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Bimatroid Network and Its Application to the Structural Solvability of Systems of Equations

by

Kazuo Murota

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Kazuo Murota

Institute of Socio-Economic Planning, University of Tsukuba, Sakura, Ibaraki 305, Japan

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Abstract: A method of structural analysis of systems of equations is proposed. It is based on the observation that a network-type interconnection of bimatroids (or linking systems), named a bimatroid network, results in a bimatroid. The structural solvability of systems of equations is formulated in such a way that local dependency of parameters can be incorporated; a system of equations is structurally solvable iff there exists a complete linking in the associated bimatroid network. A hierarchical decomposition of the whole system into subsystems is proposed; it unifies the M-decomposition and the L-decomposition and can be found by efficient graph-theoretic algorithms.

Key words: bimatroid (linking system), bimatroid network, structural solvability, block-triangularization, Menger-decomposition

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Mailing address: Department of Mathematical Engineering and Instrumentation Physics, Faculty of Engineering, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

1. Introduction

There have been proposed various kinds of graph-theoretic representations of discrete systems. Among others, the signal-flow graph (and its variants) has turned out to be one of the most effective tools in capturing the structure of systems of equations. For instance, the ingenious use of a signal-flow-graph representation has been the key to the success of the chemical process simulators JUSE-GIFS and DPS developed in Japan [8], [9], [10], [16], [37], [40], [41]. A signal-flow graph expresses the flow of information, or the causal relation, in a natural and intuitively appealing manner, still admitting rigorous mathematical analyses.

This paper proposes a generalization of signal-flow graph, to be called "bimatroid network", and develops a method of structural analysis of a system of equations by means of it. Following [15] (see also [26], [28]) we consider a system of equations in the "standard form":

$$\begin{aligned} y_i &= f_i(x,u) \qquad (i=1,\ldots,M), \\ u_k &= g_k(x,u) \qquad (k=1,\ldots,K), \end{aligned} \tag{1.1}$$

where x_j $(j=1,\ldots,N)$ and u_k $(k=1,\ldots,K)$ are unknowns, y_i $(i=1,\ldots,M)$ are parameters, and f_i $(i=1,\ldots,M)$ and g_k $(k=1,\ldots,K)$ are assumed to be sufficiently smooth real-valued functions. In this form, we look at x_j $(j=1,\ldots,N)$ as primary unknown variables while u_k $(k=1,\ldots,K)$ as intermediate variables. This form is most natural and convenient when we treat a physical/engineering system represented by a set of functional relations among elementary state variables, where we want to adjust the values of x- and u-variables so as to have all the equations satisfied for arbitrarily given values of y-variables [16].

The structure of (1.1) is expressed by a kind of signal-flow graph $G=(V,A)$, called the representation graph in [15], defined as follows. The vertices of G correspond to the variables (i.e., unknowns and parameters); i.e., $V = X \cup U \cup Y$, where $X = \{x_1,\ldots,x_N\}$, $U = \{u_1,\ldots,u_K\}$ and $Y = \{y_1,\ldots,y_M\}$. The arcs of G express the direct functional dependence among variables. To be specific, the functional dependence $y_i = f_i(x,u)$ is expressed by a set of arcs coming into y_i from those x_j and u_l which effectively appear on the right-hand side. Similarly for $u_k = g_k(x,u)$.

The system (1.1) of equations is said to be structurally solvable if it has a structure which admits a unique solution for arbitrarily specified values of parameters y_i $(i=1,\ldots,M)$. (See §3 for the precise definition of the notion of structural solvability.) Under the stronger "generality assumption", labeled GA1 in [27], [28], that the partial derivatives of f_i $(i=1,\ldots,M)$ and g_k $(k=1,\ldots,K)$ can be regarded as being mutually independent, the structural solvability of (1.1) is known [15] to be equivalent to the existence of a Menger-type vertex-disjoint complete linking from X to Y in the representation graph G (see Theorem 3.1 below). This characterization of structural solvability has led to effective methods for the hierarchical decomposition (i.e., block-triangularization) of the whole system into structurally solvable subsystems [15], [22], [23], [24]. The decomposition methods, called the L-decomposition and the M-decomposition, will be described briefly in §3.

Although the graph-theoretic method explained above has been proven very effective, the generality assumption GA1 is sometimes too stringent to be satisfied. A more faithful modeling under a weaker generality assumption, labeled GA2, has been proposed in [27], [28]. It is based on the physical observation that the numbers characterizing physical systems are to be classified into two groups, one of accurate and mutually correlated numbers and the other of inaccurate and mutually independent numbers. The notion of mixed matrix has been introduced (see §3) as a useful mathematical tool. The structural solvability of (1.1) is then reduced to the nonsingularity of a mixed matrix, which can be checked by efficient matroid-theoretic algorithms. Furthermore, the combinatorial canonical form of a mixed matrix [29] (see also [26]) provides a powerful method for the hierarchical decomposition of the whole system into structurally solvable subsystems.

The method of [27], [28] under the weaker generality assumption GA2 is based on a bipartite model as compared with the signal-flow model used in the graph-theoretic method under GA1 mentioned above. By this we mean that the method of [27], [28] does not distinguish between the primary variables x_j ($j=1,\ldots,N$) and the intermediate variables u_k ($k=1,\ldots,K$) in (1.1); or alternatively, a system (1.1) with $K=0$ is considered. In other words, we may say that the matroid-theoretic method of [27], [28] under GA2 exploits the incidence relation between the unknown variables and the equations expressed by a mixed matrix, whereas the graph-theoretic method under GA1 considers the incidence relation among all the variables, including primary and intermediate variables and parameters, expressed by the representation graph G. Although the signal-flow model and the bipartite model are mathematically convertible in general, the signal-flow model might be preferred for the intuitive understanding of physical structures described by (1.1).

The objective of this paper is to develop a method of structural analysis of systems of equations which is a natural extension of the graph-theoretic method based on the signal-flow model, and which works under a weaker generality assumption than GA1. The signal-flow-type model introduced in this paper can represent in a natural manner the hierarchical structure of physical systems usually consisting of nested subsystems, such as those called devices, units, modules, multiports. A subsystem may be described by a matrix, when linearized if necessary. For physical reasons, it is plausible to assume that the entries of the matrices describing different subsystems are mutually independent whereas the entries in one matrix may be correlated. Such local dependency among physical characteristics can be treated in the proposed framework.

The contents of this paper are as follows. Section 2 introduces the terminology on graph and bimatroid, and Section 3 summarizes the previously known results on the structural solvability and the hierarchical decomposition of systems of equations. In §4, the notion of bimatroid network is defined and its significance in system modeling is discussed. Then in §5 and §6, the structural solvability of systems of equations under a generality assumption, weaker than GA1, is expressed in terms of the linking in the associated bimatroid network. A method for block-triangularization is proposed in §7. Section 8 concludes the paper.

2. Preliminaries

This subsection introduces some notations used in this paper. See [5], [11], [18], [35], [39] for precise definitions of the concepts not defined here.

Let $G=(V,A)$ be a graph with vertex set V and arc set A. For $a \in A$, $\partial^+ a$ (resp., $\partial^- a$) denotes the initial (resp., terminal) vertex of a. We sometimes write (u,v) for $a \in A$ if $\partial^+ a=u$ and $\partial^- a=v$. For $v \in V$, we put $\delta^+ v=\{a \in A \mid \partial^+ a=v\}$ and $\delta^- v=\{a \in A \mid \partial^- a=v\}$. A vertex $v \in V$ is said to be maximal (resp., minimal) if $\delta^- v=\emptyset$ (resp., $\delta^+ v=\emptyset$).

When two subsets X and Y of V are specified as "entrance" and "exit", we write $G=(V,A;X,Y)$ to indicate the two distinguished sets of vertices. (We do not assume $X \cap Y=\emptyset$.) By a <u>Menger-type linking</u> from X to Y in $G=(V,A;X,Y)$ is meant a set of pairwise vertex-disjoint directed paths from a vertex in X to a vertex in Y. (A directed path may consist of a single vertex when $X \cap Y \neq \emptyset$.) The <u>size</u> of a linking is defined to be the number of directed paths from X to Y contained in the linking. A linking of the maximum size is a <u>maximum linking</u>, and a linking of size $|X|=|Y|$ is a <u>complete linking</u>. A <u>separator</u> of (X,Y) is a subset of V which intersects any directed path from X to Y. A separator of the minimum size is a <u>minimum separator</u>. It is widely known as Menger's theorem that the maximum size of a linking is equal to the minimum size of a separator.

When $G=(V,A;X,Y)$ is a bipartite graph (with $V=X \cup Y$, $X \cap Y=\emptyset$), the linkings reduce to the matchings. The canonical decomposition of a bipartite graph due to Dulmage-Mendelsohn [1], [2], [3] will be referred to as the <u>DM-decomposition</u> (see also [21]).

The notion of bimatroid was introduced first by [35], [36] under the name of linking system, and independently, but shortly later, by [17] under the name of bimatroid.

A <u>bimatroid</u> (or <u>linking system</u>) is a triple $L=(S,T,\Lambda)$, where S and T are finite sets, and Λ is a nonempty subset of $2^S\times 2^T$ such that

(L1) if $(X,Y)\in\Lambda$, then $|X|=|Y|$;

(L2-1) if $(X,Y)\in\Lambda$ and $X'\subset X$, then $(X',Y')\in\Lambda$ for some $Y'\subset Y$;

(L2-2) if $(X,Y)\in\Lambda$ and $Y'\subset Y$, then $(X',Y')\in\Lambda$ for some $X'\subset X$;

(L3) if $(X_i,Y_i)\in\Lambda$ $(i=1,2)$, then there exists $(X,Y)\in\Lambda$ such that $X_1\subset X\subset X_1\cup X_2$, $Y_2\subset Y\subset Y_1\cup Y_2$.

We call S the <u>entrance set</u> and T the <u>exit set</u> of L. A member (X,Y) of $\Lambda=\Lambda(L)$ is called a <u>linked pair</u>; we also say that X and Y are <u>linked</u>.

The <u>birank function</u> (or <u>linking function</u>) λ: $2^S\times 2^T\to Z_+$ is defined by

$$\lambda(X,Y) = \max\{|X'| \mid (X',Y')\in\Lambda,\ X'\subset X,\ Y'\subset Y\}, \quad X\subset S,\ Y\subset T. \tag{2.1}$$

Obviously,

$$(X,Y) \in \Lambda \ \text{ iff } \ \lambda(X,Y) = |X| = |Y|. \tag{2.2}$$

With this correspondence, we may equivalently say that a bimatroid L is a triple (S,T,λ), where λ satisfies the following:

(B1) $0 \leq \lambda(X,Y) \leq \min\{|X|,|Y|\}$, $X\subset S$, $Y\subset T$;

(B2) $\lambda(X',Y') \leq \lambda(X,Y)$, $X'\subset X\subset S$, $Y'\subset Y\subset T$;

(B3) $\lambda(X\cup X',Y\cap Y')+\lambda(X\cap X',Y\cup Y') \leq \lambda(X,Y)+\lambda(X',Y')$, $X,X'\subset S$, $Y,Y'\subset T$.

By the <u>rank</u> r(L) of L, we mean the maximum size of a linked pair, i.e., $r(L) = \lambda(S,T)$. If $(S,T) \in \Lambda$, L is said to be <u>nonsingular</u>.

