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Combinatorial Dynamical System Theory

by

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Abstract

A combinatorial analogue of the dynamical system theory is developed. The combinatorial dynamical system is described by a combinatorial analogue of the state-space equation $x_{k+1} = A\,x_k + B\,u_k$; the matrices A and B are to be replaced by bimatroids (or linking systems). Related concepts such as controllability are defined and their fundamental properties are investigated.

Key words: matroid, bimatroid (linking system),
dynamical system, controllability, eigenset.

1. Introduction

In the modern control theory initiated by [Kalman 1963], a linear time-invariant dynamical system is conveniently described in the state-space equation:

$$\begin{aligned} dx/dt &= A\,x + B\,u \\ y &= C\,x \end{aligned} \tag{1.1}$$

for a continuous-time system, or

$$\begin{aligned} x_{k+1} &= A\,x_k + B\,u_k \\ y_k &= C\,x_k \end{aligned} \tag{1.2}$$

for a discrete-time system. Here x is the state vector, u the control (input) vector, y the output vector, and A, B and C are constant matrices of appropriate sizes.

In this paper we develop a combinatorial analogue of the dynamical system theory. The combinatorial analogue of the state-space equation is obtained by replacing the matrices A, B and C with the combinatorial objects of bimatroids (or linking systems). Several basic notions such as controllability and observability in the conventional dynamical system theory are recast into their combinatorial versions and then some fundamental results in the conventional dynamical system theory are given their combinatorial counterparts.

This work is partly inspired by the recent development of "structural" or "generic" arguments in the control theory. The structural approach to controllability started by [Lin 1974], for instance, considers combinatorial characterizations of the controllability (in the ordinary sense) of a dynamical system when the nonvanishing entries of the coefficient matrices A and B of (1.1) or

(1.2) are algebraically independent parameters. Graph-theoretic necessary and sufficient conditions are known for the structural controllability in the sense above and its refinements (see, e.g., [Lin 1974], [Glover-Shilverman 1976], [Shields-Pearson 1976], [Hosoe-Matsumoto 1979], [Maeda 1981], [Aoki-Hosoe-Hayakawa 1983], [Matsumoto-Ikeda 1983], [Hayakawa-Hayashi-Hosoe-Ito 1984], [Yamada-Luenberger 1985], [Murota 1985, 1986]).

It should be emphasized, however, that the notion of controllability to be introduced in this paper for the combinatorial dynamical system is different from that of the structural controllability. From the mathematical point of view, therefore, the combinatorial dynamical system theory of the present paper and the existing theory of structural controllability are independent, although both aim at capturing the structural aspects of a dynamical system from the combinatorial standpoint.

The present theory will be developed within the matroid-theoretic combinatorial framework without direct reference to the conventional dynamical system theory. However, some familiarity with the conventional dynamical system theory (e.g., [Kailath 1980], [Rosenbrock 1970], [Wolovich 1974], [Wonham 1979]) as well as with the results on structural controllability and related topics would be helpful in understanding the intuition behind the combinatorial notions introduced in this paper.

In the state-space equation (1.1) or (1.2), the matrices A, B and C behave differently under the change of the coordinate system of the state space, to which vector x belongs. That is, if the system is described in terms of $\tilde{x} = Tx$, the corresponding matrices, say $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, will be given by

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}.$$

This shows, in particular, that A is subject to similarity transformations and therefore it is meaningful to think of such concepts as the power product, the eigenvectors, and the Jordan canonical form, for the matrix A; such concepts are meaningless for B and C.

Such differences in the nature of the matrices are carried over in the combinatorial formulation of a dynamical system in terms of bimatroids. To be more concrete, the bimatroid which takes the place of the matrix A has an identical ground set both for the row set (exit set) and the column set (entrance set). For such a bimatroid as has a natural one-to-one correspondence between the row set and the column set, we can consider the power product and define an "eigenset", which is a combinatorial analogue of an eigenvector. The maximum size of an eigenset of a bimatroid will be characterized by the rank of its power product.

The contents of this paper may be outlined as follows.

In §2, basic results on matroids and bimatroids are summarized for later references.

The combinatorial analogue of a dynamical system is defined in §3 together with the related notions such as controllability.

Then in §4, the behavior of a system without inputs, which we call a free system, is considered. Specifically, some fundamental properties are shown concerning the power products and the eigensets of a bimatroid which has a natural one-to-one correspondence between the row set and the column set.

Section 5 treats the bimatroid arising from a combinatorial dynamical system in connection with controllability. It will be shown that some well-known facts about the controllability of the conventional dynamical system (1.1) or (1.2) have natural combinatorial counterparts. In particular, a criterion of controllability is established.

Another criterion of controllability is given in §6, which may be thought of as a counterpart of the structural controllability theorem of [Lin 1974], [Glover-Shilverman 1976], [Shields-Pearson 1976].

Finally in §7, some possible extensions are discussed; among others an attempt is suggested to define a combinatorial analogue of Kalman's canonical decomposition [Kalman 1962, 1963] of state space into four parts with respect to controllability and observability.

2. Mathematical Preliminaries

2.1. Matroid

The purpose of this subsection is chiefly to fix the notations. See, e.g., [Welsh 1976] for the precise definitions of matroid-theoretic concepts.

Let $M=(E,\mu)$ be a matroid defined on finite ground set E with rank function μ. The rank of M will be denoted by $r(M)$. For $X \subset E$, $M-X$ denotes the matroid obtained from M by deleting X, while M/X the matroid obtained from M by contracting X. We also say that $M-X$ is the restriction of M to $E-X$ and M/X the contraction of M to $E-X$; M/X is also written as $M\times(E-X)$. For disjoint subsets X and Y ($\subset E$), the deletion of X and the contraction of Y commute: $(M-X)/Y = (M/Y)-X$. The dual of M will be denoted by M^*.

Let $M_i=(E_i,\mu_i)$ $(i=1,2)$ be matroids. The union of M_1 and M_2, denoted as $M_1 \vee M_2$, is a matroid on $E=E_1\cup E_2$ with rank function $\mu_1\vee\mu_2$ given by

$$(\mu_1\vee\mu_2)(X) = \min\{\mu_1(Y\cap E_1)+\mu_2(Y\cap E_2)+|X-Y| \mid Y\subset X\}, \quad X\subset E. \tag{2.1}$$

The following relations are mentioned here:

$$(M_1-(X\cap E_1))\vee(M_2-(X\cap E_2)) = (M_1\vee M_2)-X, \quad X\subset E, \tag{2.2}$$

and if $X_1 \subset E_1-E_2$ and $X_2 \subset E_2-E_1$, then

$$(M_1/X_1)\vee(M_2/X_2) = (M_1\vee M_2)/(X_1\cup X_2). \tag{2.3}$$

Assume $E_1=E_2$ and further suppose a weight function $w\colon E\to Z_+$ is given. For $X \subset E$, $w(X)$ means $\Sigma\{w(x) \mid x\in X\}$. The following is the fundamental min-max relation for the matroid intersection problem.

Proposition 2.1 ([Edmonds 1970]).

$$\begin{aligned}&\max\{w(X) \mid X\subset E,\ X \text{ is independent in } M_i\ (i=1,2)\}\\ &=\min\{\sum_{i=0}^{p}\mu_1(I_i)+\sum_{j=0}^{q}\mu_2(J_j) \mid I_i\subset E,\ J_j\subset E,\\ &\qquad |\{i \mid 0\leqq i\leqq p,\ e\in I_i\}|+|\{j \mid 0\leqq i\leqq q,\ e\in J_j\}|=w(e) \text{ for } e\in E\} \qquad (2.4)\\ &=\min\{\sum_{i=0}^{p}\mu_1(I_i)+\sum_{j=0}^{q}\mu_2(J_j) \mid I_i\subset E,\ J_j\subset E,\\ &\qquad |\{i \mid 0\leqq i\leqq p,\ e\in I_i\}|+|\{j \mid 0\leqq i\leqq q,\ e\in J_j\}|=w(e) \text{ for } e\in E,\\ &\qquad I_0\supset I_1\supset\cdots\supset I_p,\ J_0\supset J_1\supset\cdots\supset J_q\}. \qquad (2.5)\end{aligned}$$

□

Let σ_i be the closure function of $M_i=(E,\mu_i)$ $(i=1,2)$. M_2 is said [Kung 1986] to be a quotient of M_1, to be denoted as $M_1 \rightarrow M_2$ or $M_2 \leftarrow M_1$, iff

$$\mu_1(X)-\mu_1(Y) \geqq \mu_2(X)-\mu_2(Y), \quad Y\subset X\subset E. \qquad (2.6)$$

This is equivalent to

$$\sigma_1(X) \subset \sigma_2(X), \quad X\subset E. \qquad (2.7)$$

It is also said that $M_1 \rightarrow M_2$ is a strong map.

Proposition 2.2. (1) If $M_1 \rightarrow M_2$ and $r(M_1) = r(M_2)$, then $M_1 = M_2$.

(2) ([Kung 1986]) $M_1 \vee M_2 \rightarrow M_1$.

(Proof) (1) Putting $Y=\emptyset$ in (2.6), we obtain

$$\mu_1(X) \geqq \mu_2(X), \quad X\subset E.$$

On the other hand, it follows from (2.6) with $X=E$ and $\mu_1(E)=\mu_2(E)$ that

$$\mu_1(Y) \leqq \mu_2(Y), \quad Y\subset E.$$

□

2.2. Bimatroid

The notion of bimatroid was introduced first by [Schrijver 1978, 1979] under the name of linking system, and shortly later by [Kung 1978] under the name of bimatroid.

A <u>bimatroid</u> (or <u>linking system</u>) is a triple $L=(S,T,\Lambda)$, where S and T are finite sets, and Λ is a nonempty subset of $2^S\times 2^T$ such that

(L1) if $(X,Y)\in\Lambda$, then $|X|=|Y|$;

(L2-1) if $(X,Y)\in\Lambda$ and $X'\subset X$, then $(X',Y')\in\Lambda$ for some $Y'\subset Y$;

(L2-2) if $(X,Y)\in\Lambda$ and $Y'\subset Y$, then $(X',Y')\in\Lambda$ for some $X'\subset X$;

(L3) if $(X_i,Y_i)\in\Lambda$ $(i=1,2)$, then $(X,Y)\in\Lambda$, $X_1\subset X\subset X_1\cup X_2$, $Y_2\subset Y\subset Y_1\cup Y_2$ for some $X\subset S$ and $Y\subset T$.

We call S the <u>row set</u> (or <u>exit set</u>) and T the <u>column set</u> (or <u>entrance set</u>) of L; we write $S = \mathrm{Row}(L)$ and $T = \mathrm{Col}(L)$. A member (X,Y) of $\Lambda=\Lambda(L)$ is called a <u>linked pair</u>; we also say that X and Y are <u>linked</u>.

