minimum-mean cycle in N_Φ. The following theorem establishes a connection between ε-optimality and minimum cycle means.

Theorem 4.2. For any submodular flow Φ, $\varepsilon(\Phi) = \max\{0, -\mu(\Phi)\}$.

Proof: This theorem easily follows from the definition of $\mu(\Phi)$ and is a direct adaptation of a result in [7]. □

Now, we analyze the primal algorithm with minimum-mean cycle selection rule (*). Let Φ be an arbitrary submodular flow in network N, Q be a feasible minimum-mean cycle in G_Φ, and Φ' be the submodular flow after changing Φ along Q by (3.1). Then we have the following lemma.

Lemma 4.1. $\varepsilon(\Phi) \leq \varepsilon(\Phi')$.

Proof: Let $\varepsilon = \varepsilon(\Phi)$ and p be a potential with respect to which Φ is ε-optimal Before Φ is changed, every arc in G_Φ satisfies (4.6) by ε-optimality and every arc a on Q satisfies $\gamma_{\Phi p}(a) = -\varepsilon$ by the definitions of ε and Q. Consider any new arc a_0 created by changing flow Φ along Q, i.e., $a_0 \notin A_\Phi$ and $a_0 \in A_{\Phi'}$. If $a_0 \notin \tilde{A}_{\partial\Phi'}$, then arc a_0 is the reorientation of an arc on Q and hence $\gamma_{\Phi' p}(a_0) = \varepsilon$. If $a_0 \in \tilde{A}_{\partial\Phi'}$, then there is an arc $a \in \tilde{A}_{\partial\Phi}$ on Q such that $(\partial^+ a, \partial^- a_0)$, $(\partial^+ a_0, \partial^- a) \in \tilde{A}_{\partial\Phi}$ (cf. [8, p.184]). By the definition of ε-optimality, we have $p(\partial^+ a) - p(\partial^- a_0) \geq -\varepsilon$, $p(\partial^+ a_0) - p(\partial^- a) \geq -\varepsilon$, and $p(\partial^+ a) - p(\partial^- a) = -\varepsilon$, which implies $p(\partial^+ a_0) - p(\partial^- a_0) \geq -\varepsilon$. It follows that every arc in $G_{\Phi'}$ remains to satisfy (4.5) and hence $\varepsilon(\Phi) \leq \varepsilon(\Phi')$. □

Let Φ (Φ') be the submodular flow before (after) the execution of an iteration in the primal algorithm with minimum-mean cycle selection rule (*), $\mu(\Phi)$ ($\mu(\Phi')$) be the minimum cycle mean in G_Φ($G_{\Phi'}$), and q_Φ($q_{\Phi'}$) be the function defined by (3.2). Then, we have

Lemma 4.2.

(1) $\mu(\Phi) \leq \mu(\Phi')$

(2) If $\mu(\Phi) = \mu(\Phi')$, then $q_\Phi \leq q_{\Phi'}$ and there exists a vertex $v \in V$ such that $q\ (v) < q_{\Phi'}(v)$.

Proof: (1) is immediate from Theorem 4.2 and Lemma 4.1. We show (2). Let Q be the minimum-mean cycle selected by selection rule (*) in G_Φ and p be a potential with respect to which Φ is $\varepsilon(\Phi)$-optimal. Suppose $\mu(\Phi) = \mu(\Phi')$. First we show that if there is no minimum-mean cycle containing vertex $x \in V$ in G_Φ, then this is also true in $G_{\Phi'}$. For, suppose that there exists a minimum-mean cycle Q^* containing such a vertex x in $G_{\Phi'}$. Then, Q^* must contain at least one new arc (w,z) in $\tilde{A}_{\partial\Phi'}$ but not in A_Φ which is created by changing Φ along Q. Therefore, there exist arc (u,v) of Q such that arcs (u,z), $(w,v) \in \tilde{A}_{\partial\Phi}$. By $\varepsilon(\Phi)$-optimality and the definitions of Q and Q^*, we have $p(w)-p(z)=\mu(\Phi)$, $p(u)-p(v)=\mu(\Phi)$, $p(u)-p(z) \geq \mu(\Phi)$ and $p(w)-p(v) \geq \mu(\Phi)$, which implies $p(u)-p(z)=\mu(\Phi)$, $p(w)-p(v)=\mu(\Phi)$. It is easy to see that

(4.7) $$Q \cup \{(u,z),(w,v)\} \cup (Q^*-\{(w,z)\})$$

contains a minimum-mean cycle through x. This contradicts the fact that no minimum-mean cycle contains x in G_Φ.

Next, we consider any minimum-mean cycle Q^* containing some new arc (w,z) in $G_{\Phi'}$. Similarly as in the above argument there exists an arc (u,v) in Q such that (u,z), $(w,v) \in \tilde{A}_{\partial\Phi}$ and it is easy to see that arcs (u,z) and (w,v) lie on minimum-mean cycles whose arcs belong to (4.7). The definition of Q and definition of q_ϕ imply that $\pi(v)<\pi(z)$ and hence $q_\phi(w)\leq\pi(v)<\pi(z)$. Therefore, the appearence of any new arc does not make q_ϕ decrease. We thus have $q_\phi\leq q_{\phi'}$. Because at least one arc in Q is deleted, there exists at least one vertex v such that $q_\phi(v)<q_{\phi'}(v)$. □

From Lemma 4.2, we have

Theorem 4.3. The primal algorithm for submodular flow problem with minimum-mean selection rule (*) terminates after a finite number of steps.

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