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An Extension of the Nash Bargaining Problem and the Nash Social Welfare Function *)

by

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2. An Extension of the Nash Bargaining Problem

The set of all individuals is $N = \{1,\ldots,n\}$ ($n \geqq 2$), and the utility space of the individuals is the nonnegative orthant E^n_+ of the n-dimensional Euclidean space E^n. Let $\mathcal{B}$ be the class of all compact subsets S of E^n_+ which satisfy

(2.1) $a > 0$ for some a in S .

Here we denote $a_i > b_i$ for $\forall i \in N$ by $a > b$, $a_i \geqq b_i$ for $\forall i \in N$ by $a \geqq b$ and $a \geqq b$ & $a \neq b$ by $a \geq b$.

A pair (β,ϕ) of a subclass β of $\mathcal{B}$ and a function ϕ on β which assigns to each S in β a nonempty subset $\phi(S)$ of S with $\phi(S) > 0$ is called a bargaining problem. We call β a domain , a set in β a game-situation and ϕ a solution-function .

In this paper, game-situations are always O-normalized, i.e., threat-points are always denoted by the O-vector . Hence a game-situation S means a pair (S,0) of an attainable set and a threat-point . This normalization does not lose any generality because von Neumann-Morgenstern utility functions are unique up to positive linear transformations .

This formulation is different from Nash's in that a domain may contain nonconvex sets and that a solution functon ϕ may be a set-valued function . The necessity of the set-valuedness of ϕ is caused by the assumption that β may contain nonconvex sets, which will be clarified in the following .

We impose rationality criteria upon (β, ϕ) as follows:

N.1. (Pareto Optimality): If $y \in S \in \beta$ and $y \geqq x$ for some $x \in \phi(S)$, then $y = x$.

N.2. (Independence of Irrelevant Alternatives): If $S \supset T$ and $\phi(S) \cap T$ is nonempty, then $\phi(T) = \phi(S) \cap T$.

N.3. (Independence of Positive Linear Transformations): For any $a = (a_1, \ldots, a_n) > 0$, $\phi(a \cdot S) = a \cdot \phi(S)$ for $\forall S \in \beta$. Here we define $a \cdot b$ by $a \cdot b = (a_1 b_1, \ldots, a_n b_n)$ for a, b in E^n and $a \cdot S$ by $a \cdot S = \{a \cdot b : b \in S\}$ for a in E^n and $S \subset E^n$.

N.4. (Symmetricity): Let π be a permutation of N . If $\pi \cdot x = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \in S$ for any x in S, then $\pi \cdot x \in \phi(S)$ for any x in $\phi(S)$.

N.5. (Continuity): Let $\{S^k\}$ be any sequence of sets in β which converges to S^o in β in the sense of the Hausdorf metric for subsets.[1] Let $\{x^k\}$ be any sequence such that $\lim_{k \to \infty} x^k = x^o$ and $x^k \in \phi(S^k)$ for $\forall k \geqq 1$. Then x^o belongs to $\phi(S^o)$.

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1). The definition of the Hausdorf metric will be given in the following. For precise definition , see [2, Chap.3] .

Axioms N.1 to N.4 are generalizations of the axioms of Nash's bargaining problem to the case where ϕ may be a set-valued function. It is easily verified that the axioms are reduced to Nash's if ϕ is a single-valued function. Axiom N.5 is not necessary in Nash's bargaininig problem, because the other axioms imply this condition there. Axioms N.2 and N.5 are conditons provided by J.F.Nash in an informal note on utility theory dated August 8, 1950.[2)]

We are in a position to show that our extension of Nash's bargaining problem is really a mathematical generalization of it. To do this, we prove that when $\beta = \mathcal{B}_c = \{ S \in \mathcal{B} : S \text{ is convex} \}$, our problem is reduced to Nash's.

<u>Lemma 1</u>. Let $(\mathcal{B}_c, \phi)$ satisfy axioms N.1 to N.4. Then $\phi(S) = \{(a_1/n, \ldots, a_n/n)\}$ for $\forall S = [a_1e^1, \ldots, a_ne^n, 0]$ ($a \in E^n$, $a > 0$). Here $[a_1e^1, \ldots, a_ne^n, 0]$ is the convex hull of $\{a_1e^1, \ldots, a_ne^n, 0\}$ and e^k ($k = 1, \ldots, n$) is the vector in E^n such that $e^k_k = 1$ and $e^k_j = 0$ if $j \neq k$.