The <u>birank function</u> (or <u>linking function</u>) λ: $2^S\times 2^T\to Z_+$ is defined by

$$\lambda(X,Y) = \max\{|X'| \mid (X',Y')\in\Lambda,\ X'\subset X,\ Y'\subset Y\}, \quad X\subset S,\ Y\subset T. \tag{2.8}$$

Obviously,

$$(X,Y) \in \Lambda \text{ iff } \lambda(X,Y) = |X| = |Y|. \tag{2.9}$$

With this correspondence, we may equivalently say that a bimatroid L is a triple (S,T,λ), where λ satisfies the following:

(B1) $0 \le \lambda(X,Y) \le \min\{|X|,|Y|\}$, $X\subset S$, $Y\subset T$;

(B2) $\lambda(X',Y') \le \lambda(X,Y)$, $X'\subset X\subset S$, $Y'\subset Y\subset T$;

(B3) $\lambda(X\cup X',Y\cap Y')+\lambda(X\cap X',Y\cup Y') \le \lambda(X,Y)+\lambda(X',Y')$, $X,X'\subset S$, $Y,Y'\subset T$.

The property (B3) is referred to as <u>bisubmodularity</u>.

By the <u>rank</u> $r(L)$ of L, we mean the maximum size of a linked pair, i.e., $r(L) = \lambda(S,T)$. L is called a <u>trivial bimatroid</u> if $r(L) = 0$.

The underlying bipartite graph (S,T,Δ) of $L=(S,T,\Lambda)$ is a bipartite graph with vertex set $S\cup T$ (disjoint union) and edge set $\Delta \subset 2^S \times 2^T$ such that

$$(x,y) \in \Delta \text{ iff } (\{x\},\{y\}) \in \Lambda. \tag{2.10}$$

Proposition 2.3 ([Schrijver 1978, 1979]). Suppose $(X,Y) \in \Lambda$. For any $Y' \subset Y$ there exists $X' \subset X$ such that $(X',Y')\in\Lambda$ and $(X-X',Y-Y')\in\Lambda$. In particular, there exists a matching between X and Y in the underlying bipartite graph (S,T,Δ). □

A bimatroid $L=(S,T,\lambda)$ determines a matroid $M(L)=(S\cup T,\lambda^{\pm})$ with ground set $S\cup T$ (disjoint union) and rank function $\lambda^{\pm}$ defined by

$$\lambda^{\pm}(X\cup Y) = \lambda(X,T-Y)+|Y|, \quad X\subset S,\ Y\subset T. \tag{2.11}$$

Note that $T=\mathrm{Col}(L)$ is a base of $M(L)$. The restriction of $M(L)$ to $S=\mathrm{Row}(L)$ is named the row matroid of L and denoted by $RM(L)=(S,\lambda^-)$, where

$$\lambda^-(X) = \lambda(X,T), \quad X\subset S.$$

Similarly, the restriction of $M(L)^*$ to $T=\mathrm{Col}(L)$ is the column matroid of L and denoted by $CM(L)=(T,\lambda^+)$, where

$$\lambda^+(Y) = \lambda(S,Y), \quad Y\subset T.$$

Obviously, we have

$$r(L) = r(RM(L)) = r(CM(L)).$$

For $X \subset S$ and $Y \subset T$, $L[X,Y]$ will mean a bimatroid such that $\mathrm{Row}(L[X,Y])=X$, $\mathrm{Col}(L[X,Y])=Y$, $\Lambda(L[X,Y])=\Lambda'$, where

$$\Lambda' = \{(X',Y') \mid X'\subset X,\ Y'\subset Y,\ (X',Y')\in\Lambda(L)\}.$$

In parallel with the multiplication of a matrix with a vector, we write

$$L\cdot Y = \{X\subset S \mid (X,Y)\in\Lambda(L)\}, \quad Y\subset T. \tag{2.12}$$

Note that $L\cdot Y$ agrees with the family of bases of $RM(L[S,Y])$ if Y is independent in $CM(L)$, and that $L\cdot Y = \emptyset$ if not.

The dual of L, denoted by L^*, is a bimatroid such that $Row(L^*)=Col(L)$, $Col(L^*)=Row(L)$, $\Lambda(L^*)=\Lambda^*$, where

$$\Lambda^* = \{(Y,X) \mid (X,Y)\in\Lambda(L)\}.$$

If $(S,T)\in\Lambda$, L is said to be nonsingular. Then the inverse of L, denoted by L^{-1}, is defined to be a bimatroid such that $Row(L^{-1})=Col(L)$, $Col(L^{-1})=Row(L)$, $\Lambda(L^{-1})=\Lambda^{-1}$, where

$$\Lambda^{-1} = \{(Y,X) \mid (S-X,T-Y)\in\Lambda(L)\}.$$

Let $L_i=(S_i,T_i,\Lambda_i)$ $(i=1,2,3)$ be bimatroids. The union of L_1 and L_2, denoted by $L_1\vee L_2$, is a bimatroid such that $Row(L_1\vee L_2)=Row(L_1)\cup Row(L_2)$, $Col(L_1\vee L_2)=Col(L_1)\cup Col(L_2)$, $\Lambda(L_1\vee L_2)=\Lambda_1\vee\Lambda_2$, where

$$\Lambda_1\vee\Lambda_2 = \{(X_1\cup X_2,Y_1\cup Y_2) \mid X_1\cap X_2=\emptyset,\ Y_1\cap Y_2=\emptyset,\ (X_1,Y_1)\in\Lambda_1,\ (X_2,Y_2)\in\Lambda_2\}. \tag{2.13}$$

That is, $L_1\vee L_2=(S_1\cup S_2,T_1\cup T_2,\Lambda_1\vee\Lambda_2)$. The birank function $\lambda_1\vee\lambda_2$ of $L_1\vee L_2$ is given by

$$(\lambda_1\vee\lambda_2)(X,Y) = \min\{\lambda_1(X',Y')+\lambda_2(X',Y')+|X-X'|+|Y-Y'| \mid X'\subset X,\ Y'\subset Y\},$$
$$X\subset S_1\cup S_2,\ Y\subset T_1\cup T_2. \tag{2.14}$$

If $T_1\cap T_2=\emptyset$, then

$$M(L_1\vee L_2) = M(L_1)\vee M(L_2). \tag{2.15}$$

If $Col(L_1) = Row(L_2)$ (i.e., $T_1=S_2$), the product of L_1 and L_2, denoted by L_1*L_2, can be defined; it is a bimatroid such that $Row(L_1*L_2)=Row(L_1)$, $Col(L_1*L_2)=Col(L_2)$, $\Lambda(L_1*L_2)=\Lambda_1*\Lambda_2$, where

$$\Lambda_1*\Lambda_2 = \{(X,Z) \mid (X,Y)\in\Lambda_1,\ (Y,Z)\in\Lambda_2 \text{ for some } Y\subset T_1\}. \tag{2.16}$$

That is, $L_1*L_2=(S_1,T_2,\Lambda_1*\Lambda_2)$. The birank function $\lambda_1*\lambda_2$ of L_1*L_2 is given by

$$(\lambda_1*\lambda_2)(X,Z) = \min\{\lambda_1(X,T_1-Y)+\lambda_2(Y,Z) \mid Y\subset T_1\}, \quad X\subset S_1,\ Z\subset T_2. \tag{2.17}$$

We see

$$M(L_1*L_2) = (M(L_1)\vee M(L_2))/T_1. \tag{2.18}$$

If $\mathrm{Col}(L_1) = \mathrm{Row}(L_2)$ and $\mathrm{Col}(L_2) = \mathrm{Row}(L_3)$, we have the associative law:

$$(L_1*L_2)*L_3 = L_1*(L_2*L_3),$$

which permits the notation: $L_1*L_2*L_3$.

Suppose a matroid $M=(T,\mu)$ is defined on the column set $T=\mathrm{Col}(L)$ of a bimatroid $L=(S,T,\lambda)$. Then another matroid, denoted by $L*M$, is induced on $S=\mathrm{Row}(L)$. The rank function $\lambda*\mu$ of $L*M$ is given by

$$(\lambda*\mu)(X) = \min\{\lambda(X,T-Y)+\mu(Y) \mid Y\subset T\}, \quad X\subset S. \tag{2.19}$$

The following relation plays important roles later.

Proposition 2.4 ([Kung 1978, 1986]). $L*M$ is a quotient of the row matroid of L:

$$RM(L) \to L*M.$$

Or equivalently,

$$RM(L_1) \to RM(L_1*L_2) \quad \text{and} \quad CM(L_2) \to CM(L_1*L_2)$$

for two bimatroids L_i $(i=1,2)$ such that L_1*L_2 can be defined. □

Suppose there is a one-to-one correspondence ψ: $S\to T$ between $S=\mathrm{Row}(L)$ and $T=\mathrm{Col}(L)$. L is said to be symmetric iff

$$(X,Y) \in \Lambda \quad \text{implies} \quad (\psi^{-1}(Y),\psi(X)) \in \Lambda.$$

A bi-polymatroid (or poly-linking system) is a triple $L=(S,T,\lambda)$, where S and T are finite sets and λ: $2^S \times 2^T \to Z_+$ (or R_+) satisfies (B2), (B3) and

(B1') $\lambda(\emptyset,Y) = \lambda(X,\emptyset) = 0$, $X \subset S$, $Y \subset T$.

As with bimatroids, we call λ the birank function and $\lambda(S,T)=r(L)$ the rank of L. Much of the notions for bimatroids can be naturally generalized for bi-polymatroids. In particular, for two bi-polymatroids $L_i=(S_i,T_i,\lambda_i)$ (i=1,2) with $T_1=S_2$, the product of L_1 and L_2 is a bi-polymatroid $(S_1,T_2,\lambda_1*\lambda_2)$, where $\lambda_1*\lambda_2$ is given by (2.17).

3. Combinatorial Dynamical System

A <u>combinatorial dynamical system</u> is a triple (A,B,C) of bimatroids such that Row(A)=Col(A)=Row(B)=Col(C) (=S) and that S, Col(B) and Row(C) are mutually disjoint. If we write Col(B) = P and Row(C) = T, we have

$$A = (S,S,\Lambda(A)),\quad B = (S,P,\Lambda(B)),\quad C = (T,S,\Lambda(C)). \tag{3.1}$$

The bimatroids A, B and C will be called respectively the <u>transition bimatroid</u>, the <u>input bimatroid</u> and the <u>output bimatroid</u>. The birank functions of A, B and C are denoted by α, β and γ, respectively. The set S is called the <u>state space</u>, whereas P is the <u>input space</u> and T the <u>output space</u>.

Up to §6, we consider a system without outputs, i.e., a system (A,B,C), where C is a trivial bimatroid. In other words, we investigate the properties of a pair (A,B), which is also referred to as a combinatorial dynamical system.

An <u>input</u> (or a <u>control</u>) is a sequence $(U_k|k=0,1,\ldots,K-1)$ such that $U_k \subset P$. We sometimes call K the <u>length</u> of the input.

When given an input $(U_k|k=0,1,\ldots,K-1)$, we say that a sequence $(X_k|k=0,1,\ldots,K)$, $X_k \subset S$, is a <u>trajectory compatible with</u> $(U_k|k=0,1,\ldots,K-1)$ iff

$$(X_{k+1}, X_k \cup U_k) \in \Lambda(A \vee B) \tag{3.2}$$

for $k=0,1,\ldots,K-1$. This condition can be expressed in a form similar to the conventional state-space equation:

$$X_{k+1} \in A \cdot X_k \vee B \cdot U_k \tag{3.3}$$

for $k=0,1,\ldots,K-1$, if we introduce, in addition to (2.12), the notation

$$F_1 \vee F_2 = \{X_1 \cup X_2 |\ X_1 \cap X_2 = \emptyset,\ X_i \in F_i\ (i=1,2)\}$$

for $F_i \subset 2^S$ (i=1,2) in general. The formula (3.3) will be referred to as the state-space equation for the combinatorial dynamical system (A,B).

