No. 308

NICE DEMANDS AND CONCAVIFIABLE SMOOTH PREFERENCES — An Extensive Review

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ABSTARCT

By constructing a simple, common framework of the local demand analysis, this paper aims, first, at giveing a concise, formal review of the results so far obtained in the literature, both on the conditions for differentiable demands and concavifiable smooth preferences, and on the related local properties of demands generated by strongly convex preferences. Then, in this framework and adpapting these results, we shall prove that sets of Pareto-Edgeworth complements, substitutes and independents will restrict the generated differentiable demand functions, in such a way that complements to a normal (resp.inferior) good are normal (resp. inferior) whereas substitutes for a normal (resp. inferior) are inferior (resp.normal), and, independents of a normal (or inferior) good are neither normal nor inferior, in the local neighborhood of a certain demand point where the income derivative of the marginal utility of money vanishes. This result still holds for the asymptotic income effects at the critical point where demands are not differentiable.

Giffen Paradox and Related Properties of Demands and Preferences: A Preliminary Survay

One of the less obvious phenomenon in the consumer's behavior is the occurrence of the so called Giffen goods (Giffen[1879]), that is, more goods (resp. less services) are demanded (resp. supplied) as their prices go up. Such a paradoxical phenomenon was pointed out in a Paretian framework of the consumer theory. Marshall[1890 109-110] also noted the possibility of this paradox in a conceptually different but essentially the same way (Hicks[1956 12-15]). The demand $d_i(p,\lambda)$ of a Marshallian consumer may behave like a Giffen good if demand d_i price p_i and the marginal utility of money λ, all change in the same direction and if the price deirvative of λ is large enough.

Several decades have ever since passed away, but little literature is available to explain any conditions that restrict the consumer' preferences that yield this phenomenon. A source of the ambiguity arises from a lack of the full knowledge of how underlying preferences R^* are related to each of the two primitive concepts; the income effects on the Paretian demands due to a change in price, and, the indirect effects on the Marshallian demands. The former involves the income derivatives of demand, which equal the price derivatives of the marginal utility of money (in the absolute values)involved in the latter effects. Samuelson[1946 9-195] has seen a restricted class of utility functions alone consistent with the constancy of the marginal utility of money in price or income.

To name only a few that rule out such a paradox, Stigler[1951] gavea a verbal proof for the old case of an additive utility function with diminishing marginal utilities. Georgescu-Roegen[1952] explained how the old concept of complementarity does affect the shape of indifference curves.

In a recent review [1974], Samuelson reconsidered his own skeptical remarks on complementarity [1947 183-189]. Dhrymes[1967 proposition 7] and most recently Chipman[1977] gave proofs to establishing the law of demand and both dealt with a strictly concave utility function which has nonnegative cross derivatives.

However, these results premised the existence of a particular or "determinate" utility function. This hypothetical utility (and its existence in the equivalent class of utility indeces, each representing equally the consumer's preferences R*) itself was an open question. In this sense, their results were partial.

Not a few number of authors, however, have been concerned with some related (but important for its own sake) properties that <u>concave</u> utilities yield on the generated demands but not shared by demands generated by (convex preferences representable by) quasi-concave utilities.

More than a decade ago Rader[1973] has argued that a concave utility function yields <u>almost everywhere</u> differentiable demand functions from which most quantity (Slutsky) variations are derived,*1 provided that the demand functions satisfy a uniform (local) Lipschitz condition in income. This result was partially conjectured by Hicks[1956] and McKenzie[1957] who initiated the minimum income approach usually doing without a utility index involved.*2

<u>A Mathematical Review</u>:

<u>1. Postulates</u>:

Assume that the consumption space Y be the interior of the positive cone of R^n. Then, Debreu[1972] studied three ways of approaching the questions of smooth preferences and showed the following three postulates are equivalent. (i) A monotone, continuous, complete and transitive binary relation R* on Y of class C^2. (ii) A function (vector field) $g(x)>0$ from Y to the unit sphere of R^n of class C^1, satisfying the (local) integrability conditions on Y. (iii) A real-valued function u(x) on Y of class C^2, satisfying $Du(x)>0$ for every $x \varepsilon Y$. We may take one, any one of the above postulates hereafter.

Assume additionally that R* is strongly convex on an open convex subset X of Y. Let $\bar{p}$ be an n dimensional price vector and $\bar{m}$ be a positive income (real number), such that, with respect to R*, the unique maximal demand point $\bar{x}$ in the budget set for the price-income system $(\bar{p},\bar{m})$ (which is a multi-valued function in prices and income) is contained in the interior of X. Then, there are two open neighborhoods N of $\bar{p}$ and M of $\bar{m}$ such that, for each (p,m) in a cartisian product N x M, a single valued function d(p,m) is well defined and continuous in N x M. However, the assumption made so far on R*, including the representability of R* by a twice continuously differentiable utility function defined on X, does not imply the everywhere differentiability of the demand functions. Katzner [1969] shows an example. See also Dhrymes[1967] for an earlier published work.

2. Nice Demand and Concavifiable Smooth Preferences:

Debreu[1972] proved d(p,m) is differentiable with respect to price at (p,m), if and only if the Gaussian curvature of the

$x=d(p,m)$.[*3] Here the curvature, denoted $c(x)$, is the Jacobian determinant $|\partial g_i/\partial x_j+(\partial g_i/\partial x_n)(-g_j/g_n)|$ of an n-1 dimensinal vector-valued function which maps $(x_1,\ldots,x_{n-1})$ into $(g_1(x),\ldots,g_{n-1}(x))$, where x_n is determined by $u(x)=\bar{u}$ as a function of $(x_1,\ldots,x_{n-1})$. The condition of non-zero curvature becomes equivalent to the non-zero bordered Hessian determinant of utility $u(x)$ at x shown by Dhrymes and Katzner as the condition for everywhere differentiability. Note here that from the homogeneity of demands, that is, for a fixed $m°$, $d(p,m)=d(pm°/m,m°)$, if $d(p,m)$ is differentiable with respect to price at (p,m), then, it is differentiable with respect also to income m. The strong convex preference, hence a strict quasi-concave utility function[*4] does not imply that the bordered Hessian determinant does not vanish anywhere. But the strong convexity does imply the non-vanishing Gaussian curvatures on a dense open subset of the indifference surface (Smale[1974]). This property plays an important role in a local demand analysis.

The problem of finding conditions, under which a quasi-concave function can be transformed into a concave function by means of an increasing real valued function, was initiated perhaps by deFinetti[1949] and Fenchel[1953], including which the mathematical references are given in Arrow and Hurwicz[1977 67], and extensively investigated by Kannai[1977,81].

For any and each ordinal utility function u representing R*, the vector field $g(x)$ is proportional to the gradient vector $Du(x)$. Denote the proportionality factor by $\lambda(x)$at x. Then,

$$Du(x)=\lambda(x)g(x),\ DDu(x)=\lambda(x)Dg(x)+[D\lambda(x)]^T g(x). \tag{1}$$

The symmetry of the Hessian DDu(x), whose elements are $\partial^2 u/\partial x_i \partial x_j = \partial g_i/\partial x_j + (\partial\lambda/\partial x_j)g_i$, i, j=1,..,n,[*5] does not imply the symmetry of the Jacobian Dg(x). The unit normal vector g implies that Dg(x) is of less than the maximum rank n at x.

