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# Limit Properties of Equilibrium Allocations of Walrasian Strategic Games

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## I. Introduction

As shown by Hurwicz (1972). honest reporting of preferences or demand mappings is not a Nash equilibrium of a Walrasian strategic game where agents can manipulate either preferences or demand mappings. As in our previous paper (Otani and Sicilian (1982)), we will be interested in characterizing the set of equilibrium allocations of a Walrasian strategic game instead of a question on whether strategies employed by agents are true preferences or true demand mappings.

In our previous paper, we have provided a fairly complete characterization of the set of equilibrium allocations, when either the number of agents is two, or the number of commodities is two. We also note that a similar results was obtained by Hurwicz (1979) for the case of two agents and two commodities. After providing notations, definitions and basic assumptions in Section II, we provide a partial generalization of our previous results in Otani and Sicilian (1982) to the case with arbitrary finite numbers of agents and commodities so that the set of Cournot-Nash (C-N) equilibrium allocations of a Walrasian strategic game stays large and, in the case of a replica economy, it does not shrink when the number of agents of each type becomes larger. In Section IV, we first show that in the economy with a continuum of agents, the set of C-N equilibrium allocations coincides with the set of Walrasian equilibrium allocations with true demand mappings. Therefore the set of C-N equilibrium allocations will not in general possess a closed graph property as a mapping from distributions on agent characteristics to distributions on the set of feasible allocations. See remarks by Dubey. Mas-Colell and Shubik (1980, p.348) on this point.

In Section IV. we will introduce a sequence of economies and try to find out whether or not a true Walrasian allocation of a limit economy may possess a nicer property than other limiting C-N equilibrium allocations. We will argue that if preferences and demand mappings strategically employed possess smoothness and if the limiting aggregate excess demand map is regular at the limit of C-N equilibria. then the limiting C-N equilibrium allocation must be a true Walrasian equilibrium allocation in the limit economy.

## II. Notations. Definitions and Basic Assumptions

The commodity space will be $R^{\ell}$. The set of agent names will be denoted by I which is either a finite set with a counting measure or [0, 1] with Lebesque measure. We use $\lambda$ to indicate a measure on I. The consumption set $X_t$ of agent t will be assumed to be the nonnegative orthant $R_+^{\ell}$. Agents will be characterized by a pair $(\omega, \succsim)$ where $\omega \in R_+^{\ell}$, $\omega \gg 0$ and $\succsim$ is a complete and transitive preference relation on $R_+^{\ell}$ which will be assumed to be continuous. monotone and strictly convex on $R_+^{\ell}$. The set of agent's characteristics is denoted by $\mathcal{A}$ considered as a measurable space as in Hildenbrand (1974). An economy will be a measurable map $\mathcal{E} : I \to \mathcal{A}$ as in Aumann (1963) and $\mathcal{E}(t)$ will be denoted by $(\omega_t, \succsim_t)$. The price space will be the $(\ell-1)$ dimensional standard simplex denoted by $\Delta$.

For each $t \in I$, the true demand function generated by his characteristic $(\omega_t, \succsim_t)$ is a function $\hat{f}_t : \Delta \to R_+^\ell$ such that $\hat{f}_t(p)$ is a $\succsim_t$ - maximal element on his budget and $\hat{f}_t$ satisfies the budget identity, i.e., $p \cdot \hat{f}_t(p) = p \cdot \omega_t$ for every $p \in \Delta$. Let S be a collection of demand maps (possibly multivalued) $f : \Delta \to R_+^\ell$ such that f is upper hemi-continuous, convex-valued on $\Delta$ and f satisfies Richter's congruence axion. (See Richter (1966).) In our economy, the allocation $(\omega_t)_{t \in I}$ of initial endowments is assumed to be common knowledge to every agent, but each agent can disguise his true demand changing his equilibrium consumption allocation to his benefit. Thus we suppose that the set of strategies available to agent t is given by

$$S_t = \{f \in S \mid p \cdot f(p) = p \cdot \omega_t\}$$

and the collection of strategy assignments $(f_t)_{t \in I}$ will be denoted by $F = \prod_{t \in I} S_t$.

The set of feasible allocations will be denoted by

$$\mathcal{X} = \{ \underset{\sim}{x} \mid \underset{\sim}{x} : I \to R_+^\ell, \underset{\sim}{x} \text{ is integrable and } \int_I (x_t - \omega_t) d\lambda = 0\}.$$

Given $f \in F$, $\nu(f)$ indicates the distribution of $f_t$ on S. Since $f \in S$ has convex-values, the aggregate mean demand map depends only on $\nu(f)$ as shown by Hildenbrand (1974, p.114) and we can write

$$\Phi(p, \nu(f)) = \int_I (f_t(p) - \omega_t)\, d\lambda.$$

The Walrasian equilibrium map $W : F \to \Delta \times \mathcal{X}$ will be defined by :

$$W(f) = \{(p,\underset{\sim}{x}) \mid p \in \Delta,\ \underset{\sim}{x} \in \mathcal{X} \text{ and } x_t \in f_t(p) \text{ a.e. on } I\}$$

$$= \{(p,\underset{\sim}{x}) \mid 0 \in \Phi(p,\nu(f)) \text{ and } x_t \in f_t(p) \text{ a.e. on } I\}.$$

Projections of $W$ on $\Delta$ and on $\mathcal{X}$ will be denoted respectively by $\pi(\cdot)$ and $w(\cdot)$. Note that $\pi(f)$ in fact depends only on $\nu(f)$ since $\pi(f) = \{p \in \Delta \mid 0 \in \Phi(p,\nu(f))\}$. Hence we can write $\pi(\nu(f))$.
Given $f = (f_t)_{t\in I} \in F$, the set of allocations agent $\tau$ can obtain using a demand map $f_\tau^\cdot \in S_\tau$ can be written as :

$$w_\tau(f_\tau^\cdot,\nu(f/f_\tau^\cdot)) = \{x \mid x \in f_\tau^\cdot(p) \text{ and } p \in \pi(\nu(f/f_\tau^\cdot))\}.$$

We can now define a Cournot-Nash (CN) equilibrium for our economy as follows.