The underlying digraph of (A,B) is defined to be a directed graph $(S\cup P,\Delta)$ with vertex set $S\cup P$ and arc set

$$\Delta = \{(x',x) \mid (\{x\},\{x'\})\in\Lambda(A)\} \cup \{(u,x) \mid (\{x\},\{u\})\in\Lambda(B)\}. \quad (3.4)$$

$X \subset S$ is said to be reachable at time k from $X_0 \subset S$ iff there exists an input $(U_i \mid i=0,1,\ldots,k-1)$ of length k and a trajectory $(X_i \mid i=0,1,\ldots,k)$ compatible with it such that $X_k=X$. We denote by $RS_k(X_0)$ the collection of reachable sets at time k from X_0, i.e.,

$$RS_k(X_0) = \{X_k\subset S \mid (X_i \mid i=0,1,\ldots,k) \text{ is a trajectory compatible with some input } (U_i \mid i=0,1,\ldots,k-1)\},\ X_0\subset S, k\geq 0. \quad (3.5)$$

$X \subset S$ is said to be reachable from $X_0 \subset S$ iff $X \in RS_k(X_0)$ for some $k\geq 0$. We put

$$RS(X_0) = \cup\{RS_k(X_0) \mid k=0,1,\ldots\}, \quad X_0\subset S, \quad (3.6)$$

and

$$RS_k = \cup\{RS_k(X_0) \mid X_0\subset S\}. \quad (3.7)$$

A system (A,B) is said to be reachable iff $\{x\} \in RS(\emptyset)$ for each $x\in S$. This is equivalent to the condition that for each $x\in S$ there exists a directed path from P to x in the underlying digraph of (A,B).

A system (A,B) is called controllable iff

$$RS(\emptyset) = 2^S. \quad (3.8)$$

An input $(U_k \mid k=0,1,\ldots,K-1)$ will be called admissible iff there exists at least one trajectory compatible with it. An input $(U_k \mid k=0,1,\ldots,K-1)$ is said to be admissible for (X',X) $(X',X \subset S)$ iff there exists a trajectory $(X_k \mid k=0,1,\ldots,K)$ compatible with $(U_k \mid k=0,1,\ldots,K-1)$ such that $X_0=X'$ and $X_K=X$. If $X \in RS(X')$, there exists an input admissible for (X',X).

4. Free Systems

This section deals with a free system in which input bimatroid B is trivial. In other words, we investigate some fundamental properties of a bimatroid A such that Row(A)=Col(A).

4.1. Power products of a bimatroid

Since Row(A)=Col(A), we can think of the product of A with itself in the sense of (2.16). A^k is defined recursively by $A^k = A^{k-1}*A = A*A^{k-1}$ for k=1,2,..., where, for convenience, we put $A^0 = (S,S,\Lambda(A^0))$ with $\Lambda(A^0) = \{(X,X) \mid X\subset S\}$. For a free system, RS_k of (3.7) agrees with the family of independent sets of $RM(A^k)$, which we are going to study.

To investigate the rank of products of bimatroids, the following result would be of fundamental importance. In the light of a similar inequality for matrix products known as the Frobenius inequality, we name it the <u>Frobenius inequality for bi-polymatroids</u>.

<u>Theorem 4.1</u>. (Frobenius inequality for bi-polymatroids). Let L_i (i=1,2,3) be bi-polymatroid such that $L_1*L_2*L_3$ can be defined. Then

$$r(L_1*L_2*L_3)+r(L_2) \geq r(L_1*L_2)+r(L_2*L_3),$$

where r(•) denotes the rank of a bi-polymatroid.

(Proof) Put $L_i=(S_i,T_i,\lambda_i)$ (i=1,2,3), where $T_1=S_2$ and $T_2=S_3$. By (2.17), we have

$$r(L_1*L_2) = \min\{\lambda_1(S_1,T_1-X_1)+\lambda_2(X_1,T_2) \mid X_1\subset T_1\}$$

and

$$r(L_2*L_3) = \min\{\lambda_2(S_2,T_2-X_2)+\lambda_3(X_2,T_3) \mid X_2\subset T_2\}.$$

From these relations as well as from

$$\lambda_2(X_1,T_2)+\lambda_2(S_2,T_2-X_2) \leqq \lambda_2(X_1,T_2-X_2)+\lambda_2(S_2,T_2),$$

which is due to the bisubmodularity of λ_2, it follows that

$$\begin{aligned} & r(L_1 * L_2)+r(L_2 * L_3) \\ & \leqq \min\{\lambda_1(S_1,T_1-X_1)+\lambda_2(X_1,T_2-X_2)+\lambda_3(X_2,T_3) \mid X_1 \subset T_1,\ X_2 \subset T_2\} \\ & \quad + \lambda_2(S_2,T_2) \\ & = r(L_1 * L_2 * L_3)+r(L_2). \end{aligned}$$ □

As an immediate consequence of Theorem 4.1, we obtain

$$r(L_1 * L_2)+n \geqq r(L_1)+r(L_2),$$

where L_1 and L_2 are bi-polymatroids and $n=|\mathrm{Col}(L_1)|=|\mathrm{Row}(L_2)|$. This may be named the Sylvester inequality for bi-polymatroids after the analogous inequality for matrices.

Theorem 4.2.

(1) $$r(A^{k-1})-r(A^k) \geqq r(A^k)-r(A^{k+1}), \quad k=1,2,\ldots.$$

(2) There exists $\tau=\tau(A)$ ($\geqq 0$) such that

$$r(A^0) > r(A^1) > \cdots > r(A^{\tau-1}) > r(A^\tau) = r(A^k), \quad k=\tau+1,\tau+2,\ldots.$$

(Proof) (1) This follows from Theorem 4.1 with $L_1=L_3=A$, $L_2=A^{k-1}$.

(2) First note the obvious relation:

$$r(A^k) \geqq r(A^{k+1}), \quad k=0,1,\ldots.$$

Let τ ($\geqq 0$) be the smallest k such that the equality holds. Then (1) implies that the equality must hold for all $k \geqq \tau$. □

The integer $\tau=\tau(A)$ will be called the time constant of A; evidently $0 \leqq \tau \leqq |S|$. We symbolically write $r(A^\infty)$ for $r(A^\tau)$.

Since

$$r(A^k) = r(RM(A^k)) = r(CM(A^k)),$$

the latter two also satisfy the similar inequalities given in Theorem 4.2 for $r(A^k)$. The following establishes a much stronger assertion for $RM(A^k)$ and $CM(A^k)$ than is implied by the second inequality of Theorem 4.2. For two matroids M_1 and M_2, $M_1 \underset{\neq}{\rightarrow} M_2$ (or $M_2 \underset{\neq}{\leftarrow} M_1$) means that M_2 is a quotient of M_1 and not isomorphic to M_1.

Theorem 4.3. Let τ be the time constant of A. Then

$$RM(A^0) \underset{\neq}{\rightarrow} RM(A^1) \underset{\neq}{\rightarrow} \cdots \underset{\neq}{\rightarrow} RM(A^{\tau-1}) \underset{\neq}{\rightarrow} RM(A^{\tau}) = RM(A^k), \quad k=\tau+1,\tau+2,\ldots,$$

and

$$CM(A^0) \underset{\neq}{\rightarrow} CM(A^1) \underset{\neq}{\rightarrow} \cdots \underset{\neq}{\rightarrow} CM(A^{\tau-1}) \underset{\neq}{\rightarrow} CM(A^{\tau}) = CM(A^k), \quad k=\tau+1,\tau+2,\ldots.$$

(Proof) It follows from Prop.2.4 (with $L_1=A^k$ and $L_2=A$, and with $L_1=A$ and $L_2=A^k$) that $RM(A^k) \rightarrow RM(A^{k+1})$ and $CM(A^k) \rightarrow CM(A^{k+1})$. Combining these with Theorem 4.2(2) and Prop.2.2(1), we establish the theorem. □

We shall adopt the notation $RM(A^{\infty})=RM(A^{\tau})$ and $CM(A^{\infty})=CM(A^{\tau})$.

Theorem 4.2(1) allows us to define a set of characteristic indices for a bimatroid A such that Row(A)=Col(A). For a bimatroid arising from a matrix, the indices to be introduced are closely related to the Jordan canonical form of the matrix. Thus the indices might be regarded as a combinatorial characteristic of a bimatroid that corresponds to the Jordan type of a matrix.

To explain the correspondence that motivates our definition, we first consider a bimatroid A arising from a bipartite graph. (Such a bimatroid is called a deltoid in [Schrijver 1978, 1979].) In other words, A is assumed to come from a matrix, say $\bar{A}$, of which the nonzero entries are algebraically independent transcendentals.

As will be seen without difficulty, $\bar{A}$ has no nonzero multiple eigenvalues; every Jordan cell with a nonzero eigenvalue is of size 1. The number of nonzero eigenvalues of $\bar{A}$ is obviously equal to $\bar{\omega}_0=r(\bar{A}^k)$ for k sufficiently large, where r(•) denotes the rank of a matrix. On the other hand, the number of the Jordan cells of size k (≧1) with eigenvalue 0 is given by

$$\bar{\omega}_k = r(\bar{A}^{k+1}) + r(\bar{A}^{k-1}) - 2r(\bar{A}^k).$$

In this way, the Jordan type of $\bar{A}$ is completely characterized by the set of numbers $(\bar{\omega}_0;\bar{\omega}_1,\bar{\omega}_2,\ldots)$.

Based on the above observations, we shall define the <u>Jordan type</u> for a bimatroid $A=(S,S,\Lambda(A))$ as the set of numbers $(\omega_0;\omega_1,\omega_2,\ldots)$, where

$$\begin{aligned} &\omega_0=r(A^\infty), \\ &\omega_k = r(A^{k+1}) + r(A^{k-1}) - 2r(A^k), \quad k=1,2,\ldots. \end{aligned} \tag{4.1}$$

Note that Theorem 4.2 guarantees that $\omega_k \geqq 0$ for $k \geqq 1$ and $\omega_k=0$ for $k>\tau$. Obviously, we have

$$\sum_{k=0} \omega_k = |S|.$$

4.2. Eigensets of a bimatroid

In this subsection, we introduce two novel notions, eigenset and recurrent set, for a bimatroid $A=(S,S,\Lambda(A))$.

A subset $X \subset S$ is called an <u>eigenset</u> of A iff $(X,X) \in \Lambda(A)$. Using the notation (3.5) for the free system, we may alternatively say that X is an eigenset iff $X \in RS_1(X)$.

A subset $X \subset S$ is called a <u>recurrent set</u> of A iff $(X,X) \in \Lambda(A^k)$ for some $k \geq 1$. ~~It is easy to see that X is a recurrent set iff $RS_k(X) \neq \emptyset$ for all k~~.

By definition, an eigenset is a recurrent set; however, the converse is not true. It will be shown later in Theorem 4.5 that the maximum sizes of an eigenset and a recurrent set coincide. We first consider a recurrent set of maximum size.

<u>Proposition 4.4</u>.

(1) $\max\{|X| \mid X \text{ is a recurrent set of } A\} = r(A^\infty)$.

(2) If X is a recurrent set of A of maximum size, X is an eigenset of A^k for all $k \geq \tau$, where $\tau=\tau(A)$ is the time constant of A.

