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URBAN AGGLOMERATION ECONOMIES IN A LINEAR CITY (revised)

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ABSTRACT:

A model of firms' spatial allocation and location is developed by explicit incorporation of urban agglomeration benefits using accessibility measure. In a linear and one-activity city, every firm is assumed to interact each other for face-to-face transactions, and the unit construction cost of office building is considered to be proportional to firm density. It is shown that both the optimum and the equilibrium distributions of firms are cosine, quadratic or cosine-hyperbolic curves, and the latter is more dispersed, and that the equilibrium rent function is concave near the city center and convex near the boundaries.

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1. INTRODUCTION

Urban agglomeration is the most important reason for the existence of cities. With the rapid growth in the service sector, the urban agglomeration economies are essentially attributed to the concentration of office firms in central business districts (CBD). The locational decision of one activity would in general endogenously be determined, ceteris paribus, by accessibility or proximity to other activities. Since most of the literature on macro urban agglomeration [e.g., Moomaw (1983), Tabuchi (1985)] often regard a city as a spaceless point, urban agglomeration economies are usually measured by summary statistics of a city like city size. However, if a city were spaceless, transportation costs within a city would become zero while transportation costs between cities would not. In this instance, there is no reason to decentralize economic activities, in the sense that every firm will be concentrated in one point city to realize the maximum agglomeration economies.

If space concept is explicitly taken into account, it is natural to introduce congestion costs as agglomeration diseconomies. Overcongestion of firms would require high construction costs of building, oblige laborers to commute long distance in a crowded train, and presumably raise land rent. Suppose these costs be paid directly or indirectly by firms, then the agglomeration diseconomies might be expressed as a function of a density measure. The location of activities would thus be determined by balancing two kinds of "forces": the force of diversification and the force of unification by Zipf's (1949) terminology.

In the context of the above, a pioneering work is Vaughan (1975) which explicitly introduced the notion of continuous spatial interaction between every possible pair of activities. To obtain socially optimum distribution of activities in a linear city, Vaughan minimized a total rent measured by a power function of activity density (the force of diversification) plus a total travel cost between every activity (the force of unification) subject to fixed number of total activities. The functional form of the analytical solution is so complex that Vaughan used empirical values of the power parameters in London and in Sydney for 1966 and found that the density distribution of the activity (population) is inverse S shaped, i.e., concave near the CBD and convex elsewhere, which seems to fit observed distributions in those cities well.

In the past literature, the force of diversification is normally regarded as a function of a commuting cost or density. The commuting cost [Fujita and Ogawa (1982), and Imai (1982)] is out of consideration here due to the assumption of one-activity city as will be mentioned later. The density is therefore the only source of the force of diversification although there are several variations of its interpretations. Beckmann (1976), Odland (1976), and Tabuchi (1982) considered population density as a disutility of residents while Borukhov and Hochman (1977), O'Hara (1977), and Tauchen and Witte (1983, 1984) considered firm density to be related to a construction cost of building. Applying the bid rent approach, on the other hand, Fujita and Ogawa (1982) regarded rent as a disutility for households and as a cost for business firms.[1]

The force of unification is due to urban agglomeration economies which is regarded as face-to-face contacts and is usually measured by an average travel cost proportional to distance between all activities with equal probability [Beckmann (1976), Borukhov and Hochman (1977), and O'Hara (1977)]. On the other side, Tabuchi (1982) and Fujita and Ogawa (1982) considered agglomeration economies as a benefit measured by potential or accessibility to all activities. However, it is quite unlikely for firms to transact or travel to all activities with equal probability. It is well known in the regional science literature that the probability of trip is a distance decay function, which may suggests the introduction of accessibility measure in formulating the urban agglomeration economies rather than the average travel distance measure.

Following Borukhov and Hochman (1977), the objective of this paper is to choose the firm's density distribution and the physical size of a city, and to express them explicitly by the given number of firms, the land rent of alternative land use such as agriculture, the accessibility parameter, and the construction cost parameter. It differs from Tauchen and Witte (1983, 1984) in that they consider the optimal firm's distribution and the number of firms given the physical size of a city,[2] and treat the trip patterns endogenous. Such an analysis is not conducted here because a major concern is to obtain an explicit analytical solution of the density distribution and the physical city size.

This paper attempts to examine the social optimum and market equilibrium distribution of firm density in a central business district based on the accessibility benefit as urban agglomeration economies and the density cost as a building construction cost. Although this paper is confined to the analysis of office firms, similar discussion on population density distribution would of course be possible assuming a certain suitable utility function of household. For analytical purposes, two simplifications will be made here: a linear rather than two-dimensional city; and one activity (i.e., office firm) rather than two activities.

The linear city assumption permits an analytical solution. Except a square city model with rectilinear distance [O'Hara (1977), Tauchen and Witte (1983, 1984)], Borukhov and Hochman (1977) would be the only paper which derived the optimal and market density distributions in a two-dimensional circular city using polar coordinates. Unfortunately, the average travel cost in their model is not weighted by the density distribution which may be a theoretical flaw in solving endogenous distribution of business firms. Note that Tabuchi (1982) also obtained the optimal density distribution in a two-dimensional plane using quadratic programming, but only in a discrete case.

In regard to the one activity city, one may conceive that neither a multicentric pattern nor a catastrophic structural transition is possible as has taken place in Fujita and Ogawa's (1982) equilibrium model under a fixed lot-size assumption. Since the concern here is office firms, the word, 'city' may be interpreted as CBD throughout this paper. Notice that the distribution of firms would presumably be more decentralized if commuting cost between two activities (firm and residence) were incorporated.