<u>Proof</u>. Let $\phi(S)$ contain $y \neq (a_1/n, \ldots, a_n/n)$. Let $\lambda_i = y_i/a_i$ for $\forall i \in N$. Then we have $\sum_{i \in N} \lambda_i = 1$ by N.1 and $\lambda_i > 0$ for $\forall i \in N$ by the assumption of ϕ. It is easily verified that λ does not satisfy $\lambda_1 = \ldots = \lambda_n$. Let $b = (1/a_1\sqrt{\lambda_1}, \ldots, 1/a_n\sqrt{\lambda_n})$. Then $b \cdot S = [e^1/\sqrt{\lambda_1}, \ldots, e^n/\sqrt{\lambda_n}, 0]$. We have ,

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2). See Shapley and Shubik [13].

by N.3 , $\phi(b \cdot S) = b \cdot \phi(S) \ni b \cdot y = (\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Let $T = \left\{ x \in E^n_+ : \sum_{i \in N} x_i^2 \leqq 1 \right\}$. It is easily verified that the Pareto surface of $b \cdot S$ is the supporting hyperplane of T at $(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$, which implies $T \subset b \cdot S$ and $b \cdot y \in T$. Hence we have $b \cdot y \in \phi(T)$ by N.2 . Since T is a strictly convex set, the intersection of the Pareto surface of $b \cdot S$ and T consists of solely $b \cdot y$. But since T is symmetric, $\pi \cdot b \cdot y$ belongs to $\phi(T)$ for any permutation π by N.4 . Hence $b_1 y_1 = \ldots = b_n y_n$, i.e., $\lambda_1 = \ldots = \lambda_n$. This is a contradiction . Q.E.D.

<u>Theorem I.</u> A bargaining problem $(\mathcal{B}_c, \phi)$ satisfies axioms N.1 through N.4 if and only if the solution function ϕ is the single-valued function such that for $\forall S \in \mathcal{B}_c$, $\phi(S) = \{x\}$ satisfies

$$(2.2) \qquad \prod_{i \in N} x_i \geqq \prod_{i \in N} y_i \qquad \text{for } \forall y \in S .$$

<u>Proof</u>. If ϕ is the single-valued function defined by (2.2) , then the problem is reduced to Nash's . It is well-known that the function satisfies N.1 through N.4 . We need to prove the necessity. As the proof is similar to Nash's , we give only the sketch of it.

For an S in $\mathcal{B}_c$, let x be the vector in S such that $\prod_{i \in N} x_i \geqq \prod_{i \in N} y_i$ for $\forall y \in S$. Let $b = (1/x_1, \ldots, 1/x_n)$, and let us consider the set $b \cdot S$. Then it is not difficult to verify by the convexity of $b \cdot S$ that $b \cdot S$ is included by $T = \left\{ y \in E^n_+ : \sum_{i \in N} y_i \leqq n \right\}$. By Lemma 1 , we have $\phi(T) = \left\{(1,\ldots,1)\right\}$. Since $(1,\ldots,1) = b \cdot x \in b \cdot S$, we have $\phi(b \cdot S) = \phi(T) \cap b \cdot S = \left\{(1,\ldots,1)\right\}$ by N.2 . By N.3, we have $\phi(S) = x \cdot \phi(b \cdot S) = \left\{ x \right\}$. Q.E.D.

Next we consider the case where a domain β contains non-convex sets . The <u>dimension</u> of S in β is defined to be the dimension of the minimal subspace which includes S . The <u>dimension</u> of β is defined to be the greatest dimension of S in β . We denote the dimension of S and that of β by dim S and dimβ , respectively . We say that (β,ϕ) is <u>regular</u> if $S \in \mathcal{B}$ and dim S $\leqq$ dimβ imply $S \in \beta$.

<u>Theorem II</u>. A regular bargaining problem (β,ϕ) with dim$\beta \geqq \min(3,n)$ satisfies axioms N.1 through N.5 if and only if the solution function ϕ satisfies

$$(2.3) \quad \phi(S) = \left\{ x \in S : \prod_{i \in N} x_i \geqq \prod_{i \in N} y_i \quad \text{for } \forall y \in S \right\}$$

$$\text{for } \forall S \in \beta .$$

Initially we should see that the solution function ϕ defined by (2.3) satisfies N.1 through N.5 . As it is, however, easy to verify that the solution function satisfies N.1 through N.4 , we prove that it satisfies N.5 .

