No. 257

Combinatorial Canonical Form of Layered Mixed Matrices and Block-Triangularization of Large-Scale Systems of Linear/Nonlinear Equations

by

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April 1985

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Abstract

With a view to obtaining an efficient procedure for solving large-scale systems of equations, a canonical block-triangular form is defined for layered mixed matrices, and some practical examples are presented. The canonical form is obtained from a straightforward application of the decomposition principle for submodular functions. The relation to the existing decomposition techniques for electrical networks, as well as to the Dulmage-Mendelsohn decomposition, is also discussed.

Keywords: block-triangularization, layered mixed matrix, submodular function, Dulmage-Mendelsohn decomposition.

1. Introduction

When solving a system of linear equations

$$A\,x = b \tag{1.1}$$

repeatedly for various values of the right-hand side vector $b=b(\theta)$ containing parameters θ, it is now standard to first decompose A (possibly with permutations of rows and columns) into LU-factors as

$$A = L\,U, \tag{1.2}$$

and then solve the triangular systems $Ly=b$, $Ux=y$ for different values of $b=b(\theta)$. It is most important here that the LU-factors of A can be determined independently of the parameters θ.

No less of interest are the cases where the coefficient A, as well as b, changes with parameters, but with its zero/nonzero pattern kept fixed. Such situations often arise in practice, for example, in solving a system of nonlinear equations by the Newton method, or in determining the frequency characteristic of an electrical network by computing its responses to inputs of various frequencies. In this case we cannot calculate the LU-factors of A in advance, so that we usually resort to the so-called graph-theoretic methods and rearrange the equations and the variables to obtain a block-triangular form (see, e.g., [8], [15], [16], [17]). In particular, the block-triangularization based on the structure theory of bipartite graphs has proved to be effective, and is known as the Dulmage-Mendelsohn decomposition (abbreviated to <u>DM-decomposition</u>) [2], [3], [4], [5]. Then, each time the parameter values are specified, the equations corresponding to the DM-blocks may be solved either by direct inversion through LU-decomposition or by some iterative method.

The above two approaches, the LU-decomposition and the DM-decomposition, are two extremes in that the former admits arbitrary elementary row transformations on Á and the latter restricts itself to permutations only. In other words, the LU-decomposition regards the entries of A as numbers belonging to a field in which arithmetic operations are defined, whereas the DM-decomposition treats them as if they were symbols, or indeterminates if one prefers algebraic terms. It is often the case, however, that part of the entries of A are to be regarded as numbers and the remaining as symbols.

To be more concrete, suppose a system of linear/nonlinear equations

$$f(x) = 0 \tag{1.3}$$

is to be solved by the Newton method. The equations may be devided into linear and nonlinear parts as

$$f(x) = Q\,x + g(x), \tag{1.4}$$

where Q is a constant matrix. Accordingly, the Jacobian matrix J(x) of f(x) is expressed as

$$J(x) = Q + T(x), \tag{1.5}$$

where T(x) is the Jacobian matrix of g(x). Then we may regard the nonvanishing entries of T(x) as independent symbols on which no arithmetic operations are expected, whereas the usual elimination operations could be defined for the matrix Q.

Another typical example is a system of equations describing an electrical network, which is made up of equations for conservation laws (i.e., Kirchhoff's laws) and those for element characteristics (see Example 3.1). The former, stemming from the topological incidence relations in the underlying graphs, involve only ±1 as the coefficients and hence is amenable to elimination operations. The latter, on the other hand, consist of coefficients which are contaminated by various noises and errors, and therefore may be modelled as independent transcendentals.

The present paper aims at establishing a decomposition technique for such systems of linear/nonlinear equations that the coefficients are classified into two groups as explained above. A canonical form is introduced for a matrix A of the form

$$A = \begin{pmatrix} Q \\ \hline T \end{pmatrix} , \tag{1.6}$$

where the entries of Q belong to a subfield K and the nonvanishing entries of T are transcendentals in an extension field F which are algebraically independent over K. A uniquely determined block-triangular form is obtained with the diagonal square blocks being nonsingular; for a singular A, rectangular blocks (corresponding to horizontal and vertical tails in the DM-decomposition) also appear, both being of full rank. The decomposition can be found by an efficient algorithm so that it can be applied to large-scale practical problems, of which some examples are given in §4.

In the literature on electrical network theory, it has been known that a system of equations describing an electrical network can be put in a block-triangular form if one chooses appropriate bases (tree-cotree pairs) for Kirchhoff's laws and rearranges the variables and the equations (for both Kirchhoff's laws and element characteristics). As far as the present author knows, the decomposition of a pair of current-graph and voltage-graph is investigated in [27], [28] in graph-theoretic terms for the networks involving controlled sources. Decompositions of such networks as have the admittance expression are considered in [12], [14] with the aid of the notion of minimum-cover in an independent-matching problem. An attempt has been made in [22], [23], [24] to define a block-triangularization for a system of equations describing the most general class of networks with arbitrary mutual couplings (such as those containing controlled sources, nullators and norators) by means of the theory of principal partition of matroids, or the decomposition principle for submodular functions [11], [13], [14], [21], [25], [32], [33]. Unfortunately, however, the method proposed in [22], [23], [24], as it stands, produces too fine a partition for block-triangularization, which will be demonstrated in §5 below. Nevertheless, the idea of [22], [23], [24] can readily be modified to yield a block-triangularization not only for the equations of electrical networks but also for those of more general systems. In fact, the canonical form defined in this paper is obtained by choosing a suitable submodular function associated with the matrix (1.6) and by utilizing the decomposition principle for submodular functions, just as has been done in [22], [23], [24] with another submodular function.

The relations of the proposed canonical form to the above-mentioned decompositions, as well as to the combinatorial canonical form of a matrix with respect to its pivotal transforms introduced in [10], is also discussed in §5 and §6 with special reference to the admissible row transformations on matrices.

2. Preliminary

Some results on the decomposition principle for submodular functions [11], [13], [14], [21], [25], [32], [33] are briefly summarized here for later references.

Let C be a finite set, and $p: 2^C \to R$ be a submodular function defined on it, i.e.,

$$p(X \cup Y) + p(X \cap Y) \leq p(X) + p(Y) \tag{2.1}$$

for $X, Y \subset C$. The family of those subsets of C which give the minimum of p will be denoted by L(p):

$$L(p) = \{ X \mid X \subset C,\ p(X) \leq p(Y) \text{ for all } Y \subset C \}. \tag{2.2}$$

From the submodularity (2.1), it follows that

$$X \cup Y,\ X \cap Y \in L(p) \quad \text{for} \quad X, Y \in L(p) .$$

In other words, L(p) is a (distributive) sublattice [1] of the boolean lattice 2^C. Note that the length of a maximal chain in L(p) from min L(p) to max L(p) is uniquely determined.

By the structure theory of distributive lattices [1], there exists a one-to-one correspondence between sublattices of 2^C and partitions of C into partially ordered blocks. Furthermore, when a sublattice is derived from a submodular function as (2.2), "minors" are induced on the blocks. To be more specific, the following is known.

<u>Theorem 2.1</u>. Let p be a submodular function defined on a finite set C, and L(p) the family of minimizers of p. Put X_0=min L(p) and X_r=max L(p).

(1) Any maximal chain in L(p)

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r$$

determines a family of intervals (difference sets)

$$\{ C_i \mid C_i = X_i \setminus X_{i-1},\ i=1,\dots,r\},$$

which is independent of the choice of a maximal chain, and hence provides a unique partition of C into disjoint subsets (blocks)

$$\underline{\underline{P}} = \{C_0;\ C_1,\ \dots,\ C_r;\ C_\infty\},$$

where $C_0 = X_0$ and $C_\infty = C \setminus X_r$. ($C_0$ and/or C_∞ can be empty.)

(2) The "minors" defined by

$$\rho_i(Y) = \rho(X_{i-1} \cup Y) - \rho(X_{i-1}), \quad \text{for } Y \subset C_i \tag{2.3}$$

(i=1,...,r) are uniquely determined independently of the choice of a maximal chain.

(3) A partial order ($\prec$) is defined on $\underline{\underline{P}} \setminus \{C_0, C_\infty\}$ by the relation

$$C_i \prec C_j \quad \text{iff} \quad C_j \subset X \in L(p) \text{ implies } C_i \subset X,$$

where $1 \leq i,j \leq r$. The partial order is trivially extended over to $\underline{\underline{P}}$ by

$$C_0 \prec C_i \prec C_\infty \quad \text{for } i=1,\dots,r.$$

(4) The "minors" defined in (2) above are expressed also as

$$\rho_i(Y) = \rho(C_{\langle i \rangle} \cup Y) - \rho(C_{\langle i \rangle}), \quad Y \subset C_i, \tag{2.4}$$

for i=1,...,r, where

$$C_{\langle i \rangle} = \cup\{ C_j \mid C_j \prec C_i,\ C_j \neq C_i \}. \tag{2.5}$$

□

Note that a linear extension of the partial order defined in (3) above can be obtained by choosing a maximal chain in L(p) as in (1) and by defining the total order on $\underline{\underline{P}}$ by

$$C_i \leq C_j \quad \text{iff} \quad i \leq j.$$

3. Mixed Matrices and Layered Mixed Matrices

Let K be a field, which contains Q, the field of rationals, and of which F is an extension field:

$$Q \subset K \subset F. \tag{3.1}$$

The set of m×n matrices over F is denoted as $\underline{M}(F;m,n)$ or simply as $\underline{M}(F)$.

A matrix $A \in \underline{M}(F)$ can be expressed as

$$A = Q_A + T_A \tag{3.2}$$

in such a way that $Q_A \in \underline{M}(K)$ and the nonvanishing entries of T_A are in F\K. To make the decomposition unique, we will assume that $(Q_A)_{ij}=0$ if $(T_A)_{ij}\neq 0$. If, in addition, the collection of the nonvanishing entries of T_A is algebraically independent [34] over K, the matrix A is called a <u>mixed</u> <u>matrix</u> with respect to K. We denote by $\underline{MM}(F/K;m,n)$ the set of m×n mixed matrices over F with respect to K. The notion of mixed matrix is introduced in [19], [20] as a mathematical tool for dealing with structural aspects of physical/engineering systems. See [20] for detailed discussion on its physical meanings.