<u>Definition</u>. A pair $(\underset{\sim}{x}^*, f^*) \in \mathcal{X} \times F$ is said to be a <u>Cournot-Nash or C-N equilibrium</u> if (i) $\underset{\sim}{x}^* \in w(f^*)$, and (ii) for a.e. $t \in I$, and for every $f_t \in S_t$, if $x_t \in w_t(f_t,\nu(f^*/f_t))$, then $x_t^* \succsim_t x_t$.

The set of C-N equilibria for $\mathcal{E}$ will be denoted by $C(\mathcal{E})$ and projections of $C(\mathcal{E})$ on $\mathcal{X}$ and on $F$ will be respectively denoted by $C_x(\mathcal{E})$ and $C_f(\mathcal{E})$.

## III The Set of C-N Equilibrium Allocations for a Finite Economy

In this section, we would like to show that, for a finite economy, the set of C-N equilibrium allocations will be large in the sense that it generally contains an $\ell(/I/-1)$ dimensional subset where $/I/$ denotes the finite number of agents. This provides us with a partial generalization to the case with any $\ell$ and any $/I/<+\infty$ of our previous results in Otani and Sicilian (1982) where we gave a more complete characterization for $\ell=2$ or $/I/=2$.

For each $t \in I$, we define the (true) inverse demand map $g_t : R_+^{\ell} \to \Delta$ by:

$$g_t(x) = \{p \in \Delta \mid x' \succsim_t x \text{ implies } p\cdot x' \geq p\cdot x\}.$$

Then we define $\hat{\mathcal{X}}$, a subset of $\mathcal{X}$, as follows:

$$\hat{\mathcal{X}} = \{\underline{x} \in \mathcal{X} \mid (\exists p \in \text{int } \Delta)(\forall t \in I)[p \cdot (x_t - \omega_t)=0] \text{ and}$$
$$(\forall t \in I)(\exists p_t \in g_t(x_t))[p_t \cdot (x_t - \omega_t) > 0]\}.$$

<u>Lemma 1</u>: If $\underline{\omega} = (\omega_t)_{t\in I} \not\in w(\hat{f})$, then (i) $\hat{\mathcal{X}}$ is star-shaped with respect to $\underline{\omega}$, (ii) $(\hat{\underline{x}}, \underline{\omega}) = \{\underline{x} \mid \underline{x} = \lambda\hat{\underline{x}} + (1-\lambda)\underline{\omega} \text{ for } \lambda \in (0,1)\} \subseteq \hat{\mathcal{X}}$ where $\hat{\underline{x}} \in w(\hat{f})$, and (iii) if $\underline{x} \in (\hat{\underline{x}}, \underline{\omega})$, then there exists a neighborhood of $\underline{x}$ in $R^{\ell}$ denoted by $N(x)$ such that $N(\underline{x}) \subseteq \hat{\mathcal{X}}$.

Since the true demand map $\hat{f}_t$ is generated by a continuous, monotone and strictly convex preference relation and $\omega_t \gg 0$, there exists $(\hat{p}, \hat{\underline{x}}) \in W(\hat{f})$ such that $\hat{p} \in \text{int}\Delta$, $\hat{x}_t \gg 0$ and $\hat{x}_t \neq \omega_t$.

Let $u_t(x_t)$ be a continuous utility function representing $\succsim$. If $p_t \in g_t(x_t)$ and $p_t \cdot (x_t - \omega_t) \geq 0$ for $x_t \neq \omega_t$, then we claim that $\psi(\lambda) = u_t[\lambda x_t + (1-\lambda)\omega_t]$ is strictly monotone increasing in $\lambda \in [0,1]$. If not, i.e., if there exists $\lambda < \lambda'$ with $\psi(\lambda) \geq \psi(\lambda')$, then $\psi(\lambda') \leq \psi(\lambda) \leq \psi(1)$ and the continuity of $\psi$ implies that there exists $\lambda'' \in (\lambda', 1)$ with $\psi(\lambda) = \psi(\lambda'')$. This contradicts the strict convexity of $\succsim_t$. Therefore if $p_t \in g_t(x_t)$, $x \neq \omega_t$ and $p_t \cdot (x_t - \omega_t) \geq 0$, then for every $\lambda \in (0,1)$ and $x_t' = \lambda x_t + (1-\lambda)\omega_t$ we must have $p_t' \cdot (x_t' - \omega_t) > 0$ for every $p_t' \in g_t(x_t')$. This proves (i) and (ii).