The Hausdorf metric h(A,B) for subsets is defined by

$$(2.4) \quad h(A,B) = \sup\left(\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A) \right),$$

where d is a bounded distance function on $E^n \times E^n$ defining the usual Euclidean topology of E^n and $d(x,A) = \inf_{y\in A} d(x,y)$. Let $\{S^m\}$ be be any sequence of sets in β such that $\lim_{m\to\infty} h(S^m,S^o) = 0$, and let y be any vector in S^o . Let $\{y^m\}$ be a sequence such that $d(y^m,y) = d(S^m,y)$ and $y^m \in S^m$ for $\forall m \geqq 1$. Then this sequence $\{y^m\}$ converges to y . In fact, since $d(y^m,y) = d(S^m,y) \leqq \sup_{x\in S^o} d(S^m,x) \leqq h(S^m,S^o)$, $0 = \lim_{m\to\infty} h(S^m,S^o) = \lim_{m\to\infty} d(y^m,y)$. Let $\{x^m\}$ be a sequence which satisfies the supposition of N.5 . Hence $\prod_{i\in N} x_i^m \geqq \prod_{i\in N} y_i^m$ for $\forall m \geqq 1$, which implies $\prod_{i\in N} x_i^o \geqq \prod_{i\in N} y_i$. Since y is arbitrary in S^o, x^o belongs to $\phi(S^o)$.

Next we show the converse part of the theorem . We define a binary relation $\succsim$ on the positive orthant $Y = (E^n_+)^o$ of the n-dimensional Euclidean space by

$$(2.4) \quad x \succsim y \quad \text{if and only if} \quad x \in \phi(\{x,y\}) .$$

<u>Lemma 2.</u> The binary relation $\succsim$ defined by (2.4) is a weak ordering.

<u>Proof</u>. It is noted that since $\dim \beta \geqq \min(3,n)$, any set consisting of

at most three points in E_+^n belongs to β . The completeness of $\succsim$ is clear . We prove the transitivity of it .

Suppose that $x \succsim y$, $y \succsim z$ but $z \succ x$, where $z \succ x$ means not $x \succsim z$. This is equivalent to

$$x \in \phi(\{x,y\}) , \quad y \in \phi(\{y,z\}) \text{ and } \{z\} = \phi(\{z,x\}) .$$

If $x \in \phi(\{x,y,z\})$, we have ,by N.2 ,

$$\phi(\{z,x\}) = \{z,x\} \cap \phi(\{x,y,z\}) \ni x ,$$

which is a contradiction. Let $y \in \phi(\{x,y,z\})$. Since $\{x,y\} \cap \phi(\{x,y,z\}) \neq \emptyset$, we have , by N.2 ,

$$\phi(\{x,y\}) = \{x,y\} \cap \phi(\{x,y,z\}) = \{x,y\} ,$$

because $x \in \phi(\{x,y\})$ by the supposition . Here $\{x,y\} \subset \phi(\{x,y,z\})$, which implies

$$\phi(\{z,x\}) = \{z,x\} \cap \phi(\{x,y,z\}) \ni x .$$

This is a contradiction . Finally , let $z \in \phi(\{x,y,z\})$. Then we have , by N.2 ,

$$\phi(\{y,z\}) = \{y,z\} \cap \phi(\{x,y,z\}) = \{y,z\} .$$

Since $\{y,z\} \subset \phi(\{x,y,z\})$, we have $\phi(\{x,y\}) = \{x,y\}$ by the supposition and N.2 . Hence we have $\phi(\{x,y,z\}) = \{x,y,z\}$, which contradicts the fact that $\{z\} = \phi(\{z,x\}) = \{z,x\} \cap \phi(\{x,y,z\})$. Q.E.D.

<u>Lemma 3</u> . The binary relation $\succsim$ satisfies the following properties:

(<u>i</u>): For any $a \in Y = (E_+^n)^o$, $a \cdot x \succsim a \cdot y$ if and only if $x \succsim y$.

(<u>ii</u>): If $x \geq y$, then $x \succ y$.