3. A Simple Framework to Specify the Related Condtions and Show their Interrelationship[*6]

In order to specify the conditions in an equivalent and simple way, choose a new coordinates system $\{\xi_1\xi_2\xi_3\xi_3\ldots\xi_{n-1}\}$ parallel to the hyperplane $H(\bar{x})=[y\epsilon X;py=p\bar{x}]$, tangent at $\bar{x}$ to the indifference hypersurface $I(\bar{x})$, so that $\bar{x}$ may be the new origin in the new coordinates. The n-1 by n-1 (symmetric) principal minor of DDu(x) matrix, denoted $A_{n-1}(x)$, whose elements are $\partial^2 u(\bar{x})/\partial x_i \partial x_j$, i, j=1,2,...,n-1, is restricted to the subspace of R^n orthogonal to $(\bar{p}_1,\ldots,\bar{p}_n)$.[*7] This allow us to write $H(\bar{x})$ simply as $\xi_n=\bar{\xi}_n$. Then, $g_i(\bar{x})=\bar{p}_i=0$ for each i=1,2,...,n-1 and $g_n(\bar{x})=1$, where $\bar{x}=\bar{\xi}=0$ in the new coordinates. Denote, by $\alpha_i(\bar{x})$, i=1,..,n-1, the characteristic roots of $A_{n-1}(x)$ restricted to the subspace. Then, $A_{n-1}(\bar{x})$ is given in a diagonalized form, that is,

$$A_{n-1}(x)=[\alpha_i\delta_{ij}]_{i,j=1}^{n-1} \qquad \alpha_i=\alpha_i(\bar{x}). \tag{2}$$

Also let us denote $\partial g_i(\bar{x})/\partial x_n$ by $\beta_i(\bar{x})$ and $\partial\lambda(\bar{x})/\partial x_n/\lambda(\bar{x})$, by $E_n(\bar{x})$. Then, the Jacobian $Dg(\bar{x})$ is reduced to $\begin{bmatrix}\alpha_i\delta_{ij} & \beta_i \\ 0 & 0\end{bmatrix}$, where $\beta_i=\beta_i(\bar{x})$. Since the n-1 by n-1 principal minor, denoted $S(\bar{x})$, of the Jacobian, $Dg(\bar{x})$, and that bordered by a row of g_i and a column of g_j and 0 simply become, for this new system, to $S(\bar{x})=A_{n-1}(\bar{x})$ and $\begin{bmatrix}S(\bar{x}) & \beta^T & 0 \\ & 0 & 1 \\ & 1 & 0\end{bmatrix}$

where $\beta=(\beta_1,\ldots,\beta_{n-1})$, respectively,

The curvature $c(\bar{x})$ is given in a simple form,

$$c(\bar{x})=\Pi_{i=1}^{n-1}\ \alpha_i(\bar{x}) \qquad (3)$$

at $\bar{x} = d(\bar{p},\bar{m})$. Thus, the non-zero curvature condition is equivalent to $\alpha_i(\bar{x})\neq 0$ for each $i=1,2,\ldots,n-1$, or equivalently, $|S(\bar{x})|\neq 0$. Let us denote the principal curvatures by $c_k(\bar{x})$, $k=2,..,n-1$, where $c_k(\bar{x})$ is proportional to that obtained by replacing n by k in the bordered Jacobian. Then, the (strong) convex preferences assumption implies that $A_{n-1}(\bar{x})$ is negative semi-definite on the hyperplane $H(\bar{x})$, hence implying $(-1)^{k-1}\Pi_{i=1}^{k-1}\alpha_i \geqq 0$, $k=2,3,..,n$. Now the Hessian $DDu(\bar{x})$ is given by $\lambda(\bar{x})\begin{bmatrix} S(x) & \beta^T \\ \beta & E_n(\bar{x}) \end{bmatrix}$ and

$$|DDu(x)| = \lambda(\bar{x})^n\ \Pi_{i=1}^{n-1}\alpha_i(E_n(\bar{x})-\ \Sigma_{i=1}^{n-1}\beta_i^2/\alpha_i), \qquad (4)$$

provided that $\bar{x}$ is a differentiability point, that is, $\alpha_i(\bar{x})\neq 0$, $i=1,\ldots, n-1$, at $\bar{x}$. See, for this reduction, Bellman[1970,69]. Recall Smale's result, that is, under the strong convex preferences, differentiability points are dense near a non-differentiability point. If $\bar{x}$ is a non-differentiability point so that $\alpha_i(\bar{x})=0$ for some i, then, β_i/α_i is defined as the limsup of β_i/α_i taken over differentiability points, x, contained in every neighborhood (of $\bar{x}$) intersecting $I(\bar{x})$ and converging to $\bar{x}$. The famous one point conditions for concavifiability, obtained by Fenchel [1953, 133-137, 1956, 500-503] (Kannai[1977,30-31][1981,574-576]), show that R* is concavifiable if there exists a real valued function $\lambda(x)$ of class C^1 near $\bar{x}$, such that $\lambda(\bar{x})\neq 0$, $\lambda(\bar{x})\ E_i(\bar{x})=\beta_i(\bar{x})$, $i=1,2,\ldots,n-1$ and

$$E_n(\bar{x})-\Sigma_{i-1}^{n-1}\ \beta_i^2(\bar{x})/\alpha_i(\bar{x})\leqq 0. \qquad (5)$$

All the conditions, that include $\alpha_i\leqq 0$, $i=1,..,n-1$, required for concavifiability, except the last inequality(5), are already established.

The existence of $\lambda(.)=|Du(.)|$ satisfying the last condition implies u is a concave function of C^2 near $\bar{x}$. Note also the condition (5) is equivalent to that $(-1)^n|DDu(\bar{x})|\geqq 0$. This condition is also equivalent to saying that the marginal utility of income is non-increasing in income, if $\alpha_i \neq 0$, $i=1,\ldots,n-1$, since

$$\partial\lambda(\bar{x})/\partial m=\partial\lambda(\bar{x})/\partial x_n-\lambda(\bar{x})\Sigma\beta_i^2(\bar{x})/\alpha_i(x)\leqq 0 \qquad (6)$$

4. Almost Everywhere Differentiable Demands and Concavifiability Condition

Consider now all prices are fixed but income variable. We do not always suppose that $\alpha_i \neq 0$, $i=1,\ldots,n-1$ at $\bar{x}=d(\bar{p},\bar{m})$, except otherwise stated.

Kannai[1985] argued that the demand functions d(p,m) satisfy a uniform (equi-, local) Lipschitz condition in income in a neighborhood of $(\bar{p},\bar{m})$, if and only if there exists a constant r>0 such that, for all x near $\bar{x}$ and for all i for which $\alpha_i(x)\neq 0$, $|\beta_i(x)/\alpha_i(x)|\leqq r$. The former condition implies a pointwise Lipschitz condition, hence, by Rademacher's theorem (Federer [1969, p.216], Rado and Reichelderfer[1958, 336 Lemmas 1,2]), d(p,m) is differentiable in income almosteverywhere in the neighborhood. Sard's theorem [1942, p.883] is applied to the critcal values of g in the n-1 sphere. Consequently at almost all prices, demands are obtained as differentiable in prices. Furthermore their income deriatives are uniformly bounded if the latter condition $|\beta_i(x)/\alpha_i(x)|\leqq r$ near $\bar{x}$ holds. Such demand $d(\bar{p},\bar{m})$, at $(\bar{p},\bar{m})$ where $\bar{p}$ may be a critical value, is the limit of uniformly Lipschitzian demands taken over nearby regular prices. Hence, the income Lipschitz condition is established. Also this latter condition (together with the integrability condition) allow us to establish the concavifiability condition (5). In fact, from any pair $(v(x),\mu(x))$ such that $Dv(x)=\mu(x)g(x)$ where v does exist on X under postulate (ii), a pair $(u(x),\lambda(x))$ can be chosen under the condition, in such a way that (5) may hold. See deFinetti[1949, 173-174].

