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Optimal Inventory Policy under Random Selling Price and Procurement

by

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Abstract

This paper studies a single-item multi-period inventory model in which a procured item is sold with fluctuating market prices. The feature of the model is that the procurement is a random variable, which reflects the uncertainty of the production of agricultural or marine products. The paper determines the optimal policy which maximizes the total expected discounted reward over an infinite planning horizon. It also discusses the relationship between the maximum expected reward and the storage capacity.

1. Introduction

In many kinds of agricultural or marine production and in the production of certain industrial items, it is not certain how many units of the item is procured when an order, i.e. a request to replenish the stock, is placed. This paper considers the inventory problem of ordering such a commodity and selling it with fluctuating market prices so as to maximize the total expected discounted reward over an infinite planning horizon.

The model studied assumes that decisions are made at regular intervals. An order can be placed in each period, and between order and delivery we assume there is a lag of one period of time. At the beginning of each period the price quotation and (if an order has been placed) the procurement are known before a decision is made. Prices in future periods are determined by either an independent stochastic process or a Markovian stochastic process. The special assumption is that the procurement is a random variable instead of a fixed known amount. The sum of the procurement and the carryover from the previous period is the all amount which can be sold. If it is observed that the commodity should not be sold, it can be stored away as carryover to a warehouse.

When a particular price quotation and a particular procurement are obtained, the decision maker must make the following decisions:

1. How many units should he sell at the current price? (Or, equivalently, how many units should he carry over to the next period?)
2. Whether to place an order or not.

In this paper, however, we assume that an order is placed in each period, and investigate the optimal selling policy alone. The problem involving the ordering decision is so complicated that the author wishes to leave it as an open question.

The model extends the inventory problem known as " a warehouse problem " to the situations in which price and procurement are random variables instead of being known with certainty. Although a family of the warehouse problems has been studied by several authors, published work involving uncertainty is very scarce. The Charnes et al. paper [1] introduced uncertainty of price and cost offers, but it essentially removes the stochastic nature of the problem. Uncertainty of procurement was first introduced by Karlin [2] in his static inventory model. However, it seems that the extension to a dynamic model has not been made yet. If we postulate a fixed amount of replenishment in a warehouse problem, problem is equivalent to that of purchasing and stockpiling sufficient quantities of a commodity in order to satisfy a constant demand in each period. Magirou [3] investigated this kind of problem with fluctuating prices, and obtained an optimal policy which minimizes the total expected discounted cost over an infinite planning horizon.

The problem will be approached by means of the functional equation technique of the theory of dynamic programming. The model is defined precisely in Section 2. From Section 2 through Section 4, we assume that the successive prices are stochastically independent. Section 3 presents the dynamic programming formulation and derives the optimal value function. The optimal policy and its characteristics are discussed in Section 4 in light of the structure of the optimal value function. The case in which the sequence of price quotations forms a Markov chain is discussed in Section 5.

Remarks.

(1) The model involves no holding cost. It will be noticed that our approach is also applicable to the situation where a convex increasing holding cost function is assumed.

(2) An constant ordering cost is implicitly assumed. Since it is postulated that an order is placed in each period, the cost is treated as naught.

2. Definition of the Model

This section formulates the model more precisely. First, the variables P_t, X_t, y_t and a_t are defined as follows:

$P_t \equiv$ discrete random variable representing the market price of a unit of commodity in period t, whose value becomes known at the beginning of the period. The realization is denoted by p_t and is bounded, $0 < p_t \leq \overline{p}$.

$X_t \equiv$ discrete random variable representing procurement that occurs in period t. The realization x_t is bounded, $0 \leq x_t \leq \overline{x}$.

$y_t \equiv$ the sum of x_t and a_{t-1}, i.e. the all amount which can be sold in period t.

$a_t \equiv$ the carryover to period t+1.

It is assumed that the commodity argued in this paper is countable, and hence x_t, y_t, and a_t are integers.

The couples of price and procurement (p_t,x_t), t=1,2,3,..., are determined by an independent stochastic process which is defined by

$f_{P,X}(.,.) \equiv$ the joint probability function of P and X,

$f_X(.) \equiv$ the marginal probability function of X,

$f_{P|X}(.) \equiv$ the conditional probability function of P given X=x.

