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A Dual Interior Primal Simplex Method for Linear Programming

by

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Assumption 3 The dual feasible region Y has a nonempty interior, i.e., there exists y such that $A^T y > c$.

The well-known duality theorem ensures that both of the problems P and D have optimal solutions with a common optimal value λ^*.

A dual-interior primal-simplex method (DIPS method), which we propose in this paper, has some interpretations. First it can be viewed as an interior point algorithm for solving the dual problem D. Suppose that the steepest descent direction $-b$ of the objective value of the dual problem D coincides with the gravitational direction and that the vertical axis represents the objective value $b^T y$ (See Figure 1). It will be convenient for our discussion here to compare the dual feasible region Y to a vessel. We fill the vessel with the water to some level λ and make a hole at the bottom of the vessel which corresponds to a minimal solution of the dual problem D with the objective value λ^*. Then the level λ of the water is coming down until the vessel is empty. For each level λ of the water, we consider a maximal ball S with a center y under the water. As the level λ of the water is coming down, the maximal ball S is shrinking with the center y moving down, and finally the center y will reach the bottom (the minimal solution of the dual problem D) when the vessel becomes empty (or the level λ attains the minimal value λ^* of the problem D). In this process the locus of the center forms a piecewise linear curve. See Figures 2, 3 and 4. Given an initial maximal ball with a center y^1 and a water level λ^1, the DIPS method traces the locus of the center to attain a minimal solution of the dual problem D.

The DIPS method may be regarded as a modification of the gravitational method given by Murty [7] and Chang and Murty [2]. Their method is outlined as follows. If we pick up a small heavy ball and release it at some point in the interior of the vessel (the dual feasible region Y), the ball is falling and rolling down by the gravitational force and stops when it minimizes the potential energy (or equivalently its center minimizes the dual objective function). In this physical process, the center of the ball draws a piecewise linear locus with each piece of line parallel to either the gravitational direction or some of the facets of Y. Their method traces this piecewise linear locus. If the ball is sufficiently small, we have an approximate minimal solution. Otherwise, we replace the ball by a smaller one and release it at some interior point under the end of the locus

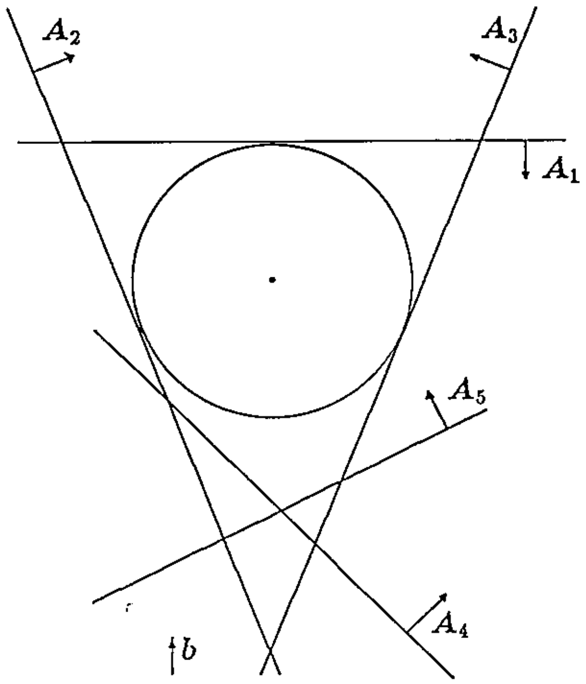

Figure 1. Dual feasible region.

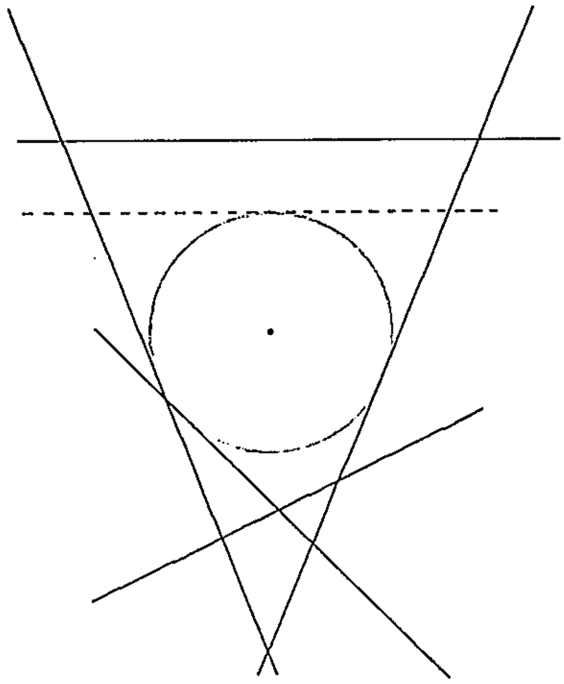

Figure 2.

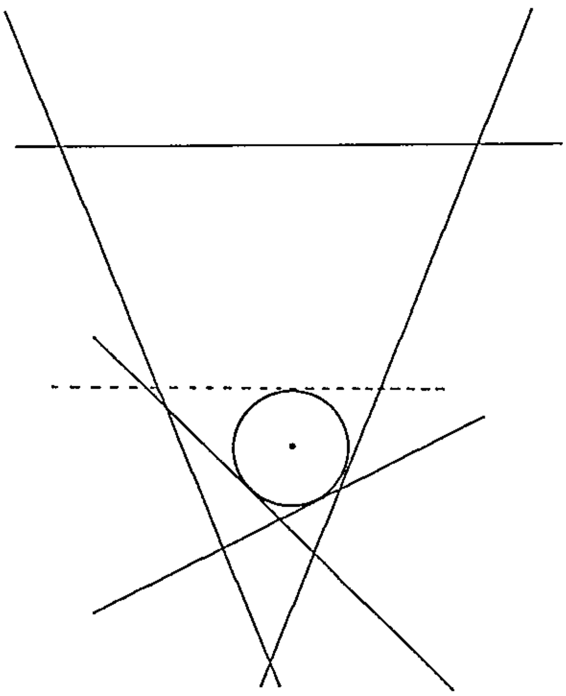

Figure 3.