To prove (iii), fix $\lambda \in (0,1)$ and let $x_t(\lambda) = \lambda \hat{x}_t + (1-\lambda)\omega_t \gg 0$. First we claim that the map $G_t(x) = g_t(x) \cdot (x - \omega_t)$ is upper hemicontinuous.[1] Hence $\{x \in R_+^\ell | G_t(x) \subseteq R_{++}\}$ is open in $R_+^\ell$ and so contains a neighborhood in $R^\ell$ of $x_t(\lambda)$. Thus it remains to show that $p \cdot (x_t - \omega) = 0$ $(t \in I)$ can be solved for $p \in \text{int}\Delta$ in a neighborhood of $(x_t(\lambda))_{t \in I}$. Let $z_t(\lambda) = x_t(\lambda) - \omega_t$ and $Z = \{(z_t)_{t \in I} | \sum_{t \in I} z_t = 0\}$. Then $\dim Z = \ell(/I/-1)$. Define a mapping $F : \Delta \times Z \to R^{/I/}$ by $F(p, (z_t)_{t \in I}) = (p \cdot z_1, \ldots, p \cdot z_t, \ldots, p \cdot z_{/I/})'$. Since $(\hat{p}, \hat{\underline{x}}) \in W(\hat{f})$, $\hat{p} \cdot z_t(\lambda) = \hat{p} \cdot (x_t(\lambda) - \omega_t) = \lambda \hat{p} \cdot (\hat{x}_t - \omega_t) = 0$ for every $t \in I$, i.e., $F(\hat{p}, (z_t(\lambda))_{t \in I}) = (z_1(\lambda), \ldots, z_{/I/}(\lambda))' \hat{p} = 0$. Since $\hat{p} \in \text{int}\Delta$, the rank of $(z_1(\lambda), \ldots, z_{/I/}(\lambda))$ must be at most $(\ell-1)$, say $(\ell_0-1)$. Thus by the implicit function theorem, in a neighborhood of $(\hat{p}, (z_t(\lambda))_{t \in I})$, $(\ell_0-1)$ prices can be solved for as functions of the rest of prices and $(z_t)_{t \in I}$ proving our claim.

For any arbitrary finite economy $\mathcal{E}$, we will show that any allocation in $\hat{\mathcal{X}}$ can be supported as a Cournot-Nash equilibrium where agents try to manipulate their demand functions, i.e.,

Proposition 1: For any finite economy, $\hat{\mathcal{X}} \subseteq C_x(\mathcal{E})$.

In order to prove the above proposition, we fix $x^* \varepsilon \hat{\mathcal{X}}$ and choose $p^* \varepsilon \operatorname{int}\Delta$ and $p_t^* \varepsilon \Delta$ so that for every $t \varepsilon I$, $p^* \cdot (x_t^* - \omega_t) = 0$, $p_t^* \varepsilon q_t(x_t^*)$ and $p_t^* \cdot (x_t^* - \omega_t) > 0$. Several lemmas are needed to prove the above proposition.

Lemma 2: There exists $\xi : R_{++} \to (0,1]$ such that (a) $\xi$ is continuous and homogeneous of degree zero in $R_{++}$, (b) $\xi(p^*) = 1$, (c) for every $t \varepsilon I$ and for every $p \varepsilon \operatorname{int}\Delta$, $p_t^* \cdot f_t^*(p) \leq p_t^* x_t^*$ where

$$f_{th}^*(p) = \omega_{th} + \xi(p)(x_{th}^* - \omega_{th}) \quad (h = 1,2,\ldots,\ell-1)$$

and

$$f_{t\ell}^*(p) = (1/p_\ell)p\cdot\omega_t - \sum_{h=1}^{\ell-1} (p_h/p_\ell) f_{th}^*(p),$$

and (d) for every $p \varepsilon \operatorname{int}\Delta$, $f_t^*(p) \varepsilon R_+$ and $p \cdot f_t^*(p) = p \cdot \omega_t$.

First, note that it $f_t^*$ is defined as above, then

$$p_t \cdot f_t^*(p) = p_t \cdot \omega_t + p_{t\ell}\xi(p)\left(\frac{p_t}{p_{t\ell}} - \frac{p}{p_\ell}\right)\cdot(x_t^* - \omega_t)$$

and

$$f_{t\ell}^*(p) = \xi(p)x_{t\ell}^* + (1 - \xi(p))\omega_{t\ell} - \xi(p)\frac{p}{p_\ell}\cdot(x_t^* - \omega_t).$$

Therefore $p_t \cdot f_t^*(p) \leq p_t \cdot x_t^*$ if and only if

$\xi(p)\left(\frac{p_t}{p_{t\ell}} - \frac{p}{p_\ell}\right)\cdot(x_t^* - \omega_t) \leq \frac{p_t}{p_{t\ell}}\cdot(x_t^* - \omega_t)$. Define $A_t = \frac{p_t}{p_{t\ell}}\cdot(x_t^* - \omega_t)$

and $B_t(p) = \frac{p}{p_\ell} \cdot (x_t^* - \omega_t)$. Then $A_t > 0$ for every $t \in I$ and $\sum_{t \in I} B_t(p) = 0$. Hence $\{ t \mid B_t(p) \leq 0 \} \neq \Phi$.

Assume for a while that $\xi(p) \in (0.1]$ for every $p \in \mathrm{int}\Delta$. Then $p_t \cdot f_t^*(p) \leq p_t \cdot x_t^*$ if and only if $\xi(p)(A_t - B_t(p)) \leq A_t$. If $B_t(p) > 0$, then $\xi(p)(A_t - B_t(p)) \leq \xi(p) A_t \leq A_t$ and if $B_t(p) \leq 0$, then $p_t \cdot f_t^*(p) \leq p_t \cdot x_t^*$ if and only if $\xi(p) \leq A_t/(A_t - B_t(p))$. Therefore we can obtain that for every $t \in I$ and for every $p \in \mathrm{int}\Delta$, $p_t \cdot f_t^*(p) \leq p_t \cdot x_t^*$ if

$\xi(p) \leq \min_{t \in I} \{ A_t/(A_t - B_t(p)) \mid B_t(p) \leq 0 \}$.

