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Applications of M-series and M-arrays
to spectral analysis and multislit imagings

by

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Abstract It is well known that M-series ($\rightarrow \S 2$) have various applications such as shift registers, cyclic codes, pseudo random numbers, modulation analysis, memory wheels and position-detecting of rolling wheel [4] [5] [7].

But as far as the author knows the application to spectral analysis has not been known, although the principle is almost same as those of mentioned applications. This is based on the two beautiful properties of M-series; 1-cycle property and quasi orthogonal property ($\rightarrow \S 2$).

Natural extension of M-series to 2-dimensional case leads us to the concept of M-arrays, whose structure also have been widely and deeply researched [3] [6]. Of course M-arrays also have beautiful properties analogous to M-series.

But so far applications of M-arrays have been confined to rather narrow fields such as 2-dimensional codes or 2-dimensional random numbers. Here we show a new application to multislit imagers [2], which is the 2-dimensional extension of spectral analysis. The author thinks, the 2-dimensionality of multisplit imager must have wider application in the field such as television, phototelegraphy, pattern recognition and satellites.

The application of Hadamard matrices to multislit imagers has been already known [2]. But the method by M-array is much more efficient than the one by Hadamard matrices mainly because of the 1-cycle property of M-array.

Beside these we propose a simple algorithm to construct M-array with period rectangle ($\rightarrow \S 2$) for applications to multislit imagers.

§ 1 M-series

Let $GF(q)$ be a Galois field of size q , where q is a prime power. And let us consider a t -th order 1-dimensional linear difference equation on $GF(q)$

$$u_i + f_1 u_{i-1} + f_2 u_{i-2} + \dots + f_t u_{i-t} = 0, \quad i \in Z, \quad f_j \in GF(q) \quad (1.1)$$

where Z is the set of all integers. We call a solution $\{u_i\} = \{u_i \in GF(q) \mid i \in Z\}$ of (1.1) a linear series, then periods of linear series are $L = q^t - 1$ at most. And a linear series having the maximal period $q^t - 1$ is called M-series of order t .

Of course $\{0\}$ is a solution of (1.1) which is called a trivial solution. It is well known that a non trivial solution of (1.1) is M-series iff the characteristic polynomial of (1.1)

$$f(x) = x^t + f_1 x^{t-1} + \dots + f_{t-1} x + f_t \quad (1.2)$$

is a primitive irreducible polynomial [5].

Let $f(x)$ be t -th order irreducible polynomial on $GF(q)$ then a root (or zero) α of $f(x)$ is in the extension field $GF(q^t)$. Let θ be a primitive element of $GF(q^t)$ and let

$$\alpha = \theta^a \quad (1.3)$$

then $f(x)$ is primitive iff

$$\gcd(a, q^t - 1) = 1 \quad (1.4)$$

where \gcd means greatest common divisor.

Thus a linear series or a solution of (1.1) is a M-series iff the characteristic equation $f(x) = 0$ of (1.1) has a root $\alpha = \theta^a$ satisfying (1.3) in $GF(q^t)$.

The following beautiful properties of M-series are well known [5],

1-cycle property : Let a non trivial solution $\{u_i \mid i \in Z\}$ of (1.1) is a M-series. We consider subseries (for $i = 1, 2, \dots, L$) of $\{u_i\}$ of length $L (= q^t - 1)$, $u_1 u_2 \dots u_L$, and from this we construct the following subseries of length L cyclically,

$$\left. \begin{array}{cccccc}
 u_1 & u_2 & \cdots & u_{L-1} & u_L & \\
 u_2 & u_3 & \cdots & u_L & u_1 & \\
 | & | & & | & | & \\
 u_L & u_1 & \cdots & u_{L-2} & u_{L-1} &
 \end{array} \right\} \quad (1.5)$$

then these L subseries are different from each other. And subseries (for $i = 1, 2, \dots, L$) of any non trivial solution of (1.1) can be found in (1.5).

quasi orthogonal property : Any two subseries in (1.5)

$$\begin{array}{l}
 v_1 \ v_2 \ \cdots \ v_L \quad (v_i = u_{i+k}) \\
 w_1 \ w_2 \ \cdots \ w_L \quad (w_i = u_{i+1})
 \end{array}$$

have the following property; each 2-dimensional vector of $GF(q)$ other than $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ occurs q^{t-2} times in $\begin{pmatrix} v_i \\ w_i \end{pmatrix}$ ($i = 1, 2, \dots, L$), and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ occurs $q^{t-2}-1$ times.

example 1.1 We consider a difference equation

$$u_i + 0 \times u_{i-1} + u_{i-2} + u_{i-3} = 0 \quad (1.6)$$

on $GF(2)$. It's characteristic polynomial $f(x) = x^3 + x + 1$ is a primitive irreducible, so a solution of (1.6) is a M-series. We choose $u_1=0, u_2=0, u_3=1$ as initial values, then we have the subseries

$$0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1$$

from which we can construct $L=7$ subseries cyclically which are shown in table 1.1. We can easily check the quasi orthogonal property in table 1.1. We can see later the meaning of w_i, y_i in the table.

table 1.1

W₁ W₂ W₃ W₄ W₅ W₆ W₇

0	0	1	0	1	1	1	y ₁
0	1	0	1	1	1	0	y ₂
1	0	1	1	1	0	0	y ₃
0	1	1	1	0	0	1	y ₄
1	1	1	0	0	1	0	y ₅
1	1	0	0	1	0	1	y ₆
1	0	0	1	0	1	1	y ₇

§ 2 application of M-series to spectral analysis

In this section we consider only binary case, i.e. only GF(2).