In the next section, the optimal distribution of firms will be solved using the calculus of variations, and the same will be compared with the market equilibrium solution obtained in the third section. The final section summarizes the concluding remarks.

2. OPTIMAL DISTRIBUTION OF FIRMS

For the sake of analytical transparency, imagine a linear city locating a fixed number of identical firms N. Firms are infinitesimally divisible and their type of industry is the same. The existence of households and hence their commuting costs are not incorporated here.[3]

Mathematically, one may formulate the firm's net profit $\psi(x)$ consisting of the accessibility benefits to all other firms, the unit construction cost of office building, and the opportunity cost at its location x as

$$\psi(x) = \int_{-a}^{a} e^{-\alpha|x-t|} y(t)dt - \beta y(x) - r_A/y(x) , \tag{1}$$

where x and t are the location points at the city, $y(x)$ is the firm density at x, a is the city length divided by 2, α is the "distance friction" or "accessibility" parameter, β is the "firm density" or "construction cost" parameter, and r_A is the opportunity cost of alternative land use.

The first term on the right hand side of (1) is a negative exponential-type accessibility function rather than an inverse power function ($\int_{-a}^{a} |x-t|^{-\alpha} y(t)dt, \alpha>0$) utilized in Tabuchi (1982) since this power function is not defined at x=t which gives rise to mathematical difficulties in such a continuous model. Thus, one may infer in this case that it would be optimal to concentrate every firm in a single point because the power function gets infinity. The second term on the right hand side of (1) shows that the unit construction cost at location x is proportional to firm density there.[4] If there is a nonlinear relationship between the unit construction cost and the firm density, one should add another parameter to $y(x)$ (e.g., $-\beta[y(x)]^{\gamma}$), which is however not conducted here to gain analytical advantages. The last term of (1) is usually referred to the agricultural land rent. This is the benefit which can be obtained by agriculture instead of city activities, i.e., the office activities.

Now suppose a city government plans to maximize the net social benefit of all firms (Φ) by summing up the net benefit of individual firms locating at x ($\psi(x)$) under the constraint of a fixed total size of firms. Mathematically, the problem is to:

$$\underset{y(x),a}{\text{maximize}} \quad \Phi = \int_{-a}^{a} \psi(x)y(x)dx$$

$$= \int_{-a}^{a}\int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)y(x)dtdx - \beta\int_{-a}^{a}\{y(x)\}^2dx - \int_{-a}^{a} r_A dx \tag{2}$$

$$\text{subject to} \quad \int_{-a}^{a} y(x)dx = N, \text{ and } y(x) \geq 0, \tag{2'}$$

where N is the total number of firms. Equation (2') simply states that the total number of firms is fixed in the linear city.

Utilizing the method of Lagrange multiplier, the maximization problem of (2) and (2') are to be rewritten as

$$\Phi(y(x),a) = \int_{-a}^{a} [y(x)e^{-\alpha x}\int_{-a}^{x} e^{\alpha t}y(t)dt + y(x)e^{\alpha x}\int_{x}^{a} e^{-\alpha t}y(t)dt - \beta\{y(x)\}^2$$

$$- r_A + \lambda y(x)]dx - \lambda N, \tag{3}$$

where λ is the Lagrange multiplier.

Equation (3) is ascribed to the problem of the standard calculus of variations with respect to y(x). By use of the Euler's equation $F_y - \frac{d}{dx} F_{y'} = 0$, where F is the functional, the first-order condition for maximum is given by

$$2\int_{-a}^{a} e^{-\alpha|x-t|} y(t)dt - 2\beta y(x) + \lambda = 0. \tag{4}$$

This shows that the accessibility is proportional to density at every location x in optimum, assuming existence of an interior solution. More generally, if the unit construction cost is a power function of firm density $-\beta[y(x)]^{\gamma}$ and $\gamma > 1$, then one can say that the accessibility is positively related to density.

In the next place, differentiating (3) with respect to the city length a , another first-order condition is obtained as:

$$\begin{aligned} & y(a)\{2\int_{-a}^{a} e^{-\alpha|a-t|} y(t)dt - \beta y(a) + \lambda\} \\ & \quad + y(-a)\{2\int_{-a}^{a} e^{-\alpha|-a-t|} y(t)dt - \beta y(-a) + \lambda\} -2r_A = 0. \end{aligned} \tag{5}$$

However, substituting both -a and a into (4), and using them, (5) can be reduced to

$$\beta[\{y(a)\}^2 + \{y(-a)\}^2] - 2r_A = 0. \tag{5'}$$

It should be pointed out that zero agricultural rent leads to $\partial\Phi/\partial a = 0$, which implies the infinite city length in optimum, i.e., nonexistence of the optimum distribution of office firms.