A subclass of mixed matrices is defined here. We call a mixed matrix $A \in \underline{MM}(F/K;m,n)$ a <u>layered</u> <u>mixed</u> <u>matrix</u> with respect to K, if the sets of nonzero rows of Q_A and T_A are disjoint in the expression (3.2) for a mixed matrix A, i.e., if A can be put in a partitioned matrix of the form

$$A = \begin{pmatrix} Q \\ \hline T \end{pmatrix}, \tag{3.3}$$

where $Q \in \underline{M}(K;m_Q,n)$, $T \in \underline{M}(F;m_T,n)$ $(m_Q+m_T=m)$, and the collection of the nonvanishing entries of T are algebraically independent over K. The set of $m \times n$ layered mixed matrices consisting of $m_Q + m_T$ rows as above will be designated by $\underline{\underline{LM}}(F/K;m_Q,m_T,n)$ or simply by $\underline{\underline{LM}}(F/K)$. Obviously we have

$$\underline{\underline{LM}}(F/K;m_Q,m_T,n) \subset \underline{\underline{MM}}(F/K;m_Q+m_T,n). \tag{3.4}$$

Consider a system of equations (1.1) where the coefficient matrix $A \in \underline{\underline{MM}}(F/K;m,n)$ is of the form (3.2). Introducing an auxiliary vector $w \in R^m$, we can express it equivalently as

$$\begin{pmatrix} I_m & Q_A \\ -I_m & T_A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}. \tag{3.5}$$

It may be assumed that we can choose m numbers in F, say $t_1, \ldots, t_m$, that are algebraically independent over the subfield of F to which the entries of T_A belong. Then, multiplying each of the last m equations by the transcendentals $t_1, \ldots, t_m$, we obtain an augmented system of equations

$$\begin{pmatrix} I_m & Q_A \\ -D_m & D_m T_A \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \tag{3.6}$$

$$D_m = \mathrm{diag}(t_1, \ldots, t_m), \tag{3.7}$$

which is still equivalent to the original system (1.1). The coefficient matrix of (3.6) is a layered mixed matrix with respect to K since the nonvanishing entries of $[-D_m | D_m T_A]$ are algebraically independent over K. In the case of a system of linear/nonlinear equations (1.4), the above tranformation from (1.1) to (3.5) may be interpreted as assigning w to the nonlinear part g(x) to obtain

$$\begin{aligned} w + Q\,x &= 0, \\ -w + g(x) &= 0, \end{aligned} \tag{3.8}$$

which is equivalent to (1.4).

In general, with a mixed matrix $A \in \underline{\underline{MM}}(F/K;m,n)$ we will associate a layered mixed matrix $\tilde{A} \in \underline{\underline{LM}}(F/K;m,m,m+n)$:

$$\tilde{A} = \begin{pmatrix} I_m & Q_A \\ -D_m & D_m T_A \end{pmatrix}, \qquad (3.9)$$

where D_m is given by (3.7). Note that the column-set of $\tilde{A}$ has a natural one-to-one correspondence with the union of the column- and the row-set of A. Since we have the obvious identity

$$\text{rank}\ \tilde{A} = \text{rank}\ A + m, \qquad (3.10)$$

we may restrict ourselves to layered mixed matrices when we deal with the unique solvability of a system of equations having an mixed matrix as its coefficient.

For a matrix G over a field in general, we will denote by M(G) the linear matroid [35] defined on the column-set of G with respect to the linear dependence of the column-vectors. The rank of a layered mixed matrix A of (3.3) is known [35] to be expressed as follows in terms of the rank of the union of two matroids M(Q) and M(T). Both M(Q) and M(T) are defined on the column-set, say C, of the matrix A, and their rank functions will be denoted by ρ and τ, respectively.

<u>Theorem 3.1</u>. Let $A \in \underline{\underline{LM}}(F/K)$ be a layered mixed matrix of the form (3.3). Then

$$\begin{aligned} \text{rank}\ A &= \text{rank}(\ M(Q) \vee M(T)\ ) \\ &= \min \{\ \rho(X)+\tau(X)-|X|\ \mid X \subset C\ \} + n\ . \end{aligned}$$

□

Note that the rank of the union of two matriods can be found by an efficient practical algorithm [6].

Corollary 3.2 ([20]). Let $A \in \underline{\underline{MM}}(F/K;m,n)$ be a mixed matrix of the form (3.2). Then

$$\text{rank } A = \text{rank}(\ M(I_m|Q_A)\ ) \vee M(I_m|T_A)\ ) - m.$$

(Proof) Immediate from (3.10) and Theorem 3.1. □

Example 3.1 ([22], Example 4.1.3). Consider the free electrical network of Fig. 3.1, which is taken from [22]. It consists of 6 resistors of resistances r_i (branch i) (i=1,...,6), and 3 voltage-controlled current sources (branch i) with mutual conductances γ_i (i=7,8,9); the current sources of branches 7, 8, 9 are controlled respectively by the voltages across branches 2, 4, 5. Then the current ξ^i in and the voltage η_i across branch i (i=1,...,9) are to satisfy the structural equations (Kirchhoff's laws) and the constitutive equations, which altogether are expressed as in (1.1) with $x = (\xi^1,\ldots,\xi^9\ ;\ \eta_1,\ldots,\eta_9)$ and

$$
A = \left|\begin{array}{cccccccccccccccccc}
\xi^1 & \xi^2 & \xi^3 & \xi^4 & \xi^5 & \xi^6 & \xi^7 & \xi^8 & \xi^9 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 & \eta_9 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & & & & & & & & & \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & & & & & & & & & \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & & & & & & & & & \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & & & & & & & & & \\
 & & & & & & & & & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\
\hdashline
r_1 & & & & & & & & & -1 & & & & & & & & \\
 & r_2 & & & & & & & & & -1 & & & & & & & \\
 & & r_3 & & & & & & & & & -1 & & & & & & \\
 & & & r_4 & & & & & & & & & -1 & & & & & \\
 & & & & r_5 & & & & & & & & & -1 & & & & \\
 & & & & & r_6 & & & & & & & & & -1 & & & \\
 & & & & & & -1 & & & & \gamma_7 & & & & & & & \\
 & & & & & & & -1 & & & & & \gamma_8 & & & & & \\
 & & & & & & & & -1 & & & & & \gamma_9 & & & & \\
\hline
\end{array}\right| \qquad (3.11)
$$

The unique solvability of the network reduces to the nonsingularity of the matrix A.

It may be justified for physical reasons (see, e.g., [20]) to regard r_i (i=1,...,6) and γ_i (i=7,8,9) as real numbers which are collectively algebraically independent over the field of rationals. Then we have $A \in \underline{\underline{MM}}(R/Q;18,18)$, and the unique solvability of the network can be determined efficiently by Corollary 3.2. Or alternatively, we may directly apply Theorem 3.1 with Q being the first 9 rows of A and T being the last 9 rows of A, since we can put A in the form of a layered mixed matrix by multiplying the last 9 rows by independent transcendentals, just as we did for (3.5) to get (3.6). This example will be taken up again in Example 4.2.

4. Combinatorial Canonical Form of Layered Mixed Matrices

This section defines a block-triangular cononical form for an $m \times n$ layered mixed matrix $A \in \underline{\underline{LM}}(F/K; m_Q, m_T, n)$ of the form (3.3), where $m = m_Q + m_T$. For A of (3.3), we consider the transformation of the form

$$P_r \begin{pmatrix} S_Q & O \\ O & P_T \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_c, \tag{4.1}$$

where S_Q is an $m_Q \times m_Q$ nonsingular matrix over K (i.e., $S_Q \in GL(m_Q, K)$); P_T, P_r and P_c are permutation matrices of orders m_T, m and n, respectively. The transformed matrix of (4.1) also belong to $\underline{\underline{LM}}(F/K; m_Q, m_T, n)$ and is equivalent to A in the ordinary sense in linear algebra. We will say that two matrices are LM-equivalent if they are connected by the transformation above. In the following, we will look for a canonical block-triangular matrix among the matrices LM-equivalent to A. The canonical form to be considered should reduce to the DM-decomposition when $m = m_T$ and $m_Q = 0$.

Let R and C denote the row- and the column-set of A, respectively; the former is the disjoint union of the row-sets, say R_Q and R_T, of Q and T:

$$R = R_Q \cup R_T. \tag{4.2}$$

For $I \subset R$ and $J \subset C$, $A[I,J]$ means the submatrix of A with row-set I and column-set J.

Theorem 3.1 states that the rank of $A[R,J]$ ($J \subset C$) can be expressed by $\rho(X) = \text{rank}\ Q[R_Q, X]$ and $\tau(X) = \text{rank}\ T[R_T, X]$ ($X \subset J$). On account of the algebraic independence of the nonvanishing entries of T, the rank $\tau(X)$ equals the term-rank [26] of $T[R_T, X]$, which is known [35] to be expressed by the adjacency associated with T; namely we have

$$\tau(X) = \min\{ \gamma(Y)+|X\backslash Y| \mid Y\subset X \}, \quad X \subset C, \tag{4.3}$$

where

$$\Gamma_T(Y) = \{ i\in R_T \mid T_{ij}\neq 0 \text{ for some } j \in Y\}, \ Y \subset C, \tag{4.4}$$

$$\gamma(Y) = |\Gamma_T(Y)|, \quad Y\subset C. \tag{4.5}$$

We consider two functions:

$$p_\tau(X) = \rho(X) + \tau(X) - |X|, \quad X \subset C, \tag{4.6}$$

$$p_\gamma(X) = \rho(X) + \gamma(X) - |X|, \quad X \subset C. \tag{4.7}$$

Since $\tau(X) \leqq \gamma(X)$ by definition, we have the obvious inequality

$$p_\tau(X) \leqq p_\gamma(X). \tag{4.8}$$

These functions, however, share a common minimum value when restricted to 2^J for any $J \subset C$.