We disperse a beam through a prism to a plate on which the intensity of each color (or wave length) is measured by a detector (Fig 2.1). For simplicity we divide the plate into $L=2^t-1$, say, 7 unit intervals or cells, and let w_i be the light intensity of i -th cell ($i=1\sim 7$). Usually we measure w_1, w_2, \dots, w_7 by the detector one by one (Fig 2.1 (a)).

The new method by M-series is the following. First we construct a mask having $2 \times 7=14$ (two period length) cells some of which are transparent and other are opaque. A transparent cell corresponds to "1" and opaque one to "0" in M-series. In example 1.1 we have M-series of order $t=3$ with cycle of length $L=2^3-1=7$, whose two periods' portion is

$$0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1 \quad (2.1)$$

The mask in Fig 2.1(b) is constructed corresponding to (2.1). The dispersed beam goes through first one period portion of the mask, and is gathered by the lens to the detector which measures the total intensity of the passing beam (Fig 2.1(b)). Let us denote this by y_1 . Next we move up the mask by one cell, and denote the measured total intensity by y_2 , and so on.

Then clearly we have

$$\left. \begin{aligned} y_1 &= w_3 + w_5 + w_6 + w_7 + e_1 \\ y_2 &= w_2 + w_4 + w_5 + w_6 + e_2 \\ &| \\ y_7 &= w_4 + w_6 + w_7 + w_1 + e_7 \end{aligned} \right\} (2.2)$$

where e_i is error of i -th measurement of the detector, whose variance is $V(e_i) = \sigma^2$. And we can estimate $w_1 \dots w_7$ by the least square estimators.

$$\left. \begin{aligned} \hat{w}_1 &= (y_3 + y_5 + y_6 + y_7)/2 - Y/4 \\ \hat{w}_2 &= (y_2 + y_4 + y_5 + y_6)/2 - Y/4 \\ &| \\ \hat{w}_7 &= (y_4 + y_6 + y_7 + y_1)/2 - Y/4 \end{aligned} \right\} (2.3)$$

$$Y = y_1 + y_2 + \dots + y_7$$

Note that the coefficients in (2.2) correspond to the elements in table 1.1, but note that the former are in GF(2) and the latter are real numbers.