In period t, a selling decision is to be made on the basis of p_t and y_t. Since there exists a storage capacity denoted by M, the following constraint is effective:

$$0 \leq a_t \leq (y_t \wedge M)^{\dagger}.$$

Assuming a discount factor $\alpha \in (0,1)$, the discounted infinite horizon reward is

$$\sum_{t=1}^{\infty} \alpha^t \{ p_t(y_t - a_t) \}.$$

The objective of the decision-maker is to find a policy $d = \{d_1, d_2, \ldots\}$ which maximizes the total expected discounted reward over an infinite planning horizon, where d_t specifies what act to choose in the period t as a function of the history $h_t \equiv (p_1, y_1, a_1, \ldots, p_{t-1}, y_{t-1}, a_{t-1}, p_t, y_t)$ of the system up to period t. For an initial amount of commodity y, price p and policy d, the expected reward is denoted by

$$V_d(y,p) \equiv \underset{d}{E} \sum_{t=1}^{\infty} \alpha^t \{ P_t(y_t - a_t) \},$$

where $y_1 = y$, $P_1 = p$, $a_t = d_t(p_1, y_1, a_1, \ldots, y_t, p_t)$, $y_{t+1} = a_t + X_{t+1}$, and $\underset{d}{E}$ indicates the conditional expectation with respect to P_t and X_t given that the policy d is employed.

$\dagger$ $(y_t \wedge M) = \min \{y_t, M\}$

The maximum reward is defined as

$$V(y,p) \equiv \sup_{d} V_d(y,p).$$

Let d^* represent the optimal policy if the above supremum is achieved by the policy d^*, i.e. $V(y,p) = V_{d^*}(y,p)$.

3. DP Formulation

Using Theorem 6.3 and Corollary 6.6 in Ross [4], the following holds:

1. $V(y,p)$ is the unique solution to

(1) $$V(y,p) = \max_{0 \leq a \leq (y \wedge M)} \{ (y-a)p + \alpha EV(X+a,P) \},$$

2. If we define the following recursion

(2) $$V_{n+1}(y,p) = \max_{0 \leq a \leq (y \wedge M)} \{ (y-a)p + \alpha EV_n(X+a,P) \},$$

for any bounded real-valued function $V_0(y,p)$, we have $V_n(y,p) \to V(y,p)$ as $n \to \infty$.

3. The stationary policy which selects an action maximizing the right-hand side of (1) attains $V(y,p)$.

In order to specify the figure of a function defined on integers, we introduce the following definition.

Definition A function $f(i)$ defined on $\{ 0,1,2,\ldots,k \}$ is called concave iff $f(i) - f(i-1) \geq f(i+1) - f(i)$ for $i=1,2,\ldots,k-1$. //

Lemma 1. The optimal value function $V(y,p)$ is nondecreasing in y and p, and is concave in y for any fixed p. //

Proof. The nondecreasing properties are evident from (1). We shall establish the concavity by showing that $V_n(y,p)$ is concave in y for all n. Let $V_0(y,p) \equiv 0$, which is concave in y. Suppose $V_n(y,p)$ is concave in y. Let us define $\Delta(y) \equiv V_{n+1}(y,p) - V_{n+1}(y-1,p)$. For $y \geq M+1$, it is evident that $\Delta(y) = \Delta(y+1)$. If $y = M$, $\Delta(y+1) = p$ and

$$\begin{aligned}\Delta(y) &= V_{n+1}(M,p) - V_{n+1}(M-1,p)\\ &= \max_{0 \leq a \leq M} \{ (M-a)p + \alpha EV_n(X+a,P) \} - \max_{0 \leq a \leq M-1} \{ (M-1-a)p + \alpha EV_n(X+a,P) \}\\ &\geq p.\end{aligned}$$

Hence, $\Delta(y) \geq \Delta(y+1)$. For any fixed $1 \leq y \leq M-1$, define

$$M(s) = \max_{0 \leq a \leq s} \{ (y-a)p + \alpha EV(X+a,P) \}.$$

Then, we have

$$\Delta(y) = M(y) + p - M(y-1),$$

$$\Delta(y+1) = p + M(y+1) - M(y).$$

Since $\alpha EV_n(X+y,P)$ is concave in y from the induction hypothesis, it is sufficient for us to consider three cases:

Case 1. Maxima of $M(y-1)$, $M(y)$ and $M(y+1)$ are attained by a common a such that $0 \leq a \leq y-1$. Then, $\Delta(y) = p = \Delta(y+1)$.

Case 2. Maximum of $M(y-1)$ is attained by $a = y-1$, and maxima of $M(y)$ and $M(y+1)$ are attained by $a = y$. We have $\Delta(y) \geq 2p > p = \Delta(y+1)$. The first inequality holds from $M(y) \geq p + \alpha EV_n(X+y-1,P)$.

Case 3. Maxima of $M(y-1)$, $M(y)$ and $M(y+1)$ are attained by $a = y-1$, $a = y$ and $a = y+1$, respectively. Then, we get

$$\Delta(y) = \alpha EV_n(X+y,P) - \alpha EV_n(X+y-1,P),$$

$$\Delta(y+1) = \alpha EV_n(X+y+1,P) - \alpha EV_n(X+y,P).$$

Since $\alpha EV_n(X+y,P)$ is concave in y by the induction hypothesis, we have $\Delta(y) \geq \Delta(y+1)$. Therefore, it is established that $V_{n+1}(y,p)$ is concave in y. //

Let us define

$$S_x(z) = E_x[\ \max\{P,z\}\ ],$$

where E_x represents the conditional expectation with respect to P given X=x. The following properties of $S_x(.)$ are easily verified.