The <u>underlying bipartite graph</u> (S,T,Δ) of $L=(S,T,\Lambda)$ is a bipartite graph with vertex set $S\cup T$ (disjoint union) and arc set $\Delta \subset 2^S\times 2^T$ such that

$$(x,y) \in \Delta \quad \text{iff} \quad (\{x\},\{y\}) \in \Lambda. \tag{2.3}$$

Proposition 2.1 ([35], [36]). If $(X,Y) \in \Lambda$, there exists a matching (of size $|X|=|Y|$) between X and Y in the underlying bipartite graph (S,T,Δ). □

Let $L_i=(S_i,T_i,\Lambda_i)$ $(i=1,2)$ be bimatroids. The union of L_1 and L_2, denoted by $L_1 \vee L_2$, is a bimatroid $(S_1 \cup S_2, T_1 \cup T_2, \Lambda_1 \vee \Lambda_2)$, where

$$\Lambda_1 \vee \Lambda_2 = \{(X_1 \cup X_2, Y_1 \cup Y_2) \mid X_1 \cap X_2=\emptyset,\ Y_1 \cap Y_2=\emptyset,\ (X_1,Y_1)\in\Lambda_1,\ (X_2,Y_2)\in\Lambda_2\}. \tag{2.4}$$

If $T_1=S_2$, the product of L_1 and L_2, denoted by $L_1 * L_2$, can be defined; it is a bimatroid $(S_1,T_2,\Lambda_1 * \Lambda_2)$, where

$$\Lambda_1 * \Lambda_2 = \{(X,Z) \mid (X,Y)\in\Lambda_1,\ (Y,Z)\in\Lambda_2 \text{ for some } Y\subset T_1\}. \tag{2.5}$$

The birank function $\lambda_1 * \lambda_2$ of $L_1 * L_2$ is given by

$$(\lambda_1 * \lambda_2)(X,Z) = \min\{\lambda_1(X,T_1-Y)+\lambda_2(Y,Z) \mid Y\subset T_1\}, \quad X\subset S_1,\ Z\subset T_2. \tag{2.6}$$

With a matrix A over a field F is associated a bimatroid $L(A)=(S,T,\Lambda)$, where S and T correspond to the column-set and the row-set of A, and $(X,Y)\in\Lambda$ iff the submatrix of A with column indices in X and row indices in Y is nonsingular. Such a bimatroid is said to be linearly represented over F.

A poly-linking system [35], which we call a bi-polymatroid here, is a triple $L=(S,T,\lambda)$, where S and T are finite sets and λ: $2^S \times 2^T \to R_+$ satisfies (B2), (B3) and

(B1') $\lambda(\emptyset,Y) = \lambda(X,\emptyset) = 0, \quad X\subset S,\ Y\subset T.$

As with bimatroids, we call λ the birank function.

A pair (x,y) of vectors $x \in R_+^S$ and $y \in R_+^T$ is said to be <u>linked</u> iff $(x,y) \in \Lambda$, where

$$\Lambda = \{(x,y) \mid x \in R_+^S,\ y \in R_+^T,\ x(S)=y(T),$$
$$x(X)+y(Y) \leq \lambda(X,Y)+x(S) \text{ for all } X \subset S,\ Y \subset T\}.$$

Here $x(X)=\Sigma\{x(e) \mid e \in X\}$ for $x \in R_+^S$, etc. Λ is a nonempty compact subset of $R_+^S \times R_+^T$ such that

(L1') if $(x,y) \in \Lambda$, then $x(S)=y(T)$;

(L2-1') if $(x,y) \in \Lambda$ and $0 \leqq x' \leqq x$, then $(x',y') \in \Lambda$ for some $y' \leqq y$;

(L2-2') if $(x,y) \in \Lambda$ and $0 \leqq y' \leqq y$, then $(x',y') \in \Lambda$ for some $x' \leqq x$;

(L3') if $(x_i,y_i) \in \Lambda$ $(i=1,2)$, then there exists $(x,y) \in \Lambda$ such that

$$x_1 \leqq x \leqq x_1 \vee x_2,\ y_2 \leqq y \leqq y_1 \vee y_2.$$

Here for two vectors u and v, $u \leqq v$ means $u(e) \leqq v(e)$ for all e, and $u \vee v$ denotes the vector with $(u \vee v)(e)=\max\{u(e),v(e)\}$. Conversely, a bi-polymatroid (S,T,λ) is obtained from Λ by putting

$$\lambda(X,Y) = \max\{x(X)+y(Y)-x(S) \mid (x,y) \in \Lambda\}.$$

Let $L_i=(S_i,T_i,\Lambda_i)$ be bi-polymatroids with birank functions λ_i $(i=1,2)$. When $T_1=S_2$, the <u>product</u> of L_1 and L_2, denoted by $L_1 * L_2$, is a bi-polymatroid $(S_1,T_2,\Lambda_1 * \Lambda_2)$ with

$$\Lambda_1 * \Lambda_2 = \{(x,z) \mid (x,y) \in \Lambda_1,\ (y,z) \in \Lambda_2 \text{ for some } y\}$$

and the birank function of (2.6). When $S_1=S_2$ $(=S)$ and $T_1=T_2$ $(=T)$, the <u>sum</u> of L_1 and L_2, denoted by L_1+L_2, is a bi-polymatroid $(S,T,\Lambda_1+\Lambda_2)$ with

$$\Lambda_1+\Lambda_2 = \{(x_1+x_2,y_1+y_2) \mid (x_1,y_1) \in \Lambda_1,\ (x_2,y_2) \in \Lambda_2\}$$

and birank function $\lambda_1+\lambda_2$.

3. Previous Results on Structural Solvability

We describe briefly the results of [15], [16], [27], [28] on the structural solvability of systems of equations. More detailed accounts may be found in [26], [28]. We also mention some methods for the hierarchical decomposition of the whole system into structurally solvable subsystems.

First of all, it is postulated that the collection (multiset) D of the partial derivatives of f_i ($i=1,\dots,M$) and g_k ($k=1,\dots,K$) of (1.1) is a subset of a field F, which is an extension of the rational number field Q. Then the Jacobian matrix $J=J(x,u)$ of (1.1) with respect to x_j ($j=1,\dots,N$) and u_k ($k=1,\dots,K$):

$$J = \begin{pmatrix} J[f,x] & J[f,u] \\ J[g,x] & J[g,u]-I_K \end{pmatrix}, \tag{3.1}$$

where

$$J[f,x] = (\partial f_i/\partial x_j), \quad J[f,u] = (\partial f_i/\partial u_l),$$
$$J[g,x] = (\partial g_k/\partial x_j), \quad J[g,u] = (\partial g_k/\partial u_l),$$

can be regarded as a matrix over F. The system (1.1) of equations is said to be <u>structurally solvable</u> iff J is nonsingular as a matrix over F.

As has been mentioned in §1, the <u>representation graph</u> of (1.1) is a graph $G=(V,A;X,Y)$ with entrance X and exit Y defined as follows. The vertex set V consists of three disjoint parts as

$$V = X \cup U \cup Y,$$

where $X = \{x_1,\dots,x_N\}$, $U = \{u_1,\dots,u_K\}$ and $Y = \{y_1,\dots,y_M\}$. The arc set A is

$$A = (\bigcup_{i=1}^{M} \delta^- y_i) \cup (\bigcup_{k=1}^{K} \delta^- u_k),$$

where

$$\delta^- y_i = \{(x_j, y_i) \mid \partial f_i/\partial x_j \neq 0,\ 1 \leq j \leq N\} \cup \{(u_l, y_i) \mid \partial f_i/\partial u_l \neq 0,\ 1 \leq l \leq K\},$$
$$\delta^- u_k = \{(x_j, u_k) \mid \partial g_k/\partial x_j \neq 0,\ 1 \leq j \leq N\} \cup \{(u_l, u_k) \mid \partial g_k/\partial u_l \neq 0,\ 1 \leq l \leq K\}. \tag{3.2}$$

In the case where the strong "generality assumption"

GA1: The collection of the nonvanishing elements of D ($\subset F$) is algebraically independent over Q,

can be accepted, the structural solvability of (1.1) is expressed by G as follows.

Theorem 3.1 ([15]). Assume GA1. The system (1.1) is structurally solvable iff there exists a Menger-type complete linking from X to Y in G. □

On the basis of this theorem, two kinds of decompositions of the representation graph with respect to maximum linkings have been considered for the block-triangularization of the system (1.1) of equations. The one is the L-decomposition of [15], [16], and the other the M-decomposition (abbreviation of Menger-decomposition) of [22], [23], [24].

The L-decomposition is defined as follows. Suppose there exists a Menger-type complete linking from X to Y in G=(V,A;X,Y), and fix one. Let σ:X→Y denote the one-to-one correspondence between X and Y with respect to the linking fixed, i.e., $\sigma(x) \in Y$ is the vertex linked with x $\in$ X. Consider an augmented graph G'=(V,A') with

$$A' = A \cup \{(\sigma(x),x) \mid x \in X\}.$$

The partition of V, as well as the partial order on it, induced by the strong components of G' is called the L-decomposition of G. It is known that the L-decomposition does not depend on the choice of the complete linking in G.

Let $B=(X,Y,\Delta)$ be the bipartite graph with

$$\Delta = \{(x,y) \mid \text{there is a directed path from } x \in X \text{ to } y \in Y \text{ in } G\},$$

which agrees with the underlying bipartite graph of the bimatroid defined in terms of the Menger-type linkings from X to Y in $G=(V,A;X,Y)$. The following may be interpreted as a statement that the L-decomposition yields the finest decomposition of (1.1) into structurally solvable subsystems as long as u_k $(k=1,\ldots,K)$ are treated as intermediate variables to be eliminated.

<u>Proposition 3.2</u> (Prop.3.2 of [22] and [24], Th.8.6 of [26]). Suppose G has a Menger-type complete linking from X to Y. The decomposition of $X \cup Y$ induced by the L-decomposition of G agrees with the DM-decomposition of B.

□

See [24], [26] for more on the L-decomposition.

The M-decomposition of $G=(V,A;X,Y)$ $(V=X \cup U \cup Y,\ X \cap Y=\emptyset)$ is defined as follows. For simplicity we assume $\delta^- v=\emptyset$ for $v \in X$ and $\delta^+ v=\emptyset$ for $v \in Y$. We associate with G a capacitated network $N = (\tilde{V},\tilde{A},c;s^+,s^-)$ with underlying graph $(\tilde{V},\tilde{A})$, capacity function $c\colon \tilde{A} \to R_+$, source s^+ and sink s^-, where

$$\tilde{V} = \{s^+, s^-\} \cup X_* \cup U_* \cup U^* \cup Y^*, \tag{3.3}$$

$$X_* = \{x_*^1, \ldots, x_*^N\},\ U_* = \{u_*^1, \ldots, u_*^K\},$$

$$U^* = \{u_1^*, \ldots, u_K^*\},\ Y^* = \{y_1^*, \ldots, y_M^*\},$$

$$\tilde{A} = \tilde{A}_o \cup \tilde{A}_d, \tag{3.4}$$

$$\tilde{A}_o = \{(v_*, w^*) \mid (v,w) \in A\},$$

$$\tilde{A}_d = \{(s^+, x_*) \mid x \in X\} \cup \{(u^*, u_*) \mid u \in U\} \cup \{(y^*, s^-) \mid y \in Y\},$$

$$c(a) = \begin{cases} 1, & a \in \tilde{A}_d, \\ +\infty, & a \in \tilde{A}_o. \end{cases} \tag{3.5}$$

There exists a one-to-one correspondence between Menger-type maximum linkings from X to Y in G and integral maximum flows from s^+ to s^- in N which have no circulation (i.e., flow along a cycle). On the other hand, minimum separators of (X,Y) in G correspond to minimum cuts with respect to (s^+, s^-) in N.

The <u>cut function</u> κ of N is defined by

$$\kappa(W) = \Sigma\{c(a) \mid a \in \tilde{A},\ \partial^+ a \in W,\ \partial^- a \in \tilde{V} - W\}, \quad W \subset \tilde{V}. \tag{3.6}$$

The maximum size of a linking from X to Y in G is equal to

$$\min\{\kappa(W) \mid W \subset \tilde{V},\ s^+ \in W,\ s^- \notin W\}. \tag{3.7}$$

Since κ is submodular, the minimizers W of (3.7) constitute a sublattice, say $L(\kappa)$, of the boolean lattice $2^{\tilde{V}}$. The lattice $L(\kappa)$ determines a partition of $\tilde{V}$ into partially ordered blocks, and hence determines a family, say $\{N^{(i)} \mid i \in \{0,\infty\} \cup \tilde{I}\}$, of subnetworks of N (cf., e.g., [26], [31]).

Roughly speaking, the M-decomposition of G is a family of subgraphs, say $\{G^{(i)} = (V^{(i)}, A^{(i)}; X^{(i)}, Y^{(i)}) \mid i \in \{0,\infty\} \cup I\}$, of G induced by $\{N^{(i)}\}$. The partial order is naturally induced on $\{G^{(i)}\}$. A concrete example is given in Example 7.1 of §7. See [24], [26] for the precise definition. Each subgraph $G^{(i)}$ is called an M-component. $G^{(i)}=(V^{(i)},A^{(i)};X^{(i)},Y^{(i)})$ $(i\in I)$ has a complete linking from $X^{(i)}$ to $Y^{(i)}$, and a maximum linking from X to Y in G is obtained as the "direct sum" of maximum linkings in $G^{(0)}$ and $G^{(\infty)}$ and complete linkings in $G^{(i)}$ $(i\in I)$.