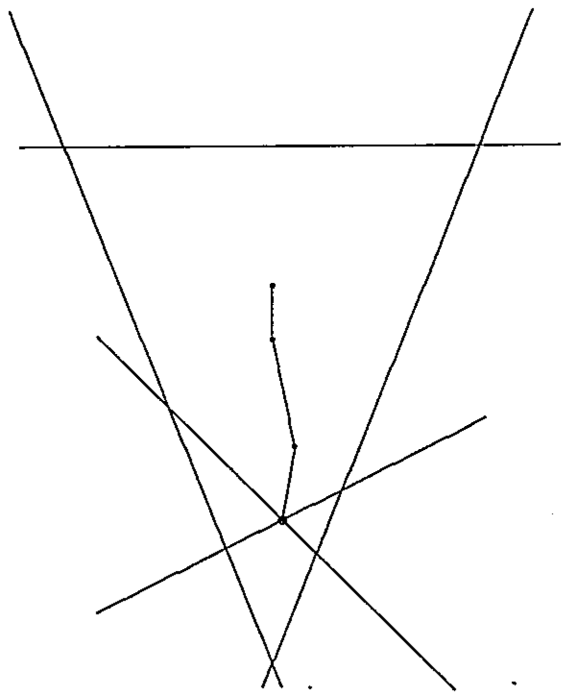

Figure 4. Locus of center.

to repeat the same process. They have also devised an additional technique for finding an exact solution of D in a finite number of steps.

In the gravitational method, the ball keeps its size as long as it can decrease the potential energy, while the DIPS method deal with a ball which can not decrease the potential energy unless it gets smaller; thus the ball shrinks continuously to decrease the potential energy.

The DIPS method can also be viewed as a primal simplex method with a certain column selection rule. In Section2, it will be shown the sequence $\{\boldsymbol{y}^k \mid k = 1, 2, \ldots\}$ of the nodal points of the piecewise linear locus of the center of the maximal ball induces a sequence $\{\boldsymbol{x}^k \mid k = 1, 2, \ldots\}$ of basic feasible solutions of the primal problem P such that $\boldsymbol{c}^T\boldsymbol{x}^k \leq \boldsymbol{c}^T\boldsymbol{x}^{k+1}$; the DIPS method generates an interior feasible solution $\boldsymbol{y}^k$ of the dual problem D and a basic feasible solution $\boldsymbol{x}^k$ of the primal problem P interdependently. Therefore,

(a) we have a lower bound $\boldsymbol{b}^T\boldsymbol{y}^{k+1}$ and an upper bound $\boldsymbol{c}^T\boldsymbol{x}^k$ for the unknown optimal value λ^* throughout the iteration,

(b) the duality gap $\boldsymbol{b}^T\boldsymbol{y}^{k+1} - \boldsymbol{c}^T\boldsymbol{x}^k$ decreases as the iteration proceeds,

(c) both of primal and dual optimal solutions are generated in at most a finite number of steps.

In Section 3, we give another interpretation to the DIPS method in terms of parametric linear programs (see, for example, [3]). From this interpretation, we will derive an interesting geometrical property on the sequence $\{\boldsymbol{x}^k \mid k = 1, 2, \ldots\}$ of basic feasible solutions of P generated by the DIPS method.

A drawback of the DIPS method is that we need in advance a maximal ball with a certain additional property in the dual feasible region to start the iteration. We can utilize the gravitational method referred above to prepare such a maximal ball; we can switch from the gravitational method to the DIPS method when the ball with a fixed size used in the former minimizes the potential energy and stops falling. In Section 4, the DIPS method is modified so that it can start from any pair of a dual interior feasible solution and a primal basic feasible solution. This modification will make it easier to

Here λ moves from a sufficiently large positive number to the minimum value λ^* of the dual problem D. It is easily verified under Assumptions 1, 2 and 3 that $Y(\lambda)$ is bounded for every $\lambda \geq \lambda^*$.

For every $\bar{y} \in R^m$ and $r > 0$, let $S(\bar{y}, r)$ denote the ball with the center $\bar{y}$ and the radius r, i.e., $S(\bar{y}, r) = \{y \in R^m \mid \|y - \bar{y}\| \leq r\}$. For simplicity of notation, we shall assume that

$$\|A_i\| = 1 \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad \|b\| = 1.$$

We can always rescale the primal problem P and the dual problem D so that the above equalities are satisfied. Then a ball $S(\bar{y}, r)$ is included in $Y(\lambda)$ if and only if the inequalities

$$A_i^T \bar{y} - c_i \;\geq\; r \qquad (i = 1, 2, \ldots, n)$$

and

$$\lambda - b^T \bar{y} \;\geq\; r$$

hold. For every ball $S(\bar{y}, r)$ in the dual feasible region Y, let $U(S(\bar{y}, r))$ be the collection of the boundary hyperplanes $\{y \in R^m \mid A_i^T y - c_i = 0\}$ of Y which support the ball, i.e.,

$$U(S(\bar{y}, r)) \;=\; \{i \in \{1, \ldots, n\} \mid A_i^T \bar{y} - c_i = r\}. \tag{2.2}$$

Since $Y(\lambda)$ is bounded for any $\lambda \geq \lambda^*$, there exists a ball which is included in $Y(\lambda)$ and has the maximum radius. We call such a ball a *maximal ball* at level λ or λ-*maximal ball*. We say that a λ-maximal ball $S(\bar{y}, r)$ is *proper* if it satisfies the following two conditions:

Condition 1 $\bar{y}$ minimizes the objective function of the dual problem D among the centers of λ-maximal balls;

Condition 2 $U(S(\bar{y}, r))$ is maximal among all $U(S(y, r))$'s such that $S(y, r)$ is a λ-maximal ball satisfying Condition 1.