(Proof) (1) Suppose X is a recurrent set. Then $(X,X) \in \Lambda(A^k)$ for some $k \geq 1$. This implies that $(X,X) \in \Lambda(A^{km})$ for any m. Choosing m so that $km \geq \tau$, we see

$$|X| \leq r(A^{km}) = r(A^\infty)$$

by Theorem 4.2(2).

To show the converse, we consider A^m with $m \geq 2^{|S|}$. Since $r(A^m) = r(A^\infty)$, by Theorem 4.2(2), there exist X_i $(\subset S)$ $(i=0,1,\ldots,m)$ such that

$$(X_i, X_{i-1}) \in \Lambda(A), \quad i=1,\ldots,m,$$

and

$$|X_i| = r(A^{\infty}), \quad i=0,1,\ldots,m.$$

All of X_i's cannot be distinct, since $m \geqq 2^{|S|}$ and, therefore, there exist i and i' such that $0 \leqq i < i' \leqq m$ and $X_i = X_{i'}$. This means that $(X_i, X_i) \in \Lambda(A^k)$ for $k=i'-i$. Hence

$$\max\{|X| \mid X \text{ is a recurrent set of } A\} \geqq |X_i| = r(A^{\infty}).$$

(2) By (1), we have $|X| = r(A^{\infty}) = r(A^k)$ for $k \geqq \tau$. Since $(X,X) \in \Lambda(A^m)$ for some $m \geqq k$, there exist Y ($\subset S$) and Z ($\subset S$) such that $(X,Y) \in \Lambda(A^k)$ and $(Z,X) \in \Lambda(A^k)$. It then follows from the property (L3) of a bimatroid that $(X', X'') \in \Lambda(A^k)$ for some $X' \supset X$ and $X'' \supset X$. We have here $X'=X''=X$, since $|X| = r(A^k)$. Hence $(X,X) \in \Lambda(A^k)$. □

The following theorem links the maximum size of an eigenset with the rank of the power products. This fact may be compared to the power method (see, e.g., [Householder 1964]) for computing the eigenvector of a matrix corresponding to the eigenvalue of maximum modulus.

Theorem 4.5.

$$\begin{aligned} &\max\{|X| \mid X \text{ is an eigenset of } A\} \\ &= \max\{|X| \mid X \text{ is a recurrent set of } A\} \\ &= r(A^{\infty}). \end{aligned}$$ □

To prove this, we first states a lemma, which is an immediate consequence of Edmonds' min-max relation mentioned in Prop.2.1. We denote by S^- and S^+ two disjoint copies of S; $\psi^-: S^- \to S$ and $\psi^+: S^+ \to S$ will designate the one-to-one correspondences. For $X \subset S$, in general, we write $X^+ = (\psi^+)^{-1}(X)$ and $X^- = (\psi^-)^{-1}(X)$, or $\psi^+(X^+) = \psi^-(X^-) = X$. $M_0=(S^-\cup S^+,\mu_0)$ will mean the partition matroid, where

$$\mu_0(X^-\cup Y^+) = |X\cup Y|, \quad X\subset S,\ Y\subset S. \tag{4.2}$$

Lemma 4.6. Let $M=(S^-\cup S^+,\mu)$ be a matroid and $M_0=(S^-\cup S^+,\mu_0)$ the partition matroid given by (4.2). Assume that M and M_0 have a common base. Then

$$\begin{aligned}&\max\{|H\cap S^-| \mid H \text{ is a common base of } M \text{ and } M_0\}\\ &= \min\{\mu(S^-\cup X_1^+)-|X_1|+\sum_{i=1}^{W-1}(\mu(X_i^-\cup X_{i+1}^+)-|X_{i+1}|)+\mu(X_W^-)\\ &\qquad \mid X_i\subset S\ (i=1,\dots,W)\},\end{aligned} \tag{4.3}$$

where $W \geq |S|$.

(Proof) First define $w\colon S^-\cup S^+ \to Z_+$ by

$$w(x^-)=W+1, \quad w(x^+)=W, \quad x\in S, \tag{4.4}$$

where $W\geq|S|$. Then a common independent set of M and M_0 of maximum weight with respect to w must be a common base of M and M_0 and

$$\begin{aligned}&\max\{|H\cap S^-| \mid H \text{ is a common base of } M \text{ and } M_0\}\\ &= \max\{w(H) \mid H \text{ is independent in } M \text{ and } M_0\} - W|S|.\end{aligned} \tag{4.5}$$

Applying (2.5) of Prop.2.1 to the right-hand side of (4.5), we obtain

$$\begin{aligned}&\max\{w(H) \mid H \text{ is independent in } M \text{ and } M_0\}\\ &= \min\{\sum_{i=0}^{p}\mu(X_i^-\cup Y_i^+)+\sum_{j=0}^{q}|Z_j\cup U_j| \mid X_i\subset S,\ Y_i\subset S,\ Z_j\subset S,\ U_j\subset S,\\ &\qquad |\{i\mid x\in X_i\}|+|\{j\mid x\in Z_j\}|=W+1 \text{ for } x\in S,\\ &\qquad |\{i\mid x\in Y_i\}|+|\{j\mid x\in U_j\}|=W \text{ for } x\in S,\\ &\qquad X_0\supset X_1\supset\cdots\supset X_p,\ Y_0\supset Y_1\supset\cdots\supset Y_p,\ Z_0\supset Z_1\supset\cdots\supset Z_q,\ U_0\supset U_1\supset\cdots\supset U_q\}.\end{aligned} \tag{4.6}$$

On the right-hand side of (4.6), we may assume that $p=W$ and $Y_W=\emptyset$. Furthermore, we may assume, in view of the nesting condition, that $q=W$

and

$$Z_i = S-X_{W-i} \quad (i=0,1,\ldots,W),$$

$$U_i = S-Y_{W-1-i} \quad (i=0,1,\ldots,W-1),$$

$$U_W = \emptyset.$$

Hence (4.6) simplifies as follows:

RHS of (4.6)

$$= \min\{ \sum_{i=0}^{W-1} \mu(X_i^- \cup Y_i^+) + \mu(X_W^-) + |S-X_0| + \sum_{i=0}^{W-1} |S-(X_{W-i} \cap Y_{W-1-i})|$$

$$\mid X_0 \supset X_1 \supset \cdots \supset X_W,\ Y_0 \supset Y_1 \supset \cdots \supset Y_{W-1} \}$$

$$= \min\{ |S-X_0| + \sum_{i=0}^{W-1} (\mu(X_i^- \cup Y_i^+) - |X_{i+1} \cap Y_i|) + \mu(X_W^-)$$

$$\mid X_0 \supset X_1 \supset \cdots \supset X_W,\ Y_0 \supset Y_1 \supset \cdots \supset Y_{W-1} \} + W|S|$$

[since we may put $Y_i = X_{i+1}$ $(i=0,1,\ldots,W-1)$]

$$= \min\{ |S-X_0| + \sum_{i=0}^{W-1} (\mu(X_i^- \cup X_{i+1}^+) - |X_{i+1}|) + \mu(X_W^-)$$

$$\mid X_0 \supset X_1 \supset \cdots \supset X_W \} + W|S|$$

[since we may put $X_0 = S$]

$$= \min\{ \mu(S^- \cup X_1^+) - |X_1| + \sum_{i=1}^{W-1} (\mu(X_i^- \cup X_{i+1}^+) - |X_{i+1}|) + \mu(X_W^-)$$

$$\mid X_1 \supset \cdots \supset X_W \} + W|S|$$

$$= \min\{ \mu(S^- \cup X_1^+) - |X_1| + \sum_{i=1}^{W-1} (\mu(X_i^- \cup X_{i+1}^+) - |X_{i+1}|) + \mu(X_W^-)$$

$$\mid X_i \subset S \ (i=1,\ldots,W) \} + W|S|. \tag{4.7}$$

The last equality follows from the alternative expression (2.4) in Prop.2.1.

Combining (4.5), (4.6) and (4.7), we obtain (4.4). □

We shall now give the proof to Theorem 4.5. The matroid corresponding to $A=(S,S,\alpha)$ will be denoted by $M(A)=(S^{-}\cup S^{+},\alpha^{\pm})$ (cf.(2.11)), where S^{-} and S^{+} are disjoint copies of S standing respectively for Row(A) and Col(A); i.e.,

$$\alpha^{\pm}(X^{-}\cup Y^{+}) = \alpha(X,S-Y)+|Y|, \quad X\subset S ,Y\subset S. \tag{4.8}$$

(Proof of Theorem 4.5) In view of Prop.4.4(1), it suffices to show

$$\max\{|X| \mid X \text{ is an eigenset of } A\} = r(A^{\infty}). \tag{4.9}$$

We first note that $X \subset S$ is an eigenset of A iff $X^{-}\cup(S^{+}-X^{+})$ is a base of $M(A)$. In other words, X is an eigenset of A iff $X^{-}=H\cap S^{-}$ for some common base $H \subset S^{-}\cup S^{+}$ of $M(A)$ and M_0, where $M_0=(S^{-}\cup S^{+},\mu_0)$ is the partition matroid defined by (4.2). It may be mentioned that S^{+} is a common base of $M(A)$ and M_0.

From the above observations and Lemma 4.6 it follows that, for $W \geq |S|$,

$$\begin{aligned}
&\max\{|X| \mid X \text{ is an eigenset of } A\}\\
&= \max\{|H\cap S^{-}| \mid H \text{ is a common base of } M(A) \text{ and } M_0\}\\
&= \min\{\alpha^{\pm}(S^{-}\cup X_1{}^{+})-|X_1|+\sum_{i=1}^{W-1}(\alpha^{\pm}(X_i{}^{-}\cup X_{i+1}{}^{+})-|X_{i+1}|)+\alpha^{\pm}(X_W{}^{-})\\
&\qquad \mid X_i\subset S \ (i=1,\dots,W)\}\\
&\hspace{20em}[\text{by } (4.8)]\\
&= \min\{\alpha(S,S-X_1)+\sum_{i=1}^{W-1}\alpha(X_i,X_{i+1})+\alpha(X_W,S) \mid X_i\subset S \ (i=1,\dots,W)\}\\
&\hspace{20em}[\text{by } (2.17)]\\
&= r(A^{W+1}) = r(A^{\infty}).
\end{aligned}$$

This establishes (4.9). □

Remark 4.1. A maximal recurrent set is not necessarily a maximum recurrent set, nor is a maximal eigenset a maximum eigenset. In fact, for the bimatroid defined by the matrix

$$\begin{array}{r|ccc|} & x & y & z \\ \hline x: & 1 & 1 & 1 \\ y: & 1 & 1 & 2 \\ z: & 1 & 1 & 1 \\ \hline \end{array},$$

$\{\{y,z\},\{x\},\{y\},\{z\},\emptyset\}$ is the family of recurrent sets and that of eigensets; $\{x\}$ is maximal and not maximum. □

Remark 4.2. A recurrent set of a bimatroid A is a common independent set of $RM(A^{\infty})$ and $CM(A^{\infty})$. The converse, however, is not true in general. Consider, for example, the bimatroid A defined by the matrix

$$\begin{array}{r|ccc|} & x & y & z \\ \hline x: & 1 & 1 & 0 \\ y: & 0 & 0 & 1 \\ z: & 0 & 0 & 1 \\ \hline \end{array}.$$

As is easily verified, $\{y\}$ is independent both in $RM(A^{\infty})$ and in $CM(A^{\infty})$, whereas it is not a recurrent set of A. □

The following theorem states that the family of recurrent sets is hereditary.