When G has a complete linking, both the L-decomposition and the M-decomposition yield a hierarchical family of subgraphs that have complete linkings; an L-component is an aggregation of a number of M-components. Both decompositions, when applied to the representation graph of (1.1), yield a hierarchical decomposition of (1.1) into smaller subsystems of equations, which are structurally solvable under GA1.

In [27], [28], the structural solvability is discussed under a weaker "generality assumption"

GA2: D-Q is algebraically independent over Q.

The argument of [27], [28] is based on the physical observation on "two kinds of numbers" and on the mathematical tool of mixed matrix.

Let K $(Q\subset K\subset F)$ be an intermediate field. A matrix $A=Q_A+T_A$ over F is called a <u>mixed matrix</u> with respect to K iff (i) Q_A is a matrix over K and (ii) the collection of nonvanishing entries of T_A is algebraically independent over K. The following relation makes it possible to compute the rank of A (a matrix over F) by means of arithmetic operations in the subfield K using efficient matroid-theoretic algorithms.

Theorem 3.3 ([27], [28]). Let $A=Q_A+T_A$ be a mixed matrix. The bimatroid defined by A is the union of bimatroids defined by Q_A and T_A:

$$L(A) = L(Q_A) \vee L(T_A). \qquad \square$$

Under the weaker assumption GA2, the Jacobian matrix J of (1.1) can be regarded as a mixed matrix $J=Q_J+T_J$ with respect to Q, and then Theorem 3.3 above provides the structural solvability condition which can be tested by efficient algorithms using rational arithmetics. Furthermore, the combinatorial canonical form of a (layered) mixed matrix [29] gives a powerful method for the hierarchical decomposition of (1.1) into structurally solvable subsystems.

By definition, the system (1.1) is structurally solvable iff the bimatroid L(J) represented over F by its Jacobian matrix J is a nonsingular bimatroid. Under GA1, L(J) reduces to a bimatroid expressed by Menger-type linkings in the representation graph of (1.1). Under GA2, on the other hand, L(J) agrees with the union of two bimatroids $L(Q_J)$ and $L(T_J)$, the former being represented over Q and the latter being transversal. In this paper, we extend this line of approach for the signal-flow-type model by generalizing the notion of representation graph. Namely, we shall express the bimatroid L(J) in terms of simpler bimatroids under a certain generality assumption weaker than GA1.

4. Bimatroid Network

A multiterminal capacitated network $N=(V,A,c;X,Y)$ with entrance X and exit Y provides a typical example of bi-polymatroid with entrance set X and exit set Y. When a number of multiterminal networks are interconnected, there arises another network, which determines a new bi-polymatroid. In this section, we observe an analogous phenomenon for interconnected bi-polymatroids. That is, we show that a network-type connection of bi-polymatroids results in another bi-polymatroid. This construction generalizes the known composition operations for bi-polymatroids (resp., bimatroids) such as sum (resp., union) and product.

Suppose a family of bi-polymatroids $\{L_p=(X_p,Y_p,\Lambda_p) \mid p\in P\}$ is given, where it is assumed that

$$X_p \cap Y_p = \emptyset \quad \text{for } p\in P, \tag{4.1}$$

and

$$X_p \cap X_q = \emptyset, \quad Y_p \cap Y_q = \emptyset \quad \text{if } p \neq q \ (\in P). \tag{4.2}$$

Also suppose a capacitated network $N=(V,A,c;X,Y)$ with entrance X and exit Y such that

$$X \cap Y = \emptyset$$

and

$$X_p \cup Y_p \subset V \quad \text{for } p\in P \tag{4.3}$$

is given. We consider a composite system in which the bi-polymatroids L_p are interconnected by the network N. The composite system $(N,\{L_p \mid p\in P\})$ will be named here a <u>bimatroid network</u>. We put

$$\hat{U} = V - (X \cup Y \cup \bigcup_{p\in P} (X_p \cup Y_p)).$$

It would be natural to define the feasibility of a flow in bimatroid network $(N,\{L_p | p\in P\})$ as follows. As is easily recognized, the feasible flow in a bimatroid network is a variant of submodular flow (or independent flow) of [4], [6], [7], [19], [20].

Without essential loss of generality we first consider the case where

$$X_p \cap Y_q = \emptyset \quad \text{if } p \neq q \ (\in P) \tag{4.4}$$

and

$$(X \cup Y) \cap (X_p \cup Y_p) = \emptyset \quad \text{for } p\in P. \tag{4.5}$$

We say $f\colon A \to R_+$ is a feasible flow of such a bimatroid network iff

$$\begin{aligned} &\text{(i) } 0 \leq f(a) \leq c(a), \quad a\in A,\\ &\text{(ii) } \partial f(v) = 0, \quad v\in \hat{U},\\ &\text{(iii) } ((-\partial f(v) | v\in X_p), (\partial f(v) | v\in Y_p)) \in \Lambda_p, \quad p\in P. \end{aligned} \tag{4.6}$$

Here $\partial f\colon V \to R$ denotes the "boundary" of f defined by

$$\partial f(v) = \Sigma\{f(a) | a\in\delta^+ v\} - \Sigma\{f(a) | a\in\delta^- v\}, \quad v\in V. \tag{4.7}$$

The third condition (iii) says that the flow flooding out of N at X_p is linked to the flow into N at Y_p by bi-polymatroid L_p. For $W \subset V$, we define the <u>cut function</u> by

$$\begin{aligned} \kappa(W) = {} & \Sigma\{c(a) | a\in A, \partial^+ a\in W, \partial^- a\in V-W\} \\ & + \Sigma\{\lambda_p(X_p \cap W, Y_p - W) | p\in P\}, \quad W\subset V. \end{aligned} \tag{4.8}$$

We observe the following fact, which might have been noted elsewhere in some disguise or other.

<u>Theorem 4.1</u>. Suppose a bimatroid network $(N,\{L_p | p\in P\})$ satisfying (4.4) and (4.5) is given. By defining

$$\Lambda = \{(x,y) \mid x=(\partial f(v) \mid v\in X)\in R_+^X,\ y=(-\partial f(v) \mid v\in Y)\in R_+^Y,$$
$$f \text{ is a feasible flow}\}$$

we obtain a bi-polymatroid (X,Y,Λ). Its birank function λ is given by

$$\lambda(X',Y') = \min\{\kappa(W) \mid X'\subset W\subset V,\ W\cap Y'=\emptyset\}, \qquad X'\subset X,\ Y'\subset Y. \tag{4.9}$$

(Proof) First we may assume $\delta^- v=\emptyset$ for $v\in X$ and $\delta^+ v=\emptyset$ for $v\in Y$ without loss of generality. For otherwise we can modify the network so as to meet this condition without changing Λ. For the same reason, we may also assume that $\delta^+ v=\emptyset$ for $v\in X_p$, $\delta^- v=\emptyset$ for $v\in Y_p$, $|\delta^- v|=1$ for $v\in X_p$, and $|\delta^+ v|=1$ for $v\in Y_p$ $(p\in P)$; and that $\{\partial^+ a,\partial^- a\}\cap\hat{U}\neq\emptyset$ for $a\in A$.

Next note that each vertex $v\in\hat{U}$ may be regarded as defining a bi-polymatroid with entrance set $\delta^- v$ and exit set $\delta^+ v$, since $\{((f(a) \mid a\in\delta^- v),(f(a) \mid a\in\delta^+ v)) \mid 0\leq f(a)\leq c(a) \text{ for } a\in\delta^- v\cup\delta^+ v;\ \partial f(v)=0\}$ satisfies (L1') to (L3'). By introducing a vertex v_a for each $a\in\delta^- v\cup\delta^+ v$, therefore, we may equivalently replace $v\in\hat{U}$ by a bi-polymatroid with entrance set $\{v_a \mid a\in\delta^- v\}$ and exit set $\{v_a \mid a\in\delta^+ v\}$. Hence we may further assume that $\hat{U}=\emptyset$ and that $c(a)=+\infty$ for all $a\in A$.

By the above argument as well as by the fact that the sum of $\{L_p \mid p\in P\}$ is again a bi-polymatroid, the proof is complete if the following lemma is established. □

Lemma 4.2. Let $L=(S,T,\Lambda)$ be a bi-polymatroid, where $S=X\cup U$ and $T=Y\cup U$ $(X\cap U=Y\cap U=\emptyset)$. Then

$$\Lambda_U = \{(x,y) \mid x\in R_+^X,\ y\in R_+^Y,\ (x\oplus u, y\oplus u)\in\Lambda \text{ for some } u\in R_+^U\}$$

determines a bi-polymatroid (X,Y,Λ_U), where $\oplus$ denotes the direct sum of vectors. The birank function λ_U of (X,Y,Λ_U) is given by

$$\lambda_U(X',Y') = \min\{\lambda(X'\cup U', Y'\cup(U-U')) \mid U'\subset U\}, \quad X'\subset X,\ Y'\subset Y.$$

(Proof) Let $L_0=(S,T,\Lambda_0)$ be a bi-polymatroid defined by

$$\Lambda_0 = \{(0_X\oplus u, 0_Y\oplus u) \mid u\in R_+^U,\ u(e)\leq C \text{ for } e\in U\},$$

where $0_X \in R_+^X$ and $0_Y\in R_+^Y$ are zero vectors, and C is a sufficiently large constant. We also define bi-polymatroids $L_S=(S,S,\Lambda_S)$ and $L_T=(T,T,\Lambda_T)$ by

$$\Lambda_S = \{(z,z) \mid z\in R_+^S,\ z(e)\leq C \text{ for } e\in S\},$$

and

$$\Lambda_T = \{(z,z) \mid z\in R_+^T,\ z(e)\leq C \text{ for } e\in T\}.$$

Then we may consider a bi-polymatroid $L_S*(L+L_0)*L_T$, which we denote by $\tilde{L}=(S,T,\tilde{\Lambda})$. It is not difficult to see that

$$(x,y) \in \Lambda_U \quad \text{iff} \quad (x\oplus w, y\oplus w) \in \tilde{\Lambda}, \tag{4.10}$$

where $w\in R_+^U$ is the vector with $w(e)=C$ for $e\in U$.

We will now show that Λ_U satisfies (L1') to (L3') of §2.

(L1'): Suppose $(x,y)\in\Lambda_U$, i.e., $(x\oplus w,y\oplus w)\in\tilde{\Lambda}$. (L1') for $\tilde{\Lambda}$ implies $x(X)+w(U)=y(Y)+w(U)$, i.e., $x(X)=y(Y)$.

(L2-1'): Suppose $(x,y)\in\Lambda_U$ and $0\leq x'\leq x$. Then $(x\oplus w,y\oplus w)\in\tilde{\Lambda}$ and $0\leq x'\oplus w\leq x\oplus w$. By (L2-1') for $\tilde{\Lambda}$, we see $(x'\oplus w,y''\oplus u)\in\tilde{\Lambda}$ for some $y''\leq y$ and $u\leq w$. Noting that $(0_X\oplus w,0_Y\oplus w)\in\tilde{\Lambda}$ and using (L3') for $\tilde{\Lambda}$, we then obtain $(x'\oplus w,y'\oplus w)\in\tilde{\Lambda}$ for some $y'\leq y''(\leq y)$.

(L2-2'): Similar to the proof of (L2-1').

(L3'): Suppose $(x_i,y_i)\in\Lambda_U$ $(i=1,2)$, i.e., $(x_i\oplus w,y_i\oplus w)\in\tilde{\Lambda}$ $(i=1,2)$. It follows from (L3') for $\tilde{\Lambda}$ that $(x\oplus w,y\oplus w)\in\tilde{\Lambda}$, i.e., $(x,y)\in\Lambda_U$, for some $x\in R_+^X$ and $y\in R_+^Y$ such that $x_1\leq x\leq x_1\vee x_2$ and $y_2\leq y\leq y_1\vee y_2$.