The following theorem is essentially due to Murty [7]. We include a proof for completeness.

Theorem 1 (Theorem 1 in [7] and Theorem 5 in [2]). *For every proper maximal ball $S(\bar{y},r)$ at level λ, there exists a feasible basis B of the primal problem P such that $B \subseteq U(S(\bar{y},r))$.*

Proof. For simplicity of notation, let $U = U(S(\bar{y},r))$. Let E be the subspace spanned by the column vectors $\boldsymbol{A}_i$ $(i \in U)$, and $E^{\perp}$ be its orthogonal complement $\{\boldsymbol{y} \in R^m \mid (\boldsymbol{A}_U)^T\boldsymbol{y} = \mathbf{o}\}$ where $\boldsymbol{A}_U$ denotes the $m\times \mid U \mid$ matrix consisting of all the column vectors $\boldsymbol{A}_i$ with $i \in U$. If the orthogonal projection $\bar{\boldsymbol{b}}$ of $\boldsymbol{b}$ onto the subspace $E^{\perp}$ were not zero, we could move the center of the ball $S(\bar{y},r)$ slightly in the direction $-\bar{\boldsymbol{b}}$ so that the ball would remain in $Y(\lambda)$ and that the value of the dual objective function $\boldsymbol{b}^T\bar{\boldsymbol{y}}$ at the center would decrease. This would contradict Condition 1. Hence $\bar{\boldsymbol{b}} = \mathbf{o}$ or $\boldsymbol{b}$ is contained in the subspace E.

Now we shall show that $\text{rank}(\boldsymbol{A}_U) = m$. Assume on the contrary that $\text{rank}(\boldsymbol{A}_U) < m$. By Assumption 1, we know that $\text{rank}(\boldsymbol{A}) = m$. Hence we can find a column vector $\boldsymbol{A}_j$, $j \notin U$, of the matrix $\boldsymbol{A}$ for which $\boldsymbol{A}_j \notin E$; hence the orthogonal projection $\boldsymbol{v}$ of $\boldsymbol{A}_j$ onto $E^{\perp}$ is nonzero. Therefore, if we move the center of the ball in the direction $-\boldsymbol{v}$ so long as the ball remains in the bounded region $Y(\lambda)$, the number of the boundary hyperplanes $\{\boldsymbol{y} \in R^m \mid \boldsymbol{A}_i^T\bar{\boldsymbol{y}} - c_i = r\}$ of Y which support the ball increases by at least one. It should be also noted that any movement of the center of the ball $S(\bar{y},r)$ in the direction $-\boldsymbol{v}$ never changes the dual objective value at the center since $\boldsymbol{b} \in E$. This contradicts Condition 2, the maximality of the index set U. Thus we have shown that $\text{rank}(\boldsymbol{A}_U) = m$.

By the Condition 1 imposed on the proper maximal ball $S(\bar{y},r)$ and the definition (2.2) of the index set U, there is no direction $\boldsymbol{v} \in R^m$ such that

$$\begin{aligned} \boldsymbol{b}^T\boldsymbol{v} &< 0 \quad \text{and} \\ \boldsymbol{A}_i^T\boldsymbol{v} &\geq 0 \quad \text{for all } i \in U. \end{aligned}$$

Hence, by applying the well-known Farkas' Lemma, we see that there is a feasible solution $\boldsymbol{x} \in X$ of the primal problem P such that $x_j = 0$ $(j \notin U)$. Since $\text{rank}(\boldsymbol{A}_U) = m$, we can choose a basic feasible solution $\bar{\boldsymbol{x}}$ from the set of such feasible solutions. Finally, letting B be the primal feasible basis associated with $\bar{\boldsymbol{x}}$, we obtain $B \subseteq U$. □

We call a ball $S(\bar{y},r) \subseteq Y$ such that $U(S(\bar{y},r))$ contains a feasible basis B of the primal problem P a *basic ball* in Y. By Theorem 1, every proper maximal ball is a basic ball. Conversely, we can prove similarly that if $S(\bar{y}, r)$ is a basic ball and we set λ to be $r + b^T\bar{y}$ then $S(\bar{y}, r)$ is a proper λ-maximal ball.

For every basic ball $S(\bar{y}, r)$, we are concerned with the following primal and dual pair of subproblems:

$$\begin{array}{lll} P_U & \text{maximize} & (c_U)^T x \\ & \text{subject to} & A_U\, x = b, \\ & & x \geq \mathrm{o}. \end{array}$$

$$\begin{array}{lll} D_U & \text{minimize} & b^T y \\ & \text{subject to} & (A_U)^T y \geq c_U. \end{array}$$

where U denotes the index set $U(S(\bar{y}, r))$. By the definition of a basic ball, we know that the problem P_U has at least one feasible basis and that $\bar{y}$ is a feasible solution of the problem D_U. Hence, by the duality theorem, the problem P_U has an optimal basis.

Theorem 2 *Let $S(\bar{y}, r)$ be a basic ball and $U = U(S(\bar{y}, r))$. Suppose that B is an optimal basis of the problem P_U. Let $v = (A_B^T)^{-1}c_B - \bar{y}$. Then*

$$-r = A_i^T v \ \leq\ A_j^T v \quad \textit{for all } i \in B \textit{ and all } j \in U \backslash B.$$

Proof. Let $z = (A_B^T)^{-1}c_B$. Then z is an optimal solution of the dual problem D_U. Assume on the contrary that B does not satisfy the condition, i.e., there exist $i \in B$ and $j \in U\backslash B$ such that

$$A_i^T v \ >\ A_j^T v. \tag{2.3}$$

Since $i, j \in U$, we have

$$A_i^T \bar{y} - c_i \ =\ A_j^T \bar{y} - c_j \ =\ r. \tag{2.4}$$

Hence

$$\begin{array}{rlr} 0 & = A_i^T z - c_i & (\text{since } i \in B) \\ & = A_i^T \bar{y} + A_i^T v - c_i & (\text{since } z = \bar{y} + v) \\ & = A_j^T \bar{y} + A_i^T v - c_j & (\text{by (2.4)}) \\ & > A_j^T \bar{y} + A_j^T v - c_j & (\text{by (2.3)}) \\ & = A_j^T z - c_j. & (\text{since } z = \bar{y} + v) \end{array}$$