Now we consider a condition on $\xi(p)$ to guarantee the nonnegativity of $f_{t\ell}^*(p)$. If $B(p) \leq 0$, then $f_{t\ell}^*(p) = \xi(p) x_{t\ell}^* + (1 - \xi(p))\omega_{t\ell} - \xi(p) B_t(p) \geq 0$ provided that $\xi(p) \in (0.1]$. On the other hand, if $B_t(p) > 0$ and

$\xi(p) \leq \min\{x_{t\ell}^*.\omega_{t\ell}\}/B_t(p)$, then

$\xi(p)\, B_t(p) \leq \min\{x_{t\ell}^*.\omega_{t\ell}\} \leq \xi(p) x_{t\ell}^* + (1 - \xi(p))\omega_{t\ell}$. Hence $f_{t\ell}(p) \geq 0$.

Now define $\xi_1(p)$ and $\xi_2(p)$ by

$\xi_1(p) = \min_{t \in I} \{ A_t/(A_t - B_t(p)) \mid B_t(p) \leq 0 \}$, and

$\xi_2(p) = \min_{t \in I} \{ \min\{x_{t\ell}^*, \omega_{t\ell}\}/B_t(p) \mid B_t(p) > 0 \}$.

Then we define $\xi(p)$ as follows:

$\xi(p) = \min\{\xi_1(p), \xi_2(p)\}$ if $\{t \mid B_t(p) > 0\} \neq \Phi$,

and $\xi(p) = \xi_1(p)$ if $\{t \mid B_t(p) > 0\} = \Phi$.

From our construction, it is clear that $\xi(p) \in (0.1]$ for every $p \in \mathrm{int}\Delta$, $f_{t\ell}^*(p) \geq 0$ for every $t \in I$ and $p \in \mathrm{int}\Delta$, and $p_t \cdot f_t^*(p) \leq p_t \cdot x_t^*$ for every $t \in I$ and $p \in \mathrm{int}\Delta$. Since $\xi(p) \in (0.1]$, for every $p \in \mathrm{int}\Delta$,

$$f^*_{th}(p) = \xi(p)x^*_{th}+(1 - \xi(p))\omega_{th} \geq 0$$

$(h = 1,2,\ldots, \ell-1)$. This concludes the proof of Lemma 2.

We will now show that the function $f^*_t$ defined in Lemma 2 can be considered as a demand function in $S_t$.

Lemma 3: The function $f^*_t$ of Lemma 2 satisties the Strong Axiom of Revealed Preference.

First note that we can write

$$f^*_{t\ell}(p) = \omega_{t\ell} - \xi(p)\sum_{h=1}^{\ell-1}(p_h/p_\ell)(x^*_{th} - \omega_{th})$$

Suppose that $f^*_t(p^1)$ is directly revealed preferred to $f^*_t(p^0)$, i.e., $p^1\cdot f^*_t(p^1) = p^1\cdot\omega_t \geq p^1\cdot f^*_t(p^0)$.

Then

$$p^1\cdot f^*_t(p^0) = p^1\cdot\omega_t + \xi(p^0)p^1_\ell\sum_{h=1}^{\ell-1}\left(\frac{p^1_h}{p^1_\ell} - \frac{p^0_h}{p^0_\ell}\right)(x^*_{th} - \omega_{th})$$

$$= p^1\cdot f_t(p^1) + \xi(p^0)p^1_\ell\left(\frac{p^1}{p^1_\ell} - \frac{p^0}{p^0_\ell}\right)\cdot(x^*_t - \omega_t).$$

Hence $p^1\cdot f^*_t(p^0) \leq p^1\cdot f^*_t(p^1)$ if and only if $\left(\frac{p^1}{p^1_\ell} - \frac{p^0}{p^0_\ell}\right)\cdot(x^*_t - \omega_t) \leq 0$, i.e.,

$\frac{p^1}{p^1_\ell}\cdot(x^*_t - \omega_t) \leq \frac{p^0}{p^0_\ell}\cdot(x^*_t - \omega_t)$.

If $f^*_t(p^{j+1})$ is directly revealed preferred to $f^*_t(p^j)$ $(j=0,1,\ldots,r-1)$, then we get $\frac{p^{j+1}}{p^{j+1}_\ell}\cdot(x^*_t - \omega_t) \leq \frac{p^j}{p^j_\ell}\cdot(x^*_t - \omega_t)$ $(j=0,1,\ldots, r-1)$.

Hence we have $\frac{p^r}{p^r_\ell} \cdot (x^*_t - \omega_t) \le \frac{p^0}{p^0_\ell} \cdot (x^*_t - \omega_t)$

which in turn implies that $f^*_t(p^r)$ is directly revealed preferred to $f^*_t(p^0)$. This proves Lemma 3.

The next lemma completes the proof of Proposition 1.