The expression of λ_U follows from (4.10) and (2.6). □

In the general case where (4.4) and (4.5) are not assumed, the notion of feasible flow may be extended (in a cumbersome way) as follows. In addition to $f: A \to R_+$, we consider $\bar{x}: V \to R_+$ and $\bar{y}: V \to R_+$. We say the triple $(f, \bar{x}, \bar{y})$ is a "feasible flow" iff

$$\begin{aligned} &\text{(i)}\ 0 \le f(a) \le c(a), \quad a \in A, \\ &\text{(ii)}\ \partial f(v) + \bar{x}(v) - \bar{y}(v) = 0, \quad v \in V-(X \cup Y), \\ &\text{(iii)}\ ((\bar{x}(v) \mid v \in X_p), (\bar{y}(v) \mid v \in Y_p)) \in \Lambda_p, \quad p \in P. \end{aligned} \tag{4.11}$$

Then Theorem 4.1 can be extended as follows.

<u>Corollary 4.3</u>. Suppose a bimatroid network $(N, \{L_p \mid p \in P\})$ is given. By defining

$$\begin{aligned} \Lambda = \{(x,y) \mid\ & x = (\partial f(v) + \bar{x}(v) - \bar{y}(v) \mid v \in X) \in R_+^{\,X}, \\ & y = (-\partial f(v) - \bar{x}(v) + \bar{y}(v) \mid v \in Y) \in R_+^{\,Y}, \\ & (f, \bar{x}, \bar{y}) \text{ is a feasible flow}\} \end{aligned}$$

we obtain a bi-polymatroid (X, Y, Λ). Its birank function λ is given by (4.9). □

We may also think of a composite system in which bimatroids are interconnected by a graph. To be specific, let $\{L_p = (X_p, Y_p, \Lambda_p) \mid p \in P\}$ be a family of bimatroids satisfying (4.1) and (4.2), and $G = (V, A; X, Y)$ be a graph with entrance X and exit Y ($X \cap Y = \emptyset$) such that (4.3) is satisfied. (The conditions (4.4) and (4.5) are not imposed.) We shall also call such a pair $(G, \{L_p \mid p \in P\})$ a bimatroid network.

By a <u>Menger-type linking</u> (or simply <u>linking</u>) from X to Y in $(G, \{L_p \mid p \in P\})$ we mean a Menger-type (vertex-disjoint) linking L from $X \cup (\bigcup_{p \in P} Y_p)$ to $Y \cup (\bigcup_{p \in P} X_p)$ in $G = (V, A)$ in the sense of §2 such that

$$(\partial^- L\cap X_p, \partial^+ L\cap Y_p) \in \Lambda_p \quad \text{for } p\in P, \tag{4.12}$$

where $\partial^- L$ (resp., $\partial^+ L$) denotes the set of terminal (resp., initial) vertices of the paths contained in the linking L. L is called a complete linking from X to Y if $\partial^+ L \supset X$ and $\partial^- L \supset Y$. The size of L, when regarded as a linking from X to Y in $(G,\{L_p|p\in P\})$, will mean $|\partial^+ L\cap X|$ $(=|\partial^- L\cap Y|)$.

Just as the Menger-type linkings in a graph (in the usual sense) can be formulated as integral flows in a network associated with the graph (cf. §3), the linkings in a bimatroid network can be treated as integral flows in a certain bimatroid network $(N,\{L_p|p\in P\})$, in which L_p are regarded as bi-polymatroids, on the basis of the integrality property of bi-polymatroids. In particular, we obtain the following as a corollary to Theorem 4.1.

Corollary 4.4. Suppose a bimatroid network $(G,\{L_p|p\in P\})$ is given. By defining

$$\Lambda = \{(\partial^+ L\cap X, \partial^- L\cap Y) \mid L \text{ is a linking in } (G,\{L_p|p\in P\})\}$$

we obtain a bimatroid (X,Y,Λ). □

As is easily seen, the problem of finding a maximum flow in a bimatroid network can be formulated as that of finding a maximum submodular flow (or independent flow) considered in [4], [6], [7], [19], [20]. (The converse is also true.) Hence efficient algorithms are available for finding a maximum flow or a maximum linking in a bimatroid network.

The notion of bimatroid network seems to be useful for the mathematical modeling of physical/engineering systems in general. A physical/engineering system is usually composed of many subsystems or modules interconnected with one another. A module may be described in terms of the input-output relation. In some cases the relevant structural aspects of the input-output relation may be represented by a bimatroid. In such a case we will be able to understand the structural properties of the whole system by recognizing the system as a bimatroid network and analyzing the combinatorial structure of the bimatroid network.

Another common feature of a large-scale physical/engineering system is that it has a hierarchical or nesting structure among modules. The whole system consists of modules, and each module is composed of a number of interconnected submodules, each of which is again made up of several modules, and so on. Submodules contained in a module are localized to that module in the sense that two modules belonging to different hierarchical levels have no direct connection. Theorem 4.1 shows that such a hierarchical structure of a physical/engineering system can naturally be incorporated into a mathematical model using bimatroid networks as the unit of construction.

It may also be mentioned that, if necessary, we can attach "cost" to arcs in a bimatroid network to represent physical characteristics.

In the succeeding sections, we shall consider a specific problem of the structural solvability of systems of equations using bimatroid networks.

5. Structural Solvability Criterion

The objective of this section is to establish an extension of Theorem 3.1 by discussing the structural solvability of (1.1) under a certain "generality assumption" weaker than GA1. The "generality assumption" adopted here is based on the physical observation that the algebraic dependency among the partial derivatives of functions f_i and g_k will be local and restricted to within subsystems, or modules. The structure of (1.1) is expressed by a generalization of the representation graph to a bimatroid network, which we call the generalized representation graph. Then the structural solvability of (1.1) is shown in Theorem 5.1 to be equivalent to the existence of a Menger-type complete linking in the generalized representation graph of (1.1).

Suppose the system (1.1) of equations describes a physical system containing several modules which are given in terms of the input-output relations. To be more concrete, let P be the family of modules and X_p and Y_p denote the sets of input and output variables of module $p \in P$. That is, we are given $\{(X_p, Y_p) \mid p \in P\}$ such that

$$X_p \subset X \cup U, \quad Y_p \subset Y \cup U, \tag{5.1}$$

$$X_p \cap Y_p = \emptyset \quad \text{for } p \in P, \tag{5.2}$$

and

$$X_p \cap X_q = \emptyset, \quad Y_p \cap Y_q = \emptyset \quad \text{if } p \neq q \ (\in P), \tag{5.3}$$

where $X=\{x_1,\ldots,x_N\}$, $U=\{u_1,\ldots,u_K\}$, $Y=\{y_1,\ldots,y_M\}$, as before. Furthermore we assume

$$X_p \cap Y_q = \emptyset \quad \text{if } p \neq q \ (\in P), \tag{5.4}$$

which states that an output variable of a module cannot be a direct input variable of another module. The condition (4.5), however, is not imposed here. We put

$$X_P = \bigcup_{p\in P} X_p, \quad Y_P = \bigcup_{p\in P} Y_p.$$

Consider the equations in (1.1) that describe the input-output relation of a module $p\in P$. Since the variables of Y_p are determined by the variables of X_p, those equations of (1.1) which correspond to module p take the form:

$$\begin{aligned} y_i &= f_i(x_p,u_p) \quad \text{for } y_i\in Y_p, \\ u_k &= g_k(x_p,u_p) \quad \text{for } u_k\in Y_p, \end{aligned} \tag{5.5}$$

where x_p and u_p are the vectors corresponding respectively to the variables of $X_p\cap X$ and $X_p\cap U$. We allow the variables of X_p to appear also as the arguments of f_i for $y_i\in Y-Y_P$ or g_k for $u_k\in U-Y_P$.

For $p\in P$, we denote by $J_p = J_p(x_p,u_p)$ the Jacobian matrix of (5.5) with respect to x_p and u_p, defined similarly to (3.1). J_p is a submatrix of J. The collection of the partial derivatives of f_i for $y_i\in Y_p$ and g_k for $u_k\in Y_p$ will be written as D_p ($\subset D$). We put

$$D_P = \bigcup_{p\in P} D_p, \tag{5.6}$$

where the right-hand side means the union of multisets. $D-D_P$ will mean the collection of the partial derivatives of f_i for $y_i\in Y-Y_P$ and g_k for $u_k\in U-Y_P$. The elements of D_p belong to a subfield of F, say F_p; the field generated by $\{F_p \mid p\in P\}$ will be denoted by F_P ($\subset F$).

We now introduce a "generality assumption" which takes into account the possible dependency among local parameters. It should be clear from (5.6) that the partial derivatives of $D-D_P$ represent the global functional relations which are not local to any module. Hence it would be physically plausible to assume that the nonvanishing elements of $D-D_P$

are independent of one another, whereas there may be algebraic dependency among partial derivatives in D_p which are local to a single module $p \in P$. Thus we are led to the "generality assumption"

> GA4: The collection of the nonvanishing elements of $D-D_P$ $(\subset F)$ is algebraically independent over F_P.

This is obviously weaker than GA1 and reduces to GA1 when $P=\emptyset$.

In accordance with the above setting, we modify the notion of representation graph as follows. Let $\hat{G}=(V,\hat{A};X,Y)$ be the representation graph of (1.1) as defined in §3 except that the arc set is replaced by

$$\hat{A} = \cup\{\delta^- y_i \mid y_i \in Y-Y_P\} \cup \{\delta^- u_k \mid u_k \in U-Y_P\},$$

where $\delta^- y_i$ and $\delta^- u_k$ are given by (3.2). With each module $p \in P$ we associate the bimatroid $L_p=L(J_p)$, which is linearly represented over F_p. Note that X_p and Y_p are the column-set and the row-set of J_p and hence $L_p=(X_p,Y_p,\Lambda_p)$. The bimatroid network $(\hat{G},\{L_p \mid p \in P\})$ will be called the <u>generalized representation graph</u> of (1.1) with respect to $\{(X_p,Y_p) \mid p \in P\}$.

By construction, $(\hat{G}=(V,\hat{A};X,Y),\{L_p=(X_p,Y_p,\Lambda_p) \mid p \in P\})$ has the following properties:

(i) A vertex of $X \cup Y_P$ is maximal in $\hat{G}$, i.e., $\delta^- v=\emptyset$ for $v \in X \cup Y_P$;

(ii) A vertex of Y is minimal in $\hat{G}$, i.e., $\delta^+ v=\emptyset$ for $v \in Y$, whereas a vertex of X_P is not necessarily minimal.

Under the assumption GA4, the submatrix $J[g,u]-I_K$ in J of (3.1) is guaranteed to be nonsingular. This means that the variables u_k $(k=1,\ldots,K)$ can be eliminated, at least locally, and therefore (1.1) reduces to

$$y_i = f_i(x,u(x)) \qquad (i=1,\ldots,M) \tag{5.7}$$

with unknowns x_j $(j=1,\ldots,N)$. The Jacobian matrix $J[y,x]$ of (5.7) is given by

$$J[y,x] = J[f,x] - J[f,u](J[g,u]-I)^{-1}J[g,x]. \tag{5.8}$$

Note here that $J[y,x]$ has row-set Y and column-set X, and $L(J[y,x])$ is a bimatroid with entrance set X and exit set Y.

The main result of the present paper is now stated. It is a direct extension of Theorem 3.1 and enables us to check the structural solvability under a weaker "generality assumption" by efficient algorithms using arithmetic operations in the subfield F_P. Its extension to a hierarchical system of equations is discussed later in §6.

<u>Theorem 5.1</u>. Assume GA4 as well as (5.2), (5.3) and (5.4).

(1) The bimatroid $L(J[y,x])$ agrees with the bimatroid (X,Y,Λ) associated with the generalized representation graph $(\hat{G},\{L_p|p\in P\})$ as in Corollary 4.4.

(2) The system (1.1) is structurally solvable iff there exists a Menger-type complete linking from X to Y in the generalized representation graph $(\hat{G},\{L_p|p\in P\})$.

(Proof) The equivalence of (1) and (2) is immediate from the fact that, provided $J[g,u]-I_K$ is nonsingular, J is nonsingular iff $J[y,x]$ is nonsingular.

We first claim that we may restrict ourselves to the case where

$$X_p \cup Y_p \subset U \quad \text{for } p\in P. \tag{5.9}$$

(This is equivalent to (4.5) since we have (5.1) here.)

Suppose, e.g., $x \in X\cap X_p$. We may introduce a new variable u, replace all the occurrences of x in (1.1) by u, and add a new equation

$$u = x \tag{5.10}$$

to (1.1); u is added to U. The augmented system of equations is structurally solvable iff so is the original system (1.1).

