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# A simple proof of the validity of Bland's anticycling rule for the simplex method

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Abstract

We show a simple proof of the validity of Bland's anticycling rule for the simplex method. The proof is made by a reduction to absurdity, assuming cycling, as Bland's original proof. However, it is based only on the structure of coefficient matrices but not on the right hand side vectors, and, therefore, the fact that the basic solutions are degenerate during the cycle is not employed.

## 1. Introduction

A nice anticycling rule for the simplex method [3] was introduced by R. G. Bland [1] in 1976 (also see the exposition [4] by M. Iri). Most of the recently published textbooks on Linear Programming or Mathematical Programming, e.g. [2], [5] and [6], take up space to explain Bland's anticycling rule. However, the proofs of the validity of Bland's rule in [2], [5] and [6] almost follow Bland's original one [1], and they are not simple enough.

In this note we shall show a very simple direct proof. The proof is made by a reduction to absurdity, assuming cycling, as Bland's original proof. However, it is based only on the structure of coefficient matrices but not on the right hand side vectors, and, therefore, the fact that the basic solutions are degenerate during the cycle is not employed.

## 2. Bland's anticycling rule

In the following, we denote the (i,j)-th component of a matrix $A$ ($A'$, $A''$, ...) by $a_{ij}$ ($a'_{ij}$, $a''_{ij}$, ...), and the i-th row of $A$ ($A'$, $A''$, ...) by $a_i$ ($a'_i$, $a''_i$, ...).

Consider a linear programming problem expressed in a canonical form:

(2.1) Maximize $x_0$

subject to $Ax = b$

$x_j \geqq 0 \quad (j=1,2,\ldots,n).$

Here, $x$ is the $(n+1)\times 1$ column vector with variables $x_0, x_1, \ldots, x_n$ as its components, $A$ is the $(m+1)\times(n+1)$ constant coefficient matrix, $b$ is the $(m+1)\times 1$ constant right hand side vector, and the row and the column index sets of $A$ are $I = \{0,1,\ldots,m\}$ and $J = \{0,1,\ldots,n\}$, respectively. Let $B = \{B_0=0,B_1,\ldots,B_m\}$ be the basis index set such that

(2.2) $a_{iB_i} = 1, \quad a_{rB_i} = 0 \quad (0\leqq r\leqq m,\ r\neq i)$

for each $i = 0, 1, \ldots, m$. We suppose that $b_i \geqq 0$ $(1\leqq i\leqq m)$, i.e., (2.1) is primal feasible. We call $(A,b)$ a <u>simplex tableau</u>. For simplex tableaux $(A',b')$ and $(A'',b'')$ we write the basis index sets as $B' = \{B'_0=0,B'_1,\ldots,B'_m\}$ and $B'' = \{B''_0=0,B''_1,\ldots,B''_m\}$, respectively, which satisfy (2.2) for corresponding matrices $A'$ and $A''$.

Bland's anticycling rule [1] for the simplex method is:

(i) among all candidates to enter the basis, select the nonbasic variable $x_j$ having the lowest index $j$;

(ii) among all candidates to leave the basis, select the basic variable $x_{B_i}$ having the lowest index $B_i$.

<u>Theorem</u>(Bland[1]): The simplex method under Bland's rule (i) and (ii) cannot cycle.

(Proof) Suppose to the contrary that cycling occurs. We shall show a contradiction. Without loss of generality, we assume that each variable $x_j$ $(1 \leqq j \leqq n)$ enters and leaves the basis during the cycle.

Let $(A', b')$ [resp. $(A'', b'')$] be the simplex tableau which we have when Bland's rule selects $x_n$ as the leaving basic [resp. entering nonbasic] variable. Then, because of rule (ii), for some integers $p$ and $q$ such that $1 \leqq p \leqq m$ and $1 \leqq q \leqq n-1$ we have

(2.3) $B'_p = n,$

(2.4) $a'_{pq} > 0, \quad a'_{iq} \leqq 0 \quad (1 \leqq i \leqq m,\ i \neq p), \quad a'_{0q} < 0.$

Moreover, because of rule (i),

(2.5) $a''_{0n} < 0, \quad a''_{0j} \geqq 0 \quad (0 \leqq j \leqq n-1).$

(Bland's proof will be simplified hereafter.) Since tableau $(A'', b'')$ is obtained by pivoting in rows $a'_i$ $(1 \leqq i \leqq m)$ of tableau $(A', b')$, the zero-th row $a''_0$ of $A''$ is expressed as

(2.6) $a''_0 = a'_0 + \sum_{i=1}^{m} \alpha_i a'_i$

for some coefficients $\alpha_i$ $(1 \leqq i \leqq m)$. Here, $\alpha_i$'s are uniquely determined from (2.6) as

(2.7) $\alpha_i = a''_{0B'_i} \quad (1 \leqq i \leqq m)$

due to (2.2) for $A'$. (2.3)~(2.7) lead us to the following contradiction:

(2.8) $0 \leqq a''_{0q} = a'_{0q} + \sum_{i=1}^{m} \alpha_i a'_{iq} = a'_{0q} + a''_{0n} a'_{pq} + \sum_{\substack{i=1 \\ i \neq p}}^{m} a''_{0B'_i} a'_{iq} < 0.$ □

The simplex tableaux $(A', b')$ and $(A'', b'')$ appearing in the proof are shown in Figure 1.