1) For any $z \leq 0$, $S_x(z) = \mu_{P|x}$, where $\mu_{P|x}$ is the mean of the conditional distribution of P given X=x.

2) For any $z \geq \bar{p}$, $S_x(z) = z$.

3) $S_x(z)$ is nondecreasing in z.

4) For any $z_1 \geq z_2$, $S_x(z_1)-S_x(z_2) \leq z_1 - z_2$.

5) For any $\beta \leq 1$, $z - \beta S_x(z)$ is nondecreasing in z.

6) For any β such that $\beta\in(0,1)$ and any real Δ,

$$Z = \beta S_x(z) + \Delta$$

is a contraction mapping from R^1 into R^1. Hence, the equation

(3) $$z = \beta S_x(z) + \Delta$$

has a unique solution.

Proposition 1. The optimal value function is expressed as follows:

$$V(y,p) = \begin{cases} \sum_{i=1}^{y} \max\{ p, c_i\} + c_0 & \text{for } 0 \le y \le M, \\ (y-M)p + \sum_{i=1}^{M} \max\{ p, c_i\} + c_0 & \text{for } M+1 \le y \le M+\bar{x}. \end{cases}$$

The sequence $c_1, c_2, \ldots, c_M$ is nonincreasing and is the unique solution of the system of equations

$$c_i = \alpha\{ \sum_{x=0}^{M-i} f_X(x) S_x(c_{x+i}) + \sum_{x=M-i+1}^{\bar{x}} f_X(x)\mu_{P|x} \} \quad \text{for } i=1,2,\ldots,M. \tag{4}$$

c_0 is given by

$$c_0 = \frac{\alpha}{1-\alpha}\{ \sum_{x=0}^{M} f_X(x) \sum_{i=1}^{x} S_x(c_i) + \sum_{x=M+1}^{\bar{x}} f_X(x) \sum_{i=1}^{M} S_x(c_i) + \sum_{x=M+1}^{\bar{x}} f_X(x)(x-M)\mu_{P|x}\}, \tag{5}$$

where $\mu_{P|x}$ is the mean of the conditional distribution of P given X=x. //

Proof. From (1), we have

$$\begin{cases} V(0,p) = \alpha EV(X,P), & \\ V(y,p) = \max\{ p + V(y-1,p), \alpha EV(X+y,P) \} & \text{for } 1 \le y \le M, \\ V(y,p) = (y-M)p + V(M,p) & \text{for } M+1 \le y \le M+\bar{x}. \end{cases} \tag{6}$$

Let us define c_i, i=0,1,2,...,M, by

$$c_0 = \alpha EV(X,P),$$

$$c_i = \alpha EV(X+i,P) - \alpha EV(X+i-1,P) \qquad \text{for } i=1,2,\ldots,M.$$

Then

$$\alpha EV(X+y,P) = \sum_{i=0}^{y} c_i, \tag{7}$$

and hence, (6) is written as

$$\begin{cases} V(0,p) = c_0, \\ V(y,p) = \max\{\, p+V(y-1,p),\ \sum_{i=0}^{y} c_i \,\} & \text{for } 1 \le y \le M, \\ V(y,p) = (y-M)p + V(M,p) & \text{for } M+1 \le y \le M+\bar{x}. \end{cases}$$

By induction, we shall show that $V(y,p)$ and c_i satisfy

$$\begin{cases} V(0,p) = c_0, \\ V(y,p) = \sum_{i=1}^{y} \max\{p,c_i\} + c_0 & \text{for } 1 \le y \le M, \\ V(y,p) = (y-M)p + V(M,p) & \text{for } M+1 \le y \le M+\bar{x}. \end{cases} \tag{8}$$

We observe that $V(1,p) = \max\{p,c_1\} + c_0$, which satisfies (8). For any $1 \le k \le M-1$, suppose

$$V(k,p) = \sum_{i=1}^{k} \max\{p,c_i\} + c_0.$$

Then

$$V(k+1,p) = \max\{\, p+\sum_{i=1}^{k} \max\{p,c_i\}+c_0,\ \sum_{i=0}^{k+1} c_i \,\}.$$

If $p \ge c_{k+1}$, (8) holds with k+1. If $p < c_{k+1}$, we get from the nonincreasing property of c_i, $i=1,2,\ldots,M$

$$V(k+1,p) = \sum_{i=0}^{k+1} c_i = \sum_{i=1}^{k+1} \max\{p,c_i\} + c_0,$$

which completes the induction. Substituting (8) into (7), we have

$$(9) \qquad \sum_{i=0}^{y} c_i = \alpha\{ \sum_{x=0}^{M-y} f_X(x) \sum_{i=1}^{x+y} S_x(c_i) + \sum_{x=M-y+1}^{\overline{x}} f_X(x) \sum_{i=1}^{M} S_x(c_i)$$

$$+ \sum_{x=M-y+1}^{\overline{x}} f_X(x)(x+y-M)\mu_{P|x}\} + \alpha c_0$$

for $y=0,1,2,\ldots,M$.