Furthermore, replace (5.10) by

$$u = tx \tag{5.11}$$

with a transcendental number $t \in F$ over F_p. Note that we may assume F to be large enough to contain such element t. This modification does not affect the structural solvability of the augmented system, since it corresponds to multiplying by t the column x of the Jacobian matrix of the augmented system. In addition, the modified augmented system satisfies GA4.

It should also be noted that the corresponding change in the generalized representation graph is to rename the vertex x as u and to add an arc from a new vertex x to u. The existence of a complete linking in the generalized representation graph is not affected by this change, either. A similar argument applies to $y \in Y \cap Y_p$ if any. Hence (5.9) is justified.

Next we claim that we may assume

$$\delta^+ u = \emptyset \quad \text{for } u \in X_p \tag{5.12}$$

in $\hat{G}$ without loss of generality. This says that an input variable of a module appears only on the right-hand side of the equations describing that module.

If $\delta^+ u_k \neq \emptyset$ for $u_k \in X_p$, $p \in P$, we may introduce a new variable u_k', replace by u_k' all the occurrences of u_k on the right-hand side of an equation that has a variable of $(Y \cup U) - Y_p$ on the left-hand side, rewrite

$u_k=g_k(x,u)$ as $u_k'=g_k(x,u)$ and add a new equation

$$u_k = u_k' \tag{5.13}$$

to (1.1); u_k' is added to U. We regard $u_k \in X_p$ and $u_k' \in U-X_p$. The augmented system of equations is structurally solvable iff so is the original system (1.1).

Furthermore, replace (5.13) by

$$u_k = tu_k' \tag{5.14}$$

with a transcendental number $t\in F$ over F_p. Taking notice of the facts that the nonvanishing partial derivatives of g_k (now corresponding to u_k') are algebraically independent, and that the nonvanishing partial derivatives with respect to u_k' are also algebraically independent, we see that this modification keeps the structural solvability of the augmented system invariant; the modification corresponds to multiplying by t the column u_k' and dividing by t the row u_k' of the Jacobian matrix of the augmented system. Then the modified augmented system meets GA4.

The corresponding change in the generalized representation graph is to rename the vertex u_k as u_k' and to add an arc from u_k' to a new vertex u_k. It is easy to check that the existence of a complete linking in the generalized representation graph is not affected by this change. Therefore, (5.12) is not restrictive, either.

Under the additional assumptions (5.9) and (5.12), the Jacobian matrix J of (1.1) (cf. (3.1)) can be written as follows:

$$J = \begin{array}{c|cccc|} & X & X_P & Y_P & \hat{U} \\ \hline Y & T_1 & 0 & T_2 & T_3 \\ X_P & T_4 & -I & T_5 & T_6 \\ Y_P & 0 & M & -I & 0 \\ \hat{U} & T_7 & 0 & T_8 & T_9-I \\ \hline \end{array}, \tag{5.15}$$

where $\hat{U}=U-(X_P \cup Y_P)$, M is a matrix over F_P that is the direct sum of the Jacobian matrices J_p of (5.5), and T_i's denote matrices over F with the nonvanishing entries being collectively algebraically independent over F_P.

Note that J is nonsingular iff

$$J' = \begin{array}{c|cccc|} & X & X_P & Y_P & \hat{U} \\ \hline Y & T_1 & 0 & T_2 & T_3 \\ X_P & T_4 & -I & T_5 & T_6 \\ Y_P & 0 & M & -I & 0 \\ \hat{U} & T_7 & 0 & T_8 & T_9-D \\ \hline \end{array} \tag{5.16}$$

is nonsingular, where D is a diagonal matrix with the diagonal entries being indeterminates over F. Then J' is a mixed matrix with respect to F_P (see §3) with the expression

$$J' = Q_{J'} + T_{J'}, \tag{5.17}$$

where

$$Q_{J'} = \begin{array}{c|cccc|} & X & X_P & Y_P & \hat{U} \\ \hline Y & 0 & 0 & 0 & 0 \\ X_P & 0 & -I & 0 & 0 \\ Y_P & 0 & M & -I & 0 \\ \hat{U} & 0 & 0 & 0 & 0 \\ \hline \end{array} , \tag{5.18}$$

$$T_{J'} = \begin{array}{c|cccc|} & X & X_P & Y_P & \hat{U} \\ \hline Y & T_1 & 0 & T_2 & T_3 \\ X_P & T_4 & 0 & T_5 & T_6 \\ Y_P & 0 & 0 & 0 & 0 \\ \hat{U} & T_7 & 0 & T_8 & T_9-D \\ \hline \end{array} . \tag{5.19}$$

By Theorem 3.3, as well as the definition of the union of bimatroids, we see J is nonsingular iff there exist $X' \subset X_P$ and $Y' \subset Y_P$ such that

the submatrix of $Q_{J'}$ with rows in $Y_P \cup (X_P - X')$ and columns in $X_P \cup (Y_P - Y')$ is nonsingular, (5.20)

and that

the submatrix of $T_{J'}$ with rows in $Y \cup X' \cup \hat{U}$ and columns in $X \cup Y' \cup \hat{U}$ is nonsingular. (5.21)

The former condition (5.20) reduces further to the following:

the submatrix of M with rows in Y' and columns in X' is nonsingular.

In terms of the bimatroids $L_p=(X_p,Y_p,\Lambda_p)$, this condition is equivalent to saying that

$$(X'\cap X_p, Y'\cap Y_p) \in \Lambda_p, \quad p\in P. \tag{5.22}$$

On the other hand, the latter condition (5.21) can be rephrased in terms of the existence of a linking (in the ordinary sense) in $\hat{G}$. Namely, it follows from Theorem 3.1, due to the algebraic independence of the nonvanishing entries of $T_{J'}$, that (5.21) is equivalent to the existence of a complete linking L from $X\cup Y'$ ($\subset X\cup Y_p$) to $Y\cup X'$ ($\subset Y\cup X_p$) in $\hat{G}$. Since $X'\cap X_p=\partial^- L\cap X_p$ and $Y'\cap Y_p=\partial^+ L\cap Y_p$ in (5.22), the linking L can be regarded as a linking in the bimatroid network $(\hat{G},\{L_p|p\in P\})$.

Thus the theorem is established. □

In this section we have discussed the structural solvability of (1.1) when it contains a number of subsystems, which we have called modules. A module p is described by (5.5) in terms of the input-output relation between X_p and Y_p. It is highly probable in practical situations that the module p itself has an internal structure, consisting of a number of submodules. Then module p is described by a system of equations in the form of (1.1), where $X=X_p$, $Y=Y_p$, and $U=U_p$ denotes the set of internal or local variables which are implicit in (5.5). In other words, we may say the input-output relation (5.5) is obtained after eliminating the variables of U_p, just as (5.7) is obtained from (1.1).

In such a case, structural analysis based on Theorem 5.1 would be particularly advantageous when the modules possess relatively small number of input-output variables while involving a large number of internal variables. When certain types of modules are employed frequently, we can determine the combinatorial structures of those modules in advance and store them in data base in some form or other, as observed in [32] for the case of multiports in electrical networks.

Example 5.1. The assumption (5.4) is essential for Theorem 5.1 to hold. Consider, for example, a system of equations

$$y = au_1 + au_2,$$
$$u_1 = bx,$$
$$u_2 = -bx.$$

$X=\{x\}$, $U=\{u_1,u_2\}$, $Y=\{y\}$, and $\{a,b\}$ is assumed to be algebraically independent over Q. This system is not structurally solvable. If we consider that this system is composed of two modules with $X_2=Y_1=\{u_1,u_2\}$, $X_1=\{x\}$ and $Y_2=\{y\}$, the "generality assumption" GA4 is trivially satisfied, while (5.4) is not.

The generalized representation graph is given by $(\hat{G}=(V,\hat{A};X,Y),\{L_p=(X_p,Y_p,\Lambda_p)\,|\,p=1,2\})$, where $V=X\cup U\cup Y$, $\hat{A}=\emptyset$, $\Lambda_1=\{(\{x\},\{u_k\})\,|\, k=1,2\}$, $\Lambda_2=\{(\{u_k\},\{y\})\,|\, k=1,2\}$. It has a complete linking L from X to Y. In fact, let L consist of three trivial paths $\{x\}$, $\{u_1\}$ and $\{y\}$. Then $\partial^+L = \partial^-L = \{x,u_1,y\}$, and L satisfies (4.12). □

Example 5.2. Consider the following system of equations of the form (1.1) with $X=\{x_1,x_2,x_3\}$, $U=\{u_1,\dots,u_{11}\}$ and $Y=\{y_1,y_2,y_3\}$:

$$\begin{aligned}
y_1 &= f_1(u_4,\ u_5),\\
y_2 &= \alpha u_9 + u_{11},\\
y_3 &= f_3(u_{10}),\\
u_1 &= g_1(x_1,\ u_4),\\
u_2 &= g_2(x_1,\ u_1),\\
u_3 &= g_3(x_1,\ x_2),\\
u_4 &= u_1 + u_2,\\
u_5 &= u_1 + u_2 - u_3,\\
u_6 &= 2u_3,\\
u_7 &= -u_3,\\
u_8 &= g_8(x_3,\ u_3,\ u_{10}),\\
u_9 &= g_9(u_7,\ u_8),\\
u_{10} &= u_9 - u_{11},\\
u_{11} &= g_{11}(u_6,\ u_7),
\end{aligned} \tag{5.23}$$

where α is a nonzero constant. We suppose the system contains two modules; one with input variables $X_1=\{u_1,u_2,u_3\}$ and output variables $Y_1=\{u_4,u_5,u_6,u_7\}$, while the other with $X_2=\{u_9,u_{11}\}$ and $Y_2=\{y_2,u_{10}\}$. That is, among the equations (5.23), module 1 is described by

$$\begin{aligned}
u_4 &= u_1 + u_2,\\
u_5 &= u_1 + u_2 - u_3,\\
u_6 &= 2u_3,\\
u_7 &= -u_3,
\end{aligned} \tag{5.24}$$

and module 2 is by

$$\begin{aligned}
y_2 &= \alpha u_9 + u_{11},\\
u_{10} &= u_9 - u_{11}.
\end{aligned} \tag{5.25}$$

The representation graph $G=(V,A;X,Y)$ in the sense of §3 and the underlying graph $\hat{G}=(V,\hat{A};X,Y)$ of the generalized representation graph $(\hat{G},\{L_1,L_2\})$ are depicted in Fig.5.1, where the arcs of $A-\hat{A}$ are drawn in broken lines. The bimatroids L_1 and L_2 are defined by the matrices as

$$L_1 \sim \begin{array}{c|ccc|} & u_1 & u_2 & u_3 \\ \hline u_4 & 1 & 1 & 0 \\ u_5 & 1 & 1 & -1 \\ u_6 & 0 & 0 & 2 \\ u_7 & 0 & 0 & -1 \\ \hline \end{array}\ , \qquad L_2 \sim \begin{array}{c|cc|} & u_9 & u_{11} \\ \hline y_2 & \alpha & 1 \\ u_{10} & 1 & -1 \\ \hline \end{array}\ . \tag{5.26}$$

Let $N=(\tilde{V},\tilde{A},c;s^+,s^-)$ and $\hat{N}=(\tilde{V},\hat{A}^*,\hat{c};s^+,s^-)$ be the networks associated respectively with G and $\hat{G}$ as (3.3) and (3.5). They are shown in Fig.5.2, where the arcs of $\tilde{A}-\hat{A}^*$ are drawn in broken lines. The linkings in $(\hat{G},\{L_1,L_2\})$ can be treated as flows in the bimatroid network $(\hat{N},\{\hat{L}_1,\hat{L}_2\})$, where, for $p=1,2$, $\hat{L}_p=(\hat{X}_p,\hat{Y}_p,\hat{\Lambda}_p)$ ($\hat{X}_p \subset X_* \cup U_*$, $\hat{Y}_p \subset Y^* \cup U^*$) is the bi-polymatroid corresponding to $L_p=(X_p,Y_p,\Lambda_p)$.