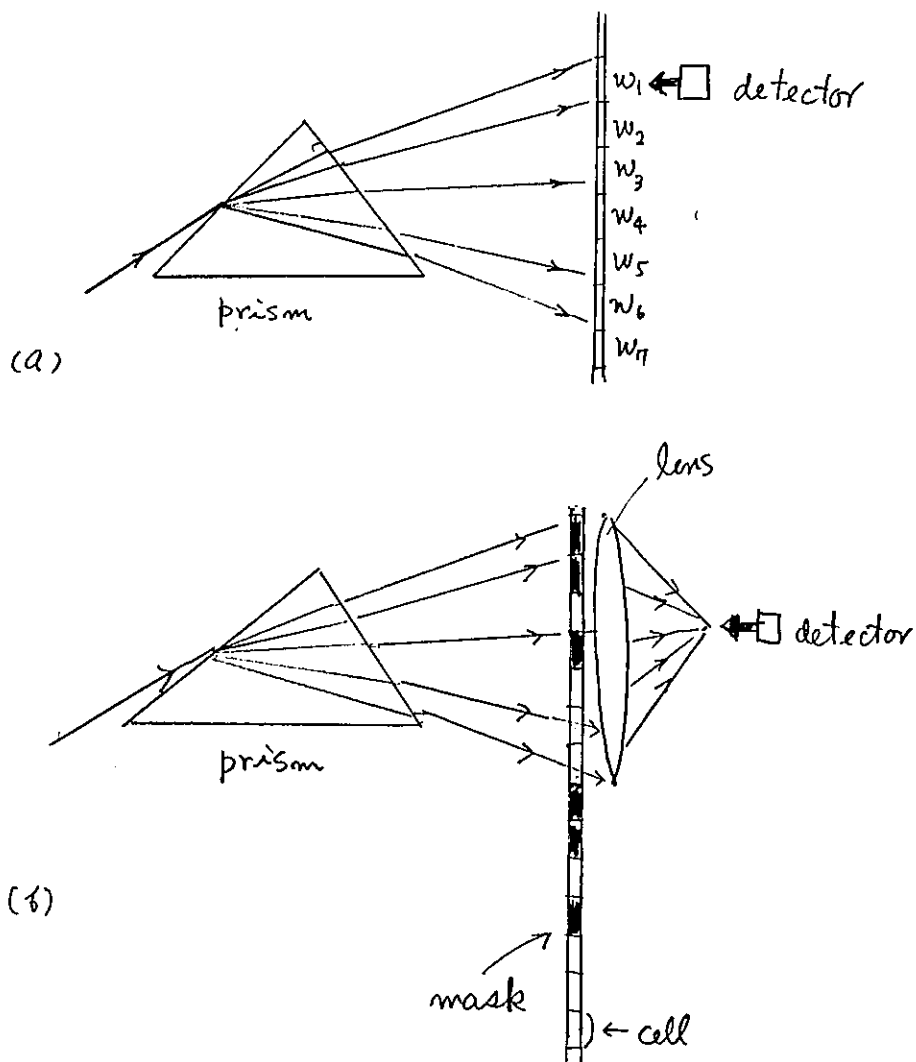


Fig. 2.1

For general t we have

$$y = Mw + e \quad (2.4)$$

where $y = (y_1 \ y_2 \ \dots \ y_L)^T$, $w = (w_1 \ w_2 \ \dots \ w_L)^T$, $e = (e_1 \ e_2 \ \dots \ e_L)^T$, and M is a $L \times L$ matrix constructed cyclically by M -series of order t as table 1.1.

From the quasi orthgonality of M -series we have

$$MM^T = 2^{t-2}I + 2^{t-2}J \quad (2.5)$$

where M^T is the transposed matrix of M , and J is all one matrix and I is unit matrix. And further we have

$$MJ = 2^{t-1}J \text{ or } M^{-1}J = 2^{1-t}J \quad (2.6)$$

Multiplying (2.5) by M^{-1} from left and viewing (2.6) we have

$$M^{-1} = 2^{2-t}M^T - 2^{1-t}J \quad (2.7)$$

Let \hat{w} be the least square estimator of w in (2.4), then because of the fact that the number of the observations is equal unknown parameters we have $\hat{w} = M^{-1}y$, so we have

$$\hat{w} = M^{-1}y = 2^{2-t}M^T y - 2^{1-t}Jy \quad (2.8)$$

For $t=3$ (2.8) results into (2.3).

By the well known property of least square the variance of \hat{w}_i is the i -th diagonal element of $(M^T M)^{-1}$ multiplied by σ^2 , and

$$(M^T M)^{-1} = 2^{2-t}I - 2^{1-t}J$$

so we have

$$V(\hat{w}_i) = 2^{1-t} \sigma^2 \quad (i=1 \sim L)$$

This shows that for large t M -series method gives far better estimation than the usual one-by-one method by which the variance of the estimator is σ^2 .

Optimality of M -series method

For any measuring of L times let $L \times L$ structure matrix be A (corresponding to M in (2.4)), then the so called information matrix is $A^T A$. Let λ and $|A^T A|$ be the minimal eigen value and determinant of $A^T A$ respectively, then we have a plenty of reasons of that the greater λ and $|A^T A|$ prove the better measurement. In this sense the M -series method is an optimal measurement based on the following theorem.

Theorem 2.1 The minimal eigen value of the information matrix $M^T M = 2^{t-2}I + 2^{t-2}J$ is equal to 2^{t-2} and the minimal eigen value λ of $A^T A$ for any A satisfies

$$\lambda \leq 2^{t-2} \quad (2.10)$$

Theorem 2.2 $|M^T M| = (L+1)2^{(t-2)L}$ and for any A we have

$$|A^T A| \leq (L+1)2^{(t-2)L} \quad (2.11)$$

For proofs of the above Theorems see [1].

§3 M-array

We consider 2-dimensional linear difference equations on $GF(q)$

$$\left. \begin{aligned} f_{00}^1 u_{ij} + f_{10}^1 u_{i-1,j} + f_{11}^1 u_{i-1,j-1} + \dots + f_{t_1, s_1}^1 u_{i-t_1, j-s_1} &= 0 \\ f_{00}^2 u_{ij} + f_{10}^2 u_{i-1,j} + f_{11}^2 u_{i-1,j-1} + \dots + f_{t_2, s_2}^2 u_{i-t_2, j-s_2} &= 0 \\ \dots & \\ f_{00}^k u_{ij} + f_{10}^k u_{i-1,j} + f_{11}^k u_{i-1,j-1} + \dots + f_{t_k, s_k}^k u_{i-t_k, j-s_k} &= 0 \end{aligned} \right\} (3.1)$$

$(i, j) \in Z^2$

where $f_{mn}^l \in GF(q)$.

