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Abstract

The principle of dimensional homogeneity, which asserts that any system of equations describing a physical phenomenon should be consistent with respect to physical dimensions, is shown to imply a kind of total unimodularity of the physically-dimensioned coefficient matrix of the (linearized) system. This fact can be utilized in the structural approach to systems analysis in a number of ways; for example, it is useful in formulating some problems concerning dynamical systems in matroid-theoretic terms as well as in reducing the computational complexity needed to solve them by combinatorial algorithms.

Keywords: physical dimensions, dimensional homogeneity, structural analysis of systems, mixed matrix, total unimodularity, computational complexity, matroid.

1. Introduction

The concept of physical dimensions would be counted among the most fundamental in recognizing the nature of physical quantities in an appropriate manner. The principle of dimensional homogeneity claims that any equation describing a physical phenomenon, if it is to be qualified as such, must be consistent with respect to physical dimensions. This principle constitutes the basis of the method usually called dimensional analysis, which has long been known to scientists and turned out to be fruitful in various fields [2], [7], [14], [16]. It is important here to notice that we cannot talk of dimensional homogeneity until we recognize the difference in the nature of quantities from the viewpoint of physical dimensions.

In connection to structural analysis of large-scale physical/engineering systems, on the other hand, another sort of distinction in the nature of numbers is made in [25]. It is pointed out there that by classifying the nonvanishing numbers into two different kinds according as they are "accurate" or "inaccurate" (in a certain algebraic sense to be explained later), we can construct those mathematical models which are fairly faithful to real situations and, at the same time, simple enough to be amenable to subsequent mathematical analysis.

In this paper, we will show how we can make use of the concept of physical dimensions in the structural analysis of large-scale systems. Suppose a physical system is described by a system of equations, which may in turn be expressed by a matrix when linearized if necessary. With each entry of the matrix is associated the physical dimensions in a physically consistent manner.

It will be pointed out first that, by virtue of the principle of dimensional homogeneity, the physically-dimensioned coefficient matrix describing a physical system enjoys a kind of total unimodularity in a certain ring defined appropriately with reference to physical dimensions. Several implications of this fact are discussed in connection to the mathematical framework developed in [25] for the structural systems analysis under the assumption that the nonvanishing numbers are distinguished into "accurate" and "inaccurate" numbers. To reflect the dual viewpoint from structural analysis and dimensional analysis, the notion of physical matrix is introduced as a mathematical model of the matrices that we encounter in real physical systems.

The concept of physical dimensions can be utilized in the structural analysis in a number of ways. In particular, (i) the consistency among physical dimensions associated with the entries of a matrix can be used for detecting descriptive errors; (ii) the structural problems of determining the solvability [20], [37], the controllability [18], [31], and the dynamical degree [6] of a dynamical system described in descriptor form [19], [20] can be formulated and solved with the aid of matroid-theoretic concepts in the framework of [25] under a physically plausible assumption concerning the physical dimensions of the "accurate" numbers; (iii) the computational complexity in solving the above problems by combinatorial algorithms can be drastically reduced to render the proposed methods applicable to real-world large-scale systems.

2. Physically-dimensioned matrices

A physical system is usually described by a set of relations among relevant physical quantities, to each of which is assigned the physical dimensions. When a set of fundamental dimensions, or equivalently, a set of fundamental quantities, is chosen, the dimensions of the remaining physical quantities can be uniquely expressed by the so-called dimensional formulas [2], [7], [14], [16]. For example, a standard choice of fundamental quantities in mechanics consists of length L, mass M and time T, and the dimensional formula for force is then given by $[LMT^{-2}] = [L][M][T]^{-2}$ or simply by LMT^{-2}. In general, the exponents of dimensions, namely the powers in the dimensional formula, may take on rational numbers other than integers.

In this paper we do not go into the philosophical arguments such as those on what the physical dimensions are and which set of physical quantities are most fundamental. Instead we assume that the fundamental quantities with the associated fundamental dimensions are given along with the dimensional formulas for other quantities.