If $\alpha=-1$ in module 2, the bimatroid network $(\hat{N},\{\hat{L}_1,\hat{L}_2\})$ has the minimum cut $\hat{W}=\{s^+\}\cup X_* \cup (U_*-\{u_*^{10}\}) \cup (U^*-\{u_{10}^*\}) \cup \{y_1^*\}$ with $\hat{\kappa}(\hat{W})=2$, where $\hat{\kappa}$ is the cut function (4.8) of $(\hat{N},\{\hat{L}_1,\hat{L}_2\})$. By Theorem 4.1, this means that no complete linking exists from X to Y in $(\hat{G},\{L_1,L_2\})$. Hence, (5.23) is not structurally solvable.

Suppose $\alpha \neq -1$, and accept GA4. Then $(\hat{G},\{L_1,L_2\})$ has a complete linking from X to Y and therefore the system (5.23) is structurally solvable by Theorem 5.1. □

6. Hierarchical System of Equations

It has been mentioned at the end of §4 that we can use bimatroid networks for modeling the hierarchical structure of a physical/engineering system in general by virtue of the facts stated in Theorem 4.1 and its corollaries. In the following, we shall explain how this general approach can be embodied for the specific problem of the structural solvability of systems of equations. We consider the situation where a system is composed of modules and each module is made up of submodules.

The following example will illustrate the notations to be introduced subsequently.

Example 6.1. Recall Example 5.2. We label system (5.23) as $S(0)$ and put $X(0)=\{x_1,x_2,x_3\}$, $U(0)=\{u_1,\dots,u_{11}\}$ and $Y(0)=\{y_1,y_2,y_3\}$. $S(0)$ contains two modules indexed by $P(0)=\{p_1,p_2\}$ and described respectively by the input-output relations (5.24) and (5.25); we put $X(1,p_1)=\{u_1,u_2,u_3\}$, $Y(1,p_1)=\{u_4,u_5,u_6,u_7\}$, $X(1,p_2)=\{u_9,u_{11}\}$, $Y(1,p_2)=\{y_2,u_{10}\}$.

Let us suppose, for example, that (5.24) and (5.25) are obtained respectively from the following systems $S(1,p_1)$ and $S(1,p_2)$ by eliminating the variables in $U(1,p_1)$ and $U(1,p_2)$:

$$
\begin{aligned}
S(1,p_1):\ u_4 &= f_4(u_{12}),\\
u_5 &= f_5(u_{12},\ u_{13}),\\
u_6 &= f_6(u_{14},\ u_{15}),\\
u_7 &= f_7(u_{14}),\\
u_{12} &= g_{12}(u_1,\ u_2),
\end{aligned}
$$

$$u_{13} = g_{13}(u_{12},\ u_{14}),$$
$$u_{14} = g_{14}(u_3,\ u_{15}),$$
$$u_{15} = g_{15}(u_{16}),$$
$$u_{16} = g_{16}(u_3,\ u_{14}),$$

where $U(1,p_1)=\{u_{12},\ldots,u_{16}\}$;

$$S(1,p_2)\colon\ y_2 = f_8(u_{11},\ u_{17}),$$
$$u_{10} = f_{10}(u_9,\ u_{17}),$$
$$u_{17} = g_{17}(u_{11},\ u_9),$$

where $U(1,p_2)=\{u_{17}\}$. We further suppose that $S(1,p_1)$ contains two submodules indexed by $P(1,p_1)=\{q_1,q_2\}$, while $S(1,p_2)$ contains none; q_1 has the input variables $X(2,q_1)=\{u_{14},u_{15}\}$ and the output variable $Y(2,q_1)=\{u_6\}$; similarly $X(2,q_2)=\{u_1,u_2\}$ and $Y(2,q_2)=\{u_{12}\}$.

The whole system $\tilde{S}$ of equations of the form (1.1) with unknown variables $\{x_1,x_2,x_3\}\cup\{u_1,\ldots,u_{17}\}$ and parameters $\{y_1,y_2,y_3\}$ is obtained by replacing in $S(0)$ the equations of (5.24) and (5.25) by $S(1,p_1)$ and $S(1,p_2)$. □

To discuss the hierarchy of modules in general, let us fix the notation as follows. The system of equations of the form (1.1) representing the whole system is denoted by $S(0)$, where X, Y and U in (1.1) are accordingly written as $X(0)$, $Y(0)$ and $U(0)$. The set of modules contained in $S(0)$ is denoted by $P(0)$. Each module $p\in P(0)$ is described again by a system of equations of the form (1.1), which we denote by $S(1,p)$; X, Y and U in (1.1) are written as $X(1,p)$, $Y(1,p)$ and $U(1,p)$. We assume (5.1) to (5.4), where we understand that $X=X(0)$, $Y=Y(0)$, $U=U(0)$, $X_p=X(1,p)$, $Y_p=Y(1,p)$. In addition, we have

$$U(1,p)\cap(X(0)\cup U(0)\cup Y(0))=\emptyset, \quad p\in P(0),$$

and (6.1)

$$U(1,p)\cap U(1,q)=\emptyset \quad \text{if } p\neq q \ (\in P(0)),$$

since $U(1,p)$ is local to module $p\in P(0)$. The set of submodules contained in $S(1,p)$ is designated by $P(1,p)$. (Evidently, $P(0)\cap P(1,p)=\emptyset$ for $p\in P(0)$, and $P(1,p)\cap P(1,q)=\emptyset$ if $p\neq q$ ($\in P(0)$).) Each submodule $q\in P(1,p)$ is assumed to be expressed by a system, say $S(2,q)$, of equations of the form (5.5) that describes the input-output relation between the input variables $X(2,q)$ ($\subset X(1,p)\cup U(1,p)$) and the output variables $Y(2,q)$ ($\subset Y(1,p)\cup U(1,p)$), although it is possible to go further into the internal structure of the submodule q by considering a system of the form (1.1) for it. (5.2) to (5.4) are assumed again with $X=X(1,p)$, $Y=Y(1,p)$, $U=U(1,p)$, $X_p=X(2,q)$ and $Y_p=Y(2,q)$.

It should be emphasized here that a module p ($\in P(0)$) is recognized as a set of relations between $X(1,p)$ ($\subset X(0)\cup U(0)$) and $Y(1,p)$ ($\subset Y(0)\cup U(0)$) in the system $S(0)$ of equations, whereas in $S(1,p)$ it is given a full description involving the internal variables $U(1,p)$. The equations for p in $S(0)$ are obtained from $S(1,p)$ by eliminating the variables of $U(1,p)$.

In accordance with the hierarchy of modules, we may consider the hierarchy of the underlying subfields of **F**. For $p\in P(0)$ and $q\in P(1,p)$, let $D(2,q)$ denote the collection of the partial derivatives of the equations in $S(2,q)$ with respect to the variables of $X(2,q)$; we put

$$D_{P(1,p)} = \bigcup_{q\in P(1,p)} D(2,q), \quad p\in P(0) \tag{6.2}$$

as in (5.6). $D(2,q)$ is contained in a subfield of F, say $F_{2,q}$; the subfield of F generated by $\{F_{2,q} \mid q\in P(1,p)\}$ will be denoted as $F_{P(1,p)}$.

In a similar manner, let $D(1,p)$ mean the collection of the partial derivatives of the equations in $S(1,p)$ with respect to the variables of $X(1,p) \cup U(1,p)$. We have

$$F_{1,p} \supset D(1,p) \supset D_{P(1,p)}, \tag{6.3}$$

where $F_{1,p}$ is a subfield of F generated by $D(1,p)$, and define $F_{P(0)}$ to be the subfield generated by $\{F_{1,p} | \; p \in P(0)\}$. By the construction, we have

$$F \supset F_{P(0)} \supset F_{1,p} \supset F_{P(1,p)} \supset F_{2,q}, \quad p \in P(0),\ q \in P(1,p). \tag{6.4}$$

Consider the system $S(0)$. The collection of the partial derivatives of the equations with respect to the variables of $X(0) \cup U(0)$ will be denoted by $D(0)$. We put

$$D_{P(0)} = \bigcup_{p \in P(0)} D_p, \tag{6.5}$$

where D_p means the subset of $D(0)$ corresponding to module $p \in P(0)$. Note the difference between D_p and $D(1,p)$. An element of D_p is obtained from $D(1,p)$ according to a formula similar to (5.8). Therefore $D_p \subset F_{1,p}$ and

$$D_{P(0)} \subset F_{P(0)}. \tag{6.6}$$

Now suppose we are interested in the structural consistency of the whole system. In other words, we are concerned with the structural solvability of the system $\tilde{S}$ of equations which is obtained from $S(0)$ by replacing the equations for the modules $p \in P(0)$ with their full descriptions $S(1,p)$, $p \in P(0)$. $\tilde{S}$ is a system of equations in the form (1.1) with $X(0) \cup U(0) \cup \{U(1,p) | p \in P(0)\}$ as unknown variables and $Y(0)$ as parameters. Our objective is to point out that the structural solvability of $\tilde{S}$ can be tested by applying Theorem 5.1(2) to $S(0)$, in which the modules $p \in P(0)$ are treated as bimatroids.

Here we assume GA4 at each level of hierarchy. That is, we assume

(i) The collection of the nonvanishing elements of $D(0)-D_{P(0)}$ $(\subset F)$ is algebraically independent over $F_{P(0)}$, (6.7)

and

(ii) For each $p\in P(0)$, the collection of the nonvanishing elements of $D(1,p)-D_{P(1,p)}$ $(\subset F_{1,p})$ is algebraically independent over $F_{P(1,p)}$. (6.8)

The former (6.7) assumes GA4 for $S(0)$, in which modules $p\in P(0)$ are described in terms of the input-output relations. The latter (6.8), on the other hand, assumes GA4 for the individual modules.

It is worth noting that the present assumptions (6.7) and (6.8) are less restrictive than assuming GA4 for the whole system $\tilde{S}$. If GA4 were stated for $\tilde{S}$, it would read as follows:

The collection of the nonvanishing elements of

$$[D(0)-D_{P(0)}]\cup[\bigcup_{p\in P(0)}(D(1,p)-D_{P(1,p)})] \ (\subset F)$$

is algebraically independent over $F_{P(1)}$, (6.9)

where $F_{P(1)}$ is the subfield of F generated by $\{F_{P(1,p)} \mid p\in P(0)\}$. It is easy to see (6.9) implies (6.7) and (6.8). The converse, however, is not true, since (6.8) does not guarantee the algebraic independence among elements chosen from $D(1,p)-D_{P(1,p)}$ with different modules $p\in P(0)$.

The structural solvability of the whole system $\tilde{S}$ can be checked in two stages as follows. First we summarize the structure of each module $p\in P(0)$ into the bimatroid $(X(1,p),Y(1,p),\Lambda(1,p))$ applying Corollary 4.4 and Theorem 5.1(1) to the generalized representation graph of $S(1,p)$. As a result the internal variables $U(1,p)$ will no longer be involved. Then we apply Theorem 5.1(2) to S(0) to test for the structural solvability of S(0) using the information about the modules $p \in P(0)$ expressed in the language of bimatroids. Such two-stage method is justified by the theorem below. We put

$$X(1) = \bigcup_{p \in P(0)} X(1,p), \quad Y(1) = \bigcup_{p \in P(0)} Y(1,p), \quad U(1) = \bigcup_{p \in P(0)} U(1,p),$$

and define a bimatroid network $(\hat{G},\{L_p | p \in P(0)\})$ as follows:

$$\hat{G} = (X(0) \cup U(0) \cup Y(0), \hat{A}; X(0), Y(0)),$$

where

$$\hat{A} = \cup\{\delta^- y_i |\ y_i \in Y(0)-Y(1)\} \cup \{\delta^- u_k |\ u_k \in U(0)-Y(1)\}$$

($\delta^- y_i$ and $\delta^- u_k$ are defined by (3.2) with respect to S(0));

$$L_p = (X(1,p), Y(1,p), \Lambda(1,p))$$

is the bimatroid defined (as in Corollary 4.4) by the linkings in the generalized representation graph of S(1,p). Note that $\hat{G}$ agrees with the underlying graph of the generalized representation graph of S(0). Moreover, the proof of the theorem reveals that $(\hat{G},\{L_p | p \in P(0)\})$ coincides with the generalized representation graph of S(0).

Theorem 6.1. Suppose (6.7) and (6.8) hold. (We assume (5.2), (5.3) and (5.4) for each of the systems S(0) and S(1,p), $p \in P(0)$; (6.1) is also assumed). Then $\tilde{S}$ is structurally solvable iff there exists a Menger-type complete linking from X(0) to Y(0) in the bimatroid network $(\hat{G},\{L_p | p \in P(0)\})$ defined above.