This point remind us of the early result of Rader[1972 Theorem 12] referred above to. Under the strong convexity assumption that Rader seems to have made [1973, 917], the concave utility assumption could have been dispensed, in so far as a locally nice demand is concerned, because, at least near $\bar{x}$, the hypothesis of income Lipschitzian demands implies the concavifiability of R*, as we have seen above.

The intended geometric interpretation for the Lipschitz condition near a differentiability point $\bar{x}$ is that the Engel Curves d(p,.) for each p near $\bar{p}$ intersect the indifference surface uniformly transversally near $\bar{x}$. If demand function $d_i(p,m)$ has a bounded income derivative vector $\partial d_i(p,m)/\partial m$ in the neighborhood N x M, then the locally uniform Lipschitz condition is satisfied. Cf-Uzawa[1960, 1971] for a role of this condition in the definition of continuous demand functions in m.

Thus, if at any point in X, $\alpha_i < 0$, i=1.2...,n-1, then the utility function can be chosen to be concave at least in compact subsets of X where $|\beta_i/\alpha_i|$, i=1,...,n-1, are bounded above. Thus, the difference between concavifiable and non-cancavifiable utilities appears only near points where demand functions are not differentiable.

5. Local Behavior of Demand at a Non-Differentiability point:

An extensive concern was naturally directed to behaviour of demands near these non-differentiability points. Let us return to the original coordinates. In view of the fact remarked in the footnote*7 about the effects of the orthogonal changes on $\partial d_i/\partial p_j, i,j=1,..,n$, applications of the formula to the knowledge of (the inverse of) the Hessian $DDu(\bar{x})$ restricted to the new coordinates $\{\xi_1,..,\xi_{n-1}\}$ with $\xi_n=0$, might lead to a simplification.*8 In the present and following subsections, however, an advantage will be taken of investigating directly the local behavior in the original coordinates $\{x_1,..,x_n\}$. Let $\bar{A}_n$ denote the usual bordered Hessian;

$$\bar{A}_n = \begin{bmatrix} u_{ij} & p_i \\ p & 0 \end{bmatrix} \quad i,j=1,..,n$$

and $|\bar{A}_n|$ the determinant of $\bar{A}_n$. Let $\bar{A}_n(i_1,i_2,..,i_k;j_1,j_2,..,j_k)$ denote the minor obtained by deleting the rows $i_1,i_2,..,i_k$ and the columns $j_1,j_2,..,j_k$ from $\bar{A}_n$, where $1\leqq i_1<i_2 ...<i_k\leqq n+1$, $1\leqq j_1<j_2<...<j_k\leqq n+1$. Then, it is well known the own-price derivative $\partial d_n(\bar{p},\bar{m})/\partial x_n$ of the n the demand at $(\bar{p},\bar{m})$ is decomposed into two terms; see Slutsky[1915,52],

$$\partial d_n(\bar{p},\bar{m})/\partial x_n = \lambda(\bar{x})|\bar{A}_{n-1}|/|\bar{A}_n| + d_n(\bar{p},\bar{m})|\bar{A}_n(n+1;n)|/|\bar{A}_n| \quad (7)$$

provided that $|\bar{A}_n| \neq 0$ at $\bar{x}=d(\bar{p},\bar{m})$. If $|\bar{A}_n|=0$, the limsup of own-price derivatives at nearby regular prices, at each of which $\bar{A}_n$ does not vanish, is taken instead. Compare the asymptotic substitution and income terms, as determinant $|\bar{A}_n|$ approaches zero. Hurwicz, Jordan and Kannai[1985] proved, by finding that $(|\bar{A}_n(n+1;n)|/|\bar{A}_n|)^2$ is bounded above by a positive scalar multiple of $-|\bar{A}_{n-1}|/|\bar{A}_n|$, that the negative asymptotic substitution term could not be dominated by the infinite asymptotic income term, if the preference ordering R* is represented by a concave, twice continuously differentiable utility function. Infinite Giffen (postive income) effects are not possible (though finite Giffen (positive income) effects may still be possible).

6. The Least Concave Utility Functions:

Whenever a numerical representation by a concave function is possible, a monotone and concave transformation, f, of the concave utility function always yield another concave utility function. Hence it is natural to ask whether there is a least concave function representing the underlying preference relation R*. Denote by U the set of continuous, concave, real-valued functions on X, each representing R*. This set is weakly ordered by the relation: v U is more concave than u U, if there is a strictly increasing, concave function of the interval u(X). Then, Debreu[1976] proved that, if U is not empty, then U has a least element. Thus, wherever u(x) is a least concave function, every concave function v(x) representing R* is given by v(x)=f(u(x)). If there are two least utility functions then, one is derived from the other by a positive affine transformation. They are unique up to a positive affin transformation. In this sense, the utility is determinate as a cardinal utility. See Lange[1934] for a further discussion.

Several ways of computing the least concave function(s) have been given by Kannai[1977]. Denote by $F(\bar{x})$ the L.H.S. of (6). Then, by (6'), $F(\bar{x})/\lambda(\bar{x})=\partial\lambda(\bar{x})/\partial m$. u(x) is concave near $\bar{x}$ if and only if $F(x)\leqq 0$. If further F(x)=0 at x in a convex neighborhood C of $\bar{x}$ intersecting $I(\bar{x})$, then, a utility function obtained near $\bar{x}$ is least concave. Because $0=F(x)/\lambda(x) = \limsup\{F(y)/\lambda(y)\}$ over differentiability points (y in the intersection of C and $I(\bar{x})$) which converge to x. This is equivalent to Fenchel's famous condition $\inf(-S_r/k^2S^*_{r-1})=0$ [1951-1953, p.135] in the case of a concave function, hence, Kannai's theorem 12 [1981, 577][1977, Theorem 5.6] applies. The necessity property can also be obtained on a convex, compact set C. One may show that $F(x)<0$ on C implies that u can not be least concave by making use of a result of Kannai[1977, proposition 2.5]. Thus, in view of expression (5), the equivalent condition $F(x)=0$ (at a point x on every indifference hypersurface in C) implies the Hessian $DDu(x)$ vanishing at x even if demands are differentiable there. From the classical determinant manipulation, we see the income derivative of the marginal utility of income vanishing at x, where $F(x)=0$ and $|\bar{A}_n|\neq 0$, that is,

$$\partial\lambda(x)/\partial m = -|\bar{A}_n(n+1;n+1)|/|\bar{A}_n| = -|DDu(x)|/|\bar{A}_n| = 0$$

7. Implication From Pareto-Edgeworth Complements Substitues and Independents

Thus, the gradual erosion of concave utility has finally reached a moment to reconsider the controversial definition of complements and substitutes by Edgeworth and Pareto, hence their economic implication at least partially anticipated by the previous authors. Does a (least) concave utility with these concepts imposes on the generated demands any hypothetical restrictions that are invariant under any choice of utility indices. Following Kannai[1980], define them as follows: Let u on X be a least concave function representing R* on X. Then, in case the utility function is twice differentiable at x, commodities i and j ($i\neq j$) are Pareto-Edgeworth complements at x, if the cross derivative