Let us consider a linear (or linearized) system represented by a system of linear equations:

$$A\,x = b\ , \tag{2.1}$$

where we assume that the entries of the m by n matrix $A=(a_{ij})$, as well as the components of $x=(x_j)$ and $b=(b_i)$, belong to some field $\mathbb{F}$, an extension of the field $\mathbb{Q}$ of rationals; namely,

$$a_{ij},\ x_j,\ b_i \in \mathbb{F} \quad (i=1,\dots,m;\ j=1,\dots,n). \tag{2.2}$$

Not only the components of x and b but also the entries of A have physical dimensions, expressed as $[Z_1]^{p_1}\cdots[Z_d]^{p_d}$ in terms of the chosen set of fundamental quantities $Z_1, \dots, Z_d$.

From algebraic point of view, we may regard $Z_1, \ldots, Z_d$ as indeterminates over $\mathbb{F}$ and consider the extension field $\mathbb{E}$ of $\mathbb{F}$ generated over $\mathbb{F}$ by all the formal fractional powers of $Z_1, \ldots, Z_d$; i.e.,

$$\mathbb{E} = \mathbb{F}(\{Z_1^{p_1} \ldots Z_d^{p_d} \mid p_k \in \mathbb{Q},\ k=1,\ldots d\}). \tag{2.3}$$

Accordingly, (2.1) may be replaced by the following system of equations in the extension field $\mathbb{E}$:

$$\tilde{A}\,\tilde{x} = \tilde{b}\ , \tag{2.4}$$

where

$$\tilde{a}_{ij} = a_{ij} \prod_{k=1}^{d} Z_k^{p_{ijk}}, \tag{2.5}$$

$$\tilde{x}_j = x_j \prod_{k=1}^{d} Z_k^{c_{jk}}, \tag{2.6}$$

$$\tilde{b}_i = b_i \prod_{k=1}^{d} Z_k^{r_{ik}} \tag{2.7}$$

with the exponents p_{ijk}, c_{jk}, r_{ik} of rational numbers representing the physical dimensions.

The principle of dimensional homogeneity now demands that the exponents should satisfy

$$p_{ijk} = r_{ik} - c_{jk} \tag{2.8}$$

for $i=1,\ldots,m$; $j=1,\ldots,n$; $k=1,\ldots,d$. Based on this observation, we will define the notion of dimensioned matrix as follows.

Definition 2.1. Let $\tilde{A}=(\tilde{a}_{ij})$ be a matrix over $\mathbb{E}$ (cf. (2.3)) which is expressed as in (2.5) with exponents $p_{ijk} \in \mathbb{Q}$. We call $\tilde{A}$ a <u>dimensioned matrix</u> if (2.8) holds for some suitably chosen r_{ik} and c_{jk} ($\in \mathbb{Q}$). |≡|

The set of dimensioned matrices (of any order) with base field $\mathbb{F}$ and fundamental quantities (i.e., indeterminates) $Z_1,\ldots,Z_d$ will be denoted by $D(\mathbb{F};Z_1,\ldots,Z_d)$. The following is a restatement of the definition, where

$$D_r = \mathrm{diag}(\prod_{k=1}^{d} Z_k^{r_{1k}},\ldots,\prod_{k=1}^{d} Z_k^{r_{mk}}), \qquad (2.9)$$

$$D_c = \mathrm{diag}(\prod_{k=1}^{d} Z_k^{c_{1k}},\ldots,\prod_{k=1}^{d} Z_k^{c_{nk}}) \qquad (2.10)$$

with $r_{ik} \in \mathbb{Q}$ and $c_{jk} \in \mathbb{Q}$ ($i=1,\ldots,m$; $j=1,\ldots,n$; $k=1,\ldots,d$).

Proposition 2.1. A matrix $\tilde{A}$ over $\mathbb{E}$ belongs to $D(\mathbb{F};Z_1,\ldots,Z_d)$ iff it can be expressed as

$$\tilde{A} = D_r\ A\ D_c^{-1},$$

where A is a matrix over $\mathbb{F}$, and D_r and D_c are diagonal nonsingular matrices of (2.9) and (2.10). |≡|

As an immediate consequence, any minor of a dimensioned matrix has a simple form, a "monomial" in Z_k's in the abuse of language, as stated below, where $\tilde{A}(I,J)$ denotes the submatrix of $\tilde{A}$ corresponding to rows in I and columns in J.