Letting $y = 0$, we get (5). Taking the difference of (9) with respect to y, we obtain (4). Since the equation (4) with $i=M$ is a kind of equation (3), c_M is uniquely determined. Similarly, it is shown recursively that $c_{M-1},\ldots,c_1$ are uniquely determined, which proves the proposition. //

Remark. The sequence $c_1, c_2,\ldots, c_M$ in Proposition 1 satisfies

$$0 \leq c_M \leq c_{M-1} \leq \cdots \leq c_2 \leq c_1 \leq \overline{p}. \qquad //$$

Proof. It suffices to show that $c_1 \leq \overline{p}$, since other inequalities are obtained by Lemma 1. Suppose $c_1 > \overline{p}$. From the properties 1), 2) and 3) of $S_x(z)$, we have

$$(10) \qquad S_x(c_{x+1}) \leq S_x(c_1) = c_1,$$

$$(11) \qquad \mu_{P|x} \leq S_x(c_1) = c_1.$$

Substituting (10) and (11) into (4) with $i=1$, we get

$$c_1 \leq \alpha\{ \sum_{x=0}^{M-1} f_X(x)c_1 + \sum_{x=M}^{\overline{x}} f_X(x)c_1 \} \leq \alpha c_1,$$

which contradicts to the assumption that $\alpha \in (0,1)$.

4. Optimal Policy

Theorem 1. When the process is in state (y,p), the optimal $a = d^*(y,p)$ is described as follows:

1) If there exists a k such that $c_{k+1} < p \leq c_k$, then

$$a = \begin{cases} k & \text{if } k < y \quad (\text{ sell } y\text{-}k \text{ units }), \\ y & \text{if } k \geq y \quad (\text{ sell nothing }). \end{cases}$$

2) If $p \leq c_M$, then

$a = (y \wedge M)$ (sell nothing or sell the excess of the commodity over the storage capacity).

3) If $p > c_1$, then

$a = 0$ (sell all). //

Proof. If there exists a k such that $c_{k+1} < p \leq c_k$, by Proposition 1, we get for $0 \leq y \leq M$

$$V(y,p) = \sum_{i=1}^{y} \max\{p,c_i\} + c_0 = \begin{cases} (y-k)p + \sum_{i=0}^{k} c_i & \text{if } k < y, \\ \sum_{i=0}^{y} c_i & \text{if } k \geq y, \end{cases}$$

and for $M+1 \leq y \leq M+\overline{x}$

$$V(k,p) = (y-M)p + \sum_{i=1}^{M} \max\{p,c_i\} + c_0$$

$$= (y-k)p + \sum_{i=1}^{k} c_i + c_0.$$

Employing equation (7), we have in either case

$$V(y,p) = \begin{cases} (y-k)p + \alpha EV(X+k,P) & \text{if } k < y, \\ \alpha EV(X+y,P) & \text{if } k \geq y. \end{cases}$$

Therefore, if $k < y$, then the maximum of (1) is attained by $a = k$, and if $k \geq y$, it is attained by $a = y$, which proves case 1) of the theorem.

Similarly, if $p \leq c_M$, then

$$V(y,p) = \begin{cases} \sum_{i=0}^{y} c_i = \alpha EV(X+y,P) & \text{if } 0 \leq y \leq M, \\ (y-M)p + \sum_{i=0}^{M} c_i = (y-M)p + \alpha EV(X+M,P) & \text{if } M+1 \leq y \leq M+\overline{x}, \end{cases}$$

and if $p > c_1$, $V(y,p) = yp + c_0 = yp + \alpha EV(X,P)$, which completes the proof of case 2) and case 3). //

The optimal carryover in state (y,p) is described by the subsets on the range of the price as shown in Figure 1.