Theorem 4.7. If X is a recurrent set of a bimatroid A and $Y \subset X$, then Y is also a recurrent set of A.

(Proof) The proof consists of three steps (i)∿(iii).

(i) First we claim that

$$\text{if } X \text{ is an eigenset and } x \in X, \text{ then } \{x\} \text{ is a recurrent set.} \tag{4.10}$$

By Prop. 2.3, there is a matching between X^- and X^+ in the underlying bipartite graph (S^-, S^+, Δ). That is, there is a one-to-one correspondence $\sigma: X \to X$ (or a permutation on X) such that $(\{\sigma(x)\}, \{x\}) \in \Lambda(A)$.

For each $x \in X$, there exists k (≥ 1) such that $\sigma^k(x) = x$. This implies that $(\{x\}, \{x\}) \in \Lambda(A^k)$, since $(\{\sigma^i(x)\}, \{\sigma^{i-1}(x)\}) \in \Lambda(A)$ for $i=1,\ldots,k$. This establishes (4.10).

(ii) Next we claim that

$$\text{if } X \text{ is an eigenset and } x \in X, \text{ then } X-\{x\} \text{ is a recurrent set.} \tag{4.11}$$

Since $(X,X) \in \Lambda(A)$, $A[X,X]$ is nonsingular (cf. §2.2 for notation). Put $A' = A[X,X]^{-1}$. Then X is an eigenset of A' and (4.10) implies that, for each $x \in X$, there exist x_0 $(=x)$, $x_1, \ldots, x_k$ $(=x)$ in X such that $(\{x_i\}, \{x_{i-1}\}) \in \Lambda(A')$ for $i=1,\ldots,k$. This is equivalent to the following:

$$(X_{i-1}, X_i) \in \Lambda(A[X,X]), \quad i=1,\ldots,k,$$

where $X_i = X-\{x_i\}$. Hence $(X-\{x\}, X-\{x\}) \in \Lambda(A^k)$, establishing (4.11).

(iii) We finally claim that

$$\text{if } X \text{ is a recurrent set and } x \in X, \text{ then } X-\{x\} \text{ is a recurrent set.} \tag{4.12}$$

If X is a recurrent set, then X is an eigenset of A^k for some $k \geq 1$. By (4.11), we see $X-\{x\}$ is a recurrent set of A^k, i.e., $(X-\{x\}, X-\{x\}) \in \Lambda(A^{km})$ for some $m \geq 1$. This shows (4.12), i.e., that $X-\{x\}$ is a recurrent set of A. Obviously, the assertion of the theorem follows from (4.12) by induction. □

<u>Remark 4.3</u>. The family of eigensets is not hereditary as seen in the bimatroid defined by the matrix

$$\begin{array}{r|cc|} & x & y \\ \hline x: & 0 & 1 \\ y: & 1 & 0 \\ \hline \end{array}\, .$$

□

Finally in this section, we mention some nice properties of a symmetric bimatroid.

Proposition 4.8. Let $A=(S,S,\Lambda(A))$ be a symmetric bimatroid.

(1) The time constant $\tau(A) \leqq 1$.

(2) The Jordan type of (4.1) is given by $\omega_0=r(A)$, $\omega_1=|S|-r(A)$, $\omega_k=0$ for $k\geqq 2$.

(3) X is a maximum eigenset of A iff X is a base of RM(A) (=CM(A)).

(Proof) Suppose $X \subset S$ is a base of CM(A). By symmetry of A, we have $(X,Y)\in\Lambda(A)$ and $(Y,X)\in\Lambda(A)$ for some $Y\subset S$. The property (L3) of a bimatroid, combined with $|X|=r(A)$, implies that $(X,X)\in\Lambda(A)$. This shows that $\tau(A) \leqq 1$, i.e., $r(A)=r(A^\infty)$. The other assertions are immediate from the above argument and Theorem 4.5. □

We shall say that a bimatroid A is definite if $A=L*L^*$ for some bimatroid L. Evidently, a definite bimatroid is symmetric.

Proposition 4.9. Let $A=L*L^*$ be a definite bimatroid.

(1) $r(A) = r(L)$.

(2) If X is an eigenset of A and $Y \subset X$, then Y is also an eigenset of A.

(Proof) (1) By (2.17) and bisubmodularity of the birank function λ of L, we have

$$r(A) = \min\{\lambda(S,Y)+\lambda(S,T-Y) \mid Y\subset T\}$$
$$= \lambda(S,T) = r(L).$$

(2) The assertion is immediate from the fact that $(X,X) \in \Lambda(A)$ iff $X \in RM(A) = RM(L)$. □

5. Reachability and Controllability

In this section we deal with the reachability and controllability of a combinatorial dynamical system (A,B).

5.1. Control bimatroid

This subsection is to introduce a bimatroid, called the control bimatroid, which is of fundamental importance in considering the controllability.

Let $S^{(i)}$ $(i=0,1,\ldots)$ denote disjoint copies of the state space S, and $P^{(i)}$ $(i=0,1,\ldots)$ denote disjoint copies of the input space P. The natural one-to-one correspondences are expressed by $\psi_i: S^{(i)}\to S$ and $\pi_i: P^{(i)}\to P$. For $i=1,2,\ldots$, we denote by $A^{(i)}=(S^{(i)},S^{(i-1)},\alpha^{(i)})$ and $B^{(i)}=(S^{(i)},P^{(i-1)},\beta^{(i)})$ the bimatroids which are respectively isomorphic to $A=(S,S,\alpha)$ and $B=(S,P,\beta)$.

For (j,k) with $0\leq j<k$, we define a bimatroid $D_{j,k}$, to be called the <u>control bimatroid</u>, by the identity

$$D_{j,k}=B^{(k)}\vee A^{(k)}*(B^{(k-1)}\vee A^{(k-1)}*\cdots*(B^{(j+2)}\vee A^{(j+2)}*B^{(j+1)})\cdots). \tag{5.1}$$

By definition, we have $\mathrm{Row}(D_{j,k}) = S^{(k)}$ and $\mathrm{Col}(D_{j,k}) = P_{j,k-1}$, where

$$P_{j,k-1} = \cup\{P^{(i)} \mid i=j,\ldots,k-1\}. \tag{5.2}$$

The birank function of $D_{j,k}$ will be written as $\delta_{j,k}$. Furthermore we put $D_k = D_{0,k}$ and $\delta_k = \delta_{0,k}$ for notational convenience.

The control bimatroids $D_{j,k}$ have the following fundamental properties.

Lemma 5.1.

(1) [translation] $D_{j,k}=(S^{(k)}, P_{j,k-1},\delta_{j,k})$ is isomorphic to $D_{k-j}=(S^{(k-j)}, P_{0,k-j-1},\delta_{k-j})$:

$$D_{j,k} \simeq D_{k-j}, \quad 0\leq j<k. \tag{5.3}$$

(2) [recurrence relation]

$$D_{j,k} = A^{(k)}*D_{j,k-1} \vee B^{(k)}, \quad k=j+1,j+2,\ldots, \tag{5.4}$$

where, for convenience, we define $D_{j,j} = (S^{(j)},\emptyset,\delta_{0,j})$ to be a trivial bimatroid. □

Since $\mathrm{Col}(A^{(k)}*D_{j,k-1})=P_{j,k-2}$ and $\mathrm{Col}(B^{(k)})=P^{(k-1)}$ are disjoint in (5.4), we see from (2.15) that

$$M(D_{j,k}) = M(A^{(k)}*D_{j,k-1}) \vee M(B^{(k)}). \tag{5.5}$$

Using (2.18) and (2.3), we see that

$$\begin{aligned} &M(A^{(k)}*D_{j,k-1}) \vee M(B^{(k)}) \\ &= ([M(A^{(k)})\vee M(D_{j,k-1})]/S^{(k-1)}) \vee M(B^{(k)}) \\ &= [M(A^{(k)})\vee M(D_{j,k-1})\vee M(B^{(k)})] / S^{(k-1)}. \end{aligned} \tag{5.6}$$

Substituting (5.6) into (5.5), we obtain the recurrence relation for $M(D_{j,k})$:

$$M(D_{j,k}) = [M(A^{(k)})\vee M(B^{(k)})\vee M(D_{j,k-1})] / S^{(k-1)}, \quad j<k. \tag{5.7}$$

From this as well as (2.3) it follows that

$$M(D_{j,k}) = [(\bigvee_{i=j+2}^{k} M(A^{(i)}))\vee(\bigvee_{i=j+1}^{k} M(B^{(i)}))] / S_{j+1,k-1}, \tag{5.8}$$

where

$$S_{j+1,k-1} = \cup\{S^{(i)} \mid i=j+1,\ldots,k-1\}. \tag{5.9}$$

Note that $S^{(k)}\cup P_{j,k-1}$ is the ground set of $M(D_{j,k})$.

5.2. Controllability criterion

The purpose of this subsection is to investigage the structure of $RS_k(\emptyset)$ of (3.5), the family of those sets which are reachable at time k from the empty set. To be specific, we show that $RS_k(\emptyset)$, for each k, constitutes the family of independent sets of a matroid, say $R_k=(S,\rho_k)$, and that R_k's grow with k, in the sense stated in Theorem 5.2. Then we see that the system (A,B) is controllable (cf. (3.8)) iff the sequence of matroids R_k grows to the free matroid on S.

We first observe, by the definitions (3.2) (or (3.3)), (3.5) and (5.1), that $X \in RS_k(\emptyset)$ iff $X^{(k)}=\psi_k^{-1}(X)$ $(\subset S^{(k)})$ is independent in $RM(D_k)=(S^{(k)},\delta_k^-)$. We denote by $R_k=(S,\rho_k)$ the matroid isomorphic to $RM(D_k)$ (with the natural correspondence of the ground sets through ψ_k: $S^{(k)}\to S$) and call it the <u>reachability matroid</u>. We shall investigate the behavior of the sequence $(R_k \mid k=0,1,\ldots)$.

The recurrence relation of Lemma 5.1(2) implies

$$RM(D_k) = A^{(k)}*RM(D_{k-1}) \vee RM(B^{(k)}), \quad k=1,2,\ldots, \tag{5.10}$$

where $A^{(k)}*RM(D_{k-1})$ means the matroid on $S^{(k)}$ induced from $RM(D_{k-1})$ by $A^{(k)}$. This indicates that ρ_k is determined by the "linear iteration":

$$\begin{aligned} &\rho_0 = 0, \\ &\rho_k = \alpha*\rho_{k-1} \vee \beta^-, \quad k=1,2,\ldots \end{aligned} \tag{5.11}$$

(see (2.19) for notation), where β^- is the rank function of RM(B). We may equivalently say that $R_k=(S,\rho_k)$ is defined by (5.11).

The properties of R_k are listed in Theorem 5.2, which is to be compared with Theorems 4.2 and 4.3.

Theorem 5.2. Let $R_k=(S,\rho_k)$, $k=0,1,\ldots$, satisfy (5.11).