(<u>iii</u>): If $x \succ y \succ z$, then there is a λ ($0 < \lambda < 1$) such that $\lambda x+(1-\lambda)z \sim y$, where $x \sim y$ means $x \succsim y$ and $y \succsim x$.

(<u>iv</u>): $x \sim \pi \cdot x$ for any permutation π such that $\pi = \pi^{-1}$ and any x in Y .

<u>Proof</u>. It is not difficult to prove (i) and (ii) .

Let us consider (iv) . Let π be a permutation such that $\pi = \pi^{-1}$. Then $\{x, \pi\cdot x\} = \{\pi^{-1}x, x\} = \pi^{-1}\cdot\{x, \pi x\}$. Hence we have , by N.4, $\phi(\{x, \pi\cdot x\}) = \pi^{-1}\phi(\{x, \pi\cdot x\}) = \{x, \pi\cdot x\}$. This implies $x \sim \pi\cdot x$.

Let us show (iii). Let $L_1 = \{\lambda : 0 < \lambda < 1 \ \& \ \lambda x+(1-\lambda)z \succsim y\}$ and $L_2 = \{\lambda : 0 < \lambda < 1 \ \& \ y \succ \lambda x+(1-\lambda)z\}$. Of course, at least one of L_1 and L_2 is not empty . Suppose that L_1 , $L_2 \neq \emptyset$. Let $\sup L_2 = \lambda$, and let $\{\lambda^m\}$ be a sequence in L_2 with $\lim_{m\to\infty} \lambda^m = \lambda$. This implies $y \in \phi(\{\lambda^m x+(1-\lambda^m)z, y\})$ for $\forall m \geqq 1$. Since $d(\lambda^m x+(1-\lambda^m)z, \lambda x+(1-\lambda)z) \geqq h(\{\lambda^m x+(1-\lambda^m)z,y\},\{\lambda x+(1-\lambda)z,y\})$ for $\forall m \geqq 1$, $\{\{\lambda^m x+(1-\lambda^m)z,y\}\}$ converges to $\{\lambda x+(1-\lambda)z,y\}$. Hence we have , by N.5 , $y \in \phi(\{\lambda x+(1-\lambda)z,y\})$. Since $x \succ y$, λ must be smaller than 1 . Hence there is a sequence $\{\mu^m\}$ in L_1 such that $\lim_{m\to\infty} \mu^m = \lambda$. Of course, it holds that

$$\mu^m x+(1-\mu^m)z \in \phi(\{\mu^m x+(1-\mu^m)z,y\}) \quad \text{for } \forall m \geqq 1 ,$$

$$\{\mu^m x+(1-\mu^m)z,y\} \longrightarrow \{\lambda x+(1-\lambda)z,y\} \quad \text{as } m \longrightarrow \infty .$$

Hence $\lambda x+(1-\lambda)z$ belongs to $\phi(\{\lambda x+(1-\lambda)z,y\})$ by N.5 . Hence we have $\phi(\{\lambda x+(1-\lambda)z,y\}) = \{\lambda x+(1-\lambda)z,y\}$, which implies $\lambda x+(1-\lambda)z \sim y$.

Let $L_1 = \emptyset$. Then $\sup L_2 = 1$. There is a sequence $\{\lambda^m\}$ in L_2 converging to 1 . Then we have $y \in \phi(\{1x+0z,y\})$ similarly to the above discussion , which is a contradiction . When $L_2 = \emptyset$, we can derive a contradiction similarly . Q.E.D.

When the weak ordering $\succsim$ on Y satisfies (ii) and (iii) of Lemma 3, there exists a real-valued function G(x) on Y representing the weak ordering $\succsim$,i.e.,

$$(2.5) \quad G(x) \geqq G(y) \quad \text{if and only if} \quad x \succsim y \ .$$

This can be easily verified similarly to Lemma 3.3 of [4] . Moreover, since the weak ordering $\succsim$ satisfies (i) and (ii) of Lemma 3 , the function G(x) is represented as

$$(2.6) \quad G(x) = V(\prod_{i \in N} x_i^{c_i}) \qquad \text{for } \forall x \in Y \ ,$$

where V is a monotonically increasing function on E^1 and c_i is a positive constant for $\forall i \in N$. This can be also obtained similarly to Sec.3 of [4] by using the strong lemma of Osborne [11] . It follows from (iv) of Lemma 3 that $c_1 = \ldots = c_n$.