To solve the integral equation (4), first of all, differentiate it with respect to x,

$$-2\alpha\int_{-a}^{x} e^{-\alpha|x-t|}\, y(t)dt + 2\alpha\int_{x}^{a} e^{-\alpha|x-t|}\, y(t)dt - 2\beta y'(x) = 0\,. \qquad (4')$$

Again differentiate (4') with respect to x,

$$2\alpha^2\int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - 4\alpha y(x) - 2\beta y''(x) = 0\,. \qquad (4'')$$

Eliminating the integral part in (4) and (4"), one can obtain

$$\beta y''(x) + (2\alpha-\alpha^2\beta)y(x) = -\frac{\alpha^2\lambda}{2}. \qquad (6)$$

Solving the differential equation (6), the optimal distribution of the firm density is then obtained as

$$y(x) = C_1 e^{\sqrt{\alpha^2-2\alpha/\beta}\,x} + C_2 e^{-\sqrt{\alpha^2-2\alpha/\beta}\,x} + \frac{\lambda\alpha}{2(\alpha\beta-2)} \qquad \text{for } \alpha\beta\neq 2, \qquad (7)$$

$$= C_1' + C_2' x - \frac{\lambda\alpha^3}{4}x^2 \qquad \text{for } \alpha\beta=2. \qquad (7')$$

Now, to eliminate three undetermined constants (C_1, C_2, λ) in (7), and (C_1', C_2', λ) in (7'), we need three equations. One equation is the constraint of fixed total number of firms which is equation (2). The other two equations are two boundary conditions of the integral equation (4), which are obtained by substituting (7) into (4) for $x = -a$ and a.

The optimum solution of the firm density $y(x)$ can be solved and expressed by the four parameters of N, r_A, α and β, and the physical city length a. In accordance with the value of the square root $\sqrt{\alpha^2 - 2\alpha/\beta}$, the solutions are then classified into the following five cases:

(i) $\alpha = 0$

$$y(x) = \frac{N}{2a}, \tag{8a}$$

(ii) $\beta = 0$

concentration on one point,

(iii) $0 < \alpha\beta < 2$

$$y(x) = \frac{N}{2a} \frac{2\cos(\sqrt{2\alpha/\beta-\alpha^2}\ x) - \alpha\beta\cos(\sqrt{2\alpha/\beta-\alpha^2}\ a) + \sqrt{2\alpha/\beta-\alpha^2}\ \beta\sin(\sqrt{2\alpha/\beta-\alpha^2}\ a)}{\dfrac{2\sin(\sqrt{2\alpha/\beta-\alpha^2}\ a)}{\sqrt{2\alpha/\beta-\alpha^2}\ a} - \alpha\beta\cos(\sqrt{2\alpha/\beta-\alpha^2}\ a) + \sqrt{2\alpha/\beta-\alpha^2}\ \beta\sin(\sqrt{2\alpha/\beta-\alpha^2}\ a)}, \tag{8b}$$

(iv) $\alpha\beta = 2$

$$y(x) = \frac{N}{2a} \frac{-x^2 + a^2 + \beta a + \beta^2/2}{2a^2/3 + \beta a + \beta^2/2}, \tag{8c}$$

(v) $2 < \alpha\beta$

$$y(x) = \frac{N}{2a}\,\frac{-2\cosh(\sqrt{\alpha^2-2\alpha/\beta}\;x)+\alpha\beta\cosh(\sqrt{\alpha^2-2\alpha/\beta}\;a)+\sqrt{\alpha^2-2\alpha/\beta}\;\beta\sinh(\sqrt{\alpha^2-2\alpha/\beta}\;a)}{\frac{-2\sinh(\sqrt{\alpha^2-2\alpha/\beta}\;a)}{\sqrt{\alpha^2-2\alpha/\beta}\;a}+\alpha\beta\cosh(\sqrt{\alpha^2-2\alpha/\beta}\;a)+\sqrt{\alpha^2-2\alpha/\beta}\;\beta\sinh(\sqrt{\alpha^2-2\alpha/\beta}\;a)}, \tag{8d}$$

where in each case, the city length a should be determined by (5'). Since, the optimal distributions (8a)-(8d) are even functions symmetrical to the axis of $x=0$, and since $y(a)$ should be nonnegative, (5') can be reduced to

$$y(a) = \sqrt{r_A/\beta}. \tag{5''}$$

To satisfy Jacobi's necessary condition for optimum [Gelfand and Fomin (1963)], one should add

$$\sqrt{2\alpha/\beta-\alpha^2}\;a \leqq \pi, \qquad \text{for } 0 < \alpha\beta < 2. \tag{9}$$

Thus, as shown in the appendix, the optimal solution of a exists and its value is uniquely determined from (5") and (9) given the parameter values of N, r_A, α and β. Only in case (i), a is explicitly solved: as $\frac{N}{2a} = \sqrt{r_A/\beta}$, the optimal a is $\frac{N}{2}\sqrt{\beta/r_A}$, and hence the optimal distribution of firms is $\sqrt{r_A/\beta}$ for $-a \leqq x \leqq a$. In cases (iii), (iv) and (v), however, the optimal value of a is obtained by numerical calculation.

(8a) is constant density, (8b) is a cosine curve, (8c) is a quadratic curve, and (8d) is a cosine-hyperbolic curve. For illustrative purposes, they are drawn together in Figure 1 by setting $\sqrt{r_A/N} = 1$, $\alpha = 4$, and several values of β. The city length of a is computed by (5") with the constraint of (9) using the Newton method. One can see that as the "construction cost" parameter β increases, the density distribution of firms becomes flatter and the city length (city size) becomes greater regardless of its functional form.

Figure 2, on the other hand, describes the effect of the change in α on the density distribution and the city length fixing $\sqrt{r_A/N} = 1$ and $\beta = 4$. One can observe first that the density at the city boundaries is constant with respect to α, which is easily ascertained by (5"). Next, the effect of the "distance friction" parameter α on the optimal city length a, is ambiguous: as α increases, firms tend to concentrate and then disperse. When α approaches zero, the first term of the right hand side of (2) disappears. This means that since the transportation costs are unimportant, the locational differential does not exist, and hence the social objective is simply to minimize the aggregate construction costs. When α becomes infinity, the first term of (2) also disappears. In this case, the transportation costs are so great that the accessibility benefits cannot be enjoyed, and so are ignored. Notice that such characteristics are attributed to the negative exponential specification of the accessibility benefits.