Lemma 4.1. For $J \subset C$, we have

$$\min\{ p_\tau(X) \mid X\subset J \} = \min\{ p_\gamma(X) \mid X\subset J \}.$$

(Proof) From (4.6) and (4.3) it follows that

$$\begin{aligned} &\min\{ p_\tau(X) \mid X\subset J \} \\ &= \min\{ \rho(X)-|X| + \min\{ \gamma(Y)+|X\backslash Y| \mid Y\subset X \} \mid X\subset J \} \\ &= \min\{ \rho(X)+\gamma(Y)-|Y| \mid Y\subset X\subset J \} \\ &= \min\{ \rho(Y)+\gamma(Y)-|Y| \mid Y\subset J \} \\ &= \min\{ p_\gamma(Y) \mid Y\subset J \}. \end{aligned}$$

□

Combined with Theorem 3.1, this gives a characterization of rank A in terms of ρ and γ, instead of ρ and τ.

Theorem 4.2. Let $A \in \underline{\underline{LM}}(F/K;m_Q,m_T,n)$ of the form (3.3). Then

$$\text{rank } A[R,J] = \min\{ p_\gamma(X) \mid X\subset J \} + |J|,$$

for $J\subset C$, where p_γ is defined by (4.7). □

The important fact is that $p_\gamma: 2^C \to Z$ of (4.7) is submodular, and hence, as explained in §2, its minimizer $L(p_\gamma)$ determines a unique partition of the column-set C of A into partially ordered blocks. To be specific, we choose (cf. Theorem 2.1 (1)) a maximal chain in $L(p_\gamma)$:

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r \tag{4.9}$$

to get the blocks:

$$C_0=X_0;\ C_j=X_j \setminus X_{j-1} \quad (j=1,\ldots,r);\ C_\infty=C\setminus X_r. \tag{4.10}$$

A partition $\{R_{Tj} \mid j=0,1,\ldots,r,\infty\}$ of the row-set R_T of T is induced from (4.9) naturally as follows:

$$R_{T0}=Y_{T0};\ R_{Tj}=Y_{Tj}\setminus Y_{T,j-1} \quad (j=1,\ldots,r);\ R_{T\infty}=R_T\setminus Y_{Tr}, \tag{4.11}$$

where

$$Y_{Tj} = \Gamma_T(X_j) \qquad (j=0,1,\ldots,r). \tag{4.12}$$

Though the partition (4.11) is defined here with reference to a particular choice of a maximal chain (4.9), it admits a direct expression in terms of the partial order among the blocks, as follows.

Lemma 4.3.

$$T[R_{Ti},C_j] = 0 \quad \text{unless } C_i \prec C_j,$$

and therefore

$$R_{Tj} = \Gamma_T(C_{<j>} \cup C_j) \setminus \Gamma_T(C_{<j>}), \quad (j=1,\ldots,r),$$

where $C_{<j>}$ is defined by (2.5).

(Proof) Suppose C_i and C_j are not ordered, i.e., that neither $C_i \subset C_{<j>}$ nor $C_j \subset C_{<i>}$. Put $X=C_{<i>} \cup C_{<j>}$. Since X, $X\cup C_i$, $X\cup C_j$ and $X\cup C_i \cup C_j$ all belong to $L(p_\gamma)$, we have $p_\gamma(X) = p_\gamma(X\cup C_i) = p_\gamma(X\cup C_j) = p_\gamma(X\cup C_i\cup C_j)$. This implies, by the submodularity, that

$$\rho(X\cup C_i\cup C_j) - \rho(X\cup C_i) = \rho(X\cup C_j) - \rho(X), \tag{4.13}$$

$$\gamma(X\cup C_i\cup C_j) - \gamma(X\cup C_i) = \gamma(X\cup C_j) - \gamma(X). \tag{4.14}$$

It is easy to see that (4.14) means

$$\Gamma_T(C_i) \cap \Gamma_T(C_j) \subset \Gamma_T(X),$$

which establishes the lemma. □

As for the matrix Q, it can be transformed to a block-triangular matrix $\bar{Q}$ by the usual elimination operations; that is, for some $S_Q \in GL(m_Q, K)$, the row-set of $\bar{Q} = S_Q Q$ is partitioned into disjoint subsets $\{R_{Qj} \mid j=0,1,\ldots,r,\infty\}$ such that

$$\begin{aligned} |R_{Q0}| &= \rho(X_0), \\ |R_{Qj}| &= \rho(X_j) - \rho(X_{j-1}) \qquad (j=1,\ldots,r), \\ |R_{Q\infty}| &= |R_Q| - \rho(X_r), \end{aligned} \tag{4.15}$$

and

$$\bar{Q}[R_{Qi}, C_j] = 0 \qquad (0 \leq j < i \leq \infty). \tag{4.16}$$

By the same argument as the proof of Lemma 4.3 (by (4.13) in particular), we may further assume that

$$\bar{Q}[R_{Qi}, C_j] = 0 \quad \text{unless} \quad C_i \prec C_j. \tag{4.17}$$

We will put

$$Y_{Qj} = \bigcup_{i=0}^{j} R_{Qi} \qquad (j=0,1,\ldots,r), \tag{4.18}$$

$$Y_j = Y_{Qj} \cup Y_{Tj} \qquad (j=0,1,\ldots,r), \tag{4.19}$$

$$R_j = R_{Qj} \cup R_{Tj} \qquad (j=0,1,\ldots,r,\infty). \tag{4.20}$$

Consider the matrix

$$\bar{A} = \begin{pmatrix} \bar{Q} \\ T \end{pmatrix} = \begin{pmatrix} S_Q Q \\ T \end{pmatrix}, \tag{4.21}$$

which is LM-equivalent to A (under the transformation (4.1)). The row-set $R=R_Q \cup R_T$ of $\bar{A}$, as well as the column-set C, is now partitioned into blocks $\{R_j | j=0,1,\dots,r,\infty\}$, on which the partial order ($\prec$) on $\{C_j | j=0,1,\dots,r,\infty\}$ can naturally be induced.

Theorem 4.4. Let $\bar{A}$ be as above, whose row-set R and column-set C are partitioned into partially ordered blocks.

(1) $\bar{A}[R_i,C_j] = 0$ unless $C_i \prec C_j$. In particular,

$$\bar{A}[R_i,C_j] = 0 \quad \text{if} \quad i>j. \tag{4.22}$$

(2) $|R_0| < |C_0|$ if $C_0 \neq \emptyset$,

$|R_j| = |C_j|$ for $j=1,\dots,r$,

$|R_\infty| > |C_\infty|$ if $C_\infty \neq \emptyset$.

(From the last relation follows a more symmetric statement:

$|R_\infty| > |C_\infty|$ if $R_\infty \neq \emptyset$.)

(3) rank $\bar{A}[Y_j,X_j]$ = rank $\bar{A}[R,X_j] = |Y_j|$ $(j=0,1,\dots,r)$.

(4) rank $\bar{Q}[Y_{Qj},X_j] = |Y_{Qj}|$ $(j=0,1,\dots,r)$,

rank $T[Y_{Tj},X_j] = |Y_{Tj}|$ $(j=0,1,\dots,r)$.

(5) rank $\bar{A}[R_0,C_0] = |R_0|$,

rank $\bar{A}[R_j,C_j] = |R_j| = |C_j|$ $(j=1,\dots,r)$,

rank $\bar{A}[R_\infty,C_\infty] = |C_\infty|$.

(6) For $j=0,1,\dots,r,\infty$, the submatrix $\bar{A}[R_j,C_j]$ ($\in$ <u>LM</u>(F/K)) is irreducible in the sense that the submodular function $\bar{p}_j$ (defined on C_j), the correspondent of p_γ of (4.7), has no minimizers distinct from $\emptyset$ and C_j.

(Proof) (1): Immediate from Lemma 4.3 and (4.16).

(2): If $C_0 \neq \emptyset$, then $0 = p_\gamma(\emptyset) > \min p_\gamma = p_\gamma(C_0) = \rho(C_0)+\gamma(C_0)-|C_0| = |R_0|-|C_0|$.

For $j=1,\ldots,r$, we have $p_\gamma(X_{j-1}) = p_\gamma(X_j)$, i.e.,

$$\rho(X_{j-1}) + \gamma(X_{j-1}) - |X_{j-1}| = \rho(X_j) + \gamma(X_j) - |X_j|.$$

By (4.11), (4.12), and (4.15), this reduces to $|C_j|=|R_j|$.

If $C_\infty \neq \emptyset$, then $p_\gamma(C) > \min p_\gamma = p_\gamma(X_r)$, which implies $|R|-|C| \geq \rho(C)+\gamma(C)-|C| > \rho(X_r)+\gamma(X_r)-|X_r| = |Y_r|-|X_r|$. Hence $|R_\infty| = |R|-|Y_r| > |C|-|X_r| = |C_\infty|$.

(3): From (1) above and Theorem 4.2, we have $\text{rank } \bar{A}[Y_j,X_j] = \text{rank } \bar{A}[R,X_j] = \text{rank } A[R,X_j] = \min\{p_\gamma(X)|X\subset X_j\} + |X_j| = p_\gamma(X_j) + |X_j| = \rho(X_j) + \gamma(X_j) = |Y_{Qj}| + |Y_{Tj}| = |Y_j|$.

(4): Immediate from (3) above.

(5): The identities for $j=0,1,\ldots,r$ are immediate from (1) and (3) above. By Theorem 4.2, we have

$$\text{rank } \bar{A}[R_\infty,C_\infty] = \min\{\bar{p}_\infty(Z)|Z\subset C_\infty\} + |C_\infty|,$$

where

$$\bar{p}_\infty(Z) = \text{rank } \bar{Q}[R_{Q\infty},Z] + |\Gamma_T(Z)\cap R_{T\infty}| - |Z|.$$

On the other hand, this turns out to be nonnegative, since

$$\begin{aligned}\bar{p}_\infty(Z) &= (\rho(X_r\cup Z)-\rho(X_r)) + (\gamma(X_r\cup Z)-\gamma(X_r)) - |Z| \\ &= p_\gamma(X_r\cup Z) - p_\gamma(X_r) \\ &= p_\gamma(X_r\cup Z) - \min p_\gamma. \qquad (4.23)\end{aligned}$$

(6): First consider the case of $j=\infty$. Recalling $X_r = \max L(p_\gamma)$, we see from (4.23) that $\bar{p}_\infty$ has the unique minimizer of $Z=\emptyset$. The second case of $j=0$ is easy, since $\bar{p}_0(Z)=p_\gamma(Z)$ has the unique minimizer $Z=C_0$. The other cases ($1\leq j\leq r$) can be treated similarly using the expression

$$\begin{aligned}\bar{p}_j(Z) &= \text{rank } \bar{Q}[R_{Qj},Z] + |\Gamma_T(Z)\cap R_{Tj}| - |Z| \\ &= p_\gamma(X_{j-1}\cup Z) - \min p_\gamma.\end{aligned}$$

□

This theorem shows that with suitable permutation matrices P_r and P_c, $P_r \bar{A} P_c$ is a block-triangular matrix which is LM-equivalent to A. The ordering of the blocks is uniquely determined up to the partial order ($\prec$). The following argument shows that it is the finest block-triangular form that is LM-equivalent to A and enjoys the properties (2) and (5) of Theorem 4.4.