We have shown that

$$(2.7)\quad \phi(\{x,y\}) = \Big\{ z : \prod_{i\in N} z_i \geqq \prod_{i\in N} x_i \,,\ \prod_{i\in N} y_i \ \& \ z = x \text{ or } y \Big\} \quad \text{for } \forall x,y \text{ in } Y \,.$$

Let S be arbitrary in β and let $x \in \phi(S)$. Since $x \in \phi(S) \cap \{x,y\} = \phi(\{x,y\})$ for $\forall y \in S$, we have $\prod_{i\in N} x_i \geqq \prod_{i\in N} y_i$. Hence we have $\prod_{i\in N} x_i \geqq \prod_{i\in N} y_i$ for $\forall y \in S$. We complete the proof .

3. The Nash Social Welfare Function and the Adjoint Choice Problem

In this section, we investigate the relation between the extension of Nash's bargaining problem provided in the previous section and the Nash social welfare function of Kaneko and Nakamura [4] . Initially we should review the Nash social welfare function.

Let X be the set of all alternatives, where X may be a finite or infinite set . We associate the origin x_o with X , and let $X^* = X \cup \{x_o\}$. The origin must be assumed to be the worst state we can imagine , economically , but it , technically , is not necessary to do so . We denote a lottery by $(\alpha_1 a_1 + \dots + \alpha_t a_t)$ which has t possible outcomes a_i in X* with probabilities α_i (i = 1,...,t) respectively . Of course, it must hold that $\sum_{i=1}^{t} \alpha_i = 1$ and $\alpha_i \geqq 0$ for $\forall i \leqq t$. We denote by m(X*) the set of all lotteries having arbitrary finite numbers of outcomes . An individual preference ordering R_i is a weak ordering on m(X*) such that

(3.1) $x \, R_i \, x_o$ for $\forall x \in X$

(3.2) R_i satisfies the von Neumann-Morgenstern utility axioms,[3)]

where a R_i b means that a is preferred to b or is indifferent to b. We denote , by a P_i b , not b R_i a , and , by a I_i b , a R_i b &

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3). See [12, Chap.6] .

$b\ R_i a$. Let $\mathcal{R}$ be the set of all individual preference orderings on $m(X^*)$, and let $\mathcal{R}^n = \mathcal{R} \times \cdots \times \mathcal{R}$. We call $p = (R_1, \ldots, R_n)$ in $\mathcal{R}^n$ a <u>profile</u> . We put $m_p(X^*) = \{a \in m(X^*) : a\ P_i\ x_o$ for $\forall i \in N\}$ for each profile $p = (R_1, \ldots, R_n) \in \mathcal{R}^n$.

A <u>social welfare function</u> $W(p)$ is a function defined on $\mathcal{R}^n$ which assigns to each profile p in $\mathcal{R}^n$ a weak ordering $R = W(p)$ on $m_p(X^*)$. If $m_p(X^*) = \emptyset$, $W(p)$ is the empty weak ordering, i.e., $W(p) = \emptyset$. Since a weak ordering $R = W(p)$ is considered to be a subset of $m_p(X^*) \times m_p(X^*)$, $W(p) = \emptyset$ is a natural setting . We call (X,W) a <u>social welfare problem</u> . The strict social preference ordering P and the social indifference relation I of $R = W(p)$ are defined in the usual way . Kaneko and Nakamura [4] provides the following four rationality criteria which a reasonable social welfare function should satisfy .

<u>A.1. (Pareto Optimality)</u>: Let p be any profile and let a,b be in $m_p(X^*)$. If $a\ R_i\ b$ for $\forall i \in N$ and $a\ P_j\ b$ for some $j \in N$, then $a\ P\ b$.