Thus we obtain $0 > A_j^T z - c_j$. This contradicts the feasibility of the optimal solution z of the problem D_U. □

For every feasible basis B of the primal problem P, let $\Sigma(B)$ denote the collection of all the basic balls $S(\bar{y}, r)$ with $B \subseteq U(S(\bar{y}, r))$, and let $C(B)$ denote the set of all the centers $\bar{y}$ of basic balls $S(\bar{y}, r)$ with $B \subseteq U(S(\bar{y}, r))$, i.e.,

$$C(B) = \{\bar{y} \in R^m \mid S(\bar{y}, r) \in \Sigma(B), \quad r \geq 0\}.$$

Theorem 3 *Let B be a feasible basis of the primal problem P. If the set $C(B)$ is nonempty, it forms a closed interval (closed convex subset) of the line*

$$L(B) = \{y \in R^m \mid y = s(A_B^T)^{-1}e + z \ \text{ for all } \ s \in R\}$$

where $e = (1, \ldots, 1)^T \in R^B$ and $z = (A_B^T)^{-1}c_B$.

Proof. Let $S(\bar{y}, r)$ be a basic ball. Since $A_i^T \bar{y} - c_i = r$ for all $i \in B$, $\bar{y} = r(A_B^T)^{-1}e + z$, i.e., $\bar{y}$ lies on the line $L(B)$.

Let $S(\bar{y}^p, r^p)$ $(p = 0, 1)$ be basic balls in $\Sigma(B)$. Then

$$A_i^T \bar{y}^0 - c_i \ = \ r^0 \quad \text{and} \quad A_i^T \bar{y}^1 - c_i \ = \ r^1 \tag{2.5}$$

hold for all $i \in B$. Suppose that $S(\bar{y}^t, r^t)$ is a ball with a center $\bar{y}^t = (1-t)\bar{y}^0 + t\bar{y}^1$ and a radius $r^t = (1-t)r^0 + t r^1$ for some $t \in [0, 1]$. By (2.5), it is clear that $A_i^T \bar{y}^t - c_i = r^t$ for all $i \in B$. We also see from the convexity of the dual feasible region Y that $S(\bar{y}^t, r^t) \subseteq Y$. Thus $S(\bar{y}^t, r^t)$ is a basic ball with $B \subseteq U(S(\bar{y}^t, r^t))$. Thus we have shown that $C(B)$ is convex. The closedness of $C(B)$ follows from the closedness of Y. □

Now we are ready to explain the detail of the DIPS method. The method generates a sequence $\{S(\bar{y}^k, r^k)\}$ of basic balls, a sequence $\{B^k\}$ of primal feasible bases and a sequence $\{x^k\}$ of primal basic solutions which satisfy

$$r^{k+1} < r^k, \tag{2.6}$$

$$r^{k+1} = \min\{r \mid S(\bar{y}, r) \in \Sigma(B^k)\}, \tag{2.7}$$

$$B^k \subseteq U(S(\bar{y}^k, r^k)), \tag{2.8}$$

$$B^k \subseteq U(S(\bar{y}^{k+1}, r^{k+1})), \tag{2.9}$$

$$A_i^T v \ \leq \ A_j^T v \quad \text{for all } i \in B^k \text{ and all } j \in U(S(\bar{y}^k, r^k))\backslash B^k, \tag{2.10}$$

$$b^T \bar{y}^{k+1} < b^T \bar{y}^k \quad \text{and} \tag{2.11}$$

$$c^T x^{k+1} \geq c^T x^k \tag{2.12}$$

for every $k = 1, 2, \ldots,$ where $v = (A_{B^k}^T)^{-1} c_{B^k} - \bar{y}^k$.

The iteration starts with an initial basic ball $S(\bar{y}^1, r^1)$, which is assumed to be available in advance. Let $U = U(S(\bar{y}^1, r^1))$, and solve the subproblem P_U for this U to obtain an optimal solution x^1 and the corresponding optimal basis B^1. Let $v = (A_{B^1}^T)^{-1} c_{B^1} - \bar{y}^1$. Then, by Theorem 2, we have

$$A_i^T v \ \leq \ A_j^T v \quad \text{for all } i \in B^1 \text{ and all } j \in U(S(\bar{y}^1, r^1))\backslash B^1.$$

Now we suppose that we have obtained a basic ball $S(\bar{y}^k, r^k)$ and a primal feasible B^k satisfying (2.8) and (2.10), and show how to generate a new basic ball $S(\bar{y}^{k+1}, r^{k+1})$, a primal basic feasible solution x^{k+1} and a primal basis B^{k+1} at the kth iteration.

Let $z^k = (A_{B^k}^T)^{-1} c_{B^k}$. Then v can be rewritten as $v = z^k - \bar{y}^k$. For each $t \in [0, 1]$, we consider a ball with the center $\bar{y}^k(t) = \bar{y}^k + tv$ and the radius $r^k(t) = a^T \bar{y}^k(t) - c = (1-t) r^k$ where $a = A_i$ and $c = c_i$ for some $i \in B^k$. (Note that $r^k(t)$ is independent of any selection of $i \in B^k$). Then

$$A_i^T \bar{y}^k(t) - c_i \ = \ r^k(t) \qquad \text{for all } t \in [0,1] \text{ and all } i \in B^k. \tag{2.13}$$

For any s and t with $0 \leq s < t \leq 1$ and $i \in B^k$, we also have

$$r^k(s) - r^k(t) = (t - s) r^k > 0. \tag{2.14}$$

Hence, as t increases from 0, the ball $S(\bar{y}^k(t), r^k(t))$ shrinks and its center $\bar{y}^k(t)$ moves linearly toward z^k. Specifically,

$$\bar{y}^k(1) = z^k \quad \text{and} \quad r^k(1) = 0. \tag{2.15}$$

In other words, the ball $S(\bar{y}^k(t), r^k(t))$ shrinks into the point z^k when $t = 1$.