(Proof) Put

$$X(2,p) = \bigcup_{q \in P(1,p)} X(2,q), \quad Y(2,p) = \bigcup_{q \in P(1,p)} Y(2,q), \quad p \in P(0);$$

$$X(2) = \bigcup_{p \in P(0)} X(2,p), \quad Y(2) = \bigcup_{p \in P(0)} Y(2,p).$$

Just as (5.9) in the proof of Theorem 5.1, it may be assumed that

$$X(1) \cup Y(1) \subset U(0),$$

$$X(2,p) \cup Y(2,p) \subset U(1,p), \quad p \in P(0).$$

Put

$$\hat{U}(0) = U(0) - (X(1) \cup Y(1)),$$

$$\hat{U}(1,p) = U(1,p) - (X(2,p) \cup Y(2,p)),$$

$$\hat{U}(1) = U(1) - (X(2) \cup Y(2)) = \bigcup_{p \in P(0)} \hat{U}(1,p).$$

Furthermore, we may forbid an input variable of a module to appear elsewhere on the right-hand side (cf. (5.12)). Then the Jacobian matrix $\tilde{J}$ of $\tilde{S}$ with respect to the variables of $X(0) \cup U(0) \cup U(1)$ $(= X(0) \cup X(1) \cup Y(1) \cup X(2) \cup Y(2) \cup \hat{U}(1) \cup \hat{U}(0))$ will look as follows:

$$\tilde{J} = \begin{array}{c|ccccccc|}
 & X(0) & \overbrace{X(1) \quad Y(1) \quad \underbrace{X(2) \quad Y(2) \quad \hat{U}(1)}_{U(1)} \quad \hat{U}(0)}^{U(0) \cup U(1)} \\
\hline
Y(0) & T_1 & 0 & T_2 & 0 & 0 & 0 & T_3 \\
X(1) & T_4 & -I & T_5 & 0 & 0 & 0 & T_6 \\
Y(1) & 0 & M_1 & -I & 0 & M_2 & M_3 & 0 \\
X(2) & 0 & M_4 & 0 & -I & M_5 & M_6 & 0 \\
Y(2) & 0 & 0 & 0 & N & -I & 0 & 0 \\
\hat{U}(1) & 0 & M_7 & 0 & 0 & M_8 & M_9-I & 0 \\
\hat{U}(0) & T_7 & 0 & T_8 & 0 & 0 & 0 & T_9-I \\
\hline
\end{array} \quad . \qquad (6.10)$$

Here N is a matrix over $F_{P(1)}$, M_i's are matrices over $F_{P(0)}$, and T_i's are matrices over F with the nonvanishing entries being collectively algebraically independent over $F_{P(0)}$. We put

$$M = \begin{array}{c|cccc|} & X(1) & X(2) & Y(2) & \hat{U}(1) \\ \hline Y(1) & M_1 & 0 & M_2 & M_3 \\ X(2) & M_4 & -I & M_5 & M_6 \\ Y(2) & 0 & N & -I & 0 \\ \hat{U}(1) & M_7 & 0 & M_8 & M_9-I \\ \hline \end{array} . \tag{6.11}$$

By the same reasoning as in the proof of Theorem 5.1, $\tilde{J}$ can be regarded essentially as a mixed matrix with respect to $F_{P(0)}$. Therefore, $\tilde{J}$ is nonsingular iff there exist $X' \subset X(1)$ and $Y' \subset Y(1)$ such that

the submatrix of M with rows in $Y' \cup U(1)$ and columns in $X' \cup U(1)$ is nonsingular, (6.12)

and that

there exists a complete linking from $X \cup Y'$ ($\subset X \cup Y(1)$) to $Y \cup X'$ ($\subset Y \cup X(1)$) in $\hat{G}$. (6.13)

The matrix M of (6.11) splits into direct factors, each corresponding to a module $p \in P(0)$. That is, M is the direct sum of matrices M_p, $p \in P(0)$, where M_p takes the same form as (6.11), i.e.,

$$M_p = \begin{array}{c} \\ Y(1,p) \\ X(2,p) \\ Y(2,p) \\ \hat{U}(1,p) \end{array} \begin{array}{cccc} X(1,p) & X(2,p) & Y(2,p) & \hat{U}(1,p) \\ \hline M_{p,1} & 0 & M_{p,2} & M_{p,3} \\ M_{p,4} & -I & M_{p,5} & M_{p,6} \\ 0 & N_p & -I & 0 \\ M_{p,7} & 0 & M_{p,8} & M_{p,9}-I \end{array} . \tag{6.14}$$

The condition (6.12) is equivalent to saying that, for each $p \in P(0)$,

the submatrix of M_p with rows in $(Y' \cap Y(1,p)) \cup U(1,p)$ and columns in $(X' \cap X(1,p)) \cup U(1,p)$ is nonsingular. (6.15)

The assumption (6.8) implies that, for each $p \in P(0)$, the nonvanishing entries of $M_{p,i}$'s are algebraically independent over $F_{P(1,p)}$ and M_p can be regarded essentially as a mixed matrix with respect to $F_{P(1,p)}$ (when the identity matrix in the position $\hat{U}(1,p) \times \hat{U}(1,p)$ is modified as before). By applying Theorem 5.1(1) to the subsystem $S(1,p)$ we see that (6.15) is further equivalent to $(X' \cap X(1,p), Y' \cap Y(1,p)) \in \Lambda(1,p)$. □

The two-stage method explained above can be extended to a multistage method in an obvious fashion when a multileveled hierarchy among modules is to be considered.

7. Problem Decomposition

In Theorem 4.1 (and Corollary 4.3) we have observed that the maximum flow of a bimatroid network is characterized by the minimum cut. Taking notice of the submodularity of κ of (4.8) and utilizing the general decomposition principle associated with submodular functions [13], sometimes known as "principal partition", we can obtain a unique decomposition of a bimatroid network into subnetworks. One may be tempted to use the principal partition of κ associated with the generalized representation graph of a system of equations with a view to obtaining a block-triangularization of the system. It seems, however, that this yields too fine a partition of the variables to be useful for block-triangularization (see Example 7.1 below). We shall show that a variant of the M-decomposition instead gives a successful decomposition.

Suppose, as in §5, that we are given a system (1.1) of equations containing modules $p\in P$ of the form (5.5). The generalized representation graph is denoted by $(\hat{G},\{L_p|p\in P\})$, where $\hat{G}=(V,\hat{A};X,Y)$ $(V=X\cup U\cup Y)$ and $L_p=(X_p,Y_p,\Lambda_p)$, $p\in P$. The representation graph of (1.1) in the sense of §3 will be denoted by $G=(V,A;X,Y)$. If we denote by (X_p,Y_p,Δ_p) the underlying bipartite graph of L_p $(p\in P)$, we have

$$A = \hat{A} \cup \{\Delta_p|\ p\in P\}. \tag{7.1}$$

We now consider a decomposition of a bimatroid network $(\hat{G},\{L_p|p\in P\})$ in general. Besides $\hat{G}$ consider a graph $\bar{G}=(V,\bar{A};X,Y)$ such that

$$\bar{A} \supset \hat{A} \cup \{\Delta_p|\ p\in P\}. \tag{7.2}$$

Note that the set of vertices of $\bar{G}$ is identical with that of $\hat{G}$. If we apply the M-decomposition to $\bar{G}$, we obtain a family of subgraphs of $\bar{G}$:

$$\{\bar{G}^{(i)}=(V^{(i)},\bar{A}^{(i)};X^{(i)},Y^{(i)})|\ i\in\{0,\infty\}\cup I\}. \tag{7.3}$$

A partial order is defined among the subgraphs. The family (7.3) induces a family of bimatroid networks:

$$\{(\hat{G}^{(i)},\{L_p^{(i)}\mid p\in P\})\mid i\in\{0,\infty\}\cup I\}, \tag{7.4}$$

where

$$\hat{G}^{(i)}=(V^{(i)},\hat{A}^{(i)};X^{(i)},Y^{(i)}),$$
$$\hat{A}^{(i)}=\bar{A}^{(i)}\cap\hat{A},$$
$$L_p^{(i)}=(X_p^{(i)},Y_p^{(i)},\Lambda_p^{(i)}),$$
$$X_p^{(i)}=V^{(i)}\cap X_p,\quad Y_p^{(i)}=V^{(i)}\cap Y_p,$$
$$\Lambda_p^{(i)}=\{(X',Y')\mid X'\subset X_p^{(i)},\ Y'\subset Y_p^{(i)},\ (X',Y')\in\Lambda_p\}.$$

By Prop. 2.1, the maximum size of a Menger-type linking from X to Y in $\bar{G}$ is not less than that in the bimatroid network $(\hat{G},\{L_p|p\in P\})$. Note also that, by the properties of the M-decomposition, $\bar{G}^{(i)}$ has a complete linking from $X^{(i)}$ to $Y^{(i)}$, for each $i\in I$.

Theorem 7.1. If the maximum size of a Menger-type linking from X to Y in $\bar{G}$ is equal to that in the bimatroid network $(\hat{G},\{L_p|p\in P\})$, then, for each $i\in I$, the bimatroid network $(\hat{G}^{(i)},\{L_p^{(i)}\mid p\in P\})$ has a complete linking from $X^{(i)}$ to $Y^{(i)}$.

(Proof) As mentioned in §4, the linkings in $(\hat{G},\{L_p|p\in P\})$ can be treated as flows in a bimatroid network $(\hat{N},\{\hat{L}_p|p\in P\})$. To be specific, $\hat{N}=(\tilde{V},\hat{A}^*,\hat{c};s^+,s^-)$ is defined for $\hat{G}$ as in (3.3) to (3.5); and $\hat{L}_p=(\hat{X}_p,\hat{Y}_p,\hat{\Lambda}_p)$ $(\hat{X}_p\subset X_*\cup U_*,\ \hat{Y}_p\subset Y^*\cup U^*)$ is the bi-polymatroid corresponding to $L_p=(X_p,Y_p,\Lambda_p)$ in a natural manner. We denote by $\hat{\kappa}$ the cut function (4.8) associated with $(\hat{N},\{\hat{L}_p|p\in P\})$. (Cf. Example 5.2.)

As explained in §3, the M-decomposition of $\bar{G}$ is defined by means of the minimum cuts of the associated network. We denote by

$\bar{N}=(\tilde{V},\bar{A}^*,\bar{c};s^+,s^-)$ the network associated with $\bar{G}$ as (3.3) to (3.5), and by $\bar{\kappa}$ the cut function (3.6) of $\bar{N}$.

Note that $\hat{N}$ and $\bar{N}$ have the same vertex-set $\tilde{V}$, and that $\hat{A}^* \subset \bar{A}^*$. By the construction, we have $\hat{c}(a)=\bar{c}(a)$ for $a\in\hat{A}^*$ and $\hat{\kappa}(W)\leq\bar{\kappa}(W)$ for $W\subset\tilde{V}$. Since

$$\min\{\hat{\kappa}(W) \mid W\subset\tilde{V},\ s^+\in W,\ s^-\notin W\} = \min\{\bar{\kappa}(W) \mid W\subset\tilde{V},\ s^+\in W,\ s^-\notin W\}$$

by assumption, we see

$$L(\bar{\kappa}) \subset L(\hat{\kappa}),$$

where $L(\bar{\kappa})$ and $L(\hat{\kappa})$ are the lattices of the minimizers of $\bar{\kappa}$ and $\hat{\kappa}$, respectively. Then the assertion of the theorem follows from Theorem 4.1. □

As an application of Theorem 7.1 to the block-triangularization of (1.1), we see the following fact, although it could be easily derived independently of Theorem 7.1. Note that when $(\hat{G},\{L_p \mid p\in P\})$ is the generalized representation graph of (1.1), the condition (7.2) reduces to $\bar{A} \supset A$, where A is the arc-set of the representation graph G of (1.1) and expressed as in (7.1).