<u>A.2. (Independence of Irrelevant Alternatives with Neutral Property)</u>: Let $p = (R_1, \ldots, R_n)$ be a profile with $R = W(p)$, and let $p' = (R_1', \ldots, R_n')$ be any other profile with $R' = W(p')$. Let $a_1, a_2 \in m_p(X^*)$ and $b_1, b_2 \in m_{p'}(X^*)$. Suppose, for $\forall i \in N$,

$(\alpha_1 a_1+\alpha_2 a_2+\alpha_3 x_o)\ R_i\ (\beta_1 a_1+\beta_2 a_2+\beta_3 x_o)$ if and only if

$(\alpha_1 b_1+\alpha_2 b_2+\alpha_3 x_o)\ R_i'\ (\beta_1 b_1+\beta_2 b_2+\beta_3 x_o)$

for all probability distributions $(\alpha_1,\alpha_2,\alpha_3)$ and $(\beta_1,\beta_2,\beta_3)$. Then $a_1\ R\ a_2$ if and only if $b_1\ R'\ b_2$.

<u>A.3. (Anonymity)</u>: Let π be any permutation of N . Let $p = (R_1, \ldots, R_n)$ and let $\pi\cdot p = (R_{\pi(1)},\ldots,R_{\pi(n)})$. Then $W(p) = W(\pi\cdot p)$.

<u>A.4. (Continuity)</u>: Let p be any profile with $W(p) = R$ and let a,b,c in $m_p(X^*)$ satisfy $a\ P\ c\ P\ b$. Then there exists some α , $0 < \alpha < 1$, such that $(\alpha a+(1-\alpha)b)\ I\ c$.

We denote , by $U(R_i)$, the set of all von Neumann-Morgenstern utility functions u_i with $u_i(x_o) = 0$ representing an individual preference ordering R_i . Let $U(p) = U(R_1,\ldots,R_n) = U(R_1)\times\ldots\times U(R_n)$. It should be noted that if $m_p(X^*) \neq \emptyset$, then $U(p) \neq \emptyset$. We define a function $w_o(u,a) = w_o(u_1,\ldots,u_n,a)$ defined on $\bigcup_{p\in\mathcal{R}^n}\left[U(p)\times m_p(X^*)\right]$ by

$$(3.3)\qquad w_o(u,a) = \sum_{i\in N} \log u_i(a) \ .$$

We note that w_o is not defined for profiles p with $m_p(X^*) = \emptyset$.

We call the function w_o the <u>Nash social welfare index</u>. The <u>Nash social welfare function</u> is the social welfare function W_o which is defined by

$$(3.4) \quad a\, W_o(p)\, b \quad \text{if and only if} \quad w_o(u,a) \geqq w_o(u,b) ,$$

where $u \in U(p)$ and $a,b \in m_p(X^*)$.

<u>Theorem III. (Kaneko and Nakamura [4]).</u> When X contains at least three alternatives, the Nash social welfare function is the unique social welfare function which satisfies axioms A.1 through A.4 .

It appears that there exists a close relation between a bargaining problem with axioms N.1 through N.5 and a social welfare problem with axioms A.1 through A.4 . To clarify this relation , we consider an adjoint choice problem of a social welfare problem .

We call a subset S of $m(X^*)$ <u>p-compact</u> if $u(S) = \{u(a) : a \in S\}$ is a compact set of E^n for any u in $U(p)$. The <u>adjoint choice problem</u> $(\mathcal{O},C)$ of a social welfare problem (X,W) is defined by

$$(3.5) \quad \mathcal{O} = \bigcup_{p \in \mathcal{R}^n} \{(S,p) : S \subset m(X^*) ,\ S \cap m_p(X^*) \neq \emptyset \ \text{and} \ S \text{ is p-compact}\} ,$$

$$(3.6) \quad C(S,p) = \{x \in S : x\, W(p)\, y \ \text{for} \ \forall y \in S\} \quad \text{for} \ \forall (S,p) \in \mathcal{O} .$$

The <u>utility-representing adjoint choice problem</u> $(\mathcal{O}\!\mathcal{L}^*, C^*)$ of a social welfare problem (X,W) is defined by

$$(3.7) \qquad \mathcal{O}\!\mathcal{L}^* = \bigcup_{p \in \mathcal{R}^n} \Big\{ (S,u) : S \subset m(X^*),\ S \cap m_p(X^*) \neq \emptyset ,\ S \text{ is p-compact and } u \in U(p) \Big\} ,$$

$$(3.8) \qquad C^*(S,u) = u(C(S,p)) \quad \text{for } \forall (S,u) \in \mathcal{O}\!\mathcal{L}^* .$$

We say that the utility-representing adjoint choice problem $(\mathcal{O}\!\mathcal{L}^*, C^*)$ is <u>isomorphic to</u> a bargaining problem (β, ϕ) if

(3.9) $u(S)$ belongs to β for $\forall (S,u) \in \mathcal{O}\!\mathcal{L}^*$,

(3.10) for any T in β , there is a (S,u) in $\mathcal{O}\!\mathcal{L}^*$ with $T = u(S)$,

(3.11) $C^*(S,u) = \phi(u(S))$ for $\forall (S,u) \in \mathcal{O}\!\mathcal{L}^*$.