$\partial^2u(x)/\partial x_i\partial x_j>0$, and are Pareto-Edgeworth substitutes if $\partial^2u(x)/\partial x_i\partial x_j<0$. One of the intended reasons for utilizing the unique least concave utility in defining thus the concepts comes from the fact that, whenever $\partial^2v(x)/\partial x_i\partial x_j>0$ for some $v\varepsilon U$, $\partial^2u(x)/\partial x_i\partial x_j>0$ for a least concave utility $u\varepsilon U$ and whenever $\partial^2u(x)/\partial x_i\partial x_j<0$ for a least concave utility $u\varepsilon U$, $\partial^2v(x)/\partial x_i\partial x_j<0$ for any $v\varepsilon U$. Thus, we can define the concept of independents for the least concave utility function. Commodities i and j are Pareto-Edgeworth Independents at the demand point x, if they are neither complements nor substitutes. Then, we are able to establish an operationally meaningful economic theorem that connects the introspective concepts on utility representing R* with restrictions upon the generated demands. That is, "the utility analysis is meaningful only to the extent that it places hypothetical restrictions upon these demand functions" (Samuelson[1967 97]). The conclusions of our theorems, to which we shall provide rigorous proofs in the final section, are summarized as follows: Complements with a normal (resp. inferior) good are normal (resp. inferior) while substitutes for a normal (resp. inferior) good are inferior (resp. normal). As a corollary to this, immediately, if there exist no substitutes (nor independents) at all, then, all the commodities that are complements are normal (since there must be at leats one commodity normal!). This is intuitively easy to understand and would be intended to be true in general. However, the hypothesis, from which this conclusion is derived, is more or less retrictive, but most general to our best knowledge on the signed inverse of a qualitative matrix, due to Bassett, Habibagahi, Maybee and Quirk[1967 Theorem 4,5 227, 1968 Theorem 5,7 and Corollary 452-455].

The fundamental assumption required for this result is: There are two groups of commodities in such a way that any pair of commodities (i,j), both taken from the same group, are complements and any other pair (i,j), one from one group and the other from the other, are substitutes.[*9]

If the Hessian is indecomposable at the demand point $\bar{x}=d(\bar{p},\bar{m})$, where these concepts are assumed, then, the above results still hold true even with some independents involved. If the Hessian is, howver, decomposable instead, then, independents of a normal (inferior) good are neither normal nor inferior (without any further restriction).

These results may apply, as they stand, to the local behavior of demands at a non-differentiability point.

The price derivatives of the marginal utility of money, $\partial\lambda(\bar{x})/\partial p_i$, $i=1,..,n$, are closely related to the income derivatives of demands d_i, $i=1,..,n$. In fact, the Slutsky equation for λ is given at a differentiability point $\bar{x}$,

$$\partial\lambda(\bar{x})/\partial p_i=-\bar{\lambda}|\bar{A}(n;n+1)|/|\bar{A}_n|-d_n(\bar{p},\bar{m})|\bar{A}(n+1;n+1)|/|\bar{A}_n| \qquad (19)$$

where $\bar{\lambda}=\lambda(\bar{x})$. Because of symmetry, Theorem 1,2 & 3 may apply to determine the sign property of the first term. The second term, involving $F(\bar{x})\geqq 0$, is of a non positive sign by the concavity or the least concavity of u.

To a Marshallian consumer, that is, to a maximizer of $\{u(x)-\lambda px\}$ with respect to x (given (p,λ)), the demand function $d_i, i=1,..,n$, is a function of (p,λ), satisfying $u_i(d(p,\lambda))-\lambda p_i=0$, and an actual expenditure $m(p,\lambda)=\Sigma_{i=1}^{n}p_i d_i(p,\lambda)$. For this see Marshall[1890 276,XII,XIV in Mathematical Appendix]. Let $\bar{m}$ be the desired level of expenditure, a limit to which actual m converges and let $\bar{\lambda}$ be the corresponding value to which actual adusts as m approaches $\bar{m}$. Then, by Hicks[1956 12-15], the price derivative of the Marshallian demand $d_n(p,\lambda)$ can be decomposed into the direct effect, due to a change in p_n, and the indirect effect, due to the change in λ, on d_n. That is, $\bar{m}=m(\bar{p},\bar{\lambda})$,

$$\partial d_n(\bar{p},\bar{\lambda})/\partial p_n=\partial d_n(\bar{p},\bar{\lambda})/\partial p_n+\{\partial d_n(\bar{p},\bar{\lambda})/\partial\lambda\}\{\partial\lambda/\partial p_n\}$$

where if $|A_n|=|\bar{A}(n+1;n+1)|\neq 0$

$$\partial d_n(\bar{p},\bar{\lambda})/\partial p_n=\bar{\lambda}|\bar{A}(n,n+1;n,n+1)|/|A_n|$$

$$\partial d_n(\bar{p},\bar{\lambda})/\partial\lambda=-|\bar{A}(n+1;n)|/|A_n|$$

and $\partial\lambda/\partial p_n$ is given by (19) where i=n. Hence, the R.H.S. reduces to:

$$\partial d_n(\bar{p},\bar{\lambda})/\partial p_n=\bar{\lambda}|\bar{A}_{n-1}|/|\bar{A}_n|-d_n(\bar{p},\bar{\lambda})|\bar{A}(n+1;n)|/|\bar{A}_n|$$

which is equal to (7). Therefore, infinite Giffen effect never happen on the Marshallian demand, while finite Giffen effect may happen.

Theorems:

Consider the local behavior of demand in a neighborhood of a certain point $\bar{x}$ on the indifference surface, at which the(asymptotic) income derivative of the marginal utility of income vanishes.

Suppose the demand functions are differentiable in prices at a positive vector $(\bar{p},\bar{m})$. That $\bar{x}=d(\bar{p},\bar{m})$ is a differentiability point in X is equivalent to saying that $|Dd^{-1}(\bar{x})|\neq 0$ for the (inverse demand) function d^{-1}, the existence of which is due to postulate (ii). Thus, the single valued, continuous demand functions $d(p,m)$, defined on NxM in subsection 1 under the strong convex preferences R*, is, on an open, convex subset of $d^{-1}(X)$ equated here to N when $\bar{m}$ is fixed, equal tothe uniquely determined function $d(p)$ of C^1 on N, such that, for an open, convex subset C of X, $\bar{x}=d(\bar{p})$, $\bar{x}\in C$, $d^{-1}(\bar{x})=\bar{p}\in N$, $x=d(p)$ for any $x\in C, p=d^{-1}(x)\in N$, d^{-1} is one to one on N, and $d^{-1}(C)=N$.*10

Theorem 1: Suppose that the demand functions d are differentiable in prices p at $(\bar{p},\bar{m})>0$.

Suppose that, at the demand point $\bar{x}=d(\bar{p},\bar{m})$, the cross derivatives, $u_{ij}(\bar{x})$, i,j=1,..,n-1, possess the signs; $u_{ij}(\bar{x})>0$, i,j=1,.. ..,s; i,j=s+1,..,n, and $u_{ij}(\bar{x})<0$, i=1,..,s; j=s+1,..,n, where $1\leq s\leq n-1$.

Then,

$$\{\partial d_i(\bar{p},\bar{m})/\partial m\}\{\partial d_n(\bar{p},\bar{m})/\partial m\}<0,\ i=1,..,s$$

$$\{\partial d_j(\bar{p},\bar{m})/\partial m\}\{\partial d_n(\bar{p},\bar{m})/\partial m\}>0,\ j=s+1,..,n-1$$

Hence,

$$\{\partial d_i(\bar{p},\bar{m})/\partial m\}\{\partial d_j(\bar{p},\bar{m})/\partial m\}<0$$

provided that the income derivative, $\partial\lambda(\bar{p},\bar{m})/\partial m$, of $\lambda(\bar{x})$ vanishes hence so does the determinant of the Hessian $DDu(\bar{x})$.