Suppose that $\hat{A}$ is such a block-triangular matrix with the row-set R and the column-set C being partitioned as

$$R = \cup\{R'_j \mid j=0,1,\ldots,q,\infty\}, \tag{4.24}$$

$$C = \cup\{C'_j \mid j=0,1,\ldots,q,\infty\}, \tag{4.25}$$

where $\hat{A}[R'_i, C'_j] = 0$ for $i>j$. Since $\hat{A}$ is LM-equivalent to A, we have from Theorem 4.2

$$\text{rank } \hat{A} = \min\{ p_\gamma(X) \mid X \subset C \} + |C| \tag{4.26}$$

with the same p_γ for A. Put

$$X'_j = \bigcup_{i=0}^{j} C'_i \qquad (j=0,1,\ldots,q), \tag{4.27}$$

$$Y'_j = \bigcup_{i=0}^{j} R'_i \qquad (j=0,1,\ldots,q). \tag{4.28}$$

Since $\hat{A}$ is block-triangularized and has the properties of (2) and (5) of Theorem 4.4, we have

$$\text{rank } \hat{A} = |C| - |X'_j| + |Y'_j| \quad (j=0,1,\ldots,q). \tag{4.29}$$

Combining (4.26) and (4.29), we obtain

$$\min p_\gamma = |Y'_j| - |X'_j| \quad (j=0,1,\ldots,q).$$

This shows that

$$X'_j \in L(p_\gamma), \tag{4.30}$$

since $p_\gamma(X'_j) = \rho(X'_j) + \gamma(X'_j) - |X'_j| \leq |Y'_j| - |X'_j| = \min p_\gamma$. Therefore, the partition (4.25) is coarser than (or an aggregation of) $\{C_j \mid j=0,1,\ldots,r,\infty\}$ determined by $L(p_\gamma)$.

Thus, the matrix $P_r\bar{A}P_c$ with $\bar{A}$ constructed above provides the finest block-triangular form among the matrices LM-equivalent to A. It is named here the <u>combinatorial canonical form of a layered mixed matrix</u>. It is obvious that it agrees with the DM-decomposition when A=T (i.e., $m_Q=0$). In parallel with the DM-decomposition, the rectangular blocks corresponding to $R_0 \times C_0$ and $R_\infty \times C_\infty$, if any, will be called the <u>horizontal tail</u> and the <u>vertical tail</u>, respectively.

A comment on the algorithm is in order. From the point of view of practical application, it is very important that this canonical form can be constructed by an efficient matroid-theoretic algorithm that involves $O(m_Q n^3)$ arithmetic operations in the subfield K and $O((m+n)^2 n)$ operations for graph manipulations, as follows.

To be specific, we associate a graph G(V,E) with $A \in \underline{\underline{LM}}(F/K;m_Q,m_T,n)$ having the row-set $R=R_Q \cup R_T$ and the column-set C. The vertex-set V of G is given by

$$V = R_T \cup C_T \cup C_Q, \tag{4.31}$$

where C_T and C_Q are disjoint copies of C, and the arc-set E of G is defined as

$$\begin{aligned} E = \{(i,j_T) \in R_T \times C_T \mid T_{ij} \neq 0\ \} \\ \cup \{(j_T,j_Q) \in C_T \times C_Q \mid j \in C\ \}, \end{aligned} \tag{4.32}$$

where j_Q ($\in C_Q$) and j_T ($\in C_T$) are copies of j ($\in C$).

We consider the independent-flow problem [7], [14] on the network with the underlying graph G; R_T is the entrance vertex-set with the free matroid on it, C_Q is the exit vertex-set with the dual of M(Q) defined on it, and each arc of E has the infinite capacity. Any subset U of V can be expressed uniquely as

$$U = (R_T \backslash Y) \cup (C_T \backslash X_T) \cup Z_Q, \tag{4.33}$$

where $Y \subset R_T$; X, $Z \subset C$; X_T $(\subset C_T)$ and Z_Q $(\subset C_Q)$ are their copies. The capacity c(U) of U above is given by

$$c(U) = \begin{cases} |Y| + \rho^*(Z) & \text{if } Y \cap \Gamma_T(X) \text{ and } X \cup Z = C \\ +\infty & \text{otherwise,} \end{cases} \tag{4.34}$$

where

$$\rho^*(Z) = \rho(C \backslash Z) + |Z| - \rho(C) \tag{4.35}$$

is the rank function of $M(Q)^*$, the dual of M(Q). Noting $Y = \Gamma_T(X)$ for a minimizer U of c(U), we see that the family of minimizers U of c(U) determines the family $L(p_\gamma)$ of minimizers of p_γ, by the relation (4.33).

In this way, the desired partition of C for the combinatorial canonical form can be constructed by first finding the maximum independent flow and then decomposing the auxiliary graph associated with it into strongly connected components, among which the partial order can be induced. (To be more precise, the column-sets C_0 and C_∞ are determined by those vertices of C_T $(\subset V)$ which are reachable from the entrance and to the exit, respectively.) See, e.g., [14] for detail. Example 4.1 below will illustrate this procedure.

Example 4.1. Consider the following matrix $A \in \underline{\underline{LM}}(F/Q;3,6,7)$, where $\{t_i \mid i=1,\ldots,13\}$ are indeterminates over Q and F is the field of rational functions in t_i's over Q.

$$A = \begin{array}{|ccccccc|} \multicolumn{7}{c}{1 \quad\; 2 \quad\; 3 \quad\; 4 \quad\; 5 \quad\; 6 \quad\; 7} \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & -1 \\ -2 & 0 & 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 & 1 & -1 \\ \hdashline t_1 & & & & & t_2 & \\ t_3 & & & & t_4 & & \\ & t_5 & & t_6 & t_7 & & \\ & t_8 & & t_9 & t_{10} & & t_{11} \\ & & & & & t_{12} & \\ & & & & & t_{13} & \\ \hline \end{array} \qquad (4.36)$$

The graph G(V,E) for the associated independent-flow problem is depicted in Fig. 4.1. The auxiliary graph for a maximum independent flow is shown in Fig. 4.2, the decomposition of which into strongly connected components provides the partition (4.10) of the column-set C of A:

$$C = C_1 \cup C_2 \cup C_\infty, \qquad (4.37)$$

where $C_1=\{2,4,7\}$, $C_2=\{3\}$, $C_\infty=\{1,5,6\}$. Notice that $C_0=\emptyset$, $C_i \prec C_\infty$ (i=1,2), and C_1 and C_2 have no order relation. The combinatorial canonical form for A is given by

$$
\begin{array}{|ccc|c|ccc|}
2 & 4 & 7 & 3 & 1 & 5 & 6 \\
\hline
0 & 1 & -1 & & 1 & & 1 \\
t_5 & t_6 & 0 & & & t_7 & \\
t_8 & t_9 & t_{10} & & & t_{11} & \\
\hline
 & & & 1 & & & 2 \\
\hline
 & & & & 0 & 1 & 0 \\
 & & & & t_1 & 0 & t_2 \\
 & & & & t_3 & t_4 & 0 \\
 & & & & 0 & 0 & t_{12} \\
 & & & & 0 & 0 & t_{13} \\
\hline
\end{array}
\qquad (4.38)
$$

□

Example 4.2. Recall the eletrical network of Example 3.1. With the understanding, mentioned in Example 3.1, that the coefficient matrix A of (3.11) can be considered a member of LM(R/Q;9,9,18), the combinatorial canonical form of A is found as (4.39) below.

It has no tails ($C_0=R_\infty=\emptyset$) and 9 square diagonal blocks with the column-sets given by $C_1=\{\eta_7\}$, $C_2=\{\eta_1\}$, $C_3=\{\xi^1\}$, $C_4=\{\eta_8\}$, $C_5=\{\eta_9\}$, $C_6=\{\eta_6\}$, $C_7=\{\xi^6\}$, $C_8=\{\eta_5,\xi^5,\xi^9\}$, $C_9=\{\xi^2,\eta_2,\xi^3,\eta_3,\xi^4,\eta_4,\xi^7,\xi^8\}$. The partial order among them are given by:

$$C_9 \prec C_3 \prec C_2 \prec C_1;\ C_9 \prec C_8 \prec C_4;\ C_8 \prec C_7 \prec C_6 \prec C_5.$$

$$
\begin{array}{c|c|c|c|c|c|c|ccc|cccccccc}
\eta_7 & \eta_1 & \xi^1 & \eta_8 & \eta_9 & \eta_6 & \xi^6 & \eta_5 & \xi^5 & \xi^9 & \xi^2 & \eta_2 & \xi^3 & \eta_3 & \xi^4 & \eta_4 & \xi^7 & \xi^8 \\
\hline
-1 & -1 & & & & & & & & & & & & 1 & & 1 & & \\
 & -1 & r_1 & & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & & -1 & \\
 & & & -1 & & & & -1 & & & & & & & & 1 & & \\
 & & & & -1 & -1 & & -1 & & & & & & & & & & \\
 & & & & & -1 & r_6 & & & & & & & & & & & \\
 & & & & & & 1 & & & -1 & & & & & & & & \\
 & & & & & & & 0 & 1 & -1 & 1 & & & & 1 & & & \\
 & & & & & & & -1 & r_5 & 0 & & & & & & & & \\
 & & & & & & & \gamma_9 & 0 & -1 & & & & & & & & \\
\hline
 & & & & & & & & & & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 & & & & & & & & & & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & r_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & r_3 & -1 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & r_4 & -1 & 0 & 0 \\
 & & & & & & & & & & 0 & \gamma_7 & 0 & 0 & 0 & 0 & -1 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & \gamma_8 & 0 & -1
\end{array}
\qquad (4.39)
$$

This example will be considered again in Example 5.1.