(1) $$\rho_k(X) = \min\{\sum_{i=2}^{k} \alpha(Y_i,S-Y_{i-1}) + |X-Y_k| + \sum_{i=1}^{k} \beta^-(Y_i) \mid Y_1,\ldots,Y_{k-1}\subset S;\ Y_k\subset X\},\quad X\subset S.$$

(2) $$r(R_k)-r(R_{k-1}) \geqq r(R_{k+1})-r(R_k),\quad k=1,2,\ldots.$$

(3) There exists $\kappa=\kappa(A,B)$ $(\geqq 0)$ such that

$$r(R_0) < r(R_1) < \cdots < r(R_{\kappa-1}) < r(R_\kappa) = r(R_k),\quad k=\kappa+1,\kappa+2,\ldots.$$

(4) $$R_0 \lneq R_1 \lneq \cdots \lneq R_{\kappa-1} \lneq R_\kappa = R_k,\quad k=\kappa+1,\kappa+2,\ldots. \qquad \square$$

Before giving the proof to Theorem 5.2, we state a general result on a bi-polymatroid arising from the matroid union, which seems of interest in itself. This result will be crucial to the derivation of the inequality in (2).

Theorem 5.3. Let $M_i=(E,\mu_i)$ $(i\in T)$ be matroids and define

$$\lambda(I,X) = (\bigvee_{i\in I} \mu_i)(X),\quad I\subset T,\ X\subset E. \tag{5.12}$$

(That is, $\lambda(I,X)$ is the rank of $X \subset E$ in the union of M_i's $(i\in I)$.) Then λ is the birank function of a bi-polymatroid; i.e., λ satisfies (B1'), (B2) and (B3) of §2.2. In particular,

$$r(M_1\vee M_2\vee M_3)+r(M_2) \leqq r(M_1\vee M_2)+r(M_2\vee M_3) \tag{5.13}$$

for three matroids M_i $(i=1,2,3)$.

(Proof) It suffices to prove the bisubmodularity (B3) of λ. For $I,J \subset T$ and $X,Y \subset E$, we have, using (2.1),

$$\lambda(I,X)+\lambda(J,Y) = \min\{\sum_{i\in I} \mu_i(X') + \sum_{j\in J} \mu_j(Y') + |X-X'|+|Y-Y'| \mid X'\subset X,\ Y'\subset Y\}. \tag{5.14}$$

Substituting

$$\sum_{i\in I}\mu_i(X') + \sum_{j\in J}\mu_j(Y')$$

$$= \sum_{i\in I-J}\mu_i(X') + \sum_{i\in I\cap J}[\mu_i(X')+\mu_i(Y')] + \sum_{i\in J-I}\mu_i(Y')$$

$$\geq \sum_{i\in I-J}\mu_i(X'\cap Y') + \sum_{i\in I\cap J}[\mu_i(X'\cup Y')+\mu_i(X'\cap Y')] + \sum_{i\in J-I}\mu_i(X'\cap Y')$$

$$= \sum_{i\in I\cup J}\mu_i(X'\cap Y') + \sum_{i\in I\cap J}\mu_i(X'\cup Y')$$

into (5.14), we obtain

$$\lambda(I,X)+\lambda(J,Y)$$

$$= \min\{\sum_{i\in I\cup J}\mu_i(X'\cap Y') + |X\cap Y-X'\cap Y'|$$

$$+ \sum_{i\in I\cap J}\mu_i(X'\cup Y') + |X\cup Y-X'\cup Y'| \mid X'\subset X,\ Y'\subset Y\}$$

$$\geq \min\{\sum_{i\in I\cup J}\mu_i(Z') + |X\cap Y-Z'| \mid Z'\subset X\cap Y\}$$

$$+ \min\{\sum_{i\in I\cap J}\mu_i(Z'') + |X\cup Y-Z''| \mid Z''\subset X\cup Y\}$$

$$= \lambda(I\cup J,X\cap Y)+\lambda(I\cap J,X\cup Y). \qquad \square$$

We are now in the position to prove Theorem 5.2.

(Proof of Theorem 5.2) (1) This is straightforward from (5.11), combined with (2.1) and (2.19).

(2) For simplicity, we put

$$M_{j,k} = \bigvee_{i=j}^{k} M(A^{(i)}), \quad j\leq k,$$

$$N_{j,k} = \bigvee_{i=j}^{k} RM(B^{(i)}), \quad j\leq k. \qquad (5.15)$$

Then by (5.8), we have

$$\begin{aligned} R_k &\simeq RM(D_k) \\ &= [M_{2,k} \vee N_{1,k}]/S_{1,k-1} \\ &= [M_{2,k} \vee N_{1,k}] \times S^{(k)}. \end{aligned} \tag{5.16}$$

Since $S_{1,k-1}$ is a base of $M_{2,k}$, it follows that

$$r(R_k) = r(M_{2,k} \vee N_{1,k}) - (k-1)|S|. \tag{5.17}$$

On the other hand, applying (5.13) of Theorem 5.3 to $M_1=M_{2,2} \vee N_{1,1}$, $M_2=M_{3,k} \vee N_{2,k}$, $M_3=M_{k+1,k+1} \vee N_{k+1,k+1}$, we obtain

$$r(M_{2,k+1} \vee N_{1,k+1})+r(M_{3,k} \vee N_{2,k}) \leqq r(M_{2,k} \vee N_{1,k})+r(M_{3,k+1} \vee N_{2,k+1}). \tag{5.18}$$

Noting the obvious fact

$$\begin{aligned} M_{3,k} \vee N_{2,k} &\simeq M_{2,k-1} \vee N_{1,k-1}, \\ M_{3,k+1} \vee N_{2,k+1} &\simeq M_{2,k} \vee N_{1,k}, \end{aligned}$$

and using (5.17), we establish the desired inequality.

(3)&(4) Since

$$R_{k+1} \simeq [(M_{2,2} \vee N_{1,1}) \vee (M_{3,k+1} \vee N_{2,k+1})] \times S^{(k+1)}$$

and

$$R_k \simeq [M_{3,k+1} \vee N_{2,k+1}] \times S^{(k+1)},$$

we see, by Prop.2.2(2), that R_k is a quotient of R_{k+1}, i.e.,

$$R_k \leftarrow R_{k+1}. \tag{5.19}$$

In particular,

$$r(R_k) \leqq r(R_{k+1}), \quad k=0,1,\ldots.$$

Let κ ($\geqq 0$) be the smallest k such that the equality holds. Then (2) implies that the equality must hold for all $k \geqq \kappa$, establishing (3). The assertion of (4) also follows from (5.19) coupled with (3) and Prop.2.2(1). □

The integer $\kappa=\kappa(A,B)$ will be called the controllability index of (A,B); evidently, $0 \leq \kappa \leq |S|$. We symbolically write R_∞ for R_κ and call $r(R_\infty)$ the controllable dimension.

As a corollary of the above theorem, we obtain the first criterion for the controllability.

Theorem 5.4. A combinatorial dynamical system (A,B) is controllable iff

$$r([\bigvee_{i=2}^{|S|} M(A^{(i)})]\vee[\bigvee_{i=1}^{|S|} RM(B^{(i)})]) = |S|^2. \tag{5.20}$$

□

Remark 5.1. In the conventional dynamical system theory, the controllability of (1.1) or (1.2) is known (e.g., [Kailath 1980], [Rosenbrock 1970], [Wolovich 1974], [Wonham 1979]) to be expressed by the rank of the controllability matrix; i.e., the system (A,B) is controllable iff

$$r[B|AB|A^2B|\cdots|A^{|S|-1}B] = |S|.$$

As a combinatorial analogue to this criterion, one might be tempted to consider the condition:

$$r[RM(B)\vee RM(A*B)\vee RM(A^2*B)\vee\cdots\vee RM(A^{|S|-1}*B)] = |S| \tag{5.21}$$

for the controllability of a combinatorial dynamical system (A,B). This turns out to be weaker than the controllability, though it is implied by (5.20). Consider, for example, a combinatorial dynamical system with bimatroids A and B defined respectively by the following matrices:

$$A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix}, \quad B \sim \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where a, b, c, d and e are indeterminates. This pair (A,B) satisfies (5.21) and not (5.20) of Theorem 5.4. □

5.3. Controllability indices

In parallel with the conventional dynamical system theory [Wonham 1979], we can define the notion of controllability indices for a combinatorial dynamical system (A,B).

By Theorem 5.2,

$$\Delta\rho_k = r(R_k)-r(R_{k-1}), \quad k=1,2,\ldots,$$

form a nonnegative and decreasing sequence that vanishes for $k>\kappa$. We define

$$\kappa_i = |\{k|\ \Delta\rho_k \geq i\}|, \quad i=1,2,\ldots. \tag{5.22}$$

Evidently,

$$\kappa_i = 0 \quad \text{if } i > |P|\ (\geq r(R_1) = r(B));$$

$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{|P|}\ (\geq 0) \tag{5.23}$$

and

$$\sum_{i=1}^{|P|} \kappa_i = r(R_\infty). \tag{5.24}$$

The tuple $(\kappa_i|\ i=1,\ldots,|P|)$ will be named here the <u>controllability indices</u> of (A,B). It should be clear that $\kappa_1 = \kappa$, where $\kappa=\kappa(A,B)$ is the controllability index introduced in the previous subsection.

In the following we shall investigate the structure of admissible inputs. First we note that an input $(U_k|\ k=0,1,\ldots,K-1)$ is admissible iff $\cup\{U_k^{(k)}|\ k=0,1,\ldots,K-1\}$ is an independent set of $CM(D_K)$, where $U_k^{(k)}=\pi_k^{-1}(U_k)\ (\subset P^{(k)})$.

The following theorem may be compared to the similar fact [Wonham 1979, §5.7] for the conventional dynamical system. The theorem states that for a controllable dynamical system we can find a nicely nested input sequence which brings the null state $\emptyset$ to S. It also clarifies the significance of the controllability indices.

Theorem 5.5. Suppose (A,B) is controllable. Then there exists an input $(U_k \mid k=0,1,\ldots,K-1)$ admissible for $(\emptyset,S)$ such that

$$U_0 \subset U_1 \subset \cdots \subset U_{K-1} \tag{5.25}$$

and that

$$\{\tilde{\kappa}_u \mid u\in P\} = \{\kappa_i \mid i=1,\ldots,|P|\}, \tag{5.26}$$

where

$$\tilde{\kappa}_u = K - \min\{k \mid 0\leq k\leq K,\ u\in U_k\}, \quad u\in P, \tag{5.27}$$

(we put $U_K=P$ for convenience). □

Theorem 5.5 follows from Prop.5.6 below; it should be recalled that $\pi_i\colon P^{(i)}\to P$ is the bijection and $P_{j,k-1}$ is given by (5.2).

Proposition 5.6. For each k (≥ 0), there exists $\bar{U} \subset P_{0,k-1}$ such that

$$\pi_0(\bar{U}\cap P^{(0)}) \subset \pi_1(\bar{U}\cap P^{(1)}) \subset \cdots \subset \pi_{k-1}(\bar{U}\cap P^{(k-1)}) \tag{5.28}$$

and that

$$\bar{U}\cap P_{j,k-1} \text{ is a base of } CM(D_{j,k}) \text{ for } j=0,1,\ldots,k-1. \tag{5.29}$$

(Proof) The notation $U^{(j)} = \pi_j(U)$, $U\subset P$, will be used frequently.