We are in a position to state the result of this section .

<u>Theorem IV</u>. Let $k \geqq 3$. Then a social welfare problem (X,W) with $|X| = k$ satisfies axioms A.1 through A.4 if and only if the utility-representing adjoint choice problem $(\mathcal{O}\!\mathcal{L}^*, C^*)$ of (X,W) is isomorphic to a regular bargaining problem (β, ϕ) with $\dim \beta = \min(n,k)$ satisfying axioms N.1 through N.5 . Here $|X|$ means the number of members in X .

Proof. We prove this theorem in the case where $3 \leqq k \leqq n$. In other cases, it can be proved similarly .

Necessity: Let (X,W) be a social welfare problem with $|X| = k$ satisfying axioms A.1 through A.4 , and let (β,ϕ) be a regular bargaining problem with $\dim \beta = k$ satisfying axioms N.1 through N.5 .

Let $X = \{x_1,\ldots,x_k\}$. Then $u(m(X^*)) = [u(x_0),u(x_1),\ldots,u(x_k)]$ always inculdes u(S) for $\forall(S,u) \in \mathcal{O\!\!\iota}^*$, where $[u(x_0),u(x_1),\ldots,u(x_k)]$ is the convex hull of $\{u(x_0),u(x_1),\ldots,u(x_k)\}$. Since the convex hull is at most k-dimensional , u(S) is at most k-dimensional . Since u(S) is a compact subset of E^n_+ and $a > 0$ for some $a \in u(S)$ by (3.7) , u(S) belongs to β .

Let T be any set in β . Then T is a compact subset of E^n_+ which is at most k-dimensional . Hence there is a convex hull $[0,a^1,\ldots,a^k]$ in E^n_+ which includes T . Let us choose a profile p in $\mathcal{R}^n$ such that there is a $u = (u_1,\ldots,u_n) \in U(p)$ with $u(x_0) = 0$, $u(x_1) = a^1,\ldots, u(x^k) = a^k$. Let $S = \{(\alpha_0 x_0+\alpha_1 x_1+ \ldots +\alpha_k x_k) \in m(X^*) : \sum_{t=1}^{k} \alpha_t u(x_t) \in T\}$. Then u(S) = T for this (S,u) . It is clear that (S,u) belongs to $\mathcal{O\!\!\iota}^*$.

Since (X,W) satisfies A.1 through A.4 , W coincides with W_0 by Theorem III . Then we have , for any $(S,u) \in \mathcal{O\!\!\iota}^*$,

$$C^*(S,u) = \Big\{u(x) \in u(S) : \sum_{i \in N} \log u_i(x) \geqq \sum_{i \in N} \log u_i(y) \text{ for } \forall y \in S\Big\} .$$

Hence we have (3.11) by Theorem II .

<u>Sufficiency</u>: Let the utility-representing choice problem ($\mathcal{O}\!\mathcal{L}^*, C^*$) of (X,W) be isomorphic to a regular bargaining problem (β, ϕ) with dim $\beta = k$ satisfying axioms N.1 through N.5 . Then , for any $p \in \mathcal{R}^n$, $u \in U(p)$ and $a,b \in m_p(X^*)$, it holds by (3.8),(3.11) and Theorem II that

$$C^*(\{a,b\},u) = u(C(\{a,b\},p)) = \phi(\{u(a),u(b)\}) =$$

$$\{x : x = u(a) \text{ or } u(b) \text{ and } \prod_{i \in N} x_i \geqq \prod_{i \in N} u_i(a) , \prod_{i \in N} u_i(b)\} .$$

Hence W coincides with the Nash social welfare function W_o . It follows from Theorem III that (X,W) satisfies axioms A.1 through A.4 . Q.E.D.

To prove Theorem IV , we used Theorems II and III . But, we can prove this theorem without these theorems , though the proof becomes much longer . Hence Theorems II and III become a corollary of each other .