□

We now consider how the combinatorial canonical form can be applied to an efficient solution of a system of equations $A(\theta)x=b(\theta)$ for varying values of parameter θ. We express the coefficient matrix as

$$A(\theta) = Q_A + T_A(\theta) \tag{4.40}$$

and regard it as a mixed matrix, treating the nonvanishing entries of $T_A(\theta)$ as if they were algebraically independent. As discussed at the beginning of §3, we may introduce an auxiliary variable w to obtain the augmented system of equations (3.5) or (3.6) with the layered mixed matrix $\tilde{A}$ of (3.9) as the coefficient. The combinatorial canonical form of $\tilde{A}$ determines a decomposition of the whole augmented system into hierarchical smaller subsystems; we may repeatedly solve the subproblems with the diagonal blocks as the coefficients.

For the subproblems to be solved, the diagonal blocks of the combinatorial canonical form of $\tilde{A}$ must be nonsingular. If the assumption of the algebraic independence of the nonvanishing entries of $T_A(\theta)$ is literally met, the nonsingularity of the diagonal blocks is guaranteed by Theorem 4.4(5). It is obvious, however, from the block-triangular structure that even if the assumption is not satisfied, the diagonal blocks must be nonsingular if the original coefficient matrix A is nonsingular at all. Therefore the decomposition procedure above can be carried out successfully if the original system is uniquely solvable at all.

Each subproblem may be solved as follows. Let A_j be the coefficient matrix of the j-th subproblem. Its row-set is divided as (4.20) into R_{Qj} and R_{Tj}. Its column-set C_j may also be partitioned as

$$C_j = C_{wj} \cup C_{xj}, \tag{4.41}$$

where C_{wj} and C_{xj} correspond to part of the variables w and x, respectively. It is easy to see, by the irreducibility of A_j, that

$$|R_{Tj}| \geq |C_{wj}| \qquad \text{if} \quad R_{Tj} \neq \emptyset \tag{4.42}$$

(and $|C_j|=1$ if $R_{Tj}=\emptyset$) and that the submatrix $A_j[R_{Tj},C_{wj}]$ is of the simple form

$$A_j[R_{Tj},C_{wj}] = \begin{pmatrix} -I \\ 0 \end{pmatrix} \tag{4.43}$$

if $R_{Tj}\neq\emptyset$ and $C_{wj}\neq\emptyset$, where I is the identity matrix of order $|C_{wj}|$. Thus the subproblem can be expressed as

$$\begin{matrix} & C_{wj} & C_{xj} \\ R_{Qj}: & Q_1 & Q_2 \\ & -I & T_1 \\ R_{Tj}: & 0 & T_2 \end{matrix} \quad \begin{pmatrix} w_j \\ \\ x_j \end{pmatrix} = \begin{pmatrix} \tilde{b}_j \\ 0 \\ 0 \end{pmatrix}, \tag{4.44}$$

where $\tilde{b}_j=\tilde{b}_j(\theta)$ is to be computed from $b(\theta)$ each time θ is given. On eliminating the auxiliary variable w_j, we obtain the system of equations

$$\begin{pmatrix} Q_1T_1+Q_2 \\ T_2 \end{pmatrix} x_j = \begin{pmatrix} \tilde{b}_j \\ 0 \end{pmatrix} \tag{4.45}$$

in $|C_{xj}|$ variables. The amount of computation needed to determine x_j in this way may be estimated roughly by

$$|R_{Qj}||C_{wj}||C_{xj}| + |C_{xj}|^3/3. \tag{4.46}$$

Another approach may be conceivable that makes no distinction between w_j and x_j. We may assume that the subsystem is given by

$$\begin{matrix} R_{Qj}: \\ R_{Tj}: \end{matrix} \begin{pmatrix} I & Q_1 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \tilde{b}_j \\ 0 \end{pmatrix}, \tag{4.47}$$

where (z_1,z_2) is a rearrangement of (w_j,x_j). The Gaussian elimination procedure applied to (4.46), possibly with permutations of rows in R_{Tj}, can be done with at most

$$|R_{Tj}|^2|R_{Qj}| + |R_{Tj}|^3/3 \qquad (4.48)$$

arithmetic operations.

The above considerations reveal that the matrix A_j contains an identity matrix of order no smaller than $\max(|C_{wj}|,|R_{Qj}|)$ as a submatrix. Thus, we may adopt

$$\min(|C_{xj}|,|R_{Tj}|) \qquad (4.49)$$

as a rough measure for the substantial size of the subproblem.

Example 4.3. This example is based on the reactor-separator model (EV-6) of [36]. The system of linear/nonlinear equations to be solved involves 120 unknowns and as many equations. The Jacobian matrix, denoted as A, is sparse, containing 351 nonvanishing entries. The ordinary DM-decomposition yields 4 nontrivial blocks involving more than one unknown variable. The maximun size of the blocks is 25 (see Table 4.1).

Of the nonvanishing entries of A, 172 numbers are rational constants (1 or -1) and the remaining 179 entries are regarded here as algebraically independent numbers (in a field F) over Q. That is, we consider $A \in \underline{\underline{MM}}(F/Q;120,120)$. As explained above, we may then resort to the combinatorial canonical form of the corresponding layered mixed matrix $\tilde{A} \in \underline{\underline{LM}}(F/Q;120,120,240)$ to obtain a decomposition of the augmented system of equations with auxiliary variables (see (3.2) and (3.9)). The canonical form of $\tilde{A}$ has no tails and yields 5 nontrivial blocks, the maximum size of which being equal to 17. (The canonical form of $\tilde{A}$ has been found by a slightly modified version of the FORTRAN program originally coded by M. Ichikawa [9].) In Table 4.1, three different decompositions are compared, where the number of rows of T-part of each block, i.e., $|R_{T_j}|$ of (4.11), is indicated in brackets and the third decomposition will be explained in §5. □

Example 4.4. The system of equations considered here is compiled in [9] from a real-world problem that has arisen from the analysis of an industrial hydrogen production system. It involves 544 variables and equations, and the Jacobian matrix A consists of 1142 rational constants (1 or -1) and other 322 numbers which are regarded here as algebraically independent transcendentals in F over Q. Then we have $A \in \underline{\underline{MM}}(F/Q;544,544)$. The combinatorial canonical form of the corresponding layered mixed matrix $\tilde{A} \in \underline{\underline{LM}}(F/Q;544,544,1088)$, computed as in Example 4.3, has no tails and contains 23 nontrivial blocks with more than one variable. The DM-decomposition of A and the combinatorial canonical form of $\tilde{A}$ are summarized in Table 4.2. Note that the substantial sizes of the subproblems in terms of (4.49) are much smaller than the block sizes of the subproblems obtained by the DM-decomposition. □

Table 4.1. Block-triangularizations for Example 4.3

DM-decomposition of A		Combin. canon. form of $\tilde{A}$ (by p_γ)		Decomposition of of $\tilde{A}$ by p_T	
size	blocks	size	blocks	size	blocks
C_x		$C = C_w + C_x$ $[R_T]$		$C = C_w + C_x$ $[R_T]$	
25	1	17 = 8 + 9 [9]	1	16 = 8 + 8 [8]	1
10	1	15 = 6 + 9 [6]	1	14 = 6 + 8 [5]	1
9	2	14 = 4 +10 [9]	1	13 = 4 + 9 [8]	1
		8 = 0 + 8 [4]	1	8 = 0 + 8 [5]	1
		5 = 0 + 5 [5]	1		
1	67	1	181	1	189

Table 4.2. Block-triangularizations for Example 4.4

DM-decomposition of A		Combin. canon. form of $\tilde{A}$ (by p_γ)		
size	blocks	size		blocks
C_x		$C = C_w + C_x$	$[R_T]$	
104	1	114 = 75 +39	[75]	1
28	1	24 = 15 + 9	[15]	1
23	1	18 = 10 + 8	[10]	1
14	1	14 = 8 + 6	[8]	1
10	5	6 = 4 + 2	[4]	1
8	1	4 = 2 + 2	[2]	15
6	7	2 = 1 + 1	[1]	3
4	2			
3	9			
1	240	1		846

5. Relations to Other Decompositions

The first subsection clarifies the relation of the combinatorial canonical form to the decomposition considered in [22], [23], [24], as well as to the ordinary DM-decomposition. The second subsection points out that for a certain class of electrical networks considered in [12], [14] the combinatorial canonical form gives essentially the same block-triangularization as the method proposed in [12], [14] by way of the structure of minimum covers in an independent-matching problem.

5.1. Decomposition by $L(p_\tau)$ and the DM-decomposition

In [22], [23], [24], a method for block-triangularization of systems of equations, such as (3.11), for electrical networks is proposed as an application of the principal partition associated with a matroid intersection problem. The method of [22], [23], [24], which we term here the <u>principal partition of $M(Q)^* \wedge M(T)$</u>, is based on Theorem 3.1 and adopts the submodular function p_τ of (4.6) to obtain a decomposition of unknown variables (i.e., currents and voltages of branches in the case of electrical networks) into partially ordered blocks.

In the following, we compare the decompositions induced by the two submodular functions p_τ of (4.6) and p_γ of (4.7) associated with a layered mixed matrix $A \in \underline{\underline{LM}}(F/K; m_Q, m_T, n)$ of the form (3.3). Remember that $L(p)$ is defined in (2.2) as the family of minimizers of $p: 2^C \to R$ and that $L(p)$ is a distributive sublattice if p is submodular.

Lemma 5.1.

(1) $p_\tau(X) \leqq p_\gamma(X)$ for $X \subset C$.

(2) $\min p_\tau = \min p_\gamma$.

(3) $L(p_\tau) \supset L(p_\gamma)$.

(4) For $X \in L(p_\tau)$ there exists $Y \in L(p_\gamma)$ such that $Y \subset X$.

(5) $\min L(p_\tau) = \min L(p_\gamma)$.

(Proof) (1) and (2): Given in (4.8) and Lemma 4.1.

(3): Immediate from (1) and (2) above.