<u>Corollary 7.2</u>. If (1.1) is structurally solvable and a graph $\bar{G}=(V,\bar{A})$ satisfies $\bar{A}\supset A$, the M-decomposition applied to $\bar{G}$ yields a block-triangularization with structurally solvable subproblems. (No "generality assumption" is involved here.) □

If the M-decomposition is applied to the representation graph $G=(V,A)$, it may split a module into several M-components. In some cases, however, it may be desirable to get a block-triangularization such that a whole module is contained in a single subproblem. For that purpose, we may modify G by adding arcs $\{(x,y)|\ x\in X_p,\ y\in Y_p\}$ for each $p\in P$. The resulting graph $\bar{G}$ satisfies the condition (7.2), since it contains a complete bipartite graph for each module $p\in P$. Corollary 7.2 means that the M-components of $\bar{G}$ correspond to structurally solvable subsystems. It should be noted, however, that such modification can be justified only after the structural solvability is confirmed using the original (generalized) representation graph.

Example 7.1. This is continued from Example 5.2. We assume $\alpha\neq-1$ and accept GA4. Recall that $\hat{\kappa}$ is the cut function (4.8) of $(\hat{N},\{\hat{L}_1,\hat{L}_2\})$. The lattice $L(\hat{\kappa})$ of minimizers of $\hat{\kappa}$ yields a partition of $\tilde{V}-\{s^+,s^-\}$:

$$\hat{P} = \{\tilde{V}_i \mid i=1,\dots,12\}, \tag{7.5}$$

where

$$\begin{aligned}
&\tilde{V}_1 = \{x^3_*,\ u^*_8\},\ \tilde{V}_2 = \{x^2_*,\ u^*_3\},\ \tilde{V}_3 = \{x^1_*,\ u^1_*,\ u^*_1,\ u^2_*,\ u^*_2\},\\
&\tilde{V}_4 = \{y^*_1,\ u^4_*,\ u^*_4,\ u^5_*,\ u^*_5\},\ \tilde{V}_5 = \{u^8_*,\ u^*_9\},\ \tilde{V}_6 = \{u^3_*\},\\
&\tilde{V}_7 = \{u^*_{11},\ u^6_*,\ u^*_6,\ u^7_*,\ u^*_7\},\ \tilde{V}_8 = \{u^{11}_*\},\ \tilde{V}_9 = \{u^9_*\},\\
&\tilde{V}_{10} = \{y^*_2\},\ \tilde{V}_{11} = \{u^*_{10}\},\ \tilde{V}_{12} = \{u^{10}_*,\ y^*_3\}.
\end{aligned} \tag{7.6}$$

The partial order ($\prec$) on $\hat{P}$ is given by the transitive closure of the following:

$$\begin{aligned}
&\tilde{V}_1 \prec \tilde{V}_5 \prec \tilde{V}_9 \prec \tilde{V}_{11} \prec \tilde{V}_{12};\ \tilde{V}_1 \prec \tilde{V}_6;\ \tilde{V}_5 \prec \tilde{V}_7;\ \tilde{V}_8 \prec \tilde{V}_{11};\ \tilde{V}_9 \prec \tilde{V}_{10};\\
&\tilde{V}_2 \prec \tilde{V}_3 \prec \tilde{V}_4 \prec \tilde{V}_7 \prec \tilde{V}_8 \prec \tilde{V}_{10};\ \tilde{V}_2 \prec \tilde{V}_6 \prec \tilde{V}_7.
\end{aligned} \tag{7.7}$$

The minimum cuts of **N**, on the other hand, produces another partition of $\tilde{V}$:

$$P = (\hat{P}-\{\tilde{V}_i \mid i=3,4,6,7,8,9,10,11\})\cup\{\tilde{V}_3\cup\tilde{V}_4,\ \tilde{V}_6\cup\tilde{V}_7,\ \tilde{V}_8\cup\tilde{V}_9\cup\tilde{V}_{10}\cup\tilde{V}_{11}\},$$

which is an aggregation of $\hat{P}$. From this we obtain the M-decomposition $\{G^{(i)}=(V^{(i)},A^{(i)};X^{(i)},Y^{(i)})\mid i\in\{0,\infty\}\cup I\}$ of G, where

$$
\begin{aligned}
&I = \{1,2,3,5,6,8,12\},\\
&V^{(1)} = \{x_3,u_8\},\ X^{(1)}=\{x_3\},\ Y^{(1)}=\{u_8\};\\
&V^{(2)} = \{x_2,u_3\},\ X^{(2)}=\{x_2\},\ Y^{(2)}=\{u_3\};\\
&V^{(3)} = \{x_1,u_1,u_2,u_4,u_5,y_1\},\ X^{(3)}=\{x_1\},\ Y^{(3)}=\{y_1\};\\
&V^{(5)} = \{u_8,u_9\},\ X^{(5)}=\{u_8\},\ Y^{(5)}=\{u_9\};\\
&V^{(6)} = \{u_3,u_6,u_7,u_{11}\},\ X^{(6)}=\{u_3\},\ Y^{(6)}=\{u_{11}\};\\
&V^{(8)} = \{u_9,u_{11},y_2,u_{10}\},\ X^{(8)}=\{u_9,u_{11}\},\ Y^{(8)}=\{y_2,u_{10}\};\\
&V^{(12)} = \{u_{10},y_3\},\ X^{(12)}=\{u_{10}\},\ Y^{(12)}=\{y_3\};
\end{aligned}
$$

and $G^{(0)}$ and $G^{(\infty)}$ are null. The partial order is induced from (7.7) as follows:

$$V^{(1)} \prec V^{(5)} \prec V^{(6)} \prec V^{(8)} \prec V^{(12)};\ V^{(2)} \prec V^{(3)} \prec V^{(6)}.$$

The system (5.23) can be solved by successively solving the 7 structurally solvable subproblems corresponding to the M-components according to the partial order.

It is worth while noting that the partition $\hat{P}$, which is a refinement of P, is too fine for the block-triangularization. For example, the components $\tilde{V}_3$ and $\tilde{V}_4$ do not correspond to solvable subproblems; they are merged into one component in the M-decomposition of G.

Another observation is that the M-decomposition of G splits module 1 into two components, i.e., $V^{(3)}$ and $V^{(6)}$. Such phenomenon can be avoided, if necessary, by considering an augmented graph $\bar{G}=(V,\bar{A};X,Y)$ $(\bar{A} \supset A)$ with a complete bipartite graph for each module. That is, we may put

$$\bar{A} = A \cup \{(u_1,u_6),\ (u_1,u_7),\ (u_2,u_6),\ (u_2,u_7),\ (u_3,u_4)\}. \qquad \square$$

8. Concluding Remarks

In this paper we have discussed the use of "bimatroid network" in mathematical modeling of physical/engineering systems with special reference to the structural solvability of systems of equations. The method developed here will give a framework unifying the equation-based approach and the model approach in systems analysis.

In the field of electrical network theory, matroid-theoretic approach has been proven effective [12], [14], [30], [32], [33], [34]. There exist a number of established methods for discerning the unique solvability of an electrical network. We could apply the method of this paper to the problem of the unique solvability of an electrical network containing multiports, where the terminal characteristics of a multiport is to be summarized as a bimatroid. It seems, however, that this does not result in a novel method of analysis, reducing, at best, to a variant of the existing methods found in the literature.

The "generality assumption" GA4 may or may not be acceptable in practical situations. Without assuming GA4 we can say that the system (1.1) of equations is not structurally solvable if the generalized representation graph does not admit a complete linking. To assert the converse we need a certain further assumption, as demonstrated in Example 5.1. The assumption GA4 is motivated by the physical observation on "locality" of parameter dependency as well as by the mathematical result of [38] to the effect that the rank of a triple matrix product AXB is equal to the rank of the product of the bimatroids corresponding to the matrices A, X and B, provided the nonvanishing entries of X are algebraically independent.

There seem to be a lot of possible applications of the notion of bimatroid network in systems analysis. For example, the combinatorial structure of a dynamical system has been discussed in [25] by means of what can be viewed as a special type of bimatroid network.

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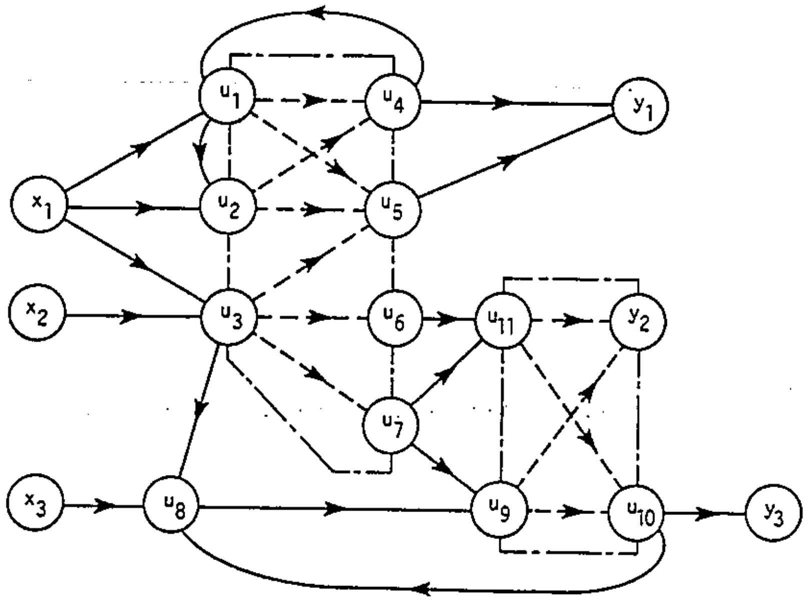

Fig. 5.1. The graphs G and $\hat{G}$ of Example 5.2.

The broken arcs are present in G and not in $\hat{G}$.

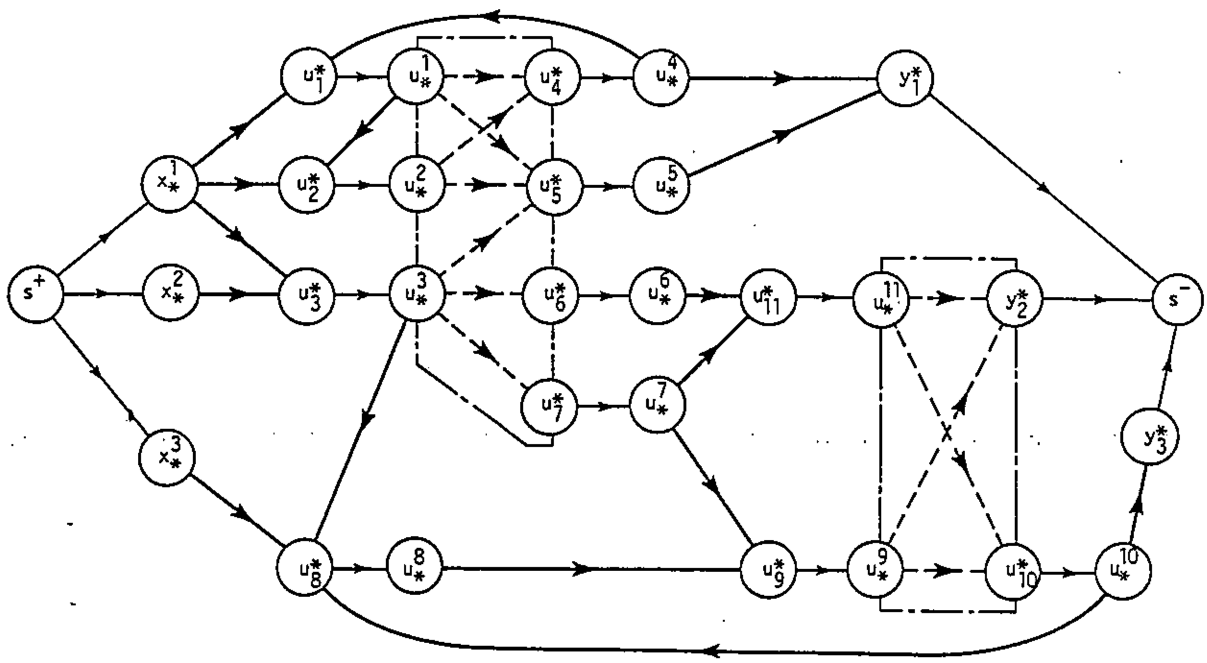

Fig. 5.2. The networks N and $\hat{N}$ of Example 5.2.
The broken arcs are present in N and not in $\hat{N}$.
The bold arcs (⟶) and the broken arcs (- -⟶- -) have infinite capacities, while the thin arcs (⟶) have unit capacities.