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by

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AN $O(m^3 \log m)$ CAPACITY-ROUNDING ALGORITHM FOR THE MINIMUM-COST CIRCULATION PROBLEM: A DUAL FRAMEWORK OF THE TARDOS ALGORITHM

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Abstract

Recently, É. Tardos has given a strongly polynomial algorithm for the minimum-cost circulation problem and solved the open problem posed in 1972 by J. Edmonds and R. M. Karp. Her algorithm runs in $O(m^2T(m,n)\log m)$ time, where m is the number of arcs, n is the number of vertices and $T(m,n)$ is the time required for solving a maximum flow problem in a network with m arcs and n vertices. In the present paper, taking an approach which is a dual of Tardos's, we also give a strongly polynomial algorithm for the minimum-cost circulation problem. Our algorithm runs in $O(m(n+m_0)^2\log m)$ time and reduces the computational complexity, where m_0 is the number of arcs with both finite lower and upper capacities.

1. Introduction

Recently, É. Tardos [9] has given an algorithm which solves the minimum-cost circulation problem in strongly polynomial time. That is, assuming that each of the fundamental arithmetic operations (addition, multiplication and division) requires unit time, the running time of her algorithm is a polynomial in the number of arcs and that of vertices of the underlying graph, and the size of any number encountered during the execution of the algorithm is polynomially bounded in the total size of the input numbers. The problem of finding a strongly polynomial algorithm for the minimum-cost circulation problem was posed by J. Edmonds and R. M. Karp [4] in 1972 and had long been open.

In the present paper, we shall also propose a strongly polynomial algorithm for the minimum-cost circulation problem, which is, in a sense, a dual of the Tardos algorithm [9].

Consider a flow network with m arcs and n vertices. The Tardos algorithm solves at most m subproblems, each of which is a minimum-cost circulation problem obtained by rounding the cost coefficients to values from among $2n\sqrt{m}$ equi-spaced quantized values and is solved by H. Röck's algorithm [8] with the scaling technique. The total running time is $O(m^2T(m,n)\log m)$, where $T(m,n)$ is the time required for solving a maximum flow problem in a network with m arcs and n vertices. On the other hand, our algorithm solves at most m subproblems, each of which is a minimum-cost circulation problem obtained by rounding the capacities to values from among $2m^2$ equi-spaced quantized values. Each subproblem is solved by the out-of-kilter method with the scaling technique given by E. L. Lawler [6]. The total running time is $O(m^3 \log m)$ and reduces the computational complexity. A precise estimation of the complexity will be given in Section 3.

Very recently, J. B. Orlin [7] has claimed that the minimum-cost circulation problem can be solved in $O(m^3(\log m)^2)$ time. (Note that Orlin considers a network with lower capacities alone, so that in his model the number of vertices may be proportional to that of arcs after transforming a given network with both lower and upper capacities for each arc.) Our algorithm shows slightly better computational complexity than Orlin's, and the two approaches are different.

2. Definitions and Prelimiaries

We denote the set of real numbers by R. Except for R, any set appearing in this paper is a finite set.

Let $N = (G=(V,A),\underline{c},\overline{c},\gamma)$ be a network, where G is the underlying connected graph with the vertex set V and the arc set A, $\underline{c}$, $\overline{c}$: $A \to R \cup \{+\infty,-\infty\}$ are, respectively, lower and upper capacity functions, and γ: $A \to R$ is the cost function. For each arc $a \in A$, $\partial^+ a$ and $\partial^- a$ are, respectively, the <u>initial</u> and the <u>terminal</u> <u>vertices</u> of a. For each vertex

$v \in V$, we define $\delta^+ v = \{a \mid a \in A, \partial^+ a = v\}$ and $\delta^- v = \{a \mid a \in A, \partial^- a = v\}$. A function $f: A \to R$ is called a <u>circulation</u> in N if

$$\partial f(v) \equiv \sum_{a \in \delta^+ v} f(a) - \sum_{a \in \delta^- v} f(a) = 0 \quad (v \in V). \tag{2.1}$$

Moreover, a circulation f in N is called <u>feasible</u> if

$$\underline{c}(a) \leqq f(a) \leqq \bar{c}(a) \qquad (a \in A). \tag{2.2}$$

The <u>cost</u> $\gamma(f)$ of a feasible circulation f is defined by

$$\gamma(f) = \sum_{a \in A} \gamma(a) f(a). \tag{2.3}$$

The <u>minimum-cost circulation problem</u> is to find a <u>minimum-cost circulation</u>, i.e., a feasible circulation of the minimum cost, in N.