To see the effect of both α and β on the degree of firm concentration, in Figure 3, a contour map of the optimal city length a is depicted on the two-dimensional plane of (α,β) for $\sqrt{r_A/N} = 1$. The city length a seems to be an increasing function of β as expected in Figure

1. It can be said that firms tend to concentrate as the unit construction cost rate β becomes relatively cheaper. On the other hand, a is not a monotonic function of α as already observed in Figure 2. When α is small, $\partial a/\partial\alpha<0$; but when α is large, $\partial a/\partial\alpha>0$. That is, $a(\alpha)$ has a turning point. Although its mathematical proof is not conducted here, it is intuitively explained above.

The optimal density y(x) is a non-increasing function of $|x|$, viz.,

$$\frac{dy}{d|x|} \leqq 0, \qquad\qquad \text{for } |x| \leqq a,$$

where the equality holds when $\alpha = 0$ or when $x = 0$. The proof is obvious due to the functional characteristics of the cosine, quadratic and cosine-hyperbolic curve. The former ($\alpha = 0$) is trivial because the optimal density is flat everywhere. The latter ($x = 0$) implies that <u>the optimal density is always smooth at the center of the city irrespective of the parameter values of α and β</u>. This is the inherent consequence of the calculus of variations where optimal solutions are necessarily continuous and smooth. This finding is in striking contrast to the classical urban economics models where the spatial interaction is limited to access to the exogenous CBD.

The convexity or concavity can be ascertained by computing the second derivative of y(x) with respect to $|x|$. As can be inferred from (8c) and (8d), the optimal density is concave when $\alpha\beta \geqq 2$. When $0 \leqq \alpha\beta < 2$, on the other hand, there is a possibility that two inflection points exist. After some manipulation, one gets

$$\frac{d^2y}{d|x|^2} = -2C''\cos(\sqrt{2\alpha/\beta-\alpha^2}|x|),$$

where C" is a positive constant with respect to x. Thus, if $\pi > \sqrt{2\alpha/\beta-\alpha^2}\, a \geqq \pi/2$ in optimum, there exist inflection points at $x = \pm\, \pi/2\sqrt{2\alpha/\beta-\alpha^2}$. That is to say, the optimal density distribution is concave near the CBD and convex apart from the CBD, whose result is similar to Vaughan (1975). On the other hand, if $\sqrt{2\alpha/\beta-\alpha^2}\, a < \pi/2$ in optimum, the above second derivative becomes non-positive for all x, which means that the optimal density is concave everywhere. It thus follows that in most of cases the optimal distribution of firm density is concave whereas the classical CBD model often yields a convex density function, such as negative exponential. Again this is the inherent consequence of the calculus of variations.

3. EQUILIBRIUM DISTRIBUTION OF FIRMS

Provided that each firm be free to move its location without any relocation costs in accordance with its profit maximization principle, the firm's net profit may be written:

$$\psi_e(x) = \int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - R(x) \qquad (10)$$

where R(x) is the building rent per unit office at location x. It is assumed that the firm's number of contacts are exogenously determined by the negative exponential accessibility function.

Following Tauchen and Witte (1984), suppose there exist many developers who construct the office buildings to rent them to the firms, and suppose under perfect competition each developer chooses building density so as to maximize its profit $\phi(x)$, i.e.,

$$\underset{y(x)}{\text{maximize}} \quad \phi(x) = y(x)R(x) - \beta\{y(x)\}^2 - r(x), \qquad (11)$$

where r(x) is the land rent at location x. Then, the first-order condition for profit maximization is:

$$R(x) - 2\beta y(x) = 0. \qquad (12)$$

If the office firms are pricetakers in the rental market, then by putting R(x) in (12) into (10), one obtains the locational equilibrium condition:

$$\psi_e = \int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - 2\beta y(x), \tag{13}$$

which should be identical everywhere in the linear city ($-a \leqq x \leqq a$) in equilibrium (and hence, (x) is dropped on the left hand side), otherwise there will exist an incentive for firms to move.

Differentiating (13) twice with respect to x, and eliminating the integral part, one can get

$$\beta y''(x) + (\alpha-\alpha^2\beta)y(x) = \alpha^2\psi_e/2 \;. \tag{14}$$

Solving the differential equation (14), the equilibrium distribution of the firm density is then given by

$$y(x) = C_3 e^{\sqrt{\alpha^2-\alpha/\beta}\; x} + C_4 e^{-\sqrt{\alpha^2-\alpha/\beta}\; x} + \frac{\psi_e \alpha}{2(1-\alpha\beta)}, \qquad \text{for } \alpha\beta \neq 1 \tag{15}$$

$$= C_3' + C_4' x + \frac{\psi_e \alpha^3}{2} x^2, \qquad \text{for } \alpha\beta = 1. \tag{15'}$$

where the constants (C_3, C_4, ψ_e) in (15) and (C_3', C_4', ψ_e) in (15') are to be determined by the boundary conditions (i.e., $r(\pm a) = r_A$) and the constraint of total number of firms (2') as before. After similar calculation as the preceding section, the equilibrium density distribution is shown to be identical to the optimal density distribution (8a)-(8d) except that β in optimum is replaced by $\beta/2$ in equilibrium.