We call a solution $\{u_{ij}\}$ of (3.1) a linear array. For a given $\{u_{ij}\}$ if we have

$$u_{i+1, j+k} = u_{ij}, \quad (i, j) \in Z^2$$

then $(k, 1) \in Z^2$ is called a period vector of $\{u_{ij}\}$. The totality of period vectors of $\{u_{ij}\}$ is a subspace of Z^2 . If the basis of the subspace consists of two independent vectors then $\{u_{ij}\}$ is called to have a double period. (The totality of period vectors of a linear array may have a basis consisting of only one vector, then the array is called to have a single period.)

If independent vectors $\mathbf{l}_1, \mathbf{l}_2 (\in \mathbb{Z}^2)$ constitute a basis of the totality of period vectors the parallelogram with \mathbf{l}_1 and \mathbf{l}_2 as two edges is called period parallelogram, and its area, which is always an integer, is called the period of $\{u_{ij}\}$. We can easily see that the period is independent of selecting way of basis $\{\mathbf{l}_1, \mathbf{l}_2\}$. If $\{u_{ij}\}$ has double period it consists of repetitions of period parallelograms thus $\{u_{ij}\}$ can be viewed to be intrinsically finite.

A linear array or a solution of (3.1) has a double period iff the characteristic equations of (3.1)

$$f^1(x, y) = 0, f^2(x, y) = 0, \dots, f^k(x, y) = 0 \quad (3.2)$$

have only finite number of solutions [6] (e. i. characteristic roots) in any extension field of $\text{GF}(q)$, where $f^i(x, y)$ is the characteristic polynomial of i -th equation of (3.1), that is,

$$f^i(x, y) = f_{00}^i x^t y^s + f_{10}^i x^{t-1} y^s + f_{11}^i x^{t-1} y^{s-1} + \dots + f_{11}^i \quad (3.3)$$

In the 2-dimensional case we cannot clarify the concept of order of difference equation nor the concept of the primitive irreducibility of polynomial without terminology of ideal theory.

Let $R = \text{GF}(q)[x, y]$ be the ring consisting of all bivariate polynomials on $\text{GF}(q)$. The ideal $\langle f^1, f^2, \dots, f^k \rangle$ generated by characteristic polynomials $f^i(x, y)$ ($i = 1 \sim k$) of (3.1) is called characteristic ideal of (3.1). Any characteristic root (x, y) also satisfies the equation $f(x, y) = 0$ for any $f(x, y) \in \langle f^1, \dots, f^k \rangle$.

The quotient ring $R/\langle f^1, \dots, f^k \rangle$ viewed as a vector space on $\text{GF}(q)$ has a finite dimension if a solution of (3.1) has double period, and let denote this dimension by

$$t = \dim(R/\langle f^1, \dots, f^k \rangle) \quad (3.4).$$

This dimension corresponds to the order of difference equation (1.1) in 1-dimensional case. Now we can define M -array as a linear array with a period equal to $q^t - 1$.

In the case of single variate polynomial if a t -th order polynomial $f(x)$ on $GF(q)$ is irreducible then the totality of roots of $f(x)$ constitutes a Frobenius cycle (FC) in $GF(q^t)$ (t -th order extension of $GF(q)$) [4] [5] [7]. Conversely if the totality of roots of $f(x)$ constitutes a FC then $f(x)$ is a power of a irreducible polynomial, which is called quasi irreducible. Thus $f(x)$ is quasi irreducible iff the totality of roots of $f(x)$ is a FC.

Now as long as we are concerned about $\{u_i\}$ with a finite period, we can confine ourselves to the quotient ring $R = GF(q)[x]/\langle x^n-1 \rangle$ instead of $GF(q)[x]$. And further we are aiming at M -series so we have only to consider the case that n and q are relatively prime. Then R becomes a semisimple ring, and any quasi irreducible polynomial becomes irreducible in R (\rightarrow [8]).

Thus we have

Theorem 3.1 A solution of (1.1) is a M -series iff the totality of characteristic roots of (1.1) constitutes a FC, and a characteristic root represented by (1.3) satisfies (1.4). ———

Now we proceed to bivariate polynomial ring $R = GF(q)[x, y]$. The concept of prime ideal in R is a natural extension of irreducible polynomial in $GF(q)[x]$. If $\langle f^1, \dots, f^k \rangle$ is a prime ideal of R then the totality of characteristic roots constitute a FC. And the primary ideal corresponds to quasi irreducible. Thus $\langle f^1, \dots, f^k \rangle$ is primary ideal iff the totality of characteristic roots of $\langle f^1, \dots, f^k \rangle$ constitutes a FC. If m and n are relatively prime to q then the $\bar{R} = R/\langle x^m-1, y^n-1 \rangle$ is semi simple ring and any primary ideal reduces to prime ideal in \bar{R} . Hereafter we consider \bar{R} instead of R (\rightarrow [8]). Thus we have

Theorem 3.2 A solution of (3.1) is a M -array iff the totality of characteristic roots of $\langle f^1, \dots, f^k \rangle$ constitutes a FC, and there exists a characteristic root (α, β) represented by

$$\alpha = \theta^a, \quad \beta = \theta^b \quad (3.5)$$

satisfying

$$\gcd(a, b, q^t - 1) = 1 \quad (3.6)$$

when θ is a primitive element of $GF(q^t)$ (\rightarrow [6]).