The existence of a feasible circulation in N can be checked by solving a maximum flow problem, and when a feasible circulation exists, the dual feasibility, i.e., the existence of a minimum-cost circulation, can be checked by solving a shortest path problem. Therefore, throughout this paper, we assume that there is a minimum-cost circulation in N and thus, without loss of generality, we also assume that for each arc $a \in A$

$$\underline{c}(a) \leqq 0 \leqq \bar{c}(a), \tag{2.4}$$

i.e., the zero function is a feasible flow. Furthermore, we assume, only for the sake of simplicity, that for each arc $a \in A$ we have $\underline{c}(a) = -\infty$ or $\bar{c}(a) = +\infty$. (If there is an arc a with both finite lower and upper capacities $\underline{c}(a)$ and $\bar{c}(a)$, then we may replace a by series arcs a' and a'' with capacities $\underline{c}(a') = \underline{c}(a)$, $\bar{c}(a') = +\infty$, $\underline{c}(a'') = -\infty$, $\bar{c}(a'') = \bar{c}(a)$.)

For lower and upper capacity functions $\underline{c}$ and $\bar{c}$, we define

$$A_\infty(\underline{c}, \bar{c}) = \{a \mid a \in A, \underline{c}(a) = -\infty, \bar{c}(a) = +\infty\}, \tag{2.5}$$

$$A_+(\bar{c}) = \{a \mid a \in A, \bar{c}(a) < +\infty\}, \tag{2.6}$$

$$A_-(\underline{c}) = \{a \mid a \in A,\ \underline{c}(a) > -\infty\}. \tag{2.7}$$

Note that, by the assumption, these three sets are disjoint and $A = A_\infty(\underline{c},\bar{c}) \cup A_+(\bar{c}) \cup A_-(\underline{c})$.

A <u>path</u> Q is a sequence

$$Q = (v_0, a_1, v_1, \ldots, v_{k-1}, a_k, v_k) \tag{2.8}$$

of vertices v_i $(0 \leqq i \leqq k)$ and arcs a_i $(1 \leqq i \leqq k)$ such that

$$\{\partial^+ a_i, \partial^- a_i\} = \{v_{i-1}, v_i\} \qquad (1 \leqq i \leqq k). \tag{2.9}$$

If $\partial^+ a_i = v_{i-1}$ and $\partial^- a_i = v_i$, then we say that arc a_i is <u>positively oriented</u> in Q, and if $\partial^+ a_i = v_i$ and $\partial^- a_i = v_{i-1}$, arc a_i is <u>negatively oriented</u> in Q. If $v_0 = v_k$ and $k \geqq 1$ in (2.8), Q is called a <u>closed path</u>.

Define a function $\varepsilon_Q\colon A \to \{0, \pm 1\}$ associated with a path Q by

$$\varepsilon_Q(a) = \begin{cases} 1 & \text{if } a \text{ lies on } Q \text{ and is positively oriented in } Q, \\ -1 & \text{if } a \text{ lies on } Q \text{ and is negatively oriented in } Q, \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

for $a \in A$. The <u>length</u> of the path Q relative to γ is defined by

$$\gamma(Q) = \sum_{a \in A} \varepsilon_Q(a)\gamma(a). \tag{2.11}$$

Given a feasible circulation f in $N = (G=(V,A), \underline{c}, \bar{c}, \gamma)$, a closed path Q in G is said to be <u>admissible</u> with respect to f and N if for each arc a lying on Q the following (i) and (ii) hold:

(i) If a is positively oriented in Q, then $f(a) < \bar{c}(a)$.

(ii) If a is negatively oriented in Q, then $\underline{c}(a) < f(a)$.