Next, the equilibrium city length of a is determined as follows. Since perfect competition prevails, the profit of every developer $\phi(x)$ is constant and equal to the agricultural land rent r_A in equilibrium. That is, substituting R(x) in (12) into (11) and setting both $\phi(a)$ and r(a) equal to r_A,

$$r_A = 2\beta\{y(a)\}^2 - \beta\{y(a)\}^2 - r_A,$$

or

$$y(a) = \sqrt{\frac{r_A}{\beta/2}}\,, \tag{16}$$

which is the same as (5") except that β in optimum is replaced by $\beta/2$ here.

Summarizing the above two results, one can say that the difference between the optimum and equilibrium solutions of y(x) and a is ascribed only to the difference in β, and that <u>the equilibrium city tends to be more dispersed than the optimum city</u> since the weight of the construction cost β in equilibrium affects the value of the exponent twice as much as that in optimum. Recall that the city length of a is an increasing function of β (Figure 3 in the previous section). This conclusion is in agreement with Borukhov and Hochman (1977) and Imai (1982) although their model assumptions differ.

Consequently, government intervention is indispensable; for instance, a locational subsidy in accordance with firm density, or possibly a proper zoning ordinance to prevent from decentralization. The conclusion obtained here is opposed to the decentralization policy which many city governments adopt. However, if costs of commuting from outside the city is taken into account, or if air pollution as a negative externality is caused owing to commuting congestion, then it would be desirable to decentralize firms compared to the optimal distribution.

It should be noted that (13) can be obtained by the Alonso's (1964) bid rent approach in residential location. Namely, substituting R(x) in (11) into (10), setting $\phi(x) = r_A$, and maximizing the land rent of r(x) in each location x with respect to y(x) and a, one arrives at the same conditions as (13) and (16). If, on the other hand, the aggregate land rent of the while city be maximized holding the net profit of the firm $\psi_e(x)$ constant at every location with given N, and holding the net profit of the developer $\phi(x)$ constant, then one arrives at the optimum conditions (4) and (5") in Section 2.

Next, consider the characteristics of the equilibrium rent functions of R(x) and r(x). Since equation (12) indicates that the unit construction cost of the office building, $\beta y(x)$ is equal to the building rent R(x) throughout the city, the characteristics of R(x) are identical to those of y(x) which has already investigated in the previous section. Using (11), (12) and $\phi(x) = r_A$, on the other hand, the land rent function of r(x) is expressed as

$$r(x) = \beta\{y(x)\}^2 - r_A. \tag{17}$$

Hence, employing the characteristics of the firms' density function as examined in the previous section, one can easily draw the following relationships:

$$\frac{dr(x)}{d|x|} \leq 0, \text{ in which the equality holds only when } x=0, \tag{18}$$

$$\frac{d^2 r(x)}{d|x|^2} < 0, \text{ near the midpoint, and} \tag{19a}$$

$$\frac{d^2 r(x)}{d|x|^2} \geqq 0, \text{ elsewhere.} \tag{19b}$$

The land rent function is therefore "bell-shaped" regardless of positive values of α and β. That is to say: from (18), <u>the land rent gradient is zero at the city center and boundaries, and the land rent decreases with distance from the city center</u>; and from (19a) and (19b), <u>the land rent function is concave near the center and convex near the boundaries</u>. In other words, the land rent function always has two inflection points between the center and two boundaries for any positive parameters of α and β.

4. CONCLUSION

This paper has investigated the implications of the accessibility benefits as urban agglomeration economies in a linear city. It is assumed that a city government maximizes aggregate net profits comprizing the accessibility benefits to all other firms, the office construction costs measured by firm density, and the opportunity costs of alternative land use.

Applying the calculus of variations, it is derived that the optimal distribution of firms' density is a constant, cosine, quadratic, or cosine-hyperbolic curve dependent upon the parameter range. While the optimal city length (a) is positively related to the rate of the building construction costs (β) as shown in Figure 1, it is negatively related to the rate of the distance friction or transportation costs (α) for small α, but is positively related to α for large α as illustrated in Figure 2. One may say that decentralization of office firms measured by a is taken place, if the construction costs become high, or if the transportation costs become extremely low or extremely high. It is also found that the optimal density of firms is always smooth at the center of the city irrespective of the parameter values, which is in contrast to the classical urban economics models. The concavity of the optimal density function is shown to hold in most of the cases, which is contrary to the well-known negative exponential density function.

The market equilibrium distribution of firm density is also obtained supposing the net profit is identical at every location under the same constraints. It is revealed that the optimal distribution of firm density is more concentrated than the market distribution, which is exhibited by the difference in the parameter value of β. In this instance, the city is in need of a pertinent government intervention, such as a locational subsidy or zoning restriction, in order to prevent from dispersion.

The equilibrium building rent $R(x)$ is shown to be proportional to the firm density (equation (13)), and hence concave in most of the cases. It is also found that the equilibrium land rent $r(x)$ is proportional to the square of the firm density (equation (17)), and is concave near the city center and convex near the city borders.

FOOTNOTES

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1. Also the force of repelling between firms like Hotelling's spatial competition is not taken into consideration in this paper.

2. It would be unrealistic for a city government to optimize the number of firms (at least in Japan). Rather it would be more realistic to optimize the physical size of a city by applying the zoning ordinance.