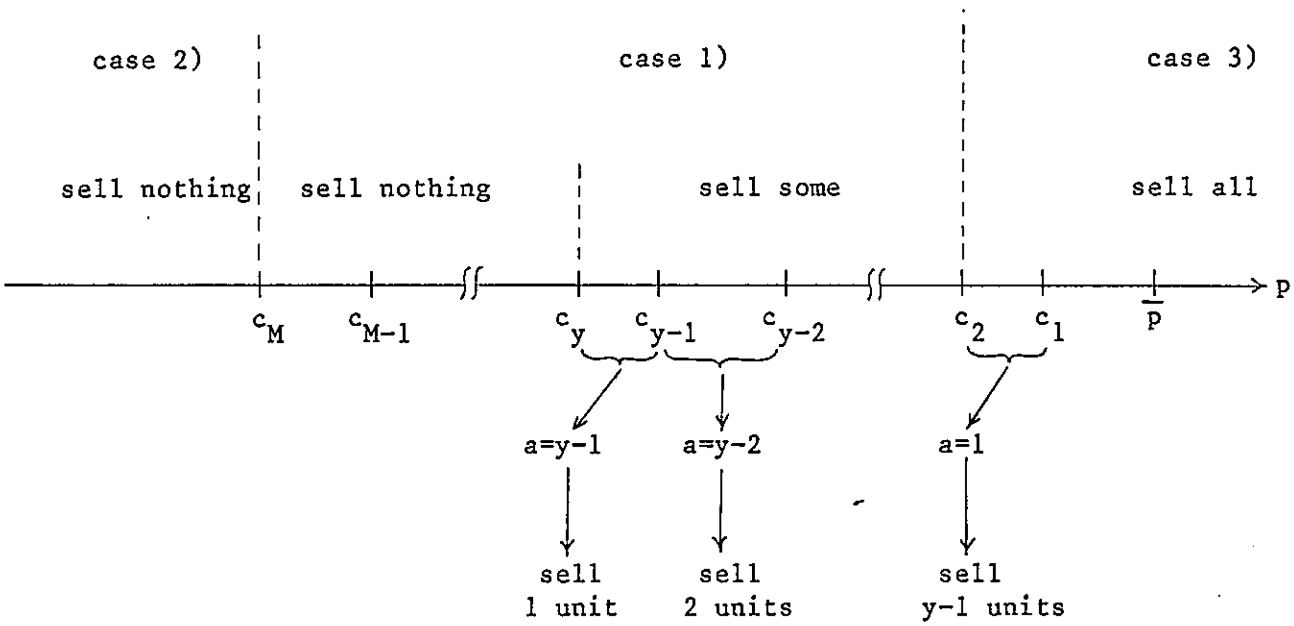


Figure 1. Optimal decision rules for a given $1 \leq y \leq M$

A numerical example is presented on the next page.

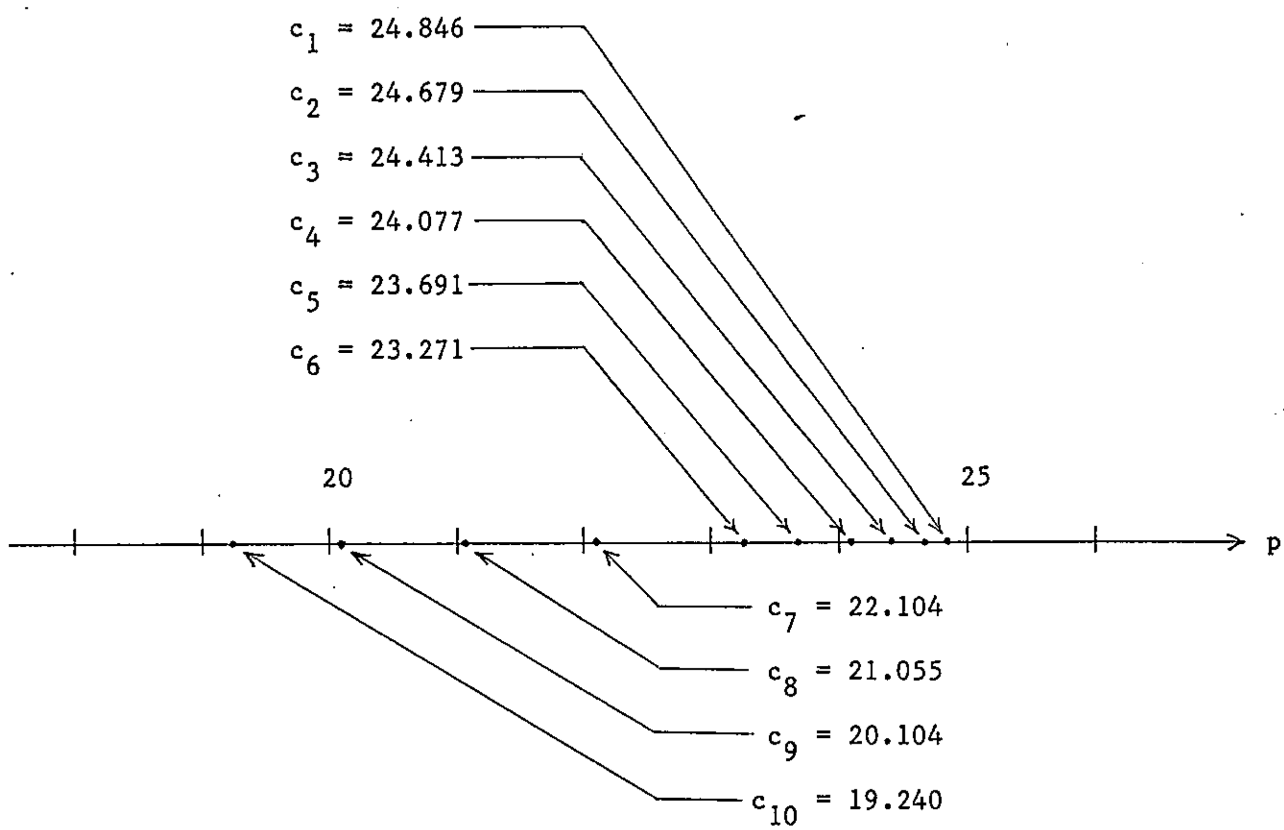
Example. Assume that X and P are independent, and that their distributions are