By (2.13), we see that if the ball $S(\bar{y}^k(t), r^k(t))$ lies in the dual feasible region Y for some $t \in [0,1]$ then it is a basic ball. By Theorem 3, we know that the set of such t's forms an interval. Let $\bar{t}^k$ be the maximum value among such t's. Then

$$A_j^T \bar{y}^k(t) - c_j \geq r^k(t) \qquad \text{for all } t \in [0, \bar{t}^k] \text{ and all } j \notin B^k. \tag{2.16}$$

We have either $\bar{t}^k = 1$ or $\bar{t}^k \in [0,1)$. Let $\bar{y}^{k+1} = \bar{y}^k(\bar{t}^k)$ and $r^{k+1} = r^k(\bar{t}^k)$.

First we consider the case that $\bar{t}^k = 1$. It follows from (2.13), (2.15) and (2.16) that $\bar{y}^{k+1} = z^k$ is a dual feasible solution with the objective value $b^T z^k = c_{B^k}^T A_{B^k}^{-1} b$. Since B^k is a primal feasible basis, we obtain, by the duality theorem, that $\bar{y}^{k+1}$ is an optimal solution of the dual problem D and B^k is an optimal basis of the primal problem P. In this case, the DIPS method stops.

Now we consider the case that $\bar{t}^k \in [0,1)$. By the definition of $\bar{t}^k$, we can find an index $e \notin B^k$ such that

$$\begin{aligned} &A_e^T \bar{y}^k(\bar{t}^k) - c_e = r^k(\bar{t}^k), \\ &A_e^T \bar{y}^k(\bar{t}^k + \varepsilon) - c_e \; < \; r^k(\bar{t}^k + \varepsilon) \qquad \text{for any } \varepsilon > 0, \\ &S(\bar{y}^{k+1}, r^{k+1}) \subseteq Y \quad \text{and} \quad B^k \cup \{e\} \subseteq U(S(\bar{y}^{k+1}, r^{k+1})). \end{aligned} \tag{2.17}$$

That is, $S(\bar{y}^k(t), r^k(t))$ bumps against the eth constraint of (2.1) when t attains $\bar{t}^k$, and then we have a new basic ball $S(\bar{y}^{k+1}, r^{k+1})$ satisfying (2.9).

Taking $\bar{t}^k + \varepsilon = 1$ in (2.17), we have

$$A_e^T z^k - c_e \; < \; r^k(1) \; = \; 0 \tag{2.18}$$

(see also (2.15)). We shall show that (2.6). It follows from (2.17) and $\bar{y}^k(t) = \bar{y}^k + tv$ $(t \in [0,1])$ that

$$r^k(\bar{t}^k + \varepsilon) - r^k(\bar{t}^k) > \varepsilon A_e^T v$$

for every sufficiently small positive ε. On the other hand, by (2.14), we have

$$0 > r^k(\bar{t}^k + \varepsilon) - r^k(\bar{t}^k) = \varepsilon A_i^T v$$

for every sufficiently small positive ε and every $i \in B^k$. Hence

$$0 > A_i^T v > A_e^T v \qquad \text{for every } i \in B^k. \tag{2.19}$$

Since B^k and v satisfy (2.10), we see $e \notin U(S(\bar{y}^k, r^k))$, i.e.,

$$A_e^T \bar{y}^k(0) - c_e > r^k = r^k(0).$$

Comparing the inequality above with (2.17), we have $\bar{t}^k > 0$, and (2.6) follows from (2.14).

To see (2.11), we evaluate the dual objective function at $\bar{y}^{k+1}$:

$$\begin{aligned} b^T \bar{y}^{k+1} &= b^T(\bar{y}^k + \bar{t}^k v) \\ &= b^T(\bar{y}^k + \bar{t}^k(z^k - \bar{y}^k)) \\ &= b^T \bar{y}^k + \bar{t}^k(b^T(A_{B^k}^T)^{-1} c_{B^k} - b^T \bar{y}^k) \\ &= b^T \bar{y}^k + \bar{t}^k\{c_{B^k}^T(A_{B^k})^{-1} b - b^T \bar{y}^k\}. \end{aligned}$$

Since B^k is a primal feasible basis and $\bar{y}^k$ is a dual interior feasible solution, by the duality theorem, we see that the term in the $\{\ \}$ of the right hand side of the last equality above is negative. Thus we obtain (2.11).

We have shown that the new ball $S(\bar{y}^{k+1}, r^{k+1})$ is a basic ball with $B^k \subseteq U(S(\bar{y}^{k+1}, r^{k+1}))$. In view of Theorem 3, both $\bar{y}^k$ and $\bar{y}^{k+1}$ lies on the line

$$L(B^k) = \{y \in R^m \mid y = s(A_{B^k}^T)^{-1} e + z^k \text{ for all } s \in R\}.$$

Furthermore, by the construction, we see that $\bar{y}^{k+1}$ is an extreme point of the interval $C(B^k)$ of all the centers of basic balls $S(\bar{y}, r) \in \Sigma(B^k)$. Since the radius of the ball $S(\bar{y}, r)$ changes linearly on $C(B^k)$, by (2.14), we obtain (2.7).