3. If households are incorporated in this model, one would conceive that due to the effect of commuting costs, the firm distribution is more decentralized as demonstrated by Fujita and Ogawa(1982) in their market equilibrium model. However, such a discussion is beyond the scope of this paper because the major objective of this paper is to explore urban agglomeration economies of firms' interaction.

4. See Tabuchi (1982) for justification of this assumption of proportionality.

APPENDIX

It will be proven that the optimal city length a in equation (5") with (9) exists and is uniquely determined.

Define

$$f(a) = \sqrt{\beta}\,y(a) = \sqrt{r_A}, \qquad 0 < a \leqq \pi \text{ for } 0 < \alpha\beta < 2,$$
$$0 < a \qquad \text{for } \alpha\beta \geqq 2.$$

Then, it is easily shown that $\lim_{a \to 0} f(a) = +\infty$ and $\lim_{a \to \pi} f(a) = 0$ for $0 < \alpha\beta < 2$, and that $\lim_{a \to 0} f(a) = +\infty$ and $\lim_{a \to +\infty} f(a) = 0$ for $\alpha\beta \geqq 2$. Also it is easily observed that $f(a)$ is continuous for the defined range. Hence, so as to demonstrate the existence and uniqueness, it is sufficient to show that $f'(a) < 0$ for the range of a.

(iii) $0 < \alpha\beta < 2$

$$\text{As } f(a) = \frac{\sqrt{\beta}N}{2} \cdot \frac{(\alpha\beta-2)\cos(2\theta a) - \theta\beta(\alpha\beta-1)\sin(2\theta a)}{\alpha\beta a\cos(\theta a) - (\theta\beta a+2/\theta)\sin(\theta a)},$$

where $\theta = \sqrt{2\alpha/\beta-\alpha^2}$,

$$f'(a) = \frac{\sqrt{\beta}N}{2} \cdot \frac{\alpha\beta(2-\alpha\beta)\cos(2\theta a) - 2(2-\alpha\beta)(1+\alpha a) + \theta\beta(\alpha\beta-1)\sin(2\theta a)}{\{\alpha\beta a\cos(\theta a) - (\theta\beta a+2/\theta)\sin(\theta a)\}^2}.$$

Let the numerator of the second term be g(a), then

$$\begin{aligned} g'(a) &= \alpha(2-\alpha\beta)\{2(\alpha\beta-1)\cos(2\theta a)-2\sqrt{1-(\alpha\beta-1)^2}\sin(2\theta a)-2\} \\ &= 2\alpha(2-\alpha\beta)\{\sin(\omega-2\theta a)-1\} \\ &\leqq 0, \end{aligned}$$

where $\omega = \sin^{-1}(\alpha\beta-1)$. Since $g(0) = -(\alpha\beta-2)^2 < 0$, $g'(a) < 0$. Then, $f'(a) < 0$, i.e., a is uniquely determined by (5") for given values of N, r_A, α and β.

(iv) $\alpha\beta = 2$

$$\text{As} \quad f(a) = \frac{\sqrt{\beta}N}{2} \cdot \frac{\beta a + \beta^2/2}{2a^2/3 + \beta a + \beta^2/2},$$

$$f'(a) = -\frac{\beta^{1.5}N}{2} \cdot \frac{4a^3/3 + 2\beta a^2 + \beta^2 a + \beta^3/4}{\{2a^3/3 + \beta a^2 + \beta^2 a/2\}^2,} < 0.$$

Therefore, $f(a)=\sqrt{r_A}$ has a unique solution, i.e., for given N, r_A, α and β, a is uniquely determined by (5").

(v) $2 < \alpha\beta$

$$\text{As} \quad f(a) = \frac{\sqrt{\beta}N}{2} \cdot \frac{(\alpha\beta-2)\cosh(\tau a) + \tau\beta\sinh(\tau a)}{\alpha\beta a\cosh(\tau a) + (\tau\beta a-2/\tau)\sinh(\tau a),}$$

where $\tau = \sqrt{\alpha^2-2\alpha/\beta}$,

$$f'(a) = \frac{\sqrt{\beta}N}{2} \cdot \frac{-\alpha\beta(\alpha\beta-2)\cosh(2\tau a) + 2(\alpha\beta-2)(1+\alpha a) + \tau\beta(\alpha\beta-1)\sinh(2\tau a)}{\{\alpha\beta a\cosh(\tau a) + (\tau\beta a-2/\tau)\sinh(\tau a)\}^2.}$$

Let the numerator of the second term be h(a), then

$$
\begin{aligned}
h'(a) &= -2\tau\alpha\beta(\alpha\beta-2)\sinh(2\tau a)-2\alpha(\alpha\beta-2)(\alpha\beta-1)\cosh(2\tau a)+2\alpha(\alpha\beta-2) \\
&\leqq -2\alpha(\alpha\beta-2)\{0+(\alpha\beta-1)-1\} \\
&= -2\alpha(\alpha\beta-2)^2 \\
&< 0.
\end{aligned}
$$

Then, $f'(a) < 0$, i.e., a is uniquely determined by (5") for given values of N, r_A, α and β.