A <u>base</u> T of a not necessarily connected graph $H = (V,B)$ is a maximal set (maximal with respect to set inclusion) of arcs which does not contain any set of arcs forming a closed path in H. A <u>basic circulation</u> associated with a base T of $G = (V,A)$ and the network $N = (G,\underline{c},\bar{c},\gamma)$ is a circulation $f\colon A \to R$ in N such that

$$f(a) = \underline{c}(a) \qquad (a \in A_{-}(\underline{c}) - T), \tag{2.12}$$

$$f(a) = \bar{c}(a) \qquad (a \in A_{+}(\bar{c}) - T). \tag{2.13}$$

Such a basic circulation is not unique if $A_{\infty}(\underline{c},\bar{c}) - T \neq \emptyset$.

For two functions $g, h\colon A \to R$, we say that g is <u>h-equisignum</u> if for each $a \in A$ (1) $g(a) > 0$ implies $h(a) > 0$ and (2) $g(a) < 0$ implies $h(a) < 0$.

For any real number x, $\lceil x \rceil$ denotes the minimum integer not less than x and $\lfloor x \rfloor$ denotes the maximum integer not greater than x. For any set S, we denote the cardinality of S by $|S|$.

The following two lemmas are well known (see, e.g., [5],[6]).

<u>Lemma 1</u>: Let f be a minimum-cost circulation in $N = (G=(V,A),\underline{c},\bar{c},\gamma)$. Then each closed path admissible with respect to f and N has a nonnegative length relative to γ. □

<u>Lemma 2</u>: There exists a function $p\colon V \to R$ such that for any minimum-cost circulation f in N and for any arc $a \in A$ we have

$$\gamma(a) + p(\partial^{+}a) - p(\partial^{-}a) > 0 \implies f(a) = \underline{c}(a), \tag{2.14}$$

$$\gamma(a) + p(\partial^{+}a) - p(\partial^{-}a) < 0 \implies f(a) = \bar{c}(a). \tag{2.15}$$

□

A real-valued function on V is called a potential in N. The function p in Lemma 2 is called an optimal potential in N. We define

$$\gamma_p(a) = \gamma(a) + p(\partial^+ a) - p(\partial^- a). \tag{2.16}$$

Now, Lemma 1 leads us to the following lemma.

Lemma 3: Let f be a feasible circulation in N. Then there exist a minimum-cost circulation f^*, closed paths Q_i $(i \varepsilon I)$ admissible with respect to f and N, and positive reals d_i $(i \varepsilon I)$ such that the following (i) $\sim$ (iii) hold:

(i) We have

$$f^* - f = \sum_{i \varepsilon I} d_i \varepsilon_{Q_i}, \tag{2.17}$$

where ε_{Q_i} for each $i \varepsilon I$ is defined by (2.10) with Q replaced by Q_i.

(ii) For each $i \varepsilon I$, ε_{Q_i} is (f^*-f)-equisignum.

(iii) For each $i \varepsilon I$, the length of Q_i relative to γ is negative.

(Proof) Let $\hat{f}$ be a minimum-cost circulation in N. Then, there exist closed paths Q_i $(i \varepsilon \hat{I})$ admissible with respect to f and N and positive reals d_i $(i \varepsilon \hat{I})$ such that the following (i') and (ii') hold:

(i') We have

$$\hat{f} - f = \sum_{i \varepsilon \hat{I}} d_i \varepsilon_{Q_i}. \tag{2.18}$$

(ii') For each $i \varepsilon \hat{I}$, ε_{Q_i} is $(\hat{f}-f)$-equisignum.

Since each closed path obtained by reversing the orientation of Q_i $(i \varepsilon \hat{I})$ is admissible with respect to $\hat{f}$, it follows from Lemma 1 that

(iii') For each $i \varepsilon \hat{I}$, the length of Q_i $(i \varepsilon \hat{I})$ relative to γ is nonpositive.

Define

$$I = \{i \mid i \varepsilon \hat{I}, \text{ the length of } Q_i \text{ is negative}\}, \qquad (2.19)$$

and also define

$$f^* = f + \sum_{i \varepsilon I} d_i \varepsilon_{Q_i}. \qquad (2.20)$$

From (i') ∿ (iii'), (2.19) and (2.20), f^* is a feasible circulation in N and

$$\gamma(f^*) = \gamma(\hat{f}). \qquad (2.21)$$

Therefore, we have a minimum-cost circulation f^*, closed paths Q_i $(i \varepsilon I)$ admissible with respect to f and N, and positive reals d_i $(i \varepsilon I)$ which satisfy (i) ∿ (iii) in the lemma. □

By the use of Lemmas 2 and 3, we can show the following lemma.