Now we will choose a new primal feasible basis $B = B^{k+1} \subseteq U(S(\bar{y}^{k+1}, r^{k+1}))$ such that

$$A_i^T v \le A_j^T v \quad \text{for all } i \in B \text{ and all } j \in U(S(\bar{y}^{k+1}, r^{k+1}))\backslash B \tag{2.20}$$

where $v = (A_B^T)^{-1} c_B - \bar{y}^{k+1}$. It should be noted that the old basis $B = B^k$ does not satisfy the above relation any more because of (2.19) and $e \in U(S(\bar{y}^{k+1}, r^{k+1}))$. Let $B = B^{k+1}$ be an optimal basis of the subproblem P_U with $U = U(S(\bar{y}^{k+1}, r^{k+1}))$, and x^{k+1} be the basic feasible solution of the problem P associated with the basis B^{k+1}. By Theorem 2, the relation (2.20) holds. Since B^k is also a feasible basis of the subproblem P_U, the inequality (2.12) follows.

Thus we have generated all the new iterates, the basic ball $S(\bar{y}^{k+1}, r^{k+1})$, the primal feasible basis B^{k+1} and the associated basic feasible solution x^{k+1} of the problem P,

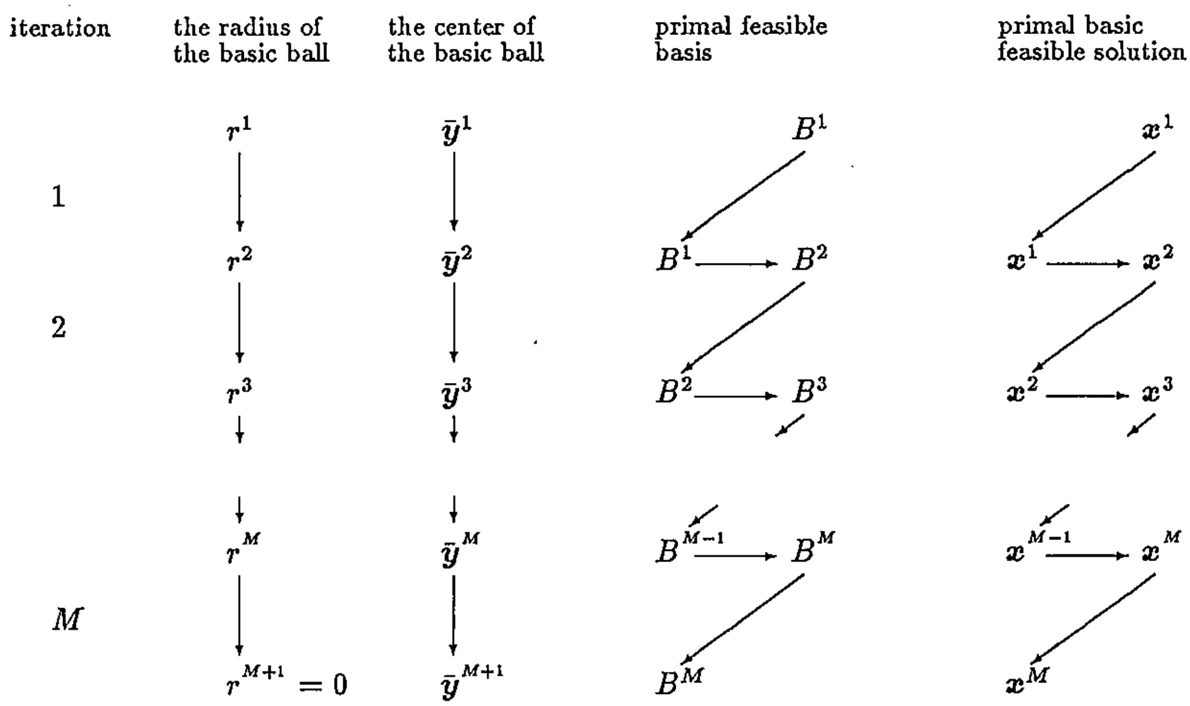

Figure 5.

and confirmed that all the relations (2.6) through (2.12) are satisfied. Replacing k by $(k+1)$, the DIPS method repeats the same process.

The relations (2.6) and (2.7) ensure that each primal feasible basis appears in the sequence $\{B^k\}$ at most once. For if $B^k = B^{k'}$ for some distinct k and k', we would have from (2.7) that $r^{k+1} = r^{k'+1}$, which would contradicts (2.6). Therefore, we can conclude that the DIPS method terminates in a finite number, say M, of iterations. Figure 5 illustrates the iterates of the DIPS method. Here $\bar{y}^{M+1}$ is an optimal solution of the dual problem D and x^M is an optimal solution of the primal problem P.

In each iteration, we need to compute an optimal basis B^{k+1} of the subproblem P_U with $U = U(S(\bar{y}^{k+1}, r^{k+1}))$. As we have observed, B^k is a feasible basis of the subproblem P_U. Hence we can apply the phase 2 of the standard primal simplex method with the initial feasible basis B^k to the subproblem P_U. This may take more than one pivot iterations generally. If the nondegeneracy assumption below is satisfied, however, only one pivot iteration is required to solve the subproblem P_U.

Assumption 4 (Dual Nondegeneracy Assumption). For any basic ball $S(\bar{y}, r)$, the index set $U(S(\bar{y}, r))$ has no more than $m+1$ elements.

Under the assumption above, the index set $U = U(S(\bar{y}^{k+1}, r^{k+1}))$ can be written as $U = B^k \cup \{e\}$ for a unique index $e \notin U(S(\bar{y}^k, r^k))$ satisfying (2.17). Hence, the subproblem P_U with $U = U(S(\bar{y}^{k+1}, r^{k+1}))$ has exactly two feasible basis, which are nondegenerate and adjacent with each other; the one is B^k and the other is B^{k+1}. Thus we need exactly one pivot operation to compute B^{k+1} from B^k. If, in addition, the primal nondegeneracy assumption below is satisfied, the value of the objective function of the problem P_U (hence P) increases by the pivot operation, i.e., $\boldsymbol{c}^T\boldsymbol{x}^{k+1} > \boldsymbol{c}^T\boldsymbol{x}^k$ since the simplex criterion $c_e - \boldsymbol{A}_e^T(\boldsymbol{A}_{B^k}^T)^{-1}\boldsymbol{c}_{B^k}$ is positive by the inequality (2.18).