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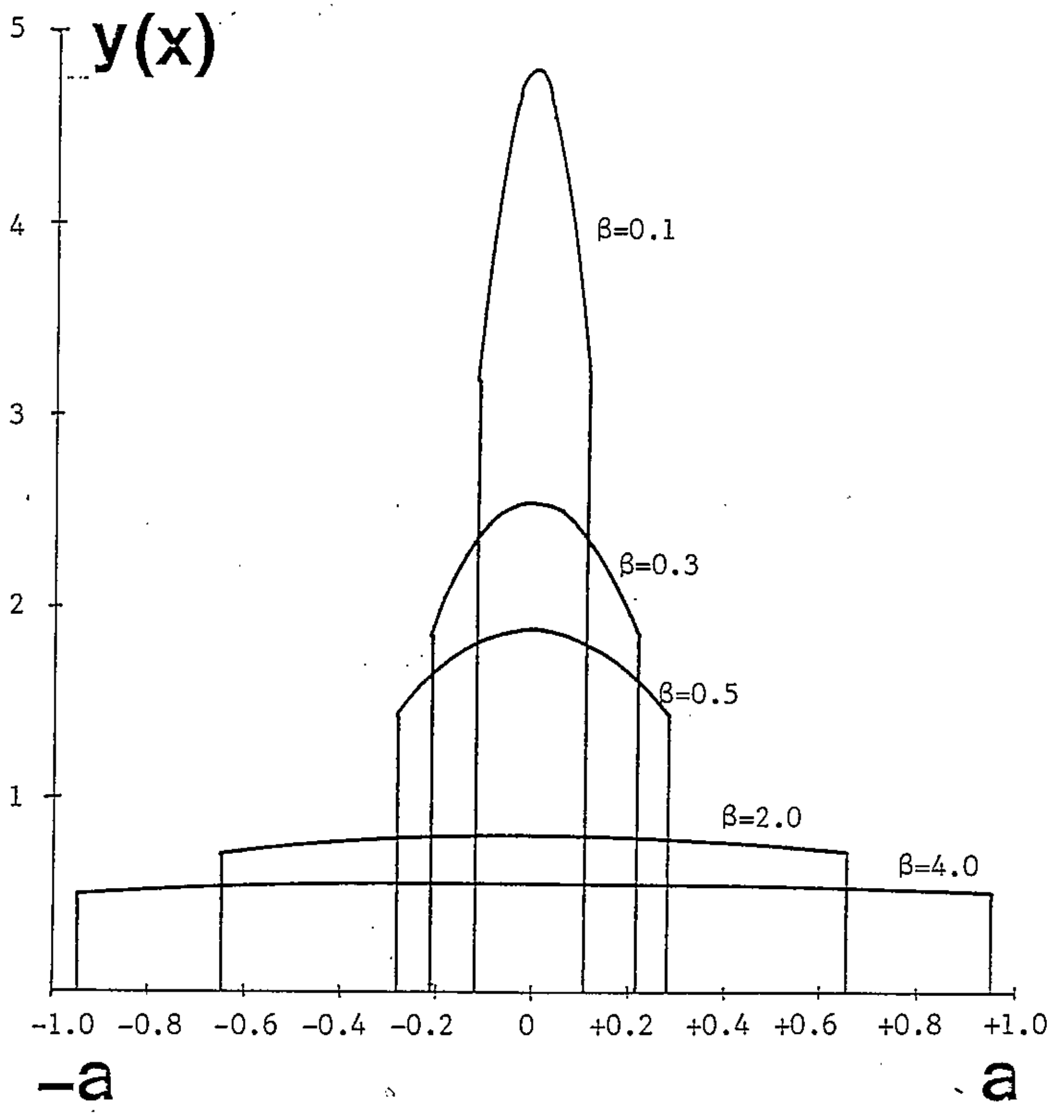