We provide another equivalent theorem. As it can be proved similarly to Theorem III . the proof is omitted .

<u>Theorem V</u>. Let $k \geqq 3$. Then a regular bargaining problem (β, ϕ) with $\dim \beta = \min(n,k)$ satisfies axioms N.1 through N.5 if and only if there is a social welfare problem (X,W) with $|X| = k$ satisfying axioms A.1 through A.4 such that the utility-representing adjoint choice problem $(\mathcal{O}\!\mathcal{L}^*, C^*)$ of (X,W) is isomorphic to (β, ϕ) .

Finally we should make a remark on a relation between the original Nash bargaining problem $(\mathcal{B}_c, \phi)$ and the Nash social welfare function. Let $\mathcal{O}\!\mathcal{L}_c = \{(m(S),p) : (S,p) \in \mathcal{O}\!\mathcal{L}\}$ and $\mathcal{O}\!\mathcal{L}_c^* = \{(m(S),u) : (S,u) \in \mathcal{O}\!\mathcal{L}^*\}$, where $m(S)$ is the set of all finite lotteries whose outcomes are members in S . C and C* are defined in the same way with (3.6) and (3.8) . We can get the following theorem, but as the proof is similar to that of Theorem IV , it is omitted .

<u>Theorem VI</u>. If a social welfare problem (X,W) with $|X| \geqq n$ satifies axioms A.1 through A.4 , then the utility-representing adjoint choice problem $(\mathcal{O}\!\mathcal{L}_c^*, C^*)$ of (X,W) is isomorphic to a bargaining problem $(\mathcal{B}_c, \phi)$ satisfying axioms N.1 through N.4 .

Thus the original Nash bargaining problem is derived from the Nash social welfare function, but the converse is not necessarily true . That is, the mathematical equivalence of the two systems of axioms does not hold , regretfully . We do not have, however, any difficulty in this because we have shown the existence of equivalence of the extension of Nash's bargaining problem and the Nash social welfare function .

4. Discussion

We have shown that the extension of Nash's bargaining problem is mathematically equivalent to the Nash social welfare function . Nash's bargaining problem is a positive theory to investigate the behavior of the outcomes of bargaining situations where each individual persues his own interests . This significance of Nash's bargaining problem is claimed by Harsanyi [3] , and is strengthened by Nash [10] and Kaneko [6] , in which Nash's cooperative solutions are derived as certain kinds of equilibrium points of gamesin extensive form .[4)] In contrast with this, the theory of the Nash social welfare function is just a normative one which intends to define a social welfare itself, being also different from Arrow's approach in which a social welfare is considered in terms of social decesion processes . Thus , though there exists the mathematical equivalence of Nash's theory and that of the Nash social welfare function , the purposes are quite different . We should give reasons for this conflict .

The significance of the Nash socail welfare function is well clarified in terms of Nash's bargaining problem . The essential difference between them is that the origin of the Nash social welfare function is fixed and must be the worst state we can imagine

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4). There are some writers ,e.g., Luce and Raiffa [8] , who interprete Nash's bargaining problem as an arbitration scheme ,which Harsanyi disagrees with . I agree with Harsanyi's point in general .

but that the threat-point of Nash's bargaining problem is determined following a bargaining situation . If the threat-point is set as the origin, then the individuals have no advantage of status and bargain on an equal footing with each other . Moreover , since the rule of Nash's bargaining problem is a unanimous one , which would be the most impartial , the outcome of the bargaining can be considered to achieve a maximal social welfare . Thus the Nash social welfare function defines the social welfare provided in terms of the unanimous bargaining with the origin as the threat-point . When the threat-point is set arbitrarily , e.g., the status-quo , the extension of Nash's bargaining problem gives a ranking on the set of alternatives yet . This can not be , however , considered to represent a social welfare . It is just a ranking of possibilities of occurrences of alternatives as the outcome of the bargaining . That is , the theory of Nash social welfare function can become a normative one only when x_o is set as the origin .

While the purpose of the Nash social welfare function is to define a social welfare , there exists the close relation between it and Nash's positive theory in the economic significance . This means that the theory of the Nash social welfare function has the same significance of Arrow's approach ,too , and that it is never inconsistent with Harsanyi's point .

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