§ 4 A simple algorithm to construct M-arrays

In this section we consider only binary case. For the sake of theorem 3.2 we can construct a M-array whose period parallelogram is a $m \times n$ rectangle [3].

First we list up m, n, t satisfying

$$2^t - 1 = m \times n \quad (4.1)$$

in table 4.1. In these we can select appropriate value of m, n for our application.

(i) Construct the set of all roots (x, y) of $x^m - 1, y^n - 1$, and decompose the set into several FCs. Find a FC whose element $(\alpha, \beta) = (\theta^a, \theta^b)$ satisfies $\gcd(a, b, 2^t - 1) = 1$.

(ii) Find polynomials $f^1(x, y), f^2(x, y), \dots, f^k(x, y)$ such that the totality of common roots of $f^1(x, y), \dots, f^k(x, y)$ just coincides with the above FC.

To do this first find two minimal order polynomials f^1, f^2 whose common root is just (α, β) . And if all other common roots of f^1, f^2 are in the FC, then f^1, f^2 are required ones (in this case $k=2$).

But if any other common root of f^1, f^2 does not belong to the FC, then we find the next minimal order polynomial f^3 whose root is (α, β) . And if all common roots of f^1, f^2, f^3 are in the FC then f^1, f^2, f^3 are required ones (in this case $k=3$). And so on.

(iii) k difference equation corresponding to f^1, \dots, f^k determine the required M-array with appropriate initial values.

table 4.1

t	2t-1	m × n
4	15	3 × 5
6	63 (3 ² ·7)	3 × 21 7 × 9
8	255 (3·5·7)	3 × 75 5 × 51 15 × 17
10	1023 (3·11·31)	3 × 341 11 × 93 31 × 33
11	2047 (23·89)	23 × 89
12	4095 (3 ² ·5·713)	3 × 1365 5 × 819 7 × 585 9 × 455 13 × 315 15 × 273 21 × 195 35 × 117 39 × 105 45 × 91 63 × 65

Example 4.1 We consider the case $t=4$, $m=3$, $n=5$. Let θ be a primitive element of $GF(2^4)$ and $\theta^4 = 1 + \theta$, then we factor x^3-1 and y^5-1 to irreducible polynomials, and we have

$$x^3-1 = (x+1)(x^2+x+1) : \text{its roots are } 1, \theta^6, \theta^{10}$$

$$y^5-1 = (y+1)(y^4+y^3+y^2+y+1) : \text{its roots are } 1, \theta^3, \theta^6, \theta^{12}, \theta^9$$

All common roots of x^3-1 , y^5-1 are shown in table 4.2, and are decomposed into five FCs, where elements of a FC are connected with lines.

table 4.2

$x \setminus y$	θ^0	θ^3	θ^6	θ^{12}	θ^9
θ^0	(θ^0, θ^0)	$(\theta^0, \theta^3) - (\theta^0, \theta^6) - (\theta^0, \theta^{12}) - (\theta^0, \theta^9)$			
θ^5	(θ^5, θ^0)	(θ^5, θ^3)	(θ^5, θ^6)	(θ^5, θ^{12})	(θ^5, θ^9)
θ^{10}	(θ^{10}, θ^0)	(θ^{10}, θ^3)	(θ^{10}, θ^6)	$(\theta^{10}, \theta^{12})$	(θ^{10}, θ^9)

Let $(\alpha, \beta) = (\theta^5, \theta^3)$ then we have $\gcd(5, 3, 15) = 1$ satisfying the condition (3.6).

To find the minimal order polynomials whose root is (θ^5, θ^3) we solve the following linear equations on $\text{GF}(2)$.

$$\begin{array}{l}
 f_{00} \ f_{10} \ f_{01} \ f_{20} \ f_{11} \ f_{02} \ f_{21} \ f_{12} \ f_{03} \ f_{22} \ f_{13} \ f_{04} \ f_{23} \ f_{14} \ f_{24} \\
 \begin{array}{l}
 1 \rightarrow \\
 \theta \rightarrow \\
 \theta^2 \rightarrow \\
 \theta^3 \rightarrow
 \end{array}
 \begin{array}{|cccccccccccccccc}
 \hline
 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
 \hline
 \end{array}
 \end{array} = 0 \quad (4.2)$$

$$\begin{array}{cccccccccccccccc}
 \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 1 & x & y & x^2 & xy & y^2 & x^2y & xy^2 & y^3 & x^2y^2 & xy^3 & y^4 & x^2y^3 & xy^4 & x^2y^4 \\
 1 & \theta^5 & \theta^3 & \theta^{10} & \theta^8 & \theta^6 & \theta^{13} & \theta^{11} & \theta^9 & \theta & \theta^{14} & \theta^{12} & \theta^4 & \theta^2 & \theta^7
 \end{array}$$

(These equations are constructed by the following way. First $x^i y^k = \theta^{5i+3k}$ is calculated to a 3-rd order polynomial of θ (by $\theta^4 = 1 + \theta$), and these coefficients are arranged in the column corresponding to $x^i y^k$.)