Figure 1. Impact of β on the Optimal Density Distribution

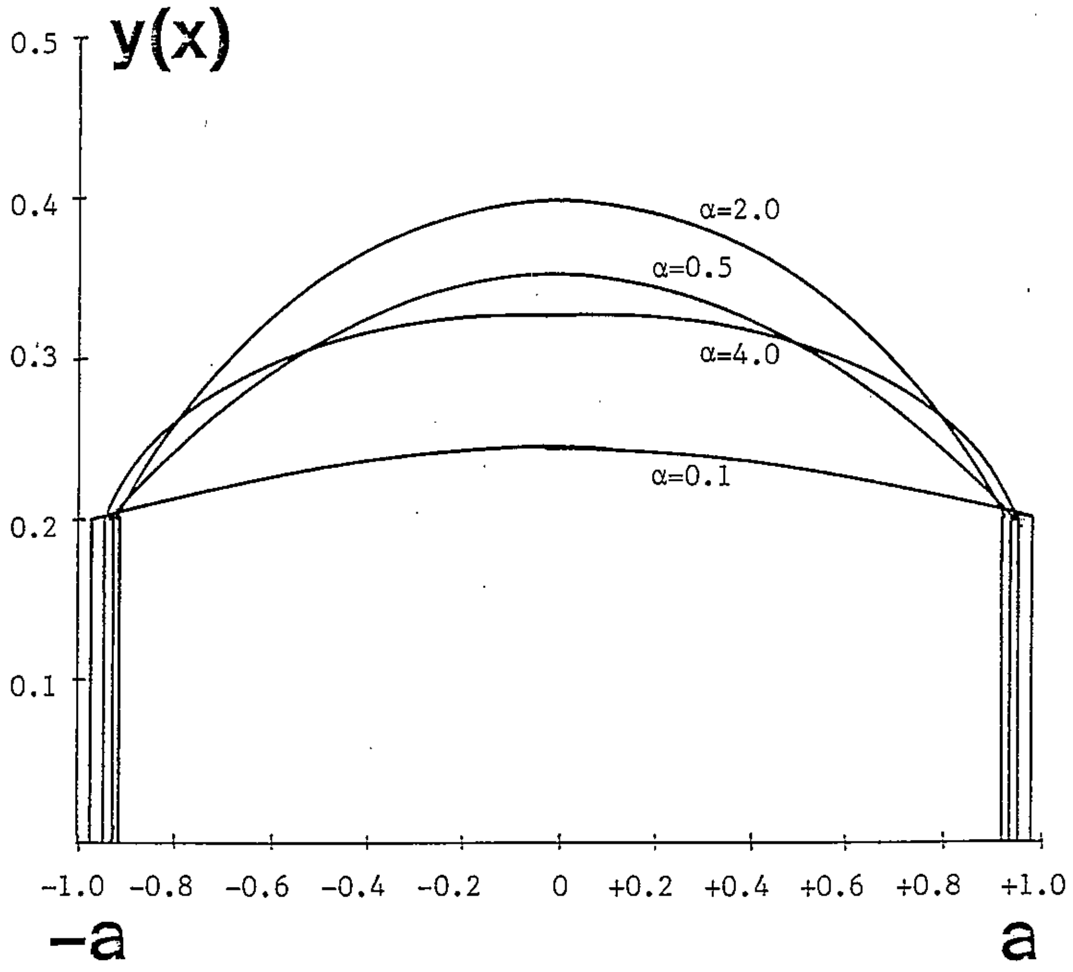

Figure 2. Impact of α on the Optimal Density Distribution

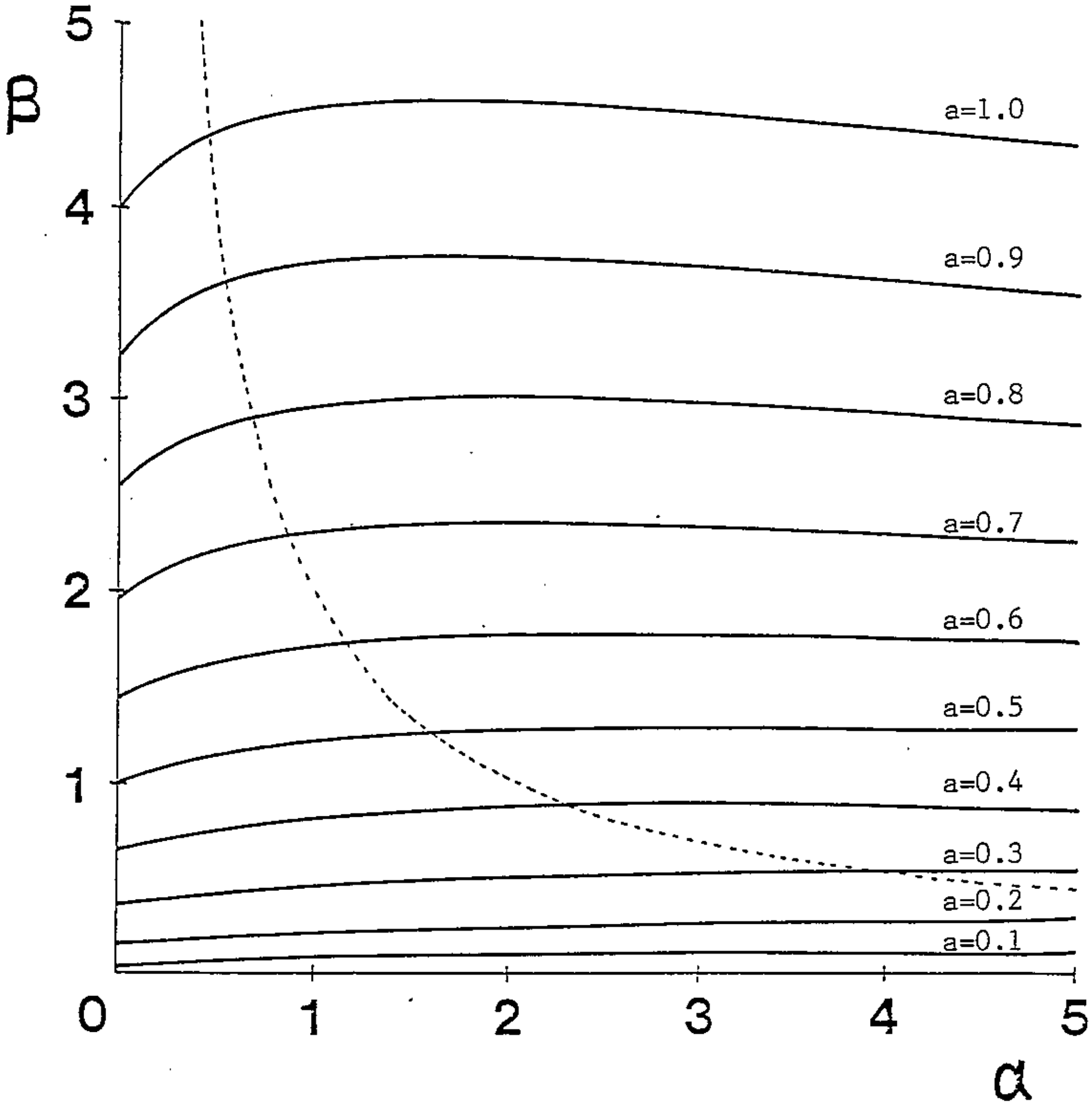

Figure 3. Impact of α and β on the Optimal City Length a

(Dotted curve is αβ=2)