Lemma 4: Let $f^* = (f^*_t)_{t \in I}$ where $f^*_t$ is defined in Lemma 2. Then $(\tilde{x}^*, f^*) \in C(\mathcal{E})$

For h=1,2, ...., $\ell$-1, we have that

$$\sum_{t \in I} f^*_{th}(p) = \sum_{t \in I} \omega_{th} + \xi(p) \sum_{t \in I} (x^*_{th} - \omega_{th}) = \sum_{t \in I} \omega_{th}$$

for every $p \in \text{int}\Delta$. Therefore $\pi(f^*) = \text{int}\Delta$ and $f^*_\tau(p) = \sum_t \omega_t - \sum_{t \ne \tau} f^*_t(p)$. Since $p^*_\tau \cdot f^*_\tau(p) \le p^*_\tau \cdot x^*_\tau$ for every $p \in \text{int}\Delta$, $p^*_\tau \cdot (\sum_t \omega_t - \sum_{t \ne \tau} f^*_t(p)) \le p^*_\tau \cdot x^*_\tau$. Hence $\sum_t \omega_t - \sum_{t \ne \tau} f^*_t(p) \precsim_\tau x^*_\tau$ for every $p \in \text{int}\Delta$.

Lemma 1 and Proposition 1 imply that the set of C-N equilibrium allocations will be large in the sense that it will contain an $\ell(/I/ - 1)$ dimensional open subset in it. For example, if we consider a replica economy with r types of agents, then regardless of the number of agents in each type, the set of C-N equilibrium allocations for the replica economy will not shrink and contain an $\ell(/I/ - 1)$ dimensional open subset in it independent of the number of agents in each type.

The construction of $\xi(p)$ in Lemma 2 does not quite work at a boundary point of $\hat{\mathcal{X}}$, in particular when $p_t \cdot (x_t - \omega_t) = 0$ for some t. Then we will get

$A_t = 0$ for some t. But as shown in our previous paper (Otani and Sicilian (1982)), it is easy to show that for any finite economy, $\hat{\underline{x}} \varepsilon C_x(\mathcal{E})$ and $\underline{w} \varepsilon C_x(\mathcal{E})$ where $\hat{\underline{x}} \varepsilon w(\hat{f})$.

## IV Limit Properties of C-N Equilibrium Allocations

In the last section, we have shown that the set of C-N equilibrium allocations stays large regardless of the number of agents in the economy. When we examine closely the nature of demand functions or offer functions constructed in Lemma 2 to support an allocation as a C-N equilibrium, we can find that the graph of an offer function exhibit a sharp kink at $x_t^*$. In this section, we try to argue that this should be a typical situation unless the allocation converges to a true Walrasian allocation. In this sense, a true Walrasian allocation possesses a nicer property than other limiting C-N equilibrium allocations.

In this section, we will consider a sequence of agent sets $I^n$ with $/I^n/<\infty$ and $/I^n/\rightarrow\infty$ and a sequence of economies $\mathcal{E}^n : I^n \rightarrow A$. An economy with a continuum of agents will be denoted by $\mathcal{E}^\infty : I^\infty \rightarrow A$ with $I^\infty = [0,1]$. Let $u^n$ and $u^\infty$ be distributions of agent characteristics in A for $\mathcal{E}^n$ and $\mathcal{E}^\infty$ respectively. Then $\mathcal{E}^n$ is said to converge to $\mathcal{E}^\infty$ if $u^n$ converges weakly to $u^\infty$. First we establish the following elementary consequenses for a continuum economy.

<u>Proposition 2</u> : For the economy $\mathcal{E}^\infty$, (a) $\hat{f}^\infty \varepsilon C_f(\mathcal{E}^\infty)$ if and only if $w(\hat{f}^\infty)$ is essentially single-valued, i.e., if $\underline{x}$ , $\underline{x}' \varepsilon w(\hat{f}^\infty)$, then for a.e. $t \varepsilon I^\infty$, $x_t \sim_t x_t'$, and (b) $C_x(\mathcal{E}^\infty) = w(\hat{f}^\infty)$.

It is easy to see that if $w(\hat{f}^{\infty})$ is not essentially single-valued, then $\hat{f}^{\infty} \notin C_{f}(\mathcal{E}^{\infty})$. Hence $\hat{f}^{\infty} \in C_{f}(\mathcal{E}^{\infty})$ implies that $w(\hat{f}^{\infty})$ is essentially single-valued. The converse to this does not appear to be trivial, but the proof of the converse becomes easy once we can establish the following technical lemma whose proof will be provided in the Appendix.

Lemma 5: If $w(\hat{f}^{\infty})$ is essentially single-valued, then there exists a measurable subset $I'$ of $I^{\infty}$ such that $\lambda(I') = 1$ and for all $\underset{\sim}{x}$, $\underset{\sim}{x}' \in w(\hat{f}^{\infty})$ and for all $t \in I'$, $x_t \sim_t x'_t$.

Suppose that $w(\hat{f}^{\infty})$ is essentially single-valued, but $\hat{f}^{\infty} \notin C_{f}(\mathcal{E}^{\infty})$. Then for any $\hat{\underset{\sim}{x}} \in w(\hat{f}^{\infty})$, $(\hat{\underset{\sim}{x}}, \hat{f}^{\infty}) \notin C(\mathcal{E}^{\infty})$, i.e., if $\lambda(I') = 1$, then there exist $t \in I'$, $f'_t \in S_t$, and $x'_t \in w(f'_t, \nu(\hat{f}^{\infty}/f'_t))$ such that $x'_t \succ_t \hat{x}_t$. If we take the above $I'$ to be the one in Lemma 5, then we get a contradiction because $\hat{\underset{\sim}{x}}/x'_t \in w(\hat{f}^{\infty})$, where $\hat{\underset{\sim}{x}}/x'_t$ denotes an allocation with $\hat{x}_t$ of $\hat{\underset{\sim}{x}}$ replaced by $x'_t$.