Sweeping out (4.2) with circled elements in (4.3) as pivots we have

$f_{00} \ f_{10} \ f_{01} \ f_{20} \ f_{11} \ f_{02} \ f_{21} \ f_{12} \ f_{03} \ f_{22} \ f_{13} \ f_{04} \ f_{23} \ f_{14} \ f_{24}$

①	0	0	1	0	1	0	0	1	1	1	1	0	1	0	= 0 (4.3)
0	①	0	1	0	0	0	1	1	1	0	1	1	0	1	
0	0	0	0	①	1	1	0	1	1	0	0	1	1	1	
0	0	①	0	0	1	1	1	1	0	1	1	0	0	1	

Setting $f_{20}=1$ (and all other non basic variables = zero) we have $f^1(x, y) = 1+x+x^2$.

Setting $f_{02}=1$ (and all other non basic variables = zero) we have $f^2(x, y) = 1+y+y^2+xy$.

The totality of common roots of $f^1(x, y)$, $f^2(x, y)$ is just the FC $=\{(\theta^5, \theta^3), (\theta^{10}, \theta^6), (\theta^5, \theta^{12}), (\theta^{10}, \theta^9)\}$. So we have the following difference equations as required,

$$u_{1,j}+u_{1+1,j}+u_{1+2,j}=0, \quad u_{1,j}+u_{1,j+1}+u_{1,j+2}+u_{1+1,j+1}=0 \quad (4.4)$$

A solution of (4.4) (with initial values shown in fig 4.1) is shown in Fig4.1.

Example 4.2 $t=6, m=7, n=9$. Primitive element θ of $GF(2^6)$, $\theta^6=1+\theta$.

$$x^7-1 = (x+1)(x^3 + x + 1) (x^3 + x^2 + 1)$$

$$\theta^{27}, \theta^{54}, \theta^{45} \quad \theta^9, \theta^{18}, \theta^{36}$$

$$y^9-1 = (y+1)(y^2 + y + 1) (y^6 + y^3 + 1)$$

$$\theta^{21}, \theta^{42} \quad \theta^7, \theta^{14}, \theta^{27}, \theta^{56}, \theta^{49}, \theta^{35}$$

$$(\alpha, \beta) = (\theta^9, \theta^7), \text{gcd}(9, 7, 63) = 1$$

The equations corresponding to (4, 2) and (4, 3) are

$f_{00} \ f_{10} \ f_{01} \ f_{20} \ f_{11} \ f_{02} \ f_{30} \ f_{21} \ f_{12} \ f_{03} \ f_{40} \ f_{31} \ f_{22} \ f_{13} \ f_{04}$

1	0	0	1	1	0	0	0	1	1	0	0	1	1	0
0	0	1	1	1	0	1	1	0	1	1	0	0	1	1
0	0	1	1	0	1	1	0	0	0	1	1	0	0	0
0	1	0	1	0	0	1	0	1	1	0	0	1	0	0
0	1	0	0	1	1	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	0	1	1	1	0	1	0	1	0

= 0

1 x y x² xy y² x³ x²y xy² y³ x⁴ x³y x²y² xy³ y⁴
 1 θ^9 θ^7 θ^{18} θ^{16} θ^{14} θ^{27} θ^{25} θ^{23} θ^{21} θ^{36} θ^{34} θ^{32} θ^{30} θ^{28}

①	0	0	0	0	1	1	0	0	1	1	0	0	0	0
0	0	0	0	①	1	0	0	1	0	0	0	0	0	1
0	0	①	0	0	1	0	0	0	0	0	1	1	1	0
0	①	0	0	0	0	0	0	1	1	1	0	0	1	0
0	0	0	①	0	0	1	0	0	0	1	0	1	1	1
0	0	0	0	0	0	0	①	1	1	0	1	0	1	0

= 0

$$f_{02} = 1 \text{ (other non basis = zero)} \quad f^1(x, y) = 1+y+y^2+xy$$

$$f_{30} = 1 \text{ (other non basis = zero)} \quad f^2(x, y) = 1+x^2+x^3$$

Corresponding difference equations are

$$u_{1,j} + u_{1,j+1} + u_{1,j+2} + u_{1+1,j+1} = 0, \quad u_{1,j} + u_{1+2,j} + u_{1+3,j} = 0$$

The desired M-array are shown in Fig4.2.