(4): Let Y_0 ($\subset X$) be a minimizer of $\min\{\gamma(Y)-|Y| \; | Y\subset X\} = \tau(X)-|X|$.
From (2), we have $\min p_\gamma = \min p_\tau = \rho(X)+\gamma(Y_0)-|Y_0| \geqq \rho(Y_0)+\gamma(Y_0)-|Y_0|$
$= p_\gamma(Y_0)$, i.e., $Y_0 \in L(p_\gamma)$.

(5): This follows from (3) and (4) above. □

In view of the correspondence between the distributive sublattices and the partition into partially ordered blocks (§2), this lemma shows that the decomposition of the column-set C (i.e., the set of variables) by the principal partition of $M(Q)^* \wedge M(T)$ is finer (including the partial order) than that of the combinatorial canonical form of the present paper. In other words, the column-set of each block of the combinatorial canonical form is an aggregation of the blocks of the principal partition of $M(Q)^* \wedge M(T)$. It is indicated by Lemma 5.1(5), however, that the column-sets of the horizontal tail are identical in both decompositions.

In §4 we have seen that the decomposition of C based on p_γ provides the finest block-triangular form under the equivalence transformation of the form (4.1). By a similar argument it can be shown that the principal partition of C associated with $M(Q)^* \wedge M(T)$

leads to the finest block-triangularization with the properties (1) to (5) of Theorem 4.4, under a wider class of transformations of the following form:

$$P_r \begin{pmatrix} S_Q & 0 \\ 0 & S_T \end{pmatrix} \begin{pmatrix} Q \\ T \end{pmatrix} P_c, \tag{5.1}$$

where $S_Q \in GL(m_Q,K)$; $S_T \in GL(m_T,F)$; and P_r and P_c are permutation matrices of orders m and n, respectively. Note that the transformed matrix no longer belongs to $\underline{\underline{LM}}(F/K;m_Q,m_T,n)$. This suggests that the block-triangularization by the principal partition of $M(Q)^* \wedge M(T)$ is more adequate when considered for a broader class of matrices. This issue will be discussed in §6.

Let Γ_A and Γ_Q be defined as (4.4) respectively for A and Q. As is well known, the DM-decomposition is induced by $L(p_{DM})$, where

$$p_{DM}(X) = |\Gamma_A(X)| - |X| \quad (X \subset C). \tag{5.2}$$

Since $|\Gamma_A(X)| = |\Gamma_Q(X)|+|\Gamma_T(X)| \geq \rho(X)+\gamma(X)$, we have

$$p_\gamma(X) \leq p_{DM}(X) \quad (X \subset C). \tag{5.3}$$

<u>Theorem 5.2</u>. If A $(\in \underline{\underline{LM}}(F/K))$ is nonsingular, then

$$\min p_\tau = \min p_\gamma = \min p_{DM} = 0$$

and

$$L(p_\tau) \supset L(p_\gamma) \supset L(p_{DM}).$$

(Proof) The relations between p_τ and p_γ follow from Lemma 5.1. By Theorem 4.2, the assumption is equivalent to $\min p_\gamma = 0$, which, combined with (5.3) and $p_{DM}(\emptyset)=0$, yields $\min p_{DM} = 0$. The inclusion $L(p_\gamma) \supset L(p_{DM})$ is then evident from (5.3). □

Example 5.1. This is continued from Examples 3.1 and 4.2. As given in [22] (cf. Fig. 5.1), the principal partition of $C=\{\xi^i,\eta_i | i=1,...,9\}$ associated with $M(Q)^*\wedge M(T)$ consists of 10 blocks; the block $C_8=\{\eta_5,\xi^5,\xi^9\}$ of the combinatorial canonical form in Example 4.2 splits into two blocks $\{\eta_5\}$ and $\{\xi^5,\xi^9\}$. It should be mentioned that, as opposed to the claim of [22], the unknown variables $\{\xi^5,\xi^9\}$ cannot be determined independently of η_5 even after the variables of $C_9 = \{\xi^2,\eta_2,\xi^3,\eta_3,\xi^4,\eta_4,\xi^7,\xi^8\}$ are fixed. □

Example 5.2. For a singular matrix the canonical form is not a refinement of the DM-decomposition. Consider, e.g., the matrix

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \left[\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix}\right] \end{array} , \qquad (5.4)$$

which may be thought of as a member of $\underline{\mathrm{LM}}(F/Q;4,0,4)$ $(F\supset Q)$. The canonical form consists of tails only; $C_0=\{1,2,3,4\}$, $|R_0|=2$, $C_\infty=\emptyset$, $|R_\infty|=2$. On the other hand, the DM-decomposition evidently decomposes A into 2 square blocks. □

Example 5.3. For the problem of Example 4.3 the decompositions based on p_γ and p_τ are compared in Table 4.1. □

5.2. Decomposition for electrical networks with admittance expression

In general, an electrical network can be described by the structural equations and the constitutive equations among currents ξ^i in and voltages η_i across the branches (cf. Example 3.1). When the branch charactristics are given in terms of self- and mutual admittances Y, the coefficient matrix A of the system of equations in (ξ, η) takes the form:

$$A = \begin{array}{|cc|} \xi & \eta \\ \hline D & 0 \\ 0 & R \\ \hline -I & Y \\ \hline \end{array}\,, \tag{5.5}$$

where D and R are the fundamental cutset and circuit matrices respectively. If the nonvanishing entries of Y are assumed to be algebraically independent over Q, the trivial scaling of the constitutive equations brings it into the class of $\underline{\underline{LM}}(R/Q)$. In this extended sense, we will regard A as a member of $\underline{\underline{LM}}(R/Q)$ of the form (3.3) with

$$Q = \begin{pmatrix} D & 0 \\ 0 & R \end{pmatrix}, \quad T = (-I \quad Y) \,. \tag{5.6}$$

The column-set C of A of (5.5) is the disjoint union of two copies, say B_ξ and B_η, of the set B of branches; i.e.,

$$C = B_\xi \cup B_\eta. \tag{5.7}$$

This allows us to identify the boolean lattice 2^C with the direct product of 2^{B_ξ} and 2^{B_η}. It may also be noted that the row-set of Y is identified with B_ξ, while its column-set is B_η.

The decomposition of C proposed in [12], [14] is as follows. Let $\mu(I)$ and $\nu(I)$ denote the rank and the nullity of the arc set I ($\subset B$) in the underlying graph. Obviously, we have

$$\mu(B\backslash J) = \nu(J) - |J| + \mu(B). \tag{5.8}$$

The nonsingularity of A of (5.5) can be formulated in terms of an independent-matching problem on the bipartite graph representing Y, where the matroid with rank function μ is attached to both B_ξ and B_η. Put

$$H = \{(I,J) \mid I\in B_\xi,\ J\in B_\eta,\ I \supset \Gamma_Y(J)\ \}, \tag{5.9}$$

where Γ_Y is defined for Y as in (4.4), and

$$p_\mu(I,J) = \mu(I) + \mu(B_\eta\backslash J) - \mu(B) \quad (I\subset B_\xi,\ J\subset B_\eta). \tag{5.10}$$

Note that $(I,J) \in H$ iff $(I,B_\eta\backslash J)$ is a cover of Y, and then $p_\mu(I,J)+\mu(B)$ is the rank of the cover in the independent-matching problem. The set of minimizers of $p_\mu|_H$, the restriction of p_μ to H, is denoted simply as $L(p_\mu)$, i.e.,

$$L(p_\mu) = \{(I,J) \in H \mid p_\mu(I,J) = \min_H p_\mu\ \}, \tag{5.11}$$

which is a sublattice of $2^{B_\xi}\times 2^{B_\eta} \simeq 2^C$ (cf. (5.7)), and hence determines a decomposition of C into partially ordered blocks.

The rest of this subsection is devoted to establishing Theorem 5.4 below, which implies that the combinatorial canonical form for A of the particular form (5.5) gives an essentially identical block-triangularization with the one provided by the method of [12], [14].

From (5.8) and (5.10) we see that

$$p_\mu(I,J) = \mu(I) + \nu(J) - |J| \qquad (I\subset B_\xi,\ J\subset B_\eta). \tag{5.12}$$

On the other hand, p_γ of (4.7) for A of (5.5) is written as

$$\begin{aligned} p_\gamma(I\cup J) &= \rho(I\cup J) + |I\cup\Gamma_Y(J)| - |I\cup J| \\ &= \mu(I) + \nu(J) - |J| + |\Gamma_Y(J)\setminus I| \qquad (I\subset B_\xi,\ J\subset B_\eta), \end{aligned} \tag{5.13}$$

since the rank ρ of M(Q) is equal to $\mu + \nu$. Combining (5.12) and (5.13), we obtain

$$p_\gamma(I\cup J) = p_\mu(I,J) + |\Gamma_Y(J)\setminus I| \qquad (I\subset B_\xi,\ J\subset B_\eta). \tag{5.14}$$

Lemma 5.3.

$$p_\gamma(I\cup J) = p_\mu(I,J) \quad \text{for } (I,J)\in H,$$

$$p_\gamma(I\cup J) > p_\mu(I,J) \quad \text{for } (I,J)\notin H.$$

(Proof) From (5.14) it follows that $p_\gamma \geqq p_\mu$, where the equality holds iff $\Gamma_Y(J)\subset I$. □

Theorem 5.4.

(1) $\min\{\ p_\gamma(I\cup J)\ |\ I\subset B_\xi,\ J\subset B_\eta\ \} = \min\{\ p_\mu(I,J)\ |\ (I,J)\in H\ \}$.

(2) $L(p_\gamma) \supset L(p_\mu)$.

(3) $\{J\subset B_\eta | I\cup J\in L(p_\gamma)\} = \{J\subset B_\eta | (I,J)\in L(p_\mu)\}$.