We now prove (b). Suppose that $\hat{\underset{\sim}{x}} \in w(\hat{f}^{\infty})$. Then there exists $\hat{p} \in \Delta$ such that $(\hat{p}, \hat{\underset{\sim}{x}}) \in W(\hat{f}^{\infty})$. Choose $f^{*} \in F^{\infty}$ so that $\{\hat{p}\} = \pi(\nu(f^{*}))$. This is possible since we can modify $\hat{f}^{\infty}$ so that $\hat{p}$ is the unique Walrasian equilibrium. Then
$w(f'_{\tau}, \nu(f^{*}/f'_{\tau})) = \{x_{\tau} | x_{\tau} \in f'_{\tau}(p) \text{ and } p \in \pi(\nu(f^{*}/f'_{\tau}))\} = f'_{\tau}(\hat{p})$ since $\pi(\nu(f^{*}/f'_{\tau})) = \pi(\nu(f^{*})) = \{\hat{p}\}$. Since $\hat{\underset{\sim}{x}} \in w(\hat{f}^{\infty})$, we have $\hat{x}_{\tau} \succsim x_{\tau}$ for every $x_{\tau} \in f'_{\tau}(\hat{p})$. Hence $\hat{\underset{\sim}{x}} \in C_{X}(\mathcal{E}^{\infty})$.

On the other hand, suppose that $\hat{\underset{\sim}{x}} \in C_{X}(\mathcal{E}^{\infty})$. Then there exists $f^{*} \in F^{\infty}$ such that $(\hat{\underset{\sim}{x}}, f^{*}) \in C(\mathcal{E}^{\infty})$, i.e., $\hat{\underset{\sim}{x}} \in w(f^{*})$ and for a.e. $\tau \in I^{\infty}$, if $f_{\tau} \in S_{\tau}$

and $x_\tau \in w(f_\tau, \nu(f^*/f_\tau))$, then $\hat{x}_\tau \succsim_\tau x_\tau$. Since $w(f_\tau, \nu(f^*/f_\tau)) = f_\tau(\hat{p})$ for some $\hat{p} \in \pi(\nu(f^*))$, we must have $\hat{x}_\tau \in \hat{f}_\tau(\hat{p})$. Then $\hat{x}_\tau \in \hat{f}_\tau(\hat{p}))$ for a.e. $\tau \in I^\infty$ and $\hat{\underline{x}} \in \mathcal{X}$ imply that $\hat{\underline{x}} \in w(\hat{f}^\infty)$.

Propositions 1 and 2 together imply that the C-N equilibrium allocation map $C_x(\mathcal{E})$ does not have a closed graph property in general. Our next proposition which is our major result of this paper is intended to provide a nicer feature which a Walrasian equilibrium allocation in a limit economy possesses distinguishing it from other limit allocations of C-N equilibrium.

Proposition 3: Assume that the following five conditions hold in addition to our previous assumptions. (i) Functions in S are further assumed to be $C^1$ on int$\Delta$ endowed with the topology of the $C^1$-uniform convergence. (ii) For every $n \in \{1, 2, \ldots\}$, for every $t \in I^n$ and for every $t \in I^\infty$, $\succsim_t$ is representable by a $C^1$-utility function denoted by $u_t$ and $D(u_t)(x_t) \gg 0$ for $x_t \gg 0$. (iii) Let $\mathcal{E}^n \to \mathcal{E}^\infty$ in distribution, $(\underline{x}^n, f^n) \in C(\mathcal{E}^n)$ and $(\underline{x}^n, f^n) \to (\underline{x}^\infty, f^\infty)$ in distribution. Also let $(\underline{x}^n, p^n) \in W(f^n)$ and $p^n$ converges to $p^\infty \in \text{int}\Delta$. (iv) For every $n \in \{1, 2, \ldots\}$, and for every $t \in I^n$, $\pi(f^n/f_\tau)$ as a mapping of $f_\tau$ is a local surjection from a neighborhood of $f_\tau^n$ to a neighborhood of $p^n$. (v) Let $\Psi^\infty(p) = \int_{I^\infty}(f_t^\infty(p) - \omega_t)d\lambda$. Then

$D(\Psi^\infty)(p^\infty) = [\partial\Psi_i^\infty(p^\infty)/\partial p_j]_{i,j=1,2,\ldots,\ell}$ has rank $(\ell-1)$.

Then we can conclude that $(p^\infty, \underline{x}^\infty) \in W(\hat{f}^\infty)$ where $\hat{f}^\infty$ is the true demand map for $\mathcal{E}^\infty$.

Let $\Psi^n(p)$ be the excess demand function defined by $\Psi^n(p) = \int_{I^n} (f_t^n(p) - \omega_t)d\lambda^n$ where $\lambda^n$ is a measure on $I^n$. By the Walras' law. we have $p \cdot \Psi^n(p) = 0$ for every $p \in \Delta$. Thus we can obtain that

$$(4.1) \qquad [D(\Psi^n\ (p)]'p + \Psi^n(p) = 0_\ell$$

where $D(\Psi^n)(p) = [\partial \Psi_i^n / \partial p_j]_{i,j=1,2,\ldots,\ell}$ and ' indicates a transpose operation.