We construct $\bar{U}$ by means of U_i ($\subset P$) $(i=0,1,\ldots,k)$, which are determined as follows. First put $U_k=P$. For $j=k-1,k-2,\ldots,1,0$ (in this order), find a base $\bar{U}_j$ ($\subset P^{(j)}$) of $U_{j+1}{}^{(j)}$ in $CM(D_{j,k})/(\cup\{\bar{U}_i \mid i=j+1,j+2,\ldots,k-1\})$ and put $U_j = \pi_j(\bar{U}_j)$. We then define

$$\bar{U} = \cup\{\bar{U}_i \mid i=0,1,\ldots,k-1\}.$$

As easily seen, $\bar{U}$ has the property (5.28) and $\bar{U}\cap P_{j,k-1}$ is an independent set of $CM(D_{j,k})$ for $j=0,1,\ldots,k-1$. It remains to be shown, therefore, that

$$\sigma_{j,k}(\bar{U}\cap P_{j,k-1}) = P_{j,k-1}, \quad j=0,1,\ldots,k-1, \tag{5.30}$$

where $\sigma_{j,k}$ denotes the closure function of $CM(D_{j,k})$. Since $CM(D_{j,k})$ is a restriction of $CM(D_k)$, we may show, instead of (5.30), that

$$\sigma_k(\bar{U}\cap P_{j,k-1}) \supset P_{j,k-1}, \quad j=0,1,\dots,k-1, \tag{5.31}$$

where σ_k is the closure function of $CM(D_k)$.

To show (5.31), we first note that (5.4), (2.2) and Prop.2.4 imply

$$CM(D_{k-1}) \to CM(D_k)-P^{(k-1)},$$

from which we see by (2.7) that

$$\sigma_{k-1}(\bar{U}') \subset \sigma_k(\bar{U}')-P^{(k-1)} \subset \sigma_k(\bar{U}'), \quad \bar{U}'\subset P_{0,k-2}. \tag{5.32}$$

By the construction, we have

$$U_i^{(i-1)} \subset \sigma_k(\bigcup_{m=i-1}^{k-1} U_m^{(m)}), \quad i=1,2,\dots,k-1, \tag{5.33}$$

from which we are going to derive

$$U_i^{(i-2)} \subset \sigma_k(\bigcup_{m=i-1}^{k-1} U_m^{(m)}), \quad i=2,3,\dots,k-1. \tag{5.34}$$

For $i\geqq 2$ it follows from (5.33) that

$$U_i^{(i-1)} \subset \sigma_{1,k}(\bigcup_{m=i-1}^{k-1} U_m^{(m)}).$$

By translation (cf. Lemma 5.1(1)), we obtain

$$U_i^{(i-2)} \subset \sigma_{k-1}(\bigcup_{m=i-1}^{k-1} U_m^{(m-1)}). \tag{5.35}$$

Using (5.32), we further obtain

$$U_i^{(i-2)} \subset \sigma_k(\bigcup_{m=i-1}^{k-1} U_m^{(m-1)}),$$

which, combined with (5.33), implies (5.34).

Repeating such arguments we may claim

$$U_i^{(j)} \subset \sigma_k(\bigcup_{m=i-1}^{k-1} U_m^{(m)}), \quad j=0,1,\dots,i,$$

which implies (5.31). □

6. Another Characterization of Controllability

This section establishes the second characterization (Theorem 6.3) of the controllability of a combinatorial dynamical system, which corresponds to the structural controllability theorem ([Lin 1974], [Glover-Shilverman 1976], [Shields-Pearson 1976]) for conventional dynamical system. The characterization of the controllability is derived from a formula (Theorem 6.2) expressing the controllable dimension $r(R_\infty)$. The formula is the combinatorial counterpart of the result of [Hosoe 1980] about the generic dimension of controllable subspaces for a conventional dynamical system.

We begin with a characterization of the reachability of (A,B). It should be remembered that α is the birank function of A and β^- the rank function of RM(B).

Proposition 6.1. (A,B) is reachable iff

$$\alpha(X,S-X)+\beta^-(X) \geq 1 \qquad (6.1)$$

for all nonempty subsets of S.

(Proof) First note that (A,B) is reachable iff for each vertex $x \in S$ in the underlying digraph there is a directed path from P to x.

If (6.1) fails for some X ($\neq\emptyset$), there is no directed path from P to X. Hence (6.1) must hold if (A,B) is reachable.

Conversely suppose (A,B) is not reachable, and let S-X be the set of vertices in S which are reachable from P by a directed path in the underlying digraph. Then $X \neq \emptyset$ and $\alpha(X,S-X) = \beta^-(X) = 0$. □

In the theorem below recall that A[X,X] denotes the bimatroid A with its row set and column set restricted to X; similarly for B[X,P]. It may be interesting to compare Theorem 4.5 and Theorem 6.2, though neither of them implies the other.

<u>Theorem 6.2</u>. Suppose (A,B) is reachable. Then

$$r(R_\infty) = \max\{|X| \mid r(A[X,X]\vee B[X,P])=|X|,\ X\subset S\}. \tag{6.2}$$

(Proof) As in §4.2, $M(A)=(S^-\cup S^+,\alpha^\pm)$ denotes the matroid that corresponds to A. We note that $r(A[X,X]\vee B[X,P])=|X|$ iff $X^-\cup(S^+-X^+)$ is a base of $M(A)\vee RM(B)$, where RM(B) in this notation is to be understood as the matroid on S^- isomorphic to RM(B) by the natural correspondence between S^- and S. Hence (6.2) is equivalent to

$$r(R_\infty) = \max\{|H\cap S^-| \mid H \text{ is a common base of } M(A)\vee RM(B) \text{ and } M_0\}, \tag{6.3}$$

where $M_0=(S^-\cup S^+,\mu_0)$ is the partition matroid of (4.2). In the following, we will show (6.3).

The rank function $\mu=\alpha^\pm\vee\beta^-$ of $M(A)\vee RM(B)$ is expressed, by (2.1), as

$$\begin{aligned}\mu(X^-\cup Z^+) &= \min\{\alpha^\pm(Y^-\cup Z^+)+\beta^-(Y)+|X-Y| \mid Y\subset X\}\\ &= \min\{\alpha(Y,S-Z)+|Z|+\beta^-(Y)+|X-Y| \mid Y\subset X\}, \quad X\subset S,\ Z\subset S. \end{aligned}\tag{6.4}$$

Substituting this in (4.3) of Lemma 4.6, and denoting by RHS the right-hand side of (6.3), we obtain

$$\begin{aligned}\text{RHS} &= \min\{\alpha(Y_0,S-X_1)+\beta^-(Y_0)+|S-Y_0|+\sum_{i=1}^{W-1}[\alpha(Y_i,S-X_{i+1})+\beta^-(Y_i)+|X_i-Y_i|]\\ &\qquad +\alpha(Y_W,S)+\beta^-(Y_W)+|X_W-Y_W| \mid Y_0\subset S;\ Y_i\subset X_i\subset S\ (i=1,\dots,W)\}\\ &\qquad\qquad [\text{since we may put } X_i=Y_i\ (i=1,\dots,W)]\\ &= \min\{\sum_{i=0}^{W-1}\alpha(Y_i,S-Y_{i+1})+\alpha(Y_W,S)+|S-Y_0|+\sum_{i=0}^{W}\beta^-(Y_i)\\ &\qquad \mid Y_i\subset S\ (i=0,1,\dots,W)\}\\ &= \min\{\alpha(Y_0,S)+\sum_{i=0}^{W-1}\alpha(Y_{i+1},S-Y_i)+|S-Y_W|+\sum_{i=0}^{W}\beta^-(Y_i)\\ &\qquad \mid Y_i\subset S\ (i=0,1,\dots,W)\},\end{aligned}\tag{6.5}$$

where $W \geqq |S|$.

On the other hand, by Theorem 5.2(1) we obtain

$$r(R_\infty) = r(R_k) \\ = \min\{\sum_{i=1}^{k-1} \alpha(Y_{i+1}, S-Y_i) + |S-Y_k| + \sum_{i=1}^{k} \beta^-(Y_i) \mid Y_i \subseteq S \ (i=1,\ldots,k)\}, \quad (6.6)$$

where $k \geqq \kappa$.

Comparing (6.5) and (6.6) with $k=W+1$, we readily see that

$$\text{RHS} \geqq r(R_\infty). \quad (6.7)$$

The reverse inequality also holds if the system is reachable, as shown below.

Choose k large enough, say $k>n+2^n$, where $n=|S|$, and let $Y_i=\tilde{Y}_i$ $(i=1,\ldots,k)$ attain the minimum on the right-hand side of (6.6). Then

$$\begin{aligned} r(R_k) &= \sum_{i=1}^{k-1} \alpha(\tilde{Y}_{i+1}, S-\tilde{Y}_i) + |S-\tilde{Y}_k| + \sum_{i=1}^{k} \beta^-(\tilde{Y}_i) \qquad (6.8) \\ &= \sum_{i=1}^{k-n} [\alpha(\tilde{Y}_{i+1}, S-\tilde{Y}_i) + \beta^-(\tilde{Y}_i)] \\ &\quad + [\sum_{i=k-n+1}^{k-1} \alpha(\tilde{Y}_{i+1}, S-\tilde{Y}_i) + |S-\tilde{Y}_k| + \sum_{i=k-n+1}^{k} \beta^-(\tilde{Y}_i)] \\ &\geqq \sum_{i=1}^{k-n} [\alpha(\tilde{Y}_{i+1}, S-\tilde{Y}_i) + \beta^-(\tilde{Y}_i)] + r(R_n). \end{aligned}$$

Since $r(R_k) = r(R_n) = r(R_\infty)$, it follows that

$$\alpha(\tilde{Y}_{i+1}, S-\tilde{Y}_i) + \beta^-(\tilde{Y}_i) = 0, \quad i=1,\ldots,k-n. \quad (6.9)$$

Since $k-n > 2^n$, there exist p and q $(1 \leqq p < q \leqq 2^n)$ such that $\tilde{Y}_p=\tilde{Y}_q$. Using the bisubmodularity of α and the submodularity of β^-, we can derive from (6.9) that

$$\alpha(\tilde{Y}, S-\tilde{Y}) + \beta^-(\tilde{Y}) = 0,$$

where

$$\tilde{Y} = \bigcup_{i=p}^{q-1} \tilde{Y}_i.$$

Because of the assumed reachability of the system, this implies $\tilde{Y}=\emptyset$ by Prop.6.1. In particular, $\tilde{Y}_p=\emptyset$.

Substituting $\tilde{Y}_p=\emptyset$ in (6.8) we obtain

$$\begin{aligned} r(R_k) &= [\sum_{i=1}^{p-2} \alpha(\tilde{Y}_{i+1},S-\tilde{Y}_i)+\sum_{i=1}^{p-1} \beta^-(\tilde{Y}_i)] \\ &\quad + [\alpha(\tilde{Y}_{p+1},S)+\sum_{i=p+1}^{k-1} \alpha(\tilde{Y}_{i+1},S-\tilde{Y}_i)+|S-\tilde{Y}_k|+\sum_{i=p+1}^{k} \beta^-(\tilde{Y}_i)] \\ &\geq \alpha(\tilde{Y}_{p+1},S)+\sum_{i=p+1}^{k-1} \alpha(\tilde{Y}_{i+1},S-\tilde{Y}_i)+|S-\tilde{Y}_k|+\sum_{i=p+1}^{k} \beta^-(\tilde{Y}_i). \end{aligned}$$

Comparing the last expression with (6.5), with the correspondence $W=k-p-1$ ($\geq n=|S|$), we see

$$r(R_\infty) = r(R_k) \geq \text{RHS},$$

which establishes (6.3) when combined with (6.7). □

As an immediate corollary we obtain the following criterion for the controllability.