(Proof) (1): By (5.13), we have

$$\begin{aligned} \min p_\gamma &= \min\{\ \min\{\mu(I)+|\Gamma_Y(J)\setminus I|\ \ |I\subset B_\xi\} + \nu(J)-|J|\ \ |J\subset B_\eta\} \\ &= \min\{\ \mu(\Gamma_Y(J))+\nu(J)-|J|\ \ |J\subset B_\eta\}, \end{aligned} \tag{5.15}$$

since $\min\{\mu(I)+|\Gamma_Y(J)\setminus I|\ \ |I\subset B_\xi\} = \min\{\mu(I)+|\Gamma_Y(J)\setminus I|\ \ |I\subset\Gamma_Y(J)\} = \mu(\Gamma_Y(J))$. This establishes (1) when combined with the rather obvious relation

$$\begin{aligned} \min_H p_\mu &= \min\{\ \mu(I)+\nu(J)-|J|\ \ |\ I\supset\Gamma_Y(J),\ J\subset B_\eta\ \} \\ &= \min\{\ \mu(\Gamma_Y(J))+\nu(J)-|J|\ \ |J\subset B_\eta\}. \end{aligned} \tag{5.16}$$

(2): Immediate from Lemma 5.3 and (1) above.

(3): From (5.15) and (5.16) it is easy to see that the families on both sides of (3) agrees with the minimizers J ($\subset B_\eta$) of $\mu(\Gamma_Y(J))+\nu(J)-|J|$. □

The Theorem 5.4(2) shows that the decomposition method of the present paper applied to (5.5) yields a finer partition of the variables $\{\xi,\eta\}$ than that proposed in [12], [14]. However, the difference is not substantial, since, as indicated by Theorem 5.4(3), they provide the identical partition for the voltage-variables η which play the primary role in (5.5); the current-variables ξ are only secondary as they are readily obtained from η by means of the admittance matrix Y. In this way, we may say that they give essentially the same decomposition. The following exemplifies that the inclusion in Theorem 5.4(2) is proper in general.

Example 5.4. For the following matrix

$$A = \begin{array}{c} \begin{array}{cccc} \xi^1 & \xi^2 & \eta_1 & \eta_2 \end{array} \\ \left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ \hline -1 & & y_{11} & 0 \\ & -1 & y_{21} & y_{22} \end{array}\right] \end{array}, \tag{5.17}$$

the combinatorial canonical form based on $L(p_\gamma)$ decomposes

$\{\xi^1, \xi^2, \eta_1, \eta_2\}$ into 4 singletons with the partial order:

$$\{\eta_2\} \prec \{\eta_1\} \prec \{\xi^1\}, \ \{\eta_2\} \prec \{\xi^2\}.$$

The decomposition of [12], [14] based on the minimum covers of Y, on the other hand, gives the partition into two blocks as

$$\{\xi^2, \eta_2\} \prec \{\xi^1, \eta_1\}.$$ □

6. Extensions and Remarks

It has been mentioned in §5.1 that the principal partition of $M(Q)*\wedge M(T)$, which corresponds to the transformation (5.1), should be considered in a wider class of matrices than $\underline{\underline{LM}}(F/K)$. Let F_0 be an intermediate field of F/K: $K \subset F_0 \subset F$, and consider a matrix $A \in \underline{\underline{M}}(F;m,n)$

$$A = \begin{pmatrix} Q \\ \hline T \end{pmatrix} , \tag{6.1}$$

such that (i) $Q \in \underline{\underline{M}}(K;m_Q,n)$, (ii) $T=Q_1T_1 \in \underline{\underline{M}}(F;m_T,n)$ where $Q_1 \in \underline{\underline{M}}(F_0;m_T,n)$ and T_1 is a diagonal matrix of order m_T with its diagonal entries being algebraically independent numbers in F over F_0. The class of such matrices A will be denoted by $\underline{\underline{LC}}(F/F_0/K;m_Q,m_T,n)$. It should be noted that $A \in \underline{\underline{LM}}(F/K;m_Q,m_T,n)$ belongs to $\underline{\underline{LC}}(F/F_0/K;m_Q,m_T,n)$ for some F_0, but not conversely.

It is known that the identity given in Theorem 3.1 still holds for $A \in \underline{\underline{LC}}(F/F_0/K)$ with ρ and τ being the rank functions of $M(Q)$ and $M(T)$ for the submatrices in (6.1). Therefore, the partition of the column-set C based on $L(p_T)$, followed by appropriate row transformations, brings about a block-triangular form with the properties (1) to (5) of Theorem 4.4. Note that the block-triangular form is obtained from A by means of the transformation (5.1), where we may assume without loss of generality that $S_T \in GL(m_T,F_0)$, and hence the transformed matrix remains in $\underline{\underline{LC}}(F/F_0/K)$.

The considerations above naturally suggest an extension to multi-layered matrices of the form

$$A = \begin{pmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_k \end{pmatrix} \tag{6.2}$$

such that

$$A_0 \in \underline{\underline{M}}(K;m_0,n),$$

$$A_i = Q_i T_i \in \underline{\underline{M}}(F_i;m_i,n) \quad (i=1,\ldots,k),$$

where

$$K \subset F_0 \subset \ldots \subset F_k \tag{6.3}$$

is a sequence of field extensions, $Q_i \in \underline{\underline{M}}(F_{i-1};m_i,n)$, and $T_i \in \underline{\underline{M}}(F_i;m_i,m_i)$ is a diagonal matrix with its diagonal entries being algebraically independent over F_{i-1} $(i=1,\ldots,k)$. Then, by Theorem 3.1, the rank of A is expressed in terms of the rank functions ρ_i of the associated matroids $M(A_i)$ $(i=0,1,\ldots,k)$ as

$$\text{rank } A = \min\{\ p(X)\ |\ X \subset C\ \} + n, \tag{6.4}$$

where

$$p(X) = \rho_0(X) + \rho_1(X) + \ldots + \rho_k(X) - |X|. \tag{6.5}$$

Based on L(p), we can obtain a block-triangular canonical form with the properties (1) to (5) of Theorem 4.4 under the transformation

$$P_r \begin{pmatrix} S_0 & & & & & \\ & S_1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & S_k \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_k \end{pmatrix} P_c \ , \tag{6.6}$$

where $S_0 \in GL(m_0,K)$; $S_i \in GL(m_i,F_{i-1})$ $(i=1,\ldots,k)$; and P_r and P_c are permutation matrices.

The canonical form for multi-layered matrix introduced above seems to have a natural meaning for electrical networks involving multi-ports, which have been investigated in [29], [30], [31]. To be specific, consider an electrical network consisting of k multi-ports, each of which is described by a set of equations with coefficient matrix A_i (i=1,...,k). Let A_0 denote the matrix (over Q) for Kirchhoff's laws. Then the coefficient matrix for the whole system is written as (6.2) (cf. Example 3.1), and the permissible transformation (6.6) reflects the locality in the sense that we can choose appropriate descriptions for each devices. Furthermore, the assumption of the algebraic independence among different devices would be fairly realistic.

Without the hierarchy of fields (6.3), we may likewise consider the block-triangularization based on p of (6.5) for a matrix of (6.2). That is, we may define a canonical form for a matrix A of (6.2) with $A_i \in \underline{\underline{M}}(F;m_i,n)$ (i=0,1,...,k) under the transformation (6.6) with $S_i \in GL(m_i,F)$ (i=0,1,...,k). In this case, however, the diagonal blocks are no longer guaranteed to be nonsingular. Two special cases may be worth mentioning. The one is the case where k=1 and $A_0=A_1$. Then the transformation (6.6), in which we may assume $S_0=S_1$, yields the combinatorial canonical form of a matrix with respect to its pivotal transforms introduced by [10]. The other is where A is nonsingular. Then it has no tails and the square blocks must necessarily be nonsingular.

The finest block-triangularization of a mixed matrix $A = Q_A + T_A \in \underline{\underline{MM}}(F/K;m,n)$ of (3.2) under the transformation

$$S\ A = S\ (Q_A+T_A), \quad S \in GL(m,K) \tag{6.7}$$

is obtained from the combinatorial canonical form of the corresponding layered mixed matrix $\tilde{A} \in \underline{\underline{LM}}(F/K;m,m,m+n)$ of (3.9). The partition of the column-set of A is induced from the partition of the column-set of $\tilde{A}$, which is identified with the union of the column- and the row-set of A, produced by the combinatorial canonical form of $\tilde{A}$. Note, however, that the transformed matrix (6.7) may not belong to $\underline{\underline{MM}}(F/K;m,n)$.

The combinatorial canonical form introduced in this paper should prove to be a useful tool in the structural analysis of systems. For example, it is reported in [18] that it plays a central role in deriving a necessary and sufficient combinatorial condition for the structural controllability of a dynamical system described in the so-called "descriptor form": Fdx/dt = Ax+Bu, where the entries of F, A and B are assumed to be classified into accurate and inaccurate numbers in the sense of [20].

Acknowledgement

The author would like to express his thanks to Professor Masao Iri of the University of Tokyo for discussions and comments. Examples 4.3 and 4.4 are based on the data provided by Junkichi Tsunekawa of the Institute of the Union of Japanese Scientists and Engineers and on the FORTRAN program originally coded in [9] by Masaki Ichikawa of the Economic Planning Agency, as well as on the kind help of Professor Yoshio Oyanagi of the University Tsukuba.

References

[1] G. Birkhoff: Lattice Theory (3rd ed.). Amer. Math. Soc. Colloq. Publ. 25, Providence, 1967.

[2] A.L. Dulmage and N.S. Mendelsohn: Coverings of bipartite graphs. Canad. J. Math., 10 (1958), 517-534.

[3] A.L. Dulmage and N.S. Mendelsohn: A structure theory of bipartite graphs of finite exterior dimension. Trans. of Royal Soc. Canada, Section III, 53 (1959), 1-13.

[4] A.L. Dulmage and N.S. Mendelsohn: On the inversion of sparse matrices. Math. Comp., 16 (1962), 494-496.

[5] A.L. Dulmage and N.S. Mendelsohn: Two algorithms for bipartite graphs. SIAM J., 11 (1963), 183-194.

[6] J. Edmonds: Minimum partition of a matroid into independent subsets. J. Nat. Bur. Stand., 69B (1965), 67-72.

[7] S. Fujishige: Algorithms for solving the independent-flow problems. J. Oper. Res. Soc. Japan, 21 (1978) 189-203.

[8] F. Harary: A graph-theoretic approach to matrix inversion by partitioning. Numer. Math., 4 (1962), 128-135.

[9] M. Ichikawa: An Application of Matroid Theory to Systems Analysis (in Japanese). Graduation thesis, Dept. Math. Eng. Instr. Phy., Univ. Tokyo, 1983.