Remark 1: The second supposition is equivalent to assuming that complements of complements and substitutes of substitutes are complements while complements of substitutes and substitutes of complements are substitutes. Cf-Morishima[1952].

The conclusion of Theorem 1 holds even with independents, if the assumed Hessian is indecomposable in the second supposition.

Remark 2: As a corollary of Theorem 1, all goods are complements and normal, if s=0. However, as shown in the proof, this holds without the proviso and even with a not least but just concave function.

Theorem 2: Suppose $u_{ij}(\bar{x})\geqq 0, i,j=1,..,n,\ i\neq j$, but the Hessian is indecomposable for a concave utility function. Then, the demand function $d_i, i=1,..,n$ behaves, net of a movement along the indifference surface, like a normal good. That is, $\partial d_i(\bar{p},\bar{m})/\partial m>0,\ i=1,..,n$.

Proofs of Theorem 1,2:

Let A_k denote the k th order principal submatrix of the Hessian, evaluated at $\bar{x}$, and $\bar{A}_k$ denote that bordered by $(p_1,..,p_k)$ and 0.

$$\bar{A}_k = \begin{bmatrix} u_{11} & u_{12}\cdots u_{1k} & p_1 \\ \cdots & \cdots & \cdots \\ u_{k1} & u_{k2}\cdots u_{kk} & p_k \\ p_1 & p_2 \cdots p_k & 0 \end{bmatrix} \quad k=2,..,n.$$

Correspond the notations to those used in subsection 5.

By both differentiability of demands at $\bar{x}$ and the concavifiability near $\bar{x}$, $A_{n-1}(\bar{x})$ has all eigen values, $\alpha_i(\bar{x}),\ i=1,..,n-1$, negative when restricted to the coordinates $\{\xi_1,..,\xi_{n-1}\}$. Assume $\alpha_1\leqq\alpha_2\leqq\alpha_3\leqq\cdots\leqq\alpha_{n-1}$, without loss of generality. Let $\alpha^*_i, i=1,..,n$, designate the eigen values of $A_n(\bar{x})(=DDu(\bar{x}))$ in the coordinates $\{x_1,..,x_n\}$ in such a way that $\alpha^*_1\leqq\alpha^*_2\leqq\alpha^*_3\leqq\cdots\leqq\alpha^*_{n-1}\leqq\alpha^*_n$. Then, by a Sturmian Separation Theorem (see Bellman[1960,117]), $\alpha^*_{h-1}\leqq\alpha_{h-1}\leqq\alpha^*_h$, h=2,..,n. Hence, $\alpha^*_i<0$, i=1,..,n-1. Thus, $A_{n-1}(\bar{x})$ is negative definite and has all diagonal $u_{ii}(\bar{x})$, i=1,..,n-1, negative. By assumption, $u_{ij}(\bar{x})>0$, i,j=1,..,s, and i,j=s+1,..,n, and $u_{ij}(\bar{x})$ <0, i=1,..,s;j=s+1,..,n, for $s\leqq n-1$. By a theorem of Bassett et.al[1967 227], the inverse of A_{n-1} exists and A_{n-1}^{-1} is of known sign pattern, such that $(-1)^{i+j}|A_{n-1}(i;j)|/|A_{n-1}|>0$, i=1,..,s;j=s+1,..,n-1, and $(-1)^{i+j}|A_{n-1}(i;j)|/$

$|A_{n-1}|<0$, $i,j=1,..,s$; $i,j=s+1,..,n-1$. This result may slightly be generalized by weakening the strict inequality to the weak one but with the indecomoposability, in the cross derivatives u_{ij}, $i,j=1,..,n-1$. See Bassett et.al [1968 Corollary, Theorem 7 454-455]. Then, by the well known relationships between the partitioned submatrices of a nonsingular matrix $\bar{A}$ and its inverse $\bar{B}$, $\bar{B}(1,..,n-1;1,..,n-1)$, where $\bar{A}=\bar{A}_n$, $\bar{B}=\bar{B}_n$, is the 2 by 2 inverse of

$$\bar{A}(1,..,n-1;1,..,n-1) - \bar{A}(1,..,n-1;n,n+1)[\bar{A}(n,n+1;n,n+1)]^{-1}\bar{A}(n,n+1;1,..,n-1) \quad (9)$$

the invertibility of which is clear by reading the formula in our notation in Gantmacher[1959 46 (I)], provided that both $\bar{A}$ and A_{n-1} are nonsingular. Then, by the famous determinant identity; $|\bar{A}(n+1;n+1)|\,|\bar{A}(n;n)| - |\bar{A}(n;n+1)|\,|\bar{A}(n+1;n)|=|\bar{A}|\,|\bar{A}(n,n+1;n,n+1)|$,

$$\bar{B}(1,..,n-1;1,..,n-1)=\begin{bmatrix} |\bar{A}(n;n)| & -|\bar{A}(n;n+1)| \\ -|\bar{A}(n+1;n)| & |\bar{A}(n+1;n+1)| \end{bmatrix}|\bar{A}|^{-1} \quad (10)$$

The second row of (10) corresponds to the income effect of the n th good and the income derivative of the marginal utility of income $\lambda(\bar{x})$, respectively. See (7) for the Slutsky equation. In (9), since $\bar{A}(n,n+1;n,n+1)=A_{n-1}$ by definition, these terms can be explained in terms of the knowledge above obtained of A_n and A_{n-1}^{-1}. By a fact that $|A_{n-1}|=|\bar{B}(1,..,n-1;1,..,n-1)|\,|\bar{A}|$, for which see Shafroth[1981,615],

$$|\bar{B}(1,..,n-1;1,..,n-1)|\neq 0.$$

This in turn implies in (10) that, if $|\bar{A}(n+1;n+1)|$ vanishes at $\bar{x}$, then,

$$|\bar{A}(n,n+1)|\neq 0. \quad (11)$$

From (9) this may be rewritten;

$$|\bar{A}(n+1;n)|= p_n|A_{n-1}|-\Sigma_{h\neq n}\Sigma_{k\neq n}(-1)^{h+k}|A_{n-1}(h;k)|u_{nh}p_k \quad (12)$$

Fisrt observe that if s=0 then $|\bar{A}(n+1;n)|/|A_{n-1}|>0$ hence $|\bar{A}(n+1;n)|/|\bar{A}|<0$. Otherwise the sign is indeterminate. This result will be used for a proof of Theorem 2 with what follows.