Evaluating (4.1) at $p^n$ yields

$$(4.2) \qquad [D(\Psi^n)(p^n)]'p^n = 0_\ell.$$

For a finite economy $\mathcal{E}^n$, the feasibility condition can be written as

$$\int_{I^n/\{\tau\}} f_t^n(p)d\lambda^n - \int_{I^n} \omega_t d\lambda^n + x_\tau \lambda(\tau) = 0_\ell$$

where $\lambda(\tau) = \lambda(\{\tau\})$. Therefore at a CN equilibrium, agent $\tau$ tries to choose $f_\tau$ so as to maximize

$$u_\tau[\frac{1}{\lambda(\tau)} \{\int_{I^n} \omega_t d\lambda^n - \int_{I^n/\{\tau\}} f_t^n(p)d\lambda^n\}]$$

subject to a constraint $p \in \pi[f^n/f_\tau]$. But, by Assumption (iv), $\pi[f^n/f_\tau]$ as a mapping of $f_\tau$ is a local surjection from a neighborhood of $f_\tau^n$ to a neighborhood of $p^n$. Therefore $p^n$ must maximize $u_\tau[\frac{1}{\lambda(\tau)} \{\int_{I^n} \omega_t d\lambda^n - \int_{I^n/\{\tau\}} f_t^n(p)d\lambda^n\}]$ on $N(p^n)$ where $N(p^n)$ is an appropriate neighborhood of $p^n$ in $\Delta$. We may assume that $N(p^n) \subseteq \text{int}\Delta$ since

$p^n$ converges to $p^\infty \varepsilon$ int$\Delta$. The first-order condition of this maximization problem yields:

$$(4.3) \qquad [\int_{I^n/\{\tau\}} D(f_t^n)(p^n)\, d\lambda^n]\cdot q_\tau^n = 0_\ell$$

where $\quad q_\tau^n = [D(u_\tau)](x_\tau^n)/ \|[D(u_\tau)](x_\tau^n)\|$.

Let $\nu^n$ and $\nu^\infty$ denote distributions of $(\underset{\sim}{x}^n, f^n)$ and $(\underset{\sim}{x}^\infty, f^\infty)$ respectively on $R_+^n \times S$ and $\nu^n(f^n)$ and $\nu^\infty(f^\infty)$ indicate marginal distributions of $\nu^n$ and $\nu^\infty$ respectively on S.

Since $f_\tau^n$ is $C^1$, we can write by changing the variable of integration to get

$$\begin{aligned} D(\Psi^n)(p^n) &= \int_{I^n} D(f_t^n)(p^n)d\lambda^n(t) = \int_S D(s)(p^n)d\nu(f^n)(s) \\ &= [\int_S D(s)(p^n)d\nu(f^n)(s) - \int_S D(s)(p^\infty)d\nu(f^n)(s)] \\ &+ \int_S D(S)(p^\infty)d\nu(f^n)(s). \end{aligned}$$

Since $\lim p^n = p^\infty$, the first term of the above converges to zero. Since $p^\infty \varepsilon$ int$\Delta$, $D(s)(p^\infty)$ is bounded on S. Thus the second term converges to $\int_S D(s)(p^\infty)d\nu^\infty(f^\infty)(s)$. Thus

$$\lim_{n\to\infty} \int_{I^n} D(f_t^n)(p^n)d\lambda^n(t) = \int_{I^\infty} D(f_\tau^\infty)(p^\infty)d\lambda(t).$$

Therefore we can conclude from (4.2) that

$$(4.4) \qquad [\int_{I^\infty} D(f_t^\infty)(p^\infty)d\lambda(t)]'p^\infty = 0_\ell$$

Let $\nu_\tau^n(f^n)$ be the distribution on S generated by $f_t^n$, $t \in I^n/\{\tau\}$. Fix $A \subseteq S$. If $f_\tau^n \notin A$, then

$$\nu_\tau^n(f^n)(A) = \lambda^n[(f^n)^{-1}(A)] = \nu^n(f^n)(A).$$

If $f_\tau^n \in A$, then

$$\nu_\tau^n(f^n)(A) = \lambda^n[(f^n)^{-1}(A/\{\tau\})]$$
$$= \lambda^n[(f^n)^{-1}(A)] - \lambda^n(\tau).$$

Since $\lambda^n(\tau) \to \lambda(\tau) = 0$. $\nu_\tau^n(f^n)$ converges weakly to $\nu^\infty(f^\infty)$. Thus we can obtain

$$\lim_{n\to\infty} \int_{I^n/\{\tau\}} D(f_t^n)(p^n)d\lambda^n(t) = \lim_{n\to\infty}\int_S D(s)(p^n)d\nu_\tau^n(f^n)(s)$$
$$= \int_S D(s)(p^\infty)d\nu^\infty(f^\infty)(s) = \int_{I^\infty} D(f_t^\infty)(p^\infty)d\lambda(t).$$