Assumption 5 (Primal Nondegeneracy Assumption). For every feasible basis B of the primal problem P, the inequality $\boldsymbol{A}_B^{-1}\boldsymbol{b} > \mathbf{o}$ holds.

Therefore, if we restrict our attention to the sequence $\{\boldsymbol{x}^k\}$ of basic feasible solutions of the primal problem P, the DIPS method may be regarded as a primal simplex method with a special column selection rule using the information on basic balls in the dual feasible region. In the next section, a certain interesting property on the sequence $\{\boldsymbol{x}^k\}$ will be shown.

In degenerate cases where Assumptions 4 and/or 5 are not satisfied, it may be necessary to incorporate some technique to avoid cycling in the simplex method for solving the subproblem P_U. For example, we can employ the smallest-subscript pivoting rule by Bland [1].

3. An Interpretation of The DIPS Method in terms of Parametric Programming.

Let $\{S(\bar{y}^k, r^k) \mid k = 1, \ldots, M+1\}$ be the sequence of basic balls, $\{B^k \mid k = 1, \ldots, M\}$ the sequence of primal feasible bases and $\{\boldsymbol{x}^k \mid k = 1, \ldots, M\}$ the sequence of primal basic solutions which are generated by the DIPS method. We will relate these sequences to the following primal-dual pair of parametric programs

$$
\begin{array}{lll}
P(r) & \text{maximize} & (\boldsymbol{c} + r\boldsymbol{e})^T\boldsymbol{x} \\
 & \text{subject to} & \boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b} \\
 & & \boldsymbol{x} \geq \mathbf{o},
\end{array}
$$

$$D(r) \qquad \text{minimize} \quad b^T y \qquad \text{subject to} \quad A^T y \geq c + re.$$

Here r is a scalar parameter and e denotes the vector of ones with the dimension n. Obviously, $P(0)$ and $D(0)$ are the same linear programs as P and D, respectively.

First we observe that each x^k is a basic feasible solution, with the basis B^k, of the problem $P(r)$ for every r. Hence we have

$$\begin{aligned} x_j^k &= 0 \quad \text{for all } j \notin B^k \text{ and} \\ x_j^{k+1} &= 0 \quad \text{for all } j \notin B^{k+1}. \end{aligned}$$

We also see that each $\bar{y}^{k+1}$ is a feasible solution of the problem $D(r^{k+1})$. By the definition of $U(S(\bar{y}^{k+1}, r^{k+1}))$, we have

$$A_i^T \bar{y}^{k+1} - (c_i + r^{k+1}) = 0 \qquad \text{for all } i \in U(S(\bar{y}^{k+1}, r^{k+1})).$$

The three equalities above, together with the relations (2.8) and (2.9), imply the complementary slackness condition holds between x^k and $\bar{y}^{k+1}$, and between x^{k+1} and $\bar{y}^{k+1}$, respectively. Hence, x^k and x^{k+1} are both optimal solutions of the primal problem $P(r^{k+1})$ and that $\bar{y}^{k+1}$ is an optimal solution of the dual problem $D(r^{k+1})$. Thus we obtain

$$(c + r^{k+1}e)^T (x^{k+1} - x^k) = 0 \qquad (k = 1, 2, \ldots). \tag{3.1}$$

The above discussion indicates that the DIPS method can be simulated by applying the parametric programming algorithm (see, for example, Gal [3]) or the complementary pivoting algorithm (Lemke [6]) to the primal problem $P(r)$ with the initial parameter value $r = r^1$.

We conclude this section by showing an interesting property of $\{x^k\}$. The maximal solution x^M of the primal problem P and the pth iterate x^p can be written as

$$x^M = \sum_{k=p}^{M-1} (x^{k+1} - x^k) + x^p$$

and

$$x^p = \sum_{k=1}^{p-1} (x^{k+1} - x^k) + x^1,$$

respectively. Let C_p denote the convex cone spanned by the set of the edge vectors $x^{k+1} - x^k$ $(k = 1, 2, \ldots, p-1)$ and let D_p denote the convex cone spanned by the set of the edge vectors $x^{k+1} - x^k$ $(k = p, p+1, \ldots, M-1)$.

Theorem 4 *For every* $p \in \{1, 2, \ldots, M-1\}$ *and every* $q \geq p$, *the vector* $x^{q+1} - x^q$ *does not lie in the interior* $(C_p)^I$ *of the cone* C_p. *Moreover, we have that*

$$(C_p)^I \cap D_p = \emptyset.$$

Proof. Since C_p is monotone increasing, i.e., $C_p \subseteq C_{p+1}$, it suffices to deal with the case when $q = p$. We have from the equality (3.1) that

$$(c + r^{p+1}e)^T(x^{p+1} - x^p) = 0.$$

That is, the hyperplane $H = \{x \in R^n \mid (c + r^{p+1}e)^T x = 0\}$ contains the vector $x^{p+1} - x^p$. On the other hand, it follows from the equality (3.1) and the inequality (2.12) that

$$e^T(x^{k+1} - x^k) \leq 0 \qquad (k = 1, 2, \ldots). \tag{3.2}$$

Hence, for $k = 1, 2, \ldots, p-1$, we have

$$\begin{aligned} & (c + r^{p+1}e)^T(x^{k+1} - x^k) \\ = & (c + r^{k+1}e)^T(x^{k+1} - x^k) + (r^{p+1} - r^{k+1})e^T(x^{k+1} - x^k) \\ \geq & 0. \end{aligned} \tag{3.3}$$

The last inequality follows since $(c+r^{k+1}e)^T(x^{k+1}-x^k) = 0$ (by (3.1)), $r^{p+1}-r^{k+1} < 0$ (by (2.6)) and $e^T(x^{k+1} - x^k) \leq 0$ (by (3.2)). Thus all the vectors $x^{k+1} - x^k$ $(k = 1, 2, \ldots, p-1)$ lie on the nonnegative side of the hyperplane H which includes $x^{p+1} - x^p$, and the first desired result follows.