Also we have a 2 by 2 matrix;

$$\bar{B}(1,..,n-1;n,n+1)=\bar{B}(1,..,n-1;1,..,n-1)\bar{A}(1,..,n-1;n,n+1)A_{n-1}^{-1} \qquad (13)$$

the second row of which corresponds to the income effects of the other goods, that is, $(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|$, j=1,..,n-1. Hence,

$$\Sigma_{h=1}^{n}(-1)^{n+1+h}|\bar{A}(n+1;h)|p_h/|\bar{A}| = 1. \qquad (14)$$

We are able to utilize the knowledge so far obtained (9) to (13) to determine the sign of each. Handling the R.H.S. of (13) to get the elements of the second row, write j th element multiplied by $|\bar{A}||A_{n-1}|$,

$$|\bar{A}(n+1;n)|\ \Sigma_{h\neq n}(-1)^{h+j}|A_{n-1}(h;j)|u_{nh}$$

$$-|\bar{A}(n+1;n+1)|\Sigma_{h\neq n}(-1)^{h+j}|A_{n-1}(h;j)|p_h \qquad (15)$$

Observe here that if s=0 then $u_{nh}>0$ and $|\bar{A}|$, $|\bar{A}(n+1;n)|$and $|\bar{A}(n+1;n+1)|$ are all of the same sign hence the uniform sign of A_{n-1}^{-1} determining the sign of (15) hence the sign of the j th element being positive. However, this is no more true if $s\geqq 1$. Then, observe $|A_{n-1}(h;j)|u_{nh}$ is of the same sign for each j. In fact, for j=1,..,s, $\Sigma_{h\neq n}(-1)^{h+j}|A_{n-1}(h;j)|u_{nh}/|A_{n-1}|>0$, and for j=s+1,..,n-1, $\Sigma_{h\neq n}(-1)^{h+j}|A_{n-1}(h;j)|u_{nh}/|A_{n-1}|<0$. But the second term is indeterminate in sign. Hence the sign of (14) through (15) is determinate if $|\bar{A}(n+1;n+1)|$ vanishes at $\bar{x}$, provided that $|\bar{A}(n+1;n)|/|\bar{A}|$ is determinate in sign. That is,

$$\text{sign}\ \{(-1)^{n+j+1}|\bar{A}(n+1;j)|/|\bar{A}|\}=\text{sign}\{-|\bar{A}(n+1;n)|/|\bar{A}|\}\ j=1,..,s$$

$$= \text{sign}\{|\bar{A}(n+1;n)|/|\bar{A}|\}\ j=s+1,..,n-1,$$

and hence for h=1,..,s and k=s+1,..,n-1,

$$\{(-1)^{n+1+h}|\bar{A}(n+1;h)|/|\bar{A}|\}\{(-1)^{n+1+k}|\bar{A}(n+1;k)|/|\bar{A}|\}<0.$$

This completes a proof of Theorem 1 for $1\leqq s\leqq n-1$. The proof of Theorem 2 follows immediately when we take s=0.

Remark 3: Two extreme cases, where the Hessian is (completely) decomposable, are pointed out.

(i) In case $u_{nh}(\bar{x})=0$, $h=1,..,n-1$, observe, in the proof above, that both the 2nd term in (12) and the 1st term in (15) are zero. Hence,

$$|\bar{A}(n+1;n)|/|\bar{A}|=p_n|A_{n-1}(\bar{x})|/|\bar{A}(\bar{x})|<0 \tag{16}$$

$$(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|= 0 \text{ if } |\bar{A}n+1;n+1)|=|DDu(\bar{x})|=0 \tag{17}$$

This still holds even if the other cross derivatives, $u_{ij}(\bar{x})$, $i,j=1,..,n-1$, have any sign patterns. In fact, when $u_{ij}(\bar{x})=0$, $i,j=1,..,n-1$ $u_{ii}(\bar{x})<0$, $i=1,..,n-1$ and $u_{nn}(\bar{x})=0$ imply more specifically

$$|\bar{A}(n+1;n)|/|\bar{A}|=-1/p_n, \quad |\bar{A}(n+1;j)|/|\bar{A}|=0$$

(ii) If $u_{nn}(\bar{x})<0$ in (i), so that $DDu(\bar{x})$ does not vanish, then, these are not true anymore, except (16). $u_{nh}=0$, $h=1,..,n-1$, imply (16), but, in stead of (17),

$$(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|=-|\bar{A}(n+1;n+1)|/|\bar{A}|$$
$$/\Sigma_{h\neq n}(-1)^{h+j}|A_{n-1}(h;j)|p_h/|A_{n-1}| \tag{18}$$

This sign is not determinate in general. In case $u_{ij}=0, i,j=1,..,n$, $i\neq j$, $|\bar{A}(n+1;n)|/|\bar{A}|=p_n/u_{nn}(-\Sigma_{i=1}^{n}p_i^2/u_{ii})<0$, $(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|=-p_j/u_{jj}(-\Sigma_{i=1}^{n}p_i^2/u_{ii})>0$.

Including these cases as special ones, the following is more than a corollary to Theorems 1&2 because of the decomposability.

Now, suppose $DDu(\bar{x})$ is (completely) decomposable so that for some s, $u_{ij}(\bar{x})=0$, $i=1,..,s$, $j=s+1,...,n$. Then, $|A_{n-1}(\bar{x})|\neq 0$ implies (iii) $|\bar{A}(n+1;j)|/|\bar{A}|=0$ if $|DDu(\bar{x})|=0$, $j=1,..,s$. Suppose further $u_{ij}(\bar{x})\geqq 0$, $i,j=1,..,s;i,j=s+1,...,n$, then, (iv) $(-1)^{n+1+j}|\bar{A}(n+1;j)|/|\bar{A}|=-|\bar{A}(n+1;n+1)|/|\bar{A}|/\Sigma_{h=1}^{s}(-1)^{h+j}|A_{n-1}(h;j)|p_h/|A_{n-1}| \geqq 0$, $j=1,..,s$, whereas, for $j=s+1,..,n-1$, $=|\bar{A}(n+1;n)|/|\bar{A}|/\Sigma_{h=s+1}^{n-1}(-1)^{h+j}|A(h;j)|u_{nh}/|A_{n-1}|-|\bar{A}(n+1;n+1)|/|\bar{A}|/\Sigma_{n=s+1}^{n-1}(-1)^{h+j}|A_{n-1}(h;j)|p_h/|A_{n-1}|>0$.

For $j=n$,

(v) $|\bar{A}(n+1;n)|/|\bar{A}|=p_n|A_{n-1}|/|\bar{A}|$

$$-\Sigma_{h=s+1}^{n-1}\Sigma_{j=s+1}^{n-1}(-1)^{h+j}\{|A_{n-1}(h;j)|/|A_{n-1}|\}u_{nh}p_j\,|A_{n-1}|/|\bar{A}|$$

which is negative if $u_{ij}(\bar{x})\geqq 0$, $i,j=1,..,s;i,j=s+1,..,n$.

To see (iii)-(v), let $A_{n-1}=A$, $B=A^{-1}$, and let the other notations correspond to those in subsection 5 and in the proofs of Theorem 1&2. Then, by assumption, $A(1,..,s;s+1,..,n-1)=A(s+1,..,n-1;1,..,s)^T=0$. Read the formula (9) for $\bar{A}$, $\bar{B}$ and $n-1$, here for A,B and s. Then,

$B(1,..,s;1,..,s)=[A(1,..,s;1,..,s)]^{-1}$

$B(s+1,..,n-1;s+1,..,n-1)=[A(s+1,..,n-1;s+1,..,n-1)]^{-1}$ likewise,

and $B(s+1,..,n-1;1,..,s)=B(1,..,s;s+1,..,n-1)^T=0$.

Thus, $|A_{n-1}(h;k)|/|A|=0$, $h=1,..,s$, $k=s+1,..,n-1$. Hence, $j=1,..,s$,

$|A(h;j)|u_{nh}=0$

since $|A(h;j)|=0$ when $u_{nh}\neq 0$, $u_{nh}=0$ when $|A(h;j)|\neq 0$. This completes the proof of (iii). By carefully adapting the above results with (12) and (15) we are able to obtain (iv) and (v), where the equality holds only when $|DDu(\bar{x})|=|A_n|=0$.