Lemma 4: Suppose that new lower and upper capacity functions $\underline{c}'$ and $\bar{c}'$ with $A_\infty(\underline{c}',\bar{c}') = A_\infty(\underline{c},\bar{c})$ satisfy

$$\underline{c}'(a) - 1 < \underline{c}(a) \leqq \underline{c}'(a) \qquad (a \varepsilon A_-(\underline{c})), \qquad (2.22)$$

$$\bar{c}'(a) \leqq \bar{c}(a) < \bar{c}'(a) + 1 \qquad (a \varepsilon A_+(\bar{c})) \qquad (2.23)$$

and that $N' = (G=(V,A),\underline{c}',\bar{c}',\gamma)$ has a feasible circulation. Let f' be a minimum-cost circulation in N', and define

$$D = A - \{a \mid a \varepsilon A, \underline{c}(a) = \underline{c}'(a), \bar{c}(a) = \bar{c}'(a)\}. \qquad (2.24)$$

Then, there exists a minimum-cost circulation f in N such that

(I) for any arc $a \in A_-(\underline{c})$ satisfying

$$f'(a) - \underline{c}'(a) \geqq |D|, \tag{2.25}$$

we have $\underline{c}(a) < f(a)$ and

(II) for any arc $a \in A_+(\bar{c})$ satisfying

$$\bar{c}'(a) - f'(a) \geqq |D|, \tag{2.26}$$

we have $f(a) < \bar{c}(a)$.

(Proof) From Lemma 3, there exists a minimum-cost circulation f in $N = (G=(V,A),\underline{c},\bar{c},\gamma)$, closed paths Q_i $(i \in I)$ admissible with respect to f' and N, and positive reals d_i $(i \in I)$ such that

(i) $$f - f' = \sum_{i\in I} d_i \varepsilon_{Q_i}, \tag{2.27}$$

(ii) for each $i \in I$, ε_{Q_i} is $(f-f')$-equisignum,

(iii) the length of each Q_i $(i \in I)$ relative to γ is negative.

From Lemma 2, we have for an optimal potential p' in N'

$$\gamma_{p'}(a) = \gamma(a) + p'(\partial^+a) - p'(\partial^-a) > 0 \implies f'(a) = \underline{c}'(a), \tag{2.28}$$

$$\gamma_{p'}(a) = \gamma(a) + p'(\partial^+a) - p'(\partial^-a) < 0 \implies f'(a) = \bar{c}'(a). \tag{2.29}$$

Since the length of a closed path relative to γ and the one relative to $\gamma_{p'}$ are the same, it follows from the above (iii) that each Q_i $(i \in I)$ must contain an arc $a \in A$ with $\gamma_{p'}(a) > 0$ which is negatively oriented in Q_i or an arc $a \in A$ with $\gamma_{p'}(a) < 0$ which is positively oriented in Q_i. Therefore, from the above (i), (ii), (2.22), (2.23), (2.28) and (2.29), we have for any $a \in A$

$$|f(a) - f'(a)| \leqq \sum_{i\in I} d_i$$

$$\leqq \sum\{\underline{c}'(a) - \underline{c}(a) \mid a \in A,\ \gamma_{p'}(a) > 0\}$$

$$+ \sum\{\bar{c}(a) - \bar{c}'(a) \mid a \in A, \gamma_{p'}(a) < 0\}$$

$$< |D|. \tag{2.30}$$

This shows that f satisfies (I) and (II) in the lemma. □

3. A Strongly Polynomial Algorithm

We show a strongly polynomial algorithm, based on Lemma 4 in the preceding section.

Algorithm

Step 0: Put $\underline{c}' \leftarrow \underline{c}$, $\bar{c}' \leftarrow \bar{c}$, $f \leftarrow 0$ (the zero function on A).