Proposition 2.2. Let $\tilde{A}$ be a dimensioned matrix, i.e., $\tilde{A} \in D(\mathbb{F};Z_1,\ldots,Z_d)$. Then

$$\det \tilde{A}(I,J) = \alpha \prod_{k=1}^{d} Z_k^{p_k}$$

for some $\alpha \in \mathbb{F}$ and $p_k \in \mathbb{Q}$ ($k=1,\ldots,d$).

(Proof) Suppose $\tilde{A}$ is expressed as in (2.5) with p_{ijk} given by (2.8). It is easy to see that

$$\det \tilde{A}(I,J) = \det A(I,J) \cdot \prod_{k=1}^{d} Z_k^{p_k},$$

where $p_k = \sum_{i\in I} r_{ik} - \sum_{j\in J} c_{jk} \in \mathbb{Q}$. Q.E.D.

3. Total unimodularity of dimensioned matrices

Let $\mathbb{R}$ be an integral domain [34], i.e., a commutative ring without zero divisors (and with a unit element). A matrix over $\mathbb{R}$ is said to be totally unimodular (over $\mathbb{R}$) if every nonvanishing minor of it is an invertible element of $\mathbb{R}$. We will denote by $U(\mathbb{R})$ the set of totally unimodular matrices (of any order) over $\mathbb{R}$. The significance of this concept lies in the fact that, if a matrix is totally unimodular over $\mathbb{R}$, not only its inverse but also its pivotal transforms are matrices over $\mathbb{R}$. In the canonical case of $\mathbb{R}$ being the ring of integers, the total unimodularity of incidence matrices of graphs play substantial roles in combinatorics [17].

Consider the ring, to be denoted as $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$, generated over $\mathbb{F}$ by all the formal fractional powers of $Z_1, \ldots, Z_d$:

$$\mathbb{F}\langle Z_1,\ldots,Z_d\rangle = \mathbb{F}[\{Z_1^{p_1}\ldots Z_d^{p_d} \mid p_k \in \mathbb{Q},\ k=1,\ldots,d\}]. \qquad (3.1)$$

It is easy to see that $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$ is an integral domain, whose quotient field agrees with $\mathbb{E}$ defined in (2.3). An element of $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$ is invertible iff it is of the form $\alpha \prod_{k=1}^{d} Z_k^{p_k}$ with $\alpha \in \mathbb{F}\setminus\{0\}$ and $p_k \in \mathbb{Q}$ for $k=1,\ldots,d$. Prop. 2.2, combined with this observation, implies that a dimensioned matrix is totally unimodular over $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$.

Proposition 3.1. $D(\mathbb{F};Z_1,\ldots,Z_d) \subset U(\mathbb{F}\langle Z_1,\ldots,Z_d\rangle)$. |≡|

Moreover, these two classes of matrices coincide with each other, as stated in Theorem 3.2 below, which, coupled with Prop. 2.1, also provides us with a concrete representation of a totally unimodular matrix over $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$.

Theorem 3.2. $D(\mathbb{F};Z_1,\ldots,Z_d) = U(\mathbb{F}\langle Z_1,\ldots,Z_d\rangle)$.

(Proof) In view of Prop. 3.1, it suffices to show that every totally unimodular matrix over $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$ is a dimensioned matrix. Furthermore, the proof can be reduced to the case of d=1 by induction on d, and the present theorem follows from Prop. 3.3 below. Q.E.D.