In the same way as above, for $k = p, \ldots, M-1$, we have

$$\begin{aligned} & (c + r^{p+1}e)^T(x^{k+1} - x^k) \\ = & (c + r^{k+1}e)^T(x^{k+1} - x^k) + (r^{p+1} - r^{k+1})e^T(x^{k+1} - x^k) \\ \leq & 0. \end{aligned}$$

Hence we have all the vectors $x^{k+1} - x^k$ $(k = p, \ldots, M-1)$ lie on the nonpositive side of the hyperplane H, and $(C_p)^I \cap D_p = \emptyset$ follows. □

Corollary 5 *Suppose that Assumptions 4 and 5 are satisfied. Then the following* (a), (b), (c) *and* (d) *hold:*

(a) x^p *and* x^{p+1} *are adjacent vertices of the primal feasible region of* $P(0)$,

(b) $c^T x^{p+1} > c^T x^p$,

(c) *the edge vector* $x^{q+1} - x^q$ *does not lie in the cone* C_p,

(d) $C_p \cap D_p = \{o\}$,

for every $p = 1, \ldots, M-1$ *and every* $q \geq p$.

Proof. We have already established the assertions (a) and (b). By (b) we have the strict inequalities in (3.2) and (3.3) of the proof above. Therefore, all the vectors $\boldsymbol{x}^{k+1} - \boldsymbol{x}^k$ $(k = 1, 2, \ldots, q-1)$ lie on the strict positive side of the hyperplane H. This ensures (c) and (d). □

Remark. The property (d) of Corollary 5 implies "distinct parallel edges $\boldsymbol{x}^{k+1} - \boldsymbol{x}^k$ and $\boldsymbol{x}^{p+1} - \boldsymbol{x}^p$ $(k \neq p)$ are never generated." This statement was originally presented by Prof. N. Tomizawa on an abstract and combinatorial linear programming model.

4. A Modification.

The discussion of the previous section will lead to a modification of the DIPS method so that it can start from any interior feasible solution $\boldsymbol{y}^1$ of the dual problem D and any basic feasible solution $\boldsymbol{x}^1$ of the primal problem P. Let B^1 be the primal feasible basis associated with $\boldsymbol{x}^1$, and $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)^T$ an n-dimensional vector such that

$$\begin{aligned} d_i &= \boldsymbol{A}_i^T \boldsymbol{y}^1 - c_i \quad \text{for every } i \in B^1 \text{ and} \\ 0 < d_j &< \boldsymbol{A}_j^T \boldsymbol{y}^1 - c_j \quad \text{for every } j \notin B^1. \end{aligned}$$

Consider the following primal-dual pair of parametric linear programs:

$$\begin{array}{lll} P'(r) & \text{maximize} & (\boldsymbol{c} + r\boldsymbol{d})^T \boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}, \\ & & \boldsymbol{x} \geq \mathbf{o}. \end{array}$$

$$\begin{array}{lll} D'(r) & \text{minimize} & \boldsymbol{b}^T \boldsymbol{y} \\ & \text{subject to} & \boldsymbol{A}^T \boldsymbol{y} \geq \boldsymbol{c} + r\boldsymbol{d}. \end{array}$$

Obviously, $P'(0)$ and $D'(0)$ coincides with the problems P and D, respectively. By the construction, $\boldsymbol{x}^1$ and $\boldsymbol{y}^1$ are feasible solutions of $P'(1)$ and $D'(1)$, and they satisfies the complementarity condition

$$(\boldsymbol{A}^T \boldsymbol{y} - \boldsymbol{c} - r\boldsymbol{d})^T \boldsymbol{x} = 0 \tag{4.1}$$

for $r = 1$. Hence they are optimal solutions of the problems $P'(1)$ and $D'(1)$, respectively. For each parameter $r \geq 0$ and each feasible solution $\boldsymbol{y}$ of $D'(r)$, define

$$U'(\boldsymbol{y}, r) = \{i \mid (\boldsymbol{A}_i^T)\boldsymbol{y} - c_i - r\, d_i = 0\}.$$

For a feasible basis B of the problem P, let $\Sigma'(B)$ denote the collection of all pairs of y and r such that y is a feasible solution of the problem $D'(r)$ and $B \subseteq U'(y,r)$. If we replace $U(S(y,r))$ by $U'(y,r)$ and $\Sigma(B)$ by $\Sigma'(B)$, we can modify the method described in Section 3 so as to generate sequences $\{r^k\}$ of parameters, $\{y^k\}$ of dual interior feasible solutions of the problem D, $\{x^k\}$ of basic feasible solutions of the problem P and $\{B^k\}$ of feasible basis of the problem P such that

$$r^{k+1} < r^k, \tag{4.2}$$

$$r^{k+1} = \min\{r \mid (y,r) \in \Sigma'(B^k)\}, \tag{4.3}$$

$$B^k \subseteq U'(y^k, r^k), \tag{4.4}$$

$$B^k \subseteq U'(y^{k+1}, r^{k+1}), \tag{4.5}$$

$$\frac{A_i^T v}{d_i} \leq \frac{A_j^T v}{d_j} \quad \text{for all } i \in B^k \text{ and all } j \in U'(y^k, r^k)\backslash B^k \tag{4.6}$$

$$b^T y^{k+1} < b^T y^k \quad \text{and} \tag{4.7}$$

$$c^T x^{k+1} \geq c^T x^k \tag{4.8}$$

for every $k = 1, 2, \ldots,$ where $v = (A_{B^k}^T)^{-1} c_{B^k} - y^k$. Theorem 4 and its Corollary 5 remains valid. The details are omitted here.

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