$$f_X(x) = \begin{cases} 0.2 & \text{for } x=0,1,\ldots,4, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_P(p) = \begin{cases} 0.025 & \text{for } p=1,2,\ldots,40, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha = 0.9$ and $M = 10$. Using the method of successive approximations, the sequence of c_i is obtained as follows:

$c_0 = 500.027$



Next proposition describes the relationship between c_i, $i=1,2,\ldots,M$, and M. Let $c_i(M)$ denote c_i under the storage capacity M. Define $S(z)= E[\max\{P,z\}]$. $S(.)$ has the same properties as $S_x(z)$ presented in Section 3, and satisfy

$$\sum_{x=0}^{\bar{x}} f_X(x)S_x(z) = S(z). \tag{12}$$

Proposition 2.

1) $c_i(M)$ is nondecreasing in M.
2) When M increases to M+1, $c_0(M)$ increases by

$$\frac{1}{1-\alpha}\{ \alpha S(c_1(M+1)) - c_1(M+1) \}.$$

3) As M tends to infinity, $c_0(M)$ converges to

$$\frac{\alpha}{1-\alpha}\sum_{x=0}^{\bar{x}} x f_X(x)S_x(c^*),$$

where $c^* = \alpha S(c^*)$.

Proof. 1) By (4), we have

$$c_{M+1}(M+1) = \alpha\{ f_X(0)S_0(c_{M+1}(M+1)) + \sum_{x=1}^{\bar{x}} f_X(x)\mu_{P|x} \},$$

$$c_M(M) = \alpha\{ f_X(0)S_0(c_M(M)) + \sum_{x=1}^{\bar{x}} f_X(x)\mu_{P|x} \}.$$

Since the equation has a unique solution, we get $c_{M+1}(M+1) = c_M(M)$. Applying (4) successively, we obtain

(13) $$c_{i+1}(M+1) = c_i(M) \qquad \text{for } i = M, M-1, \ldots, 1.$$

Thus, we have from the nonincreasing property of c_i

$$c_i(M+1) \geq c_{i+1}(M+1) = c_i(M) \qquad \text{for } i = 1, 2, \ldots, M,$$

which completes the proof of part 1) of Proposition 2. //

Proof 2) Employing (13), we have

(14) $$\sum_{i=1}^{x} S_X(c_i(M)) = \sum_{i=1}^{x} S_X(c_{i+1}(M+1)) = \sum_{i=2}^{x+1} S_X(c_i(M+1))$$

With the help of a little algebra we get from (5), (12) and (14)

$$c_0(M+1) - c_0(M)$$

$$= \frac{\alpha}{1-\alpha} \left[S(c_1(M+1)) - \left\{ \sum_{x=0}^{M} f_X(x) S_x(c_1(M+1)) + \sum_{x=M+1}^{\overline{x}} f_X(x)\, \mu_{P|x} \right\} \right]$$

$$= \frac{1}{1-\alpha} \left\{ \alpha S(c_1(M+1)) - c_1(M+1) \right\}$$

The last equality is established by (4) with M+1. //

Proof 3) From the remark in the previous section and part 1) above, $c_1(M)$ approaches to a limit, say c^+, as $M \to \infty$. Employing (13), we have

$$\lim_{M\to\infty} c_i(M) = \lim_{M\to\infty} c_1(M-i) = c^+ \qquad \text{for } i = 1, 2, \ldots, \overline{x}.$$