Thus, we are able to establish a theorem for independent commodities:

<u>Theorem 3 (Independents)</u>: <u>Suppose</u> $DDu(\bar{x})$ <u>is (completely) decomposable so that for some number</u> s, $u_{ij}(\bar{x})=0$, $i=1,..,s;j=s+1,..,n-1$. <u>Then, the differentiability assumption</u> $|A_{n-1}(x)|\neq 0$ <u>implies</u>

$$\partial d_i(\bar{p},\bar{m})/\partial m=0,\ i=1,..,s,\ \text{if}\ |DDu(\bar{x})|=0.$$

<u>Suppose further</u> $u_{ij}(\bar{x})\geqq 0, i,j=1,..,s;i,j=s+1,..,n$, <u>then, in general,</u>

$$\partial d_i(\bar{p},\bar{m})/\partial m\geqq 0,\ i=1,..,s;\quad \partial d_j(\bar{p},\bar{m})/\partial m>0,\ j=s+1,..,n,$$

<u>where, for</u> $i=1,..,s$, <u>the equality holds only when</u> $|DDu(\bar{x})|=0$, <u>but, for</u> $j=s+1,..,n-1$, <u>the inequality holds always in the strong sense</u>.

Corollary 1: Suppose that the demand functions d are not differentiable at $(\bar{p},\bar{m})$. But keep the condtion on the cross derivatives, $u_{ij}(\bar{x}), i,j=1,..n-1$, at a certain point $\bar{x}$, where the asymptotic income derivative of the marginal utility of income defined below vanishes.

$\limsup F(x^{\nu})=0$ over a sequence of differentiability points $\{x^{\nu}\}_{\nu=1}^{\infty}$, $x^{\nu}\epsilon I(\bar{x})$, $x = d(p^{\nu})$, $x^{\nu}\to\bar{x}=d(\bar{p})$. $F(x^{\nu})=|A_n(x^{\nu})|/|\bar{A}_n(x^{\nu})|$.

Then, the conclusions of Theorem 1, 2, and 3 still hold.

Proof: Note the continuity of the inverse demand functions d^{-1} implies that prices $\bar{p}$, where the demand functions d is not differentiable, are limit points of $\{p^{\nu}\}$ at each p^{ν} of which d is continuously differentiable. The strong convexity assumption on R* assures such points x^{ν}'s are dense near $\bar{x}$. Then, the second term of (15) devided by $|\bar{A}_n||A_{n-1}|$ approaches 0 as $|\bar{A}_n|$ approaches 0 on a sequence of $\{p^{\nu}\}p^{\nu}\to\bar{p}$.

$$\limsup\{|DDu(x^{\nu})|/|\bar{A}_n(x^{\nu})|\Sigma_{h\neq n}(-1)^{h+j}\ |A_{n-1}^{\nu}(h;j)|\ u_{nh}^{\nu}/|A_{n-1}^{\nu}|\}=0$$

where A_{n-1}^{ν}, p_h^{ν}, etc. show A_{n-1}, p_h etc. are evaluated at x^{ν}.

Hence,

$$\limsup\ (-1)^{n+1+j}|\bar{A}_n^{\nu}(n+1;j)|/|\bar{A}_n^{\nu}| = \limsup\ |\bar{A}_n^{\nu}(n+1;n)|/|\bar{A}_n^{\nu}| \{\Sigma_{h\neq n}(-1)^{h+j}|A_{n-1}^{\nu}(h;j)|u_{nh}^{\nu}/|A_{n-1}^{\nu}|\}.$$

Note, by continuity the sign pattern of the cross derivatives $u_{ij}(x^{\nu}), i,j=1,..,n$, do not change at nearby regular prices p^{ν}, $\nu=1,..,\infty$. Thus, the signs of the income terms attained in Theorem 1 & 2 follow at the limit point $\bar{x}$ when p^{ν}approaches the limit $\bar{p}$. Also recall the result obtained by Hurwicz, Jordan and Kannai[1985 Theorem 2.1 and its proof], $|\bar{A}_n(n+1;n)|/|\bar{A}_n|$ is, in our notation, bounded by a positive scalar multiple of $(-|\bar{A}_{n-1}|/|\bar{A}_n|)^{-1/2}$ which means, due to an arbitrary choice of n, that for j, $|\bar{A}_n(n+1;j)|/|\bar{A}_n|$ is bounded by a multiple of $(-|\bar{A}(j;j)|/|\bar{A}_n|)^{1/2}$. Hence the income term never dominates the substitution term when the income term takes an infinite value. An infinite Giffen effect never occurs.

FOOTNOTES

*1 Rader called them nice demands, because at almost all prices finite Slutsky equations hold.

*2 In fact, the concave utility was assumed to obtain a Lipschitz condition for the minimum income function ([1973 the proof of Theorem 12, Corollary F]). Consequently to obtain a transformation that maps price p, where the substitution term does not exists, into new price q, such that the set of new prices is of measure 0 whenever the set of prices p is of measure 0. Hence demands are pointwise Lipschitizian in price, applying Rademacher's theorem to obtain almost everywhere nice demands. See for this Federer[1969, 3.1.5-9 216-218] and Rado and Reichelderfer[1955 219 Lemma 1,336 Lemma 143-4]. We shall review this later in subsection 4.

*3 In the neighborhood N, demands d are studied as the inverse of the transformation, $x \to (g_1(x),\ldots,g_{n-1}(x),\Sigma_{i=1}^{n} g_i(x)x_i)$, and this transformation as the composition of two mappings; one is a transformation $x \to (g_1(x),\ldots,g_{n-1}(x),u(x))$ and the other is a transformation $(p_1,\ldots,p_{n-1},u) \to (p_1,\ldots,p_{n-1},m(u))$, where $m(u)=\min_y[\Sigma_{i=1}^{n} p_i y_i ; u(y_1,\ldots,y_n) \geq u]$. The Jacobian determinant of the inverse mapping d^{-1} is simply a multiplication of that of the first $\begin{vmatrix} \partial g_i(x)/\partial x_j \\ \partial u(x)/\partial x_j \end{vmatrix}$ $i=1,\ldots,n-1; j=1,\ldots,n$, and that of the second, $\begin{vmatrix} I & 0 \\ 0 & Dm \end{vmatrix} = Dm(u)$.
Hence $|Dd^{-1}(x)| = g_n(x)c_n(x)$, where $c_n(x)$ is the Gaussian curvature at x. The fact that $1/\|Du(x)\| = Dm(u)$ is used for this. If $c_n(x) \neq 0$, then, by the inverse function theorem (see Apostol [1957 144-145], there exists a uniquely determined function d and there are two open neighborhoods, $C \subset X$ and $N \subset P$ where $d^{-1}(x) = P$, such that $x \in C$ and $d^{-1}(x) \in N$, $N = d^{-1}(C)$, d^{-1} is one to one

on C, d is defined on N and $d(d^{-1}(y))=y$ for evey y in C and d is of class C^1 on N.

*4 The(strict) quasi-concavity implies $(-1)^k|\bar{A}_k|\geq 0$, k=2,..,n, while $(-1)^k|\bar{A}_k|>0$ implies only the quasi-concavity of u near $\bar{x}$. We shall see that $(-1)^k|\bar{A}_k|>0$ at x where demands are differentiable for k=2,..,n-1.

*6 Common to the analytical frameworks of Fenchel[1956],Kannai[1981, 1985], Hurwicz,Jordan and Kannai[1985] etc..

*5 By definition and postulates (ii)&(iii), the continuous differentiability of $\lambda(x)$ is assured. λ is of class C^1.