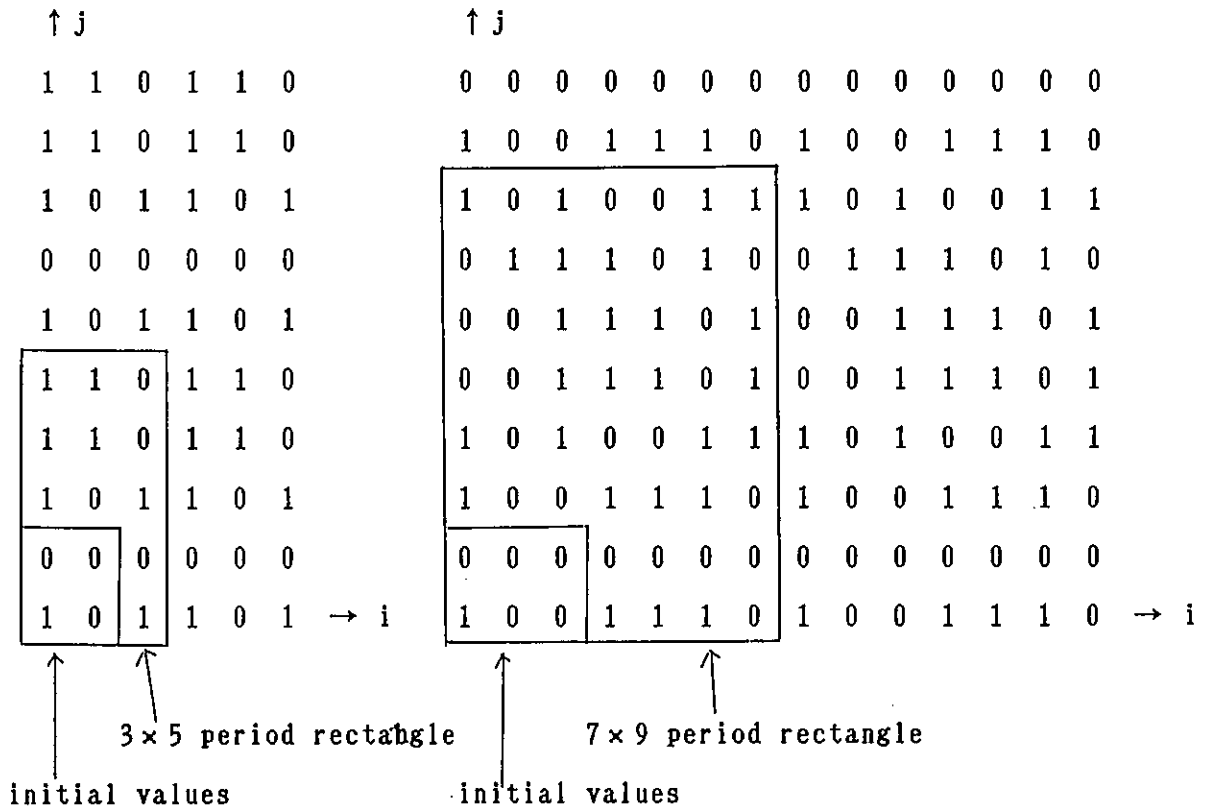


Fig4.1 3x5 M-array

Fig4.2 7x9 M-array

§ 5 Application of M-array to multislit imager

A M-array has two beautiful properties like M-series.

1-cycle property Now we have a M-array $\{u_{ij}\}$ on Z^2 , which is a solution of (4.1), and whose period parallelogram is just a $m \times n$ rectangle. We try to look into $\{u_{ij}\}$ through $m \times n$ moving rectangle window (Fig5.1).

First the left bottom corner of the window is set at origin $(0,0)$, and this moves up by one cell (keeping its vertical axis parallel to j -axis in Z^2). It continues this process $(n-1)$ times, then moves to right by one cell and moves down $(n-1)$ times, then its bottom line lies on i -axis. Thus the coordinate of the left bottom corner of the window

$$\begin{aligned} &(0, 0), (0, 1), \dots, (0, n-1); (1, n-1), (1, n-2), \dots, (1, 0); \dots \\ &(m-1, 0), (m-1, 1), \dots, (m-1, n-1) \end{aligned} \quad (5.1)$$

Let $A_{i,j}$ be $m \times n$ array (or $m \times n$ matrix whose elements are in $GF(2)$) appeared in the window whose left bottom corner's coordinate is just (i, j) . Then $A_{i,j}$ ($i = 0 \sim m-1, j = 0 \sim n-1$) are different from each other, and the $m \times n$ rectangle portion of any non trivial solution of (4.1) can be found in

$$A_{i,j} \quad (i = 0 \sim m-1, j = 0 \sim n-1) \quad (5.2)$$

quasi orthogonal property If any two different $A_{i,j}, A_{k,l}$ in (5.2) are superposed then each pair of elements (a,b) occurs 2^{t-2} time for $(a,b) = (0,1), (1,0), (1,1)$ and $(0,0)$ occurs $(2^{t-2}-1)$ times in the $m \times n$ superposed array.