Hence, we get from (4)

$$c^+ = \lim_{M\to\infty} c_1(M) = \lim_{M\to\infty} \alpha \sum_{x=0}^{\overline{x}} f_X(x) S_X(c_i(M))$$

$$= \alpha \sum_{x=0}^{\overline{x}} f_X(x) S_X(c^+) = \alpha S(c^+).$$

Uniqueness of the solution yields $c^+ = c^*$, where $c^* = \alpha S(c^+)$. Thus we obtain from (5)

$$\lim_{M\to\infty} c_0(M) = \frac{\alpha}{1-\alpha} \sum_{x=0}^{\overline{x}} x f_X(x) S_x(c^*). \quad //$$

5. Markov Chain Model

Let us consider the case in which prices in future periods are determined by a Markovian stochastic process. Assume that the set of realization of P is $\{ p_1, p_2, \ldots, p_m \}$. The conditional probability mass function of P_t given $X_t = x$ and $P_{t-1} = p_k$ is denoted by $f_{P|x,p_k}(.)$, and the conditional probability mass function of P_t given $P_{t-1} = p_k$ (the transition probability of P) is represented by $f_{P|p_k}(.)$.

Since the optimal value function and the optimal policy is derived by the same procedure used to establish Proposition 1 and Theorem 1, we shall only exhibit the results.

Proposition 3. The optimal value function is expressed as follows:

$$V(y,p_k) = \begin{cases} \sum_{i=1}^{y} \max\{p_k, c_i^{p_k}\} + c_0^{p_k} & \text{for } 0 \le y \le M, \\ (y-M)p_k + \sum_{i=1}^{M} \max\{p_k, c_i^{p_k}\} + c_0^{p_k} & \text{for } M+1 \le y \le M+\bar{x}. \end{cases}$$

The sequence $c_i^{p_k}$, $i=1,2,\ldots,M$, $k=1,2,\ldots,m$, is nonincreasing in i for any fixed p_k and is the unique solution of the system of the equation

$$(15) \qquad c_i^{p_k} = \alpha\{\sum_{x=0}^{M-i} f_X(x) \sum_{j=1}^{m} \max\{p_j, c_i^{p_j}\} f_{P|x,p_k}(p_j) + \sum_{x=M-i+1}^{\bar{x}} f_X(x)\mu_{P|x,p_k}\}, \quad \text{for } i=1,2,\ldots,M, \; k=1,2,\ldots,m.$$

$c_0^{p_k}$, $k=1,2,\ldots,m$, is given by

$$(16) \qquad c_0^{p_k} = \alpha[\sum_{x=0}^{M} f_X(x) \sum_{i=1}^{x} \sum_{j=1}^{m} \max\{p_j, c_i^{p_j}\} f_{P|x,p_k}(p_j) + \sum_{x=M+1}^{\bar{x}} f_X(x) \sum_{i=1}^{M} \sum_{j=1}^{m} \max\{p_j, c_i^{p_j}\} f_{P|x,p}(p_j) + \sum_{x=M+1}^{\bar{x}} f_X(x)(x-M)\mu_{P|x,p_k}\}] + \alpha \sum_{j=1}^{m} f_{P|p_k}(p_j) c_0^{p_j},$$

where $\mu_{P|x,p_k}$ denotes the mean of the conditional distribution of P_t given $X_t = x$ and $P_{t-1} = p_k$.

When the process is in the state (y,p_k), $0 \le y \le M$, $1 \le k \le m$, the optimal $a = d^*(y,p_k)$ is described as follows:

1) If there exists a certain j such that $c^{P}_{j+1}k < p \leq c^{P}_{j}k$, then

$$a = \begin{cases} j & \text{if } j < y \quad (\text{ sell } y-j \text{ units }), \\ y & \text{if } j \geq y \quad (\text{ sell nothing }). \end{cases}$$

2) If $p \leq c^{P}_{M}k$, then

$a = (y \wedge M)$ (sell nothing or sell the excess of the commodity over the storage capacity).

3) If $p > c^{P}_{1}k$, then

$a = 0$ (sell all). //

We will show that above two systems of equations (15) and (16) have unique solutions, respectively. Letting $i = M$ in (15), we have

$$(17) \qquad c^{P}_{M}k = \alpha\{ f_X(0) \sum_{j=1}^{m} \max\{ p_j, c^{P}_{M}j\} f_{P|0,p_k}(p_j) \} + \alpha \sum_{x=1}^{\overline{X}} f_X(x)\mu_{P|x,p_k} \qquad \text{for } k=1,2,\ldots,m.$$