Let $\mu^n$ and $\mu^\infty$ be distributions of agent characteristics on A for $\mathcal{E}^n$ and $\mathcal{E}^\infty$. Let $(u.\omega) \in \mathrm{Supp}(\mu^\infty)$. Since $\mu^n$ converges weakly to $\mu^\infty$, there exists $(u_{\tau_n}, \omega_{\tau_n}) \in \mathrm{Supp}(\mu^n)$ such that $(u_{\tau_n}, \omega_{\tau_n}) \to (u, \omega)$ in the product topology of the $C^1$ topology on u and the Euclidean topology on $\omega$.[2] Since the feasibility set is compact. we may assume that $x_{\tau_n}^n$ converges to $x^\infty$. If we let $q_{\tau_n} = Du_{\tau_n}(x_{\tau_n}^n)/\ \|Du_{\tau_n}(x_{\tau_n}^n)\|$ and $q = Du(x^\infty)/\ \|Du(x^\infty)\|$, then $q_{\tau_n}$ converges to q. Therefore using (4.3), we can conclude

$$\lim_{n\to\infty} [\int_{I^n/\{\tau_n\}} D(f_t^n)(p^n)d\lambda^n(t)]\cdot q_{\tau_n} = [\int_{I^\infty} D(f_t^\infty)(p^\infty)d\lambda(t)]\cdot q = 0 \tag{4.5}$$

for every $(u, \omega) \varepsilon \mathrm{Supp}(\mu^\infty)$ with $q = Du(x^\infty)/\|Du(x^\infty)\|$.

Comparing (4.4) and (4.5), since the rank of $D(\Psi^\infty)(p^\infty) = \int_{I^\infty} D(f_t^\infty)(p^\infty)d\lambda$ is $(\ell-1)$ and $p^\infty$, $q \varepsilon \Delta$, we can conclude that for every $(u, \omega) \varepsilon \mathrm{Supp}(\mu^\infty)$

$$Du(x^\infty)/\|Du(x^\infty)\| = p^\infty.$$

Clearly, we also have that, for a.e. $t \varepsilon I^\infty$, $p^\infty\cdot x_t^\infty = p^\infty\cdot\omega_t$ and $\int_{I^\infty} x_t^\infty d\lambda = \int_{I^\infty}\omega_t d\lambda$.

Therefore we must have $(p^\infty, \underset{\sim}{x}^\infty) \varepsilon W(\hat{f}^\infty)$.

Remark: It is certainly possible to obtain the conclusion of Proposition 3 using some other approaches. One alternative approach may be first to establish the continuity of the Walrasian equilibrium map with smooth demand functions as in Mas-Colell(1985, Chapter 5). Then we can use a similar argument as in the proof of Proposition 1 in Dubey, Mas-Colell and Shubik(1980). We believe that our approach in this paper is simpler and more direct in obtaining the result of Proposition 3.

## APPENDIX

Lemma 5: If $w(\hat{f}^{\infty})$ is essentially single-valued, then there exists a measurable subset $I'$ of $I^{\infty}$ such that $\lambda(I')=1$ and for all $\underset{\sim}{x}$, $\underset{\sim}{x}' \in w(\hat{f}^{\infty})$ and for all $t \in I'$, $x_t \sim_t x'_t$.

Proof: Let $(\hat{p}, \hat{x}) \in W(\hat{f}^{\infty})$ and $\hat{I} = \{t \in I^{\infty} \mid \hat{x}_t \in \hat{f}^{\infty}_t(\hat{p})\}$. Then $\lambda(\hat{I})=1$. For $p \in \pi(\hat{f}^{\infty})$, let $G(p) = \{t \in \hat{I} \mid x_t \not\sim_t \hat{x}_t$ for some $\underset{\sim}{x} \in \hat{f}^{\infty}(p)\}$. Define $G = \bigcup\{G(p) \mid p \in \pi(\hat{f}^{\infty})\}$ and $I' = I^{\infty}/G$.

Clearly for every $\underset{\sim}{x}$, $\underset{\sim}{x}' \in w(\hat{f}^{\infty})$ and for every $t \in I'$, $x_t \sim_t x'_t$. So it remains to show that $\lambda(I') = 1$. To show it, let $\hat{\pi}$ be a countable dense subset of $\pi(\hat{f}^{\infty})$ and let $\hat{G} = \bigcup\{G(p) \mid p \in \hat{\pi}\}$. Since $w(\hat{f}^{\infty})$ is essentially single-valued, $\lambda(G(p)) = 0$ for each $p \in \pi(\hat{f}^{\infty})$. Hence $\lambda(\hat{G}) = 0$. Let $t \in G$. Then $t \in G(\bar{p})$ for some $\bar{p} \in \pi(\hat{f}^{\infty})$ and there exists $\bar{\underset{\sim}{x}} \in \hat{f}^{\infty}(\bar{p})$ such that $\bar{x}_t \not\sim_t \hat{x}_t$. By the continuity of preference and $\hat{f}^{\infty}_t$, the set $\{p \in \pi(\hat{f}^{\infty}) \mid x_t \not\sim_t \hat{x}_t$ for some $\underset{\sim}{x} \in \hat{f}^{\infty}(p)\}$ is a neighborhood of $\bar{p}$ open in $\pi(\hat{f}^{\infty})$ and thus the above set intersets $\hat{\pi}$. Therefore $t \in \hat{G}$. Thus $G \subseteq \hat{G}$ and $\lambda(I') = 1$.

## Footnotes

1 To see this, note that $g_t : R^{\ell}_{+} \to \Delta$ is an upper hemi-continuous mapping with nonempty, compact and connex values. (See Lemma 1, p.659 in Otani (1980).)

2 As pointed out by Hildenbrand (1974, p.192), the weak convergence of the sequence $\{\mu^n\}$ to $\mu$ implies $\mathrm{Supp}(\mu) \subseteq \mathrm{Li}[\mathrm{Supp}(\mu^n)]$.

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