Proposition 3.3. Let Z be an indeterminate over $\mathbb{F}$ and $\tilde{A}=(\tilde{a}_{ij})$ be an m by n totally unimodular matrix over $\mathbb{F}\langle Z\rangle$, i.e., $\tilde{A} \in U(\mathbb{F}\langle Z\rangle)$. Then there exist $a_{ij} \in \mathbb{F}$, $r_i \in \mathbb{Q}$ and $c_j \in \mathbb{Q}$ ($i=1,\ldots,m$; $j=1,\ldots,n$) such that

$$\tilde{a}_{ij} = a_{ij}\, Z^{r_i-c_j}. \tag{3.2}$$

(Proof) By definition, $\tilde{a}_{ij}$ can be expressed as

$$\tilde{a}_{ij} = a_{ij}\, Z^{p_{ij}} \quad (a_{ij} \in \mathbb{F},\ p_{ij} \in \mathbb{Q}). \tag{3.3}$$

Consider the bipartite graph G(V,E) associated with $\tilde{A}$; the vertex set V ($|V|=m+n$) is the union of the row and column sets of $\tilde{A}$, and the edge set $E = \{(i,j) \mid \tilde{a}_{ij} \neq 0\}$. As will be evident from the subsequent arguments, we may assume without essential loss of generality that G is connected.

Fix a spanning tree T on G arbitrarily. Since T contains no cycles, we can find rational numbers r_i ($i=1,\ldots,m$) and c_j ($j=1,\ldots,n$) associated with the vertices of G such that

$$p_{ij} = r_i - c_j \tag{3.4}$$

for each edge (i,j) in T. If (3.4) holds true for every edge of G, (3.2) follows from (3.3) and (3.4).

Suppose that

$$B = \{(i,j)\in E \mid p_{ij} \neq r_i-c_j\}$$

is nonempty. By the construction, we have

$$B \cap T = \emptyset. \tag{3.5}$$

For each edge $(i,j) \in B$, we denote by $C(i,j)$ the set of all the circuits composed of that edge (i,j) and some other edges in $E \setminus B$. Note here that $C(i,j)$ is nonempty for each $(i,j) \in B$, since each edge in B is a cotree edge by (3.5) and the corresponding fundamental circuit belongs to $C(i,j)$, again by (3.5). Therefore

$$C = \cup \{ C(i,j) \mid (i,j) \in B \}$$

is nonempty if B is nonempty.

From among the circuits in C, choose the one, say C_0, having minimal number of edges. Let $i_1, j_1, i_2, j_2, \ldots, i_s, j_s (=j_0)$ be the sequence of vertices lying on C_0 and assume that $(i_1, j_1) \in B$ and that $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_s\}$ are subsets of the row-set and the column-set, respectively.

The minimal circuit C_0 has no chord, that is, $(i_p, j_q) \in E$ implies $p-q \equiv 0$ or 1 (mod s), since otherwise we are led to a contradiction to its minimality in either case where the chord belongs to B or to $E \setminus B$. Hence the determinant of the submatrix $\tilde{A}(I,J)$ is equal, up to a sign, to

$$\Delta = \prod_{r=1}^{s} \tilde{a}_{i_r j_r} + (-1)^{s-1} \prod_{r=1}^{s} \tilde{a}_{i_r j_{r-1}} = \alpha\, Z^{p+\delta} + \beta\, Z^{p},$$

where

$$\alpha = \prod_{r=1}^{s} a_{i_r j_r} \neq 0 \quad (\in \mathbb{F}),$$

$$\beta = (-1)^{s-1} \prod_{r=1}^{s} a_{i_r j_{r-1}} \neq 0 \quad (\in \mathbb{F}),$$

$$p = \sum_{r=1}^{s} r_{i_r} - \sum_{r=1}^{s} c_{j_r} \quad (\in \mathbb{Q}),$$

$$\delta = p_{i_1 j_1} - (r_{i_1} - c_{j_1}) \neq 0 \quad (\in \mathbb{Q}).$$

This contradicts the total unimodularity of $\tilde{A}$ since Δ is not invertible in $\mathbb{F}\langle Z\rangle$. Therefore B is empty and the proposition is established. Q.E.D.

The following properties of totally unimodular matrices over $F\langle Z_1,\ldots,Z_d\rangle$ are mentioned as direct consequences of Theorem 3.2.