[10] M. Iri: Combinatorial canonical form of a matrix with applications to the principal partition of a graph (in Japanese). Trans. Inst. Electr. Comm. Engin. Japan, 54A (1971), 30-37.

[11] M. Iri: A review of recent work in Japan on principal partitions of matroids and their applications. Ann. New York Acad.

Sci., 319 (1979), 306-319.

[12] M. Iri: Applications of matroid theory. Mathematical Programming ---The State of the Art (eds. A. Bachem, M. Groetschel and B. Korte), Springer, Berlin, 1983, 158-201.

[13] M. Iri: Structural theory for the combinatorial systems characterized by submodular functions. Progress in Combinatorial Optimization (eds. W.R. Pulleyblank), Academic Press, 1984, 197-219.

[14] M. Iri and S. Fujishige: Use of matroid theory in operations research, circuits and systems theory. Int. J. Systems Sci., 12 (1981), 27-54.

[15] M. Iri, J. Tsunekawa and K. Murota: Graph-theoretic approach to large-scale systems --- Structural solvability and block-triangularization (in Japanese). Trans. Infor. Process. Soc. Japan, 23 (1982), 88-95. (English translation available: Research Memorandum RMI 81-05, Dept. Math. Eng. Instr. Phy., Univ. Tokyo, 1981)

[16] K. Murota: Menger-decomposition of a graph and its application to the structural analysis of a large-scale system of equations. Kokyuroku, Res. Inst. Math. Sci., Kyoto Univ., 453 (1982), 127-173.

[17] K. Murota: Structural Solvability and Controllability of Systems. Doctor's dissertation, Dept. Math. Eng. Instr. Phys., Univ. Tokyo, 1983.

[18] K. Murota: Refined study on structural controllability of descriptor systems by means of matroids. Discussion Paper Series 258, Inst. Socio-Economic Planning, Univ. Tsukuba, 1985.

[19] K. Murota and M. Iri: Matroid-theoretic approach to the structural solvability of a system of equations (in Japanese). Trans. Infor. Process. Soc. Japan, 24 (1983), 157-164.

[20] K. Murota and M. Iri: Structural solvability of systems of equations --- A mathematical formulation for distinguishing accurate and inaccurate numbers in structural analysis of systems. Japan J. Appl. Math., 2 (1985), 247-271.

[21] M. Nakamura: Boolean sublattices connected with minimization problems on matroids. Math. Prog., 22 (1982), 117-120.

[22] M. Nakamura: Mathematical Analysis of Discrete Systems and Its Applications (in Japanese). Doctor's dissertation, Dept. Math. Eng. Instr. Phys., Univ. Tokyo, 1982.

[23] M. Nakamura: Analysis of discrete systems and its applications (in Japanese). Trans. Inst. Electr. Comm. Engin. Japan, J66A (1983), 368-373.

[24] M. Nakamura and M. Iri: Fine structures of matroid intersections and their applications. Proc. Int. Symp. Circuit and Systems, Tokyo, 1979, 996-999.

[25] M. Nakamura and M. Iri: A structural theory for submodular functions, polymatroids and polymatroid intersections. Research Memorandum RMI 81-06, Dept. Math. Eng. Instr. Phys., Univ. Tokyo, 1981.

[26] O. Ore: Graphs and matching theorems. Duke Math. J., 22 (1955), 625-639.

[27] T. Ozawa: Common trees and partition of two-graphs (in Japanese). Trans. Inst. Electr. Comm. Engin. Japan, 57A (1974),

383-390.

[28] T. Ozawa: Topological conditions for the solvability of linear active networks. Int. J. Circuit Theory and Appl., 4 (1976), 125-136.

[29] B. Petersen: Investigating solvability and complexity of linear active networks by means of matroids. IEEE Trans. Circuits and Systems, CAS-26 (1979), 330-342.

[30] A. Recski: Unique solvability and order of complexity of linear networks containing memoryless n-ports. Int. J. Circuit Theory and Appl., 7 (1979), 31-42.

[31] A. Recski and M. Iri: Network theory and transversal matroids. Disc. Appl. Math., 2 (1980), 311-326.

[32] N. Tomizawa: Strongly irreducible matroids and principal partition of a matroid into strongly irreducible minors (in Japanese). Trans. Inst. Electr. Comm. Engin. Japan, J59A (1976), 83-91.

[33] N. Tomizawa and S. Fujishige: Historical survey of extensions of the concept of principal partition and their unifying generalization to hypermatroids. Systems Science Research Report, No.5, Dept. System Sci., Tokyo Inst. Technology, 1982.

[34] B.L. van der Waerden: Algebra. Springer, Berlin, 1955.

[35] D.J.A. Welsh: Matroid Theory. Academic Press, London, 1976.

[36] K. Yajima, J. Tsunekawa and S. Kobayashi: On equation-based dynamic simulation. Proc. World Congr. Chemical Eng., Montreal, V (1981),

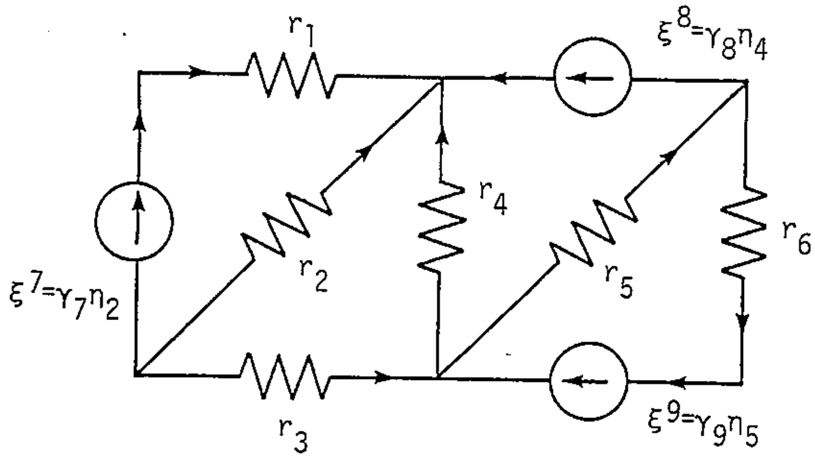

Fig.3.1. A simple electrical network of Example 3.1 (from [22])

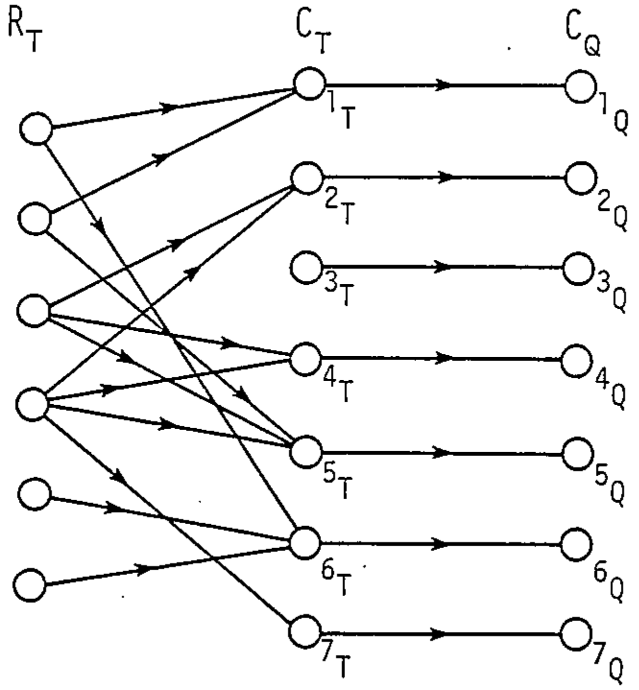

Fig.4.1. Independent-flow problem for Example 4.1

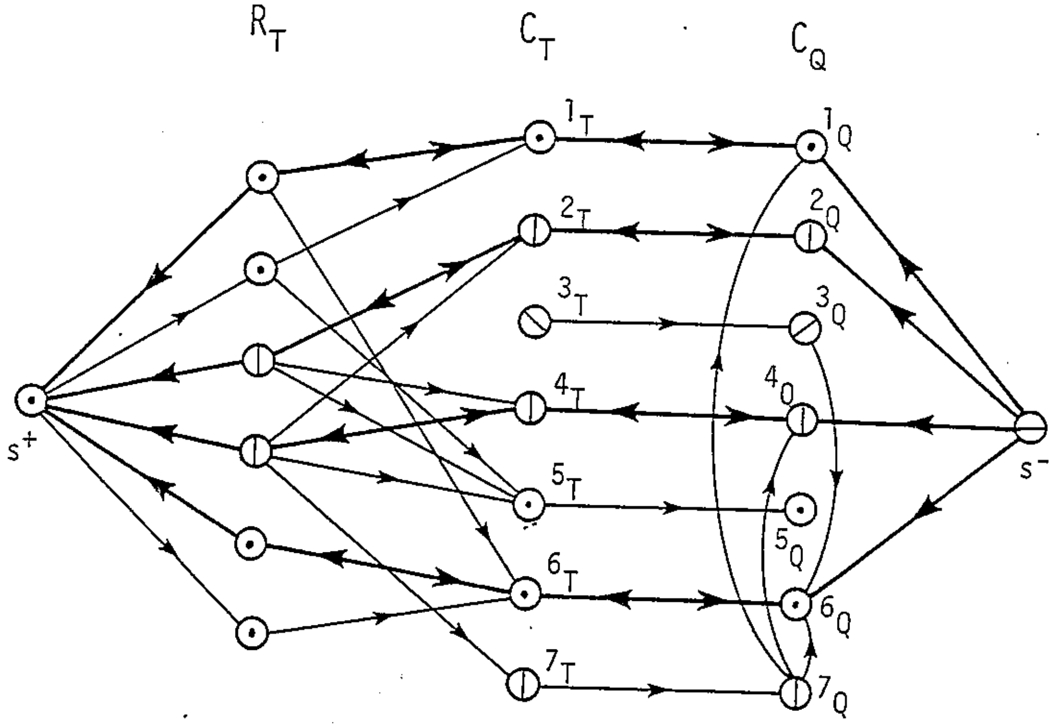

Fig.4.2. Auxiliary graph associated with a maximal independent flow for Example 4.1