Find a potential $p'': V \rightarrow R$ such that

$$\gamma_{p''}(a) \geqq 0 \qquad (a \in A_{-}(\underline{c})), \tag{3.1}$$

$$\gamma_{p''}(a) \leqq 0 \qquad (a \in A_{+}(\bar{c})), \tag{3.2}$$

$$\gamma_{p''}(a) = 0 \qquad (a \in A_{\infty}(\underline{c},\bar{c})). \tag{3.3}$$

Step 1: While $A_{\infty}(\underline{c}',\bar{c}')$ does not contain any base of G, do the following (a) ∿ (c):

(a) Find a base T of G and a feasible circulation f' in $N' = (G,\underline{c}',\bar{c}',\gamma)$ such that T contains a base of the subgraph $G' = (V,A_{\infty}(\underline{c}',\bar{c}'))$ and f' is a basic circulation, with $f'(a) = 0$ for each $a \in A_{\infty}(\underline{c}',\bar{c}') - T$, associated with T and N'.

Put

$$f \leftarrow f + f', \quad \underline{c}' \leftarrow \underline{c}' - f', \quad \bar{c}' \leftarrow \bar{c}' - f'. \tag{3.4}$$

(b) Define

$$M = \max\{\max\{\bar{c}'(a) \mid a \in A_+(\bar{c}') \cap T\}, \max\{-\underline{c}'(a) \mid a \in A_-(\underline{c}') \cap T\}\}. \tag{3.5}$$

If $M = 0$, then go to Step 3.

Otherwise, put

$$\bar{c}''(a) \leftarrow \begin{cases} \lfloor \bar{c}'(a)(|V|-1)|A|/M \rfloor & (a \in A_+(\bar{c}')) \\ +\infty & (a \in A - A_+(\bar{c}')), \end{cases} \tag{3.6}$$

$$\underline{c}''(a) \leftarrow \begin{cases} \lceil \underline{c}'(a)(|V|-1)|A|/M \rceil & (a \in A_-(\underline{c}')) \\ -\infty & (a \in A - A_-(\underline{c}')). \end{cases} \tag{3.7}$$

(c) Find a minimum-cost circulation f'' and an optimal potential, to be denoted by p'' again, in $N'' = (G, \underline{c}'', \bar{c}'', \gamma)$ by using the current potential p''.

For each $a \in A_+(\bar{c}'')$ with $\bar{c}''(a) - f''(a) \geq |V| - 1$, put $\bar{c}'(a) \leftarrow +\infty$.

For each $a \in A_-(\underline{c}'')$ with $f''(a) - \underline{c}''(a) \geq |V| - 1$, put $\underline{c}'(a) \leftarrow -\infty$.

Step 2: Let T be a base of the subgraph $G' = (V, A_\infty(\underline{c}', \bar{c}'))$, let f' be a basic (feasible) circulation, with $f'(a) = 0$ for each $a \in A_\infty(\underline{c}', \bar{c}') - T$, associated with T and $N' = (G=(V,A), \underline{c}', \bar{c}', \gamma)$, and put

$$f \leftarrow f + f', \quad \underline{c}' \leftarrow \underline{c}' - f', \quad \bar{c}' \leftarrow \bar{c}' - f'. \tag{3.8}$$

Step 3: Define

$$B = \{a \mid a \in A, \gamma_{p''}(a) = 0\}. \tag{3.9}$$

Find a feasible circulation f_0 in $N_0 = (G_0=(V,B), (\underline{c}-f)_B, (\bar{c}-f)_B)$, where $(\underline{c}-f)_B$ and $(\bar{c}-f)_B$ are lower and upper capacity functions given by restricting $\underline{c}-f$ and $\bar{c}-f$ on B.

Then, defining $f_0^A: A \to R$ as $f_0^A(a) = f_0(a)$ $(a \in B)$ and $f_0^A(a) = 0$ $(a \in A - B)$, $f + f_0^A$ is a minimum-cost circulation in $N = (G=(V,A), \underline{c}, \bar{c}, \gamma)$

and the algorithm terminates.

(End)

Let us estimate the computational complexity of the algorithm. The validity of the algorithm will be discussed in the next section. It will be shown in the next section that, in Step 1, the cycle of (a) $\sim$ (c) is repeated at most $|V| - 1$ times. We use this fact in this section to estimate the complexity.

A potential p'' satisfying (3.1) $\sim$ (3.3) can be obtained in $O(|V|^3)$ time by solving a shortest path problem, since the original network N has a minimum-cost circulation, i.e., N is dual feasible.