Now we will explain the application to multislit imager which is direct extension to 2-dimensional case of spectral analysis.

We consider the transmitting of 2-dimensional figures in such fields as satellite or space-probe surveillance or television images. Through the camera objects are reflected on the rectangular plate with, say, $m \times n$ cells. Usually the detector scans cells of the plate and measures the light intensity of each cell and these informations are transmitted one by one.

Our multislit method by M -array is as follows ; First we have $(2m-1) \times (2n-1)$ portion of the M -array as a mask pattern, in which "1" corresponds to transparent cell and "0" to opaque cell. And this mask is put on the $m \times n$ plate. We move the plate by the way of (5.1) and gather the whole light passes through the mask with the lense to the detector which measures the total light amount for each moving stay (Fig 5.1).

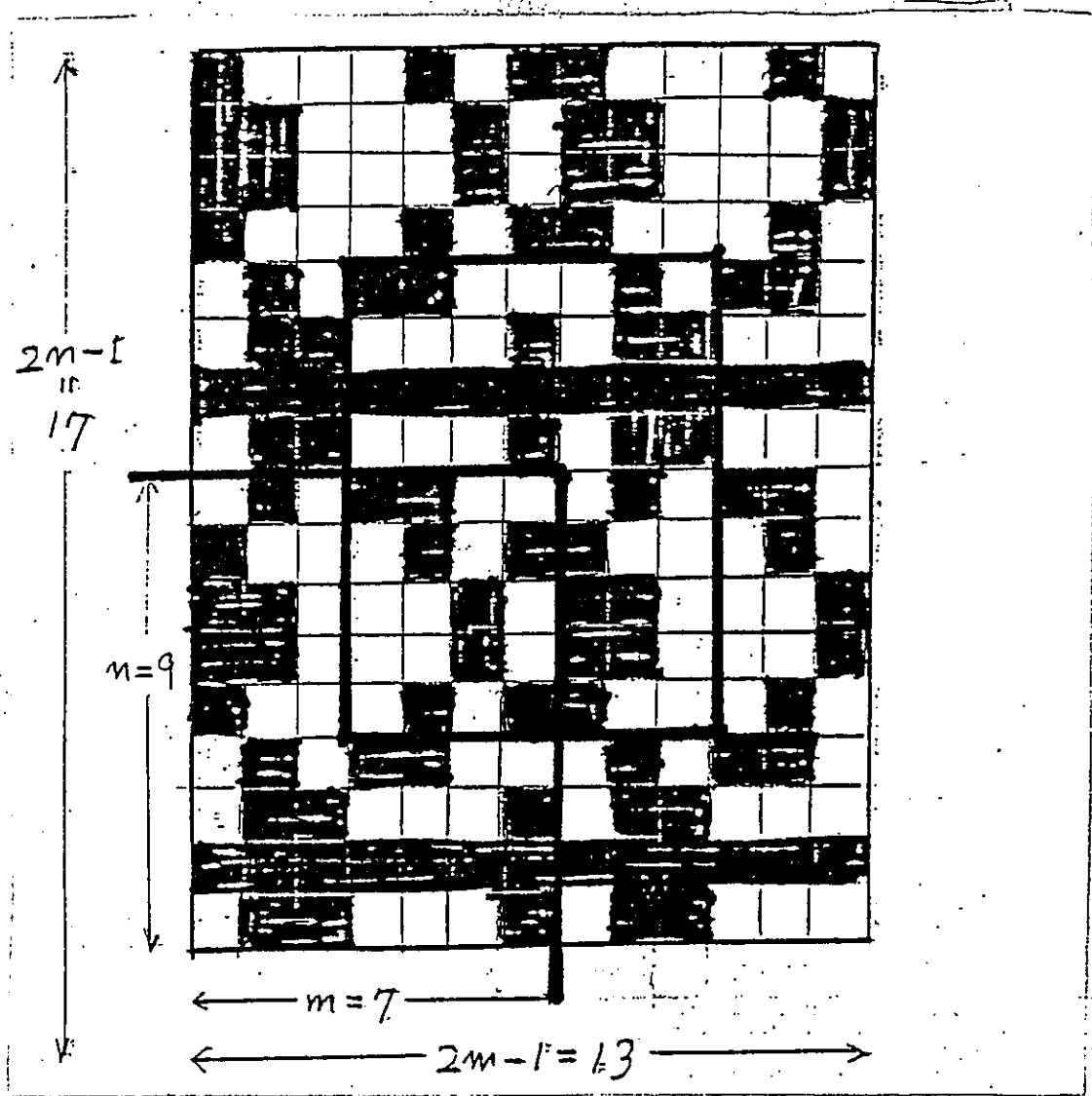


Fig 5.1 mask pattern by 7x9 M-array

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