The above proof shows that if we have a simplex tableau $(A', b')$

satisfying (2.3) and (2.4), then we will afterward never encounter any tableau (A",b") satisfying (2.5). The proof does not depend on b' and b" and hence not on the degeneracy, whereas the degeneracy is explicitly employed in [1], [2], [5] and [6].

Remarks: The author noticed the simple proof in January 1984 while reading a draft of A. Schrijver's book [7] in which the proof of anticycling also almost followed Bland's [1]. In spite of its simplicity the proof shown in this note does not seem to have been found elsewhere (cf. [2], [5], [6]). The author has decided to publish it because it will benefit the readers to further understand the essence of Bland's anticycling rule.

## References

[1] Bland, R. G.: New finite pivoting rules for the simplex method. Mathematics of Operations Research Vol. 2(1977), pp. 103~107.

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[6] Papadimitriou, C. H., and Steiglitz, K.: Combinatorial Optimization --- Algorithms and Complexity. Prentice-Hall, Englewood Cliffs, 1982.

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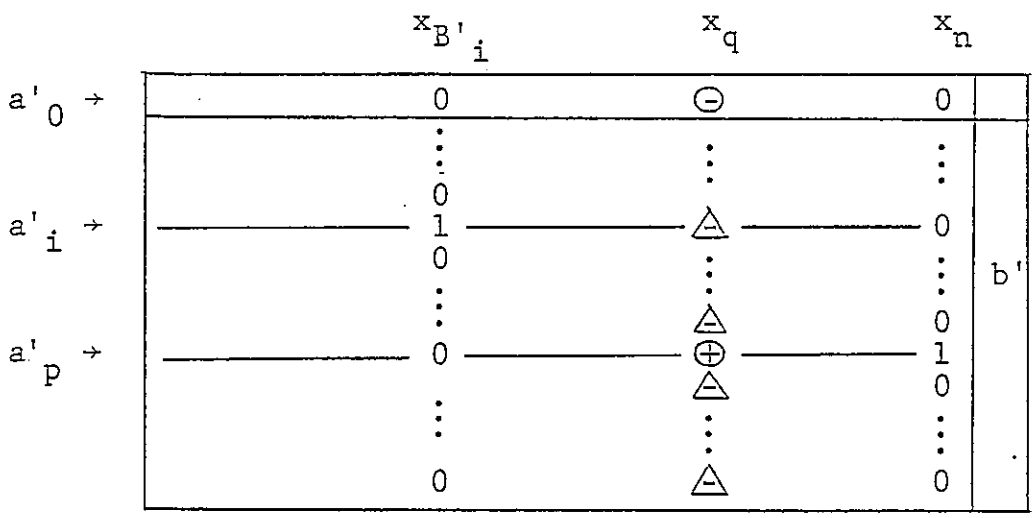
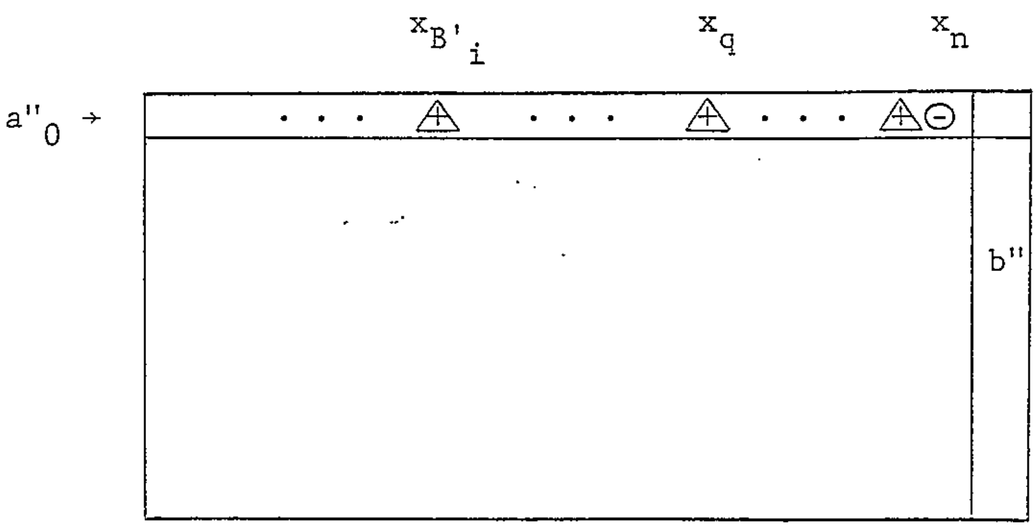
Caption

Fig. 1. The simplex tableaux (A',b') and (A'',b'') in the proof, where $\oplus$, $\ominus$, $\triangle_{+}$, and $\triangle_{-}$ denote positive, negative, nonnegative, and nonpositive components, respectively.


$x_{B'_i}$
$x_q$
$x_n$
$a'_0 \rightarrow$
$a'_i \rightarrow$
$a'_p \rightarrow$
b'

(A',b')


$x_{B'_i}$
$x_q$
$x_n$
$a''_0 \rightarrow$
b''

(A'',b'')

Fig. 1.