Theorem 6.3. A combinatorial dynamical system (A,B) is controllable iff the following two conditions are satisfied.

(i) (A,B) is reachable.

(ii) $RM(A)\vee RM(B)$ is the free matroid.

(Proof) The condition (ii) is equivalent to the fact that the right-hand side of (6.2) is equal to $|S|$. Hence the sufficiency of (i) and (ii) follows from Theorem 6.2, whereas the necessity of (i) and (ii) follows if we show the necessity of (i).

Suppose the system is controllable. i.e., $S\in RS_K(\emptyset)$ for some K. By definition, (3.2) (or (3.3)) holds for some $(U_k|\ k=0,1,\ldots,K-1)$ and $(X_k|\ k=0,1,\ldots,K)$ such that $X_0=\emptyset$ and $X_K=S$. This in turn means there exists $(Y_k|\ k=0,1,\ldots,K)$ such that $Y_{k+1}\subset X_{k+1}$,

$$(Y_{k+1},X_k)\in\Lambda(A),\quad (X_{k+1}-Y_{k+1},U_k)\in\Lambda(B),\quad k=0,1,\ldots,K-1. \qquad (6.10)$$

For $x \in S$, we can find a sequence $(x_k \mid k=j,j+1,\ldots,K)$, $x_k \in S$, and $u \in P$ such that $x_K=x$,

$$(\{x_{k+1}\},\{x_k\}) \in \Lambda(A), \quad k=j,j+1,\ldots,K-1,$$

and

$$(\{x_j\},\{u\}) \in \Lambda(B)$$

as follows. This shows the reachability (i) of the system.

To find such a sequence, first put $x_K=x$. For $k=K,K-1,\ldots$, either (a): $x_k \in Y_k$ or (b): $x_k \notin Y_k$ occurs. In case (a) we can find $x_{k-1} \in X_{k-1}$ such that $(\{x_k\},\{x_{k-1}\}) \in \Lambda(A)$, by (6.10) and Prop.2.3; in case (b), where $x_k \in X_k - Y_k$ by construction, we can find $u \in P$ such that $(\{x_k\},\{u\}) \in \Lambda(B)$, again by (6.10) and Prop.2.3. □

We will say that a bimatroid F is a <u>state feedback</u> iff Row(F)=P and Col(F)=S. The following theorem states to the effect that from a controllable system we can obtain another controllable system with smaller input space by means of a state feedback. This theorem may be compared to the similar result (e.g., [Wonham 1979, §2.1]) in the conventional dynamical system theory.

<u>Theorem 6.4</u>. Suppose (A,B) is controllable. For any $P' \subset P$ such that $B[S,P']$ is nontrivial, there exists a state feedback F which makes the system $(\tilde{A},\tilde{B})$ controllable, where

$$\tilde{A} = A \vee B[S,P-P']*F[P-P',S],$$

and

$$\tilde{B} = B[S,P'].$$

(Proof) We shall construct a deltoid (a bimatroid expressed by a bipartite graph) F which meets the requirement.

Since (A,B) is controllable, (ii) of Theorem 6.3 implies that

$$(S-X, X'') \in \Lambda(A) \qquad (6.11)$$

and

$$(X, U) \in \Lambda(B) \qquad (6.12)$$

for some X, X'' ($\subset S$) and U ($\subset P$). Put $U' = U \cap P'$ and $U'' = U - P'$. Then $(X, U' \cup U'') \in \Lambda(B)$ by (6.12) and therefore, by Prop.2.3,

$$(X', U') \in \Lambda(B) \qquad (6.13)$$

and

$$(X-X', U'') \in \Lambda(B) \qquad (6.14)$$

for some $X' \subset X$. Since $|U''| \leq |X| = |S-X''|$, there exists an injection $\sigma\colon U'' \to S-X''$. Put

$$\Delta_1 = \{(u'', \sigma(u'')) \mid u'' \in U''\}. \qquad (6.15)$$

Since $\tilde{B}$ is nontrivial by assumption. there exists $x_0 \in S$ such that an arc (u, x_0) exists for some $u \in P'$ in the underlying digraph of (A,B). Put

$$\Delta_2 = \{(u'', x_0) \mid u'' \in U''\}. \qquad (6.16)$$

The deltoid $F=(P,S,\Lambda(F))$ with the underlying bipartite graph $(P,S,\Delta_1 \cup \Delta_2)$ has the desired property, as follows. We see $S-X' \in RM(\tilde{A})$ by (6.11), (6.14) and (6.15), whereas $X' \in RM(\tilde{B})$ by (6.13). These two imply that the system $(\tilde{A},\tilde{B})$ satisfies condition (ii) of Theorem 6.3. It is easy to see that it also satisfies condition (i) of Theorem 6.3 due to (6.16). Hence $(\tilde{A},\tilde{B})$ is controllable by Theorem 6.3. □

7. Possible Extensions

So far we have concentrated ourselves on a combinatorial dynamical system (A,B), or a system (A,B,C) with C being a trivial bimatroid. This section is to suggest a possible direction of further developement of the theory which incorporates the output (or the observation) of a combinatorial dynamical system.

For a general combinatorial dynamical system (A,B,C), an <u>output</u> (or an <u>observation</u>) will mean a sequence $(Y_k | k=1,\dots,K)$ such that $Y_k \subset T$. We sometimes call K the <u>length</u> of the output.

When given an input $(U_k | k=0,1,\dots,K-1)$ and an output $(Y_k | k=1,\dots,K)$, we say that a sequence $(X_k | k=0,1,\dots,K)$, $X_k \subset S$, is a <u>trajectory compatible with</u> $(U_k | k=0,1,\dots,K-1)$ and $(Y_k | k=1,\dots,K)$ iff

$$(X_{k+1} \cup Y_{k+1}, X_k \cup U_k) \in \Lambda(A \vee B \vee C) \tag{7.1}$$

for k=0,1,...,K-1; (7.1) is equivalent to saying that there exist $X''_{k+1} \subset X_{k+1}$ and $X'_k \subset X_k$ such that

$$(X''_{k+1}, X'_k) \in \Lambda(A),$$

$$(X_{k+1} - X''_{k+1}, U_k) \in \Lambda(B),$$

and

$$(Y_{k+1}, X_k - X'_k) \in \Lambda(C).$$

The <u>underlying digraph</u> of (A,B,C) is defined to be a directed graph $(S \cup P \cup T, \Delta)$ with vertex set $S \cup P \cup T$ and arc set

$$\Delta = \{(x',x) | (\{x\},\{x'\}) \in \Lambda(A)\} \cup \{(u,x) | (\{x\},\{u\}) \in \Lambda(B)\} \cup \{(x,y) | (\{y\},\{x\}) \in \Lambda(C)\}. \tag{7.2}$$

$X \subset S$ is said to be <u>observable at time</u> k <u>from</u> $X_k \subset S$ iff there exists an output $(Y_i | i=1,\dots,k)$ of length k and a trajectory $(X_i | i=0,1,\dots,k)$ such that $X_0 = X$ and (7.1) holds with $U_i = \emptyset$ $(i=0,1,\dots,k-1)$. We denote by $OS_k(X_k)$ the collection of observable sets at time k from X_k. A system (A,B,C) is called <u>observable</u> iff

$$OS(\emptyset) = 2^S, \qquad (7.3)$$

where

$$OS(X) = \cup\{OS_k(X) \mid k=1,2,\ldots\}, \quad X\subset S. \qquad (7.4)$$

The combinatorial dynamical system (A^*,C^*,B^*) is called the dual of (A,B,C). Note that (7.1) is equivalent to

$$(X^*_{k+1}\cup Y^*_{k+1}, X^*_k \cup U^*_k) \in \Lambda(A^* \vee C^* \vee B^*),$$

where $X^*_k=X_{K-k}$, $Y^*_k=U_{K-k}$ and $U^*_k=Y_{K-k}$. Evidently, (A^*,C^*,B^*) is controllable (resp., observable) iff (A,B,C) is observable (resp., controllable). Using this relation we can translate the results established in §§5-6 for controllability into the corresponding results for observability.

In particular, we see that $OS(\emptyset)$ (cf. (7.4)) constitutes the family of independent sets of a matroid on S, just as $RS(\emptyset)$ (cf. (3.6)) yields the matroid R_∞ on S. That is, we have two matroids on S, one for observability and the other for controllability. Then we can define a decomposition of the state space S by applying the decomposition principle known as the principal partition [Iri 1979] to the pair of those matroids. The decomposition of S thus obtained may be thought of as a combinatorial analogue of Kalman's canonical decomposition [Kalman 1962, 1963], [Wonham 1979] of the state space into four parts. A similar combinatorial decomposition has been considered in [Murota 1986, §15.2] for a conventional dynamical system based on the same principle of principal partition of a pair of matroids. The detail of this decomposition as well as of further development of the theory will be reported before long.

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Addenda to "Combinatorial Dynamical System Theory"

(DPS 317, Univ. Tsukuba, September, 1986) Kazuo Murota Oct. 24, 1986

1. Prof. S. Fujishige kindly pointed out that the proof of Prop.4.4(2) implies the following characterization of maximum recurrent sets.

<u>Proposition 4.4</u>. (3) X is a recurrent set of A of maximum size iff X is a common base of $RM(A^{\infty})$ and $CM(A^{\infty})$. □

2. An alternative simpler proof of Theorem 4.5 was found.

(Alternative proof of Theorem 4.5) For $X \subset S$, let $\chi(X) \in R^S$ denote the characteristic vector, and for $x \in R^S$, let x^- and x^+ mean the corresponding vectors in R^{S^-} and R^{S^+}, respectively. For $x, y \in R^S$, we denote the direct sum of x^- and y^+ by $x^- \oplus y^+ \in R^{S^- \cup S^+}$.

Suppose X_0 is a maximum recurrent set and that $(X_i, X_{i-1}) \in \Lambda(A)$, $i=1,\ldots,m$, with $X_m = X_0$. Then

$$\chi(X_i)^- \oplus [\chi(S)^+ - \chi(X_{i-1})^+] \in B, \quad i=1,\ldots,m,$$

where B ($\subset R^{S^- \cup S^+}$) is the base polyhedron of the polymatroid P(A) associated with M(A). Since B is convex, we obtain

$$\tilde{h} = x^- \oplus [\chi(S)^+ - x^+] \in B,$$

where

$$x = \sum_{i=0}^{m-1} \chi(X_i) \;/m.$$

This shows that $\tilde{h}$ is a common base of P(A) and P_0, where P_0 is the polymatroid associated with M_0, and that $\tilde{h}(S^-) = |X_0|$.

Noting that $|H \cap S^-| \le |X_0|$ for any common base H ($\subset S^- \cup S^+$) of M(A) and M_0, and using the integrality of polymatroid intersection, we see that the maximum of $h(S^-)$ for a common base h of P(A) and P_0 is equal to $|X_0|$, which is attained by $\tilde{h}$. Then, by the integrality of polymatroid intersection, we can conclude that there exists a common base H ($\subset S^- \cup S^+$) of M(A) and M_0 such that $|H \cap S^-| = |X_0|$. □