Proposition 3.4. Put $\mathcal{U} = \mathcal{U}(\mathbb{F}\langle Z_1,\ldots,Z_d\rangle)$ for short.

(1) If $A, B \in \mathcal{U}$ (and the product AB can be defined), then there exists a diagonal matrix $D \in \mathcal{U}$ such that $ADB \in \mathcal{U}$.

(2) A square matrix $A \in \mathcal{U}$ can be decomposed, with suitable permutations of rows and columns, into LU factors, both belonging to $\mathcal{U}$; that is, there exist permutation matrices P_r, P_c, and a lower triangular matrix $L \in \mathcal{U}$, and an upper triangular matrix $U \in \mathcal{U}$ such that $P_r A P_c = LU$. |≡|

4. Physical dimensions in structural analysis

This section is devoted to demonstrating how the concept of physical dimensions and the total unimodularity of dimensioned matrices can be incorporated in the structural analysis of large-scale physical/engineering systems. We will first mention a rather obvious use of dimensions in Section 4.1 and then show more sophisticated connections with mixed matrices.

4.1. Check for dimensional consistency

When we are given a system (2.1) of equations that is supposed to represent a physical system, we can sometimes detect errors in its description by verifying the condition (2.8) for dimensional homogeneity. In case the dimensions r_{ik} and c_{jk} associated respectively with the rows and the columns of the matrix are known along with p_{ijk}'s, the test for (2.8) is straightforward. Even in the case where only p_{ijk}'s are given without information about r_{ik} and c_{jk}, we can efficiently decide whether (2.8) can be satisfied for some suitable r_{ik} and c_{jk}: Just as in the proof of Prop. 3.3, we consider a tree in the bipartite graph associated with A, and setting r_{ik} and c_{jk} so that (2.8) may be satisfied for tree edges, we check (2.8) for cotree edges.

4.2. Description of physical systems by mixed matrices

It is observed in [25] that two different kinds can be distinguished among the nonvanishing numbers characterizing physical/engineering systems, and a mathematical framework is provided there for dealing with the two kinds of numbers in systems analysis. The method developed in [25] is summarized in this subsection.

The two kinds of numbers to be distinguished from each other are as follows: (i) the numbers of one kind are accurate numbers such as incidence coefficients in electric networks, stoichiometric coefficients in chemical reactions, and coefficients appearing in some other conservation laws, and (ii) the numbers of the other kind are inaccurate and independent numbers such as resistances in electric networks and velocity constants of chemical reactions, so that they cannot be expected to be subject to any exact mathematical relations among themselves. See [25] for more detailed discussions on the distinction of numbers.

The physical consideration above is cast into a mathematical formalism as follows. Besides the field $\mathbb{F}$ to which the entries of A of (2.1) belong, we consider a subfield $\mathbb{K}$:

$$\mathbb{Q} \subset \mathbb{K} \subset \mathbb{F} , \qquad (4.1)$$

and regard the numbers in $\mathbb{K}$ as being accurate and the others as being inaccurate. Then the matrix A is decomposed as

$$A = T_A + Q_A, \qquad (4.2)$$

in such a way that the nonvanishing entries of T_A are in $\mathbb{F}\backslash\mathbb{K}$ and Q_A is a matrix over the subfield $\mathbb{K}$. If, in addition, the collection of the nonvanishing entries of T_A is algebraically independent [34] over $\mathbb{K}$, the matrix A (together with the expression (4.2)) is called a mixed matrix with respect to $\mathbb{K}$.

In dealing with structural aspects of a physical system, the first important step is to choose an appropriate mathematical description of it. A description by a collection of elementary relations among elementary variables is usually superior in this respect to a compact sophisticated representation. If a real-world system is expressed as in

(2.1) using elementary variables, it is often justified, as claimed in [25], to assume that the entries of A not belonging to $\mathbb{Q}$ are mutually algebraically independent transcendentals over $\mathbb{Q}$. (This assumption is labelled as "GA2" in [25].) Then the coefficient matrix A can be expressed as a mixed matrix with respect to $\mathbb{Q}$. Note that the generality assumption GA2 refers to a mathematical expression of a physical system, but not the system itself.