Define a mapping T from R^m to R^m such that

$$T(w) = \begin{bmatrix} \beta \sum_{j=1}^{m} \max\{ p_j, w^j \} a_{1j} + \Delta_1 \\ \vdots \\ \beta \sum_{j=1}^{m} \max\{ p_j, w^j \} a_{kj} + \Delta_k \\ \vdots \\ \beta \sum_{j=1}^{m} \max\{ p_j, w^j \} a_{mj} + \Delta_m \end{bmatrix},$$

where $w \equiv (w^1, w^2,\ldots, w^m)^t$, $a_{kj} \equiv f_{P|p_k}(p_j)$, $\beta\in(0,1)$ and Δ_k is any real. For any $u_1 \equiv (u_1^1, u_1^2,\ldots, u_1^m)^t$, $u_2 \equiv (u_2^1, u_2^2,\ldots, u_2^m)^t$, $w_1 \equiv (w_1^1, w_1^2,\ldots, w_1^m)^t$ and $w_2 \equiv (w_2^1, w_2^2,\ldots, w_2^m)^t$ such that $u_1=T(w_1)$ and $u_2=T(w_2)$, we have

$$\max_k | u_1^k - u_2^k | = \beta \max_k | \sum_{j=1}^m a_{kj}[\max\{ p_j, w_1^j\} - \max\{ p_j, w_2^j\}] |$$

$$\leq \beta \max_k \sum_{j=1}^m a_{kj}| [\max\{ p_j, w_1^j\} - \max\{ p_j, w_2^j\}] |$$

$$\leq \beta \max_k \sum_{j=1}^m a_{kj}| w_1^j - w_2^j |$$

$$\leq \beta \max_j | w_1^j - w_2^j |.$$

The first inequality holds by the triangle inequality. The third one is obtained by the fact that $a_{kj} \leq 1$. Therefore, the mapping T is a contraction mapping, so that the equation $u=T(u)$ has a unique solution, and hence the system of equations (15) has a unique solution $c_M^{p_k}$, $k=1,2,\ldots,m$. Similarly, (15) has a unique solution $c_{M-1}^{p_k}$, $c_{M-2}^{p_k},\ldots$, $c_1^{p_k}$ $k=1,2,\ldots,m$. Let us display the uniqueness of the solution of (16). We write

$$(18) \qquad c_0^{p_k} = \alpha \sum_{j=1}^m a_{kj}\, c_0^{p_j} + \Delta_k \qquad \text{for } k = 1, 2,\ldots, m,$$

where $a_{kj} = f_{P|p_k}(p_j)$ and

$$\Delta_k = \alpha[\sum_{x=0}^M f_X(x) \sum_{i=1}^X \sum_{j=1}^m \max\{ p_j, c_i^{p_j} \} f_{P|x,p_k}(p_j) + \sum_{x=M+1}^m f_X(x) \sum_{i=1}^M \sum_{j=1}^m \max\{ p_j, c_i^{p_j} \} f_{P|x,p_k}(p_j) + \sum_{x=M+1}^{\bar{X}} f_X(x)\, (x-M)\mu_{P|x,p_k}].$$

(18) is equivalent to

$$-1/\alpha\ \Delta_k = \sum_{j=1}^{m} a_{kj} c_0^{p_j} - 1/\alpha\ c_0^{p_k} \qquad \text{for}\quad k=1,2,\ldots,m,$$

which is written as

$$\text{(19)} \qquad \begin{bmatrix} 1/\alpha\ \Delta_1 \\ 1/\alpha\ \Delta_2 \\ \vdots \\ 1/\alpha\ \Delta_m \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_0 \\ \vdots \\ c_0 \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} - 1/\alpha & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} - 1/\alpha & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} - 1/\alpha \end{bmatrix}.$$

However, it is known that the absolute value of the eigenvalue of any transition probability matrix is smaller than or equal to 1, so that the determinant of A cannot be 0 from the assumption that $\alpha \in (0,1)$. Therefore, the equation (19), or the system of equations (16) has a unique solution.

6. Conclusions

An optimal selling policy of the inventory model has been presented in both the case where price quotations are i.i.d. random variables and the case where the sequence of price quotations forms a Markov chain. The policy is specified by a sequence obtained as a unique solution of a system of equations. The policy seems to be intuitively natural owing to the following properties:

1. The more we have, the more commodity should be sold.
2. A critical price which determines whether the optimal action is to sell at least one unit or to sell nothing decreases as the amount of commodity at hand becomes large.
3. As the storage capacity becomes large, the critical price stated in 2 becomes higher and the amount of commodity to be sold becomes smaller.

A further outcome is a quantitative evaluation of the maximum expected reward under a certain storage capacity. The expected reward under the optimal policy is an increasing function of the storage capacity.

Comparing the expected reward and the cost of building a warehouse, the long term problem of determining the optimal storage capacity can in principle be solved, although the calculation is rather intricate. When the successive price quotations are i.i.d. random variables, the benefits one can obtain from additional storage capacity is evaluated. This will be available to determine whether one should expand the present warehouse or not. We have also obtained the limit to which the maximum expected reward under no initial amount of commodity converges as the storage capacity tends to infinity. It will show the manager the upper bound of the cost of building the warehouse.

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