It should be noted that the zero circulation $f' = 0$ is feasible in $N' = (G, \underline{c}', \bar{c}', \gamma)$ in each (a) of Step 1. Therefore, each (a) of Step 1 can be carried out in $O(|V||A|)$ time. (At the second and later encounters with (a) of Step 1, if we modify the current base T and basic feasible circulation f' obtained at the preceding (a) of Step 1 for finding a new base and a new basic feasible circulation, the overall running time required in repeated (a) of Step 1 can be $O(|V||A|)$.)

In (b) of Step 1, taking the integer part of each scaled capacity in (3.6) and (3.7) requires $O(\log|A|)$ comparisons.

Note that $\underline{c}''(a) \leqq 0 \leqq \bar{c}''(a)$ $(a \in A)$ and that, from (3.4) $\sim$ (3.7), capacities $\underline{c}''(a)$, $\bar{c}''(a)$ $(a \in A)$ can take on finite nonzero values only for arcs a in T. Therefore, in (c) of Step 1, a minimum-cost circulation and an optimal potential in $N'' = (G, \underline{c}'', \bar{c}'', \gamma)$ can be found by the out-of-kilter method with the scaling technique (see [6]) in $O(|V||A|\log|A|)$, using the current potential p''. Here, note that the current potential p'' satisfies

$$\gamma_{p''}(a) \geqq 0 \qquad (a \in A_-(\underline{c}'')), \tag{3.10}$$

$$\gamma_{p''}(a) \leqq 0 \qquad (a \in A_+(\bar{c}'')), \tag{3.11}$$

$$\gamma_{p''}(a) = 0 \qquad (a \in A_\infty(\underline{c}'',\bar{c}'')). \tag{3.12}$$

Also, note that $A_-(\underline{c}'') = A_-(\underline{c}')$, $A_+(\bar{c}'') = A_+(\bar{c}')$ and $A_\infty(\underline{c}'',\bar{c}'') = A_\infty(\underline{c}',\bar{c}')$. After finishing (c) of Step 1 by the out-of-kilter method, we obtain an optimal potential p'' in $N'' = (G,\underline{c}'',\bar{c}'',\gamma)$ which satisfies (3.10) ∿ (3.12) at the next (c) of Step 1, since finite $\bar{c}'(a)$ becomes $+\infty$ only if $f''(a) < \bar{c}''(a)$, i.e., $\gamma_{p''}(a) = 0$, and since finite $\underline{c}'(a)$ becomes $-\infty$ only if $\underline{c}''(a) < f''(a)$, i.e., $\gamma_{p''}(a) = 0$, from Lemma 2.

Consequently, the overall running time required for finishing Step 1 is $O(|V|^2|A|\log|A|)$, since the cycle of (a) ∿ (c) is repeated at most $|V| - 1$ times.

The most time-consuming part of Steps 2 and 3 is finding a feasible circulation in N_0 in Step 3, which requires $O(|V|^2|A|)$ time by using the Dinic algorithm [2]. (This can be further reduced by using existing more sophisticated maximum flow algorithms but is already less than the time required for Step 1.)

The overall time required for the algorithm is thus $O(|V|^2|A|\log|A|)$. Throughout the algorithm we do not carry out multiplications and/or divisions repeatedly, so that the algorithm is strongly polynomial.

Recall that we have assumed that there is no arc with both finite upper and lower capacities. Suppose that we are given a network with m arcs and n vertices and that the number of arcs with both finite upper and lower capacities is equal to m_0. Then after modifying the given network to a network which satisfies the assumption, the new network has $n + m_0$ vertices. Consequently, the complexity of the algorithm is $O(m(n+m_0)^2 \log m)$. If $m_0 = m$, it becomes $O(m^3 \log m)$. If $m_0 = 0$ (e.g., in the case of the Hitchcock transportation problem (see [5])), the complexity is $O(mn^2 \log m)$.

4. The Validity of the Algorithm

In this section we show the validity of the algorithm proposed in the preceding section.