*7 The orthogonal changes do not produce any effects on the conditions in our extensive concern. For example let T denote an orthogonal matrix. Then, $Dg(\bar{\xi})=TDg(\bar{x})T'$ and $|Dg(\bar{\xi})|=|TDg(x)T'|=|Dg(\bar{x})|$. But the effects must be determined on the derivatives of demands. Since $x_i=\Sigma_{j=1}^{n}t_{ij}\xi_j$ and $\rho_j=\Sigma_{i=1}^{n}t_{ij}p_i$ ans since $px=\rho\xi$, where t_{ij},i,j =1,..,n, are i-j elements of T, $Dd(p)=TDf(\rho)T'$ where $x=d(p)$ and $\xi=f(\rho)$. For example, $\partial d_i(p)/\partial p_j=\Sigma_{h=1}^{n}\Sigma_{k=1}^{n}t_{ih}t_{jk}\,\partial f_h/\partial\rho_k$. The derivatives $\partial d_i/\partial p_j$,i,j=1,..,n correspond to the usual bordered Hessian $\bar{A}_n(x)$ defined below in subsetion 5, whereas, $\partial f_h/\partial\rho_k$ to that restricted to the subspace orthogonal to $(\bar{p}_1,..,\bar{p}_n)$, derived in subsection 3. See Hurwicz,Jordan and Kannai[1985].

*8 An application was made by Hurwicz et al.[1985 the proof of Theorem 2.2] to obtain the negative infinite limsup of the own-price derivatives in the n commodity case in which a special commodity vector is in concern, such that the vector is not perpendicular to a principal direction of the indifference surface through $\bar{x}$ for which the principal curvature vanishes.

*9 Similar to Morishima's case in the comparative statics context of the general equilibrium theory. Cf- Morishima[1952].

*10 See footnote 3. for this.

REFERENCES

Apostol, Tom M.: Mathematical Analysis: A Modern Approach to Advanced Calculus Addison-Wesley Publishing Company, Inc. 1957.

Arrow, Kenneth J. and Leonid Hurwicz: "Decentralization and Computation in Resource Allocation" in Studies in Resource Allocation Processes Cambridge: Cambridge University Press 1977.

Bassett, Lowell, Hamid Habibagahi and James Quirk: "Qualitative Economics and Morishima Matrices" Econometrica 35(1967) 221-233.

Bassett, Lowell, John Maybee and James Quirk: "Qualitative Economics and the Scope of the Correspondence Principle", Econometrica 36(1968) 544-563.

Bellman, Richard: Introduction to Matrix Analysis New York: McGraw Hill 1960.

Bernstein, B. and R.A.Toupin: "Some Properties of the Hessian Matrix of a Strictly Convex Function" Journal fur die reine und angeivandte Mathematik Bd.210 Heft 1/2 1961, 65-72.

Chipman, John: "An Empirical Implication of Auspitz-Lieben-Edgeworth-Pareto Complementarity" Journal of Economic Theory 14(1977) 228-231.

Debreu, Gerard: "Smooth Preferences" Econometrica 40(1972) 603-615.

Debreu, Gerard: "Least Concave Utility Function" Journal of Mathematical Economics 3(1976) 121-129.

De Finetti, B.: "Sulle Stratificazioni Convesse" Annali di Mathematica Pura ed Applicata 30(1949) 173-183.

Dhrymes, P.J.: "On a Class of Utility and Production Functions Yielding Everywhere Differentiable Demand Functions" Review of Economic Studies 34(1967) 399-408.

Federer, Herbert: Geometric Measure Theory Springer-Vellag Berlin, Heidelberg, New York 1969.

Fenchel, Werner: Convex Cones, Sets and Functions from notes by D.W.Blackett of Lectures at Princeton University, Spring Term 1951. Princeton University Department of Mathematics Logistics Research Project sponsored by Office of Naval Research 1953.

Fenchel, Werner: "Uber Konvexe Funktionen mit Vorgeschriebenen Niveaumannigfaitigkeiten" Mathematiische Zeitschrift 63(1956) 496-506.

Gantmacher, F.R.: The Theory of Matrices Vol.1 New York: Chelsa 1959.

Georgescu-Roegen, N.: "A Diagramatic Analysis of Complementarity" Southern Economic Journal 19(1952) 1-20.

Giffen, R.: Economic Inquiries and Studies London: George Bell and Sons 1879.

Hicks, John Richard: A Revision of Demand Theory London: Oxford University press.

Hurwicz, Leonid, James Jordan and Yakar Kannai: "On the Demand Generated by a Snooth and Concavifiable Preference Ordering" Journal of Mathematical Economics 12(1985) Forthcoming.

Jordan, James S.: "A Property of Demand Correspondence of a Concave utility Functions" Journal of Mathematical Economics 9(1982) 41-50.

Kannai, Yakar: "Concavifiability and Constructions of Concave Utility Functions" Journal of Mathematical Economics 4(1977) 1-56.

Kannai, Yakar: "The ALEP Definition of Complemenatrity and Least Concave Utility" Journal of Economic Theory 22(1980) 115-117.

Kannai, Yakar: "Concave Utility Functions-Existence, Constructions and Cardinality" in Generalized Concavity in Optimization and Economics ed. by S.Schaible and W.Ziemba New York: Academic Press 1981 543-610.

Kannai, Yakar: "Engel Curves, Marginal Utility of Income and Concavifiable Preferences" Department of Theorectical Mathematics The Weizmann Institutes of Science rehvot 76100, Israel 1985.

Katzner, Donald W.: "A Note on the Differentiability of Consumer Demand Functions" Econometrica 36(1968) 415-418.

Lange, Oscar: "The Determinateness of the Utility Function" Review of Economic Studies 1(1934) 218-225.

Marshall, A.: Principles of Economics London: Macmillan 1890, 1920.

McKenzie, Lionel W.: "Demand Theory without A Utility Index" Review of Economic Studies 24(1957) 185-189.

Morishima, Michio: "On the Laws of Change of the Price System in an Economy which Contains Complementary Commodities" Osaka Economic Papers 1(1952) 101-113.

Pareto, V.: Manual d'economie politique, 2nd ed. Giard-Paris, 1927 Manual of Political Economy translated by Schwier Augustus, M.Kelly Publisher-Clifton, New Jersey.

Rader, John Trout: "Nice Demand Functions" Econometrica 41(1973) 913-935.

Rado, T. and P.V.Reichelderfer: Continuous Transformations in Analysis Berlin, Gottingen, Heidelberg: Springer-Verlag 1985.

Samuelson, Paul A.: Foundations for Economic Analysis New York: Atheneum 1967.

Samuelson, Paul A.: "COMPLEMENATRITY An Essay on the 40th Anniversary of the Hicks-Allen Revolution in Demand Theory" Journal of Economic Literature 12(1974) 1255-1289.

Sard, Arthur: "The Measure of the Critical Values of Differentiable Maps" American Mathematical Society Bulletin 1942 883-890.

Shafroth, Chantal: "A Generalization of the Formula for Computing the Inverse of a Matrix" American Mathematical Monothly 88(1981) 614-616.

Slutsky, E.E.: "On the Theory of the Budget of the Consumer" translated by O.Ragusa, G.Stigler and K.Boulding (ed.) Readings in Price Theory Homewood III-Irwin 1952.

Smale, Steven: "Global Analysis and Economics IV" Journal of Mathematical Economics 1(1974) 119-127.

Stigler, George J.: "The Development of Utility Theory. I" Journal of Political Economy 58(1950) 307-327.