Because of the way of choosing a base T of G in (a) of Step 1, if there is an arc $a^* \in A_\infty(\underline{c}', \bar{c}') - T$, then for any arc a lying on the closed path formed by arcs in $T \cup \{a^*\}$, we must have $a \in A_\infty(\underline{c}', \bar{c}')$. Therefore, we can put $f'(a^*) = 0$ in (a) of Step 1. Moreover, we have

$$(A_+(\bar{c}') \cup A_-(\underline{c}')) \cap T \neq \emptyset, \tag{4.1}$$

and M defined by (3.5) is nonnegative. If $M = 0$, then the zero circulation $f' = 0$ is a minimum-cost circulation in $N' = (G=(V,A), \underline{c}', \bar{c}', \gamma)$, since N' has a minimum-cost circulation. If $M \neq 0$ in (b) of Step 1, let us define

$$\bar{c}_s'(a) = \bar{c}'(a)(|V|-1)|A|/M \qquad (a \in A), \tag{4.2}$$

$$\underline{c}_s'(a) = \underline{c}'(a)(|V|-1)|A|/M \qquad (a \in A). \tag{4.3}$$

Let $\hat{a}$ be an arc which attains the maximum of (3.5). The arc set $T - \{\hat{a}\}$ forms two connected subgraphs of G, and let U_1 and U_2 be the vertex sets of the two subgraphs. Define $K \subseteq A$ and η: $A \to \{0, \pm 1\}$ by

$$K = \{a \mid a \in A, \text{ either } \partial^+ a \in U_1 \text{ and } \partial^- a \in U_2, \text{ or } \partial^+ a \in U_2 \text{ and } \partial^- a \in U_1\}, \tag{4.4}$$

and for $a \in A$

$$\eta(a) = \begin{cases} 1 & (\partial^+ a \in U_1,\ \partial^- a \in U_2), \\ -1 & (\partial^+ a \in U_2,\ \partial^- a \in U_1), \\ 0 & (\text{otherwise}). \end{cases} \tag{4.5}$$

Because of the way of choosing T, we have

$$K \cap A_\infty(\underline{c}', \bar{c}') = \emptyset. \tag{4.6}$$

Moreover, for a minimum-cost circulation f'' in N'' obtained in the succeeding (c) of Step 1, we have from (4.6)

$$\begin{aligned} &\sum\{\eta(a)(\underline{c}''(a) - f''(a)) \mid a \in A_-(\underline{c}'') \cap K\} \\ &+ \sum\{\eta(a)(\bar{c}''(a) - f''(a)) \mid a \in A_+(\bar{c}'') \cap K\} \\ &= \begin{cases} \eta(\hat{a})\underline{c}''(\hat{a}) & \text{if } \hat{a} \in A_-(\underline{c}''), \\ \eta(\hat{a})\bar{c}''(\hat{a}) & \text{if } \hat{a} \in A_+(\bar{c}''). \end{cases} \end{aligned} \tag{4.7}$$

Here, use is made of the fact that $\sum\{\eta(a)f''(a) \mid a \in K\} = 0$ and that $\underline{c}''(a) = 0$ for $a \in A_-(\underline{c}'') \cap K$ and $\bar{c}''(a) = 0$ for $a \in A_+(\bar{c}'') \cap K$, except for $\hat{a}$. From (3.5) ∿ (3.7) and the fact that the absolute value of (4.7) is equal to $(|V|-1)|A|$, there exists at least one arc $a \in K \subseteq A_-(\underline{c}'') \cup A_+(\bar{c}'')$ for which

$$a \in A_-(\underline{c}''), \qquad f''(a) - \underline{c}''(a) \geq |V| - 1 \tag{4.8}$$

or

$$a \in A_+(\bar{c}''), \qquad \bar{c}''(a) - f''(a) \geqq |V| - 1. \tag{4.9}$$

Since $|T| = |V| - 1$ and for each $a \in A - T$ we have $\underline{c}''(a) = \underline{c}_s'(a)$ and $\bar{c}''(a) = \bar{c}_s'(a)$, it follows from (4.8), (4.9) and Lemma 4 with $\underline{c}$, $\bar{c}$, $\underline{c}'$, $\bar{c}'$ replaced by $\underline{c}_s'$, $\bar{c}_s'$, $\underline{c}''$, $\bar{c}''$, respectively, that there is a minimum-cost circulation f' in $N' = (G,\underline{c}',\bar{c}',\gamma)$ such that

(i) for $a \in A_-(\underline{c}'')$ with $f''(a) - \underline{c}''(a) \geqq |V| - 1$,

we have $\underline{c}'(a) < f'(a)$,

(ii) for $a \in A_+(\bar{c}'')$ with $\bar{c}''(a) - f''(a) \geqq |V| - 1$,

we have $f'(a) < \bar{c}'(a)$.

Therefore, in case of (i) (or (ii)), if we put $\underline{c}'(a) \leftarrow -\infty$ (or $\bar{c}'(a) \leftarrow +\infty$), f' remains to be a minimum-cost circulation in the new network $N' = (G,\underline{c}',\bar{c}',\gamma)$. This is a crucial point. Moreover, from (4.8), (4.9) and Lemma 4, at least one arc $a \in K \subseteqq A_-(\underline{c}'') \cup A_+(\bar{c}'')$ satisfies the requirement of the above (i) or (ii). Hence, the cardinality of a base of $G' = (V, A_\infty(\underline{c}',\bar{c}'))$ increases at least by 1. Therefore, we repeat the cycle of (a) $\sim$ (c) of Step 1 at most $|V| - 1$ times. At the beginning of Step 3, we get a network $N' = (G,\underline{c}',\bar{c}',\gamma)$ and a circulation f in $N = (G,\underline{c},\bar{c},\gamma)$. We see that the zero circulation $f' = 0$ is a minimum-cost circulation in N'. Denote the polyhedra of all the minimum-cost circulations in N' and N by $P(N')$ and $P(N)$, respectively. Then, due to the above arguments, we have

$$P(N) \subseteqq P(N') + \{f\} \equiv \{f' + f \mid f' \in P(N')\}. \tag{4.10}$$

From (4.10), any circulation in $P(N') + \{f\}$ which is feasible in the original N is a minimum-cost circulation in N. Such a feasible circulation in N is found in Step 3.

5. Concluding Remarks

We have shown an $O(m(n+m_0)^2 \log m)$ algorithm for the minimum-cost circulation problem for a network with m arcs, n vertices, and m_0 arcs with both finite lower and upper capacities, by means of capacity rounding. This algorithm is motivated by the modification of the Tardos algorithm with cost rounding, described as follows.

Tardos [9] introduced the projection of the cost vector into the circulation space (which, physically to be more precise, should be regarded as the projection of the cost vector into the orthogonal complement of the tension space). Instead of introducing such a projection, let us find a base (spanning tree) of G with a root v_0 and let $p(v)$ $(v \in V)$ be the length of the unique path in T from the root v_0 to v relative to γ. Then transform γ into γ_p modified by the potential p and define

$$M^* = \max\{|\gamma_p(a)| \mid a \in A - T\}. \qquad (5.1)$$

If $M^* = 0$, we are finished. Otherwise, suppose $\hat{a}$ is an arc which attains the maximum in (5.1). Define

$$\gamma_p'(a) = \gamma_p(a)|V|^2/M^* \qquad (a \in A). \qquad (5.2)$$

Let Q be the closed path formed by arcs in $T \cup \{\hat{a}\}$. Then, from (5.2), the length of Q relative to γ_p' is $\pm|V|^2$ and its value is invariant under any modifications of γ_p' by potentials. Therefore, for any potential $p'\colon V \to R$, we have

$$\sum_{a \in A} \varepsilon_Q(a)(\gamma_p'(a) + p'(\partial^+ a) - p'(\partial^- a)) = \pm|V|^2, \qquad (5.3)$$

where ε_Q is defined by (2.10). From (5.3), for at least one arc a

lying on Q we must have

$$|\gamma_p'(a) + p'(\partial^+ a) - p'(\partial^- a)| \geqq |V|. \tag{5.4}$$

Consequently, the Tardos algorithm modified as above works. This modified algorithm runs in $O(m^2 T(m,n) \log n)$ time, which is essentially the same as the complexity of Tardos's. But this modified version is simpler and can more easily be adapted to the submodular flow problem (cf. [1], [3]).

The algorithm proposed in the present paper is a dual of this modified version of the Tardos algorithm. The dual approach has effectively reduced the computational complexity.

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