The rank r(A) of a mixed matrix $A=T_A+Q_A$ with respect to $\mathbb{K}$ is expressed in terms of two matroids, $M(T_A)$ and $M(Q_A)$, both defined on the union of the rows R and columns C of A as follows.

Definition 4.1. For a matrix G over a field with the row-set R and the column-set C, M(G) denotes the matroid [35] defined on the set of columns, identified with $R \cup C$, of the matrix $[U|G]$ with respect to linear dependence of column vectors, where U is the unit matrix.

In the following, $M(Q_A)^*$ is the dual of $M(Q_A)$, and $M(T_A) \vee M(Q_A)$ the union of $M(T_A)$ and $M(Q_A)$ [35]; $t(T_A(I,J))$ is the term-rank [26] of the submatrix of T_A corresponding to rows in I ($\subset R$) and columns in J ($\subset C$). Note that $M(Q_A)^*$ agrees with the linear matroid defined on the set of rows of the matrix $\binom{U}{Q_A}$.

Proposition 4.1 ([25, Theorems 5.1, 5.3). For a mixed matrix $A=T_A+Q_A$,

$$r(A) = \max\{t(T_A(R\setminus I,J))+r(Q_A(I,C\setminus J)) \mid I\subset R, J\subset C\} \qquad (4.3)$$

$$= \mathrm{rank}(M(T_A) \vee M(Q_A)) - |R| \qquad (4.4)$$

$$= \text{maximum size of a common independent set of } M(T_A) \text{ and } M(Q_A)^*. \ |\Xi|$$

Based on this characterization and using the established algorithms [3], [11], [32], we can determine the rank of a matrix A over $\mathbb{F}$ with arithmetic operations in the subfield $\mathbb{K}$, as well as with graph manipulations, since $M(Q_A)^*$ is represented over the subfield $\mathbb{K}$ whereas $M(T_A)$ is a transversal matroid [35] by virtue of the algebraic independence of the nonvanishing entries of T_A. See [25] for the detailed description of the combinatorial algorithm for computing $r(A)$ by this characterization.

Example 4.1 (Electric network)

Consider the simple electric network shown in Fig. 4.1, which consists of five elements; two resistors of conductances g_i (branch i) (i=1,2), a voltage source (branch 3) controlled by the voltage across branch 1, a current source (branch 4) controlled by the current in branch 2, and an independent voltage source of voltage e (branch 5). Then the current ξ^i in and the voltage η_i across branch i (i=1,...,5) are to satisfy the structural equations (Kirchhoff's laws) and the constitutive equations, which altogether are expressed as

$$
\left[\begin{array}{ccccc|ccccc}
0 & -1 & 1 & & & & & & & \\
1 & 1 & & 1 & & & & & & \\
-1 & & & & 1 & & & & & \\
\hdashline
 & & & & & 1 & & & -1 & 1 \\
 & & & & & & 1 & 1 & -1 & \\
\hline
-1 & & & & & g_1 & & & & \\
 & -1 & & & & & g_2 & & & \\
 & & 0 & & & \alpha & & -1 & & \\
 & \beta & & -1 & & & & & 0 & \\
 & & & & 0 & & & & & -1
\end{array}\right]
\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \\ \xi^5 \\ \hline \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ e \end{pmatrix} . \qquad (4.5)
$$

The upper five equations of (4.5) are the structural equations, while the remaining five the constitutive equations.

The values of the physical parameters g_1, g_2, α and β are inaccurate numbers which are only approximately equal to their nominal values on account of various kinds of noises and errors. Therefore, we may consider them algebraically independent transcendentals in $\mathbb{F} = \mathbb{Q}(g_1,g_2,\alpha,\beta)$ over $\mathbb{Q}$. Then the coefficient matrix A of (4.5) can be expressed in an obvious way as a mixed matrix with respect to $\mathbb{K}=\mathbb{Q}$ by taking $\{g_1,g_2,\alpha,\beta\}$ for the nonvanishing entries of T_A.

The unique solvability of this network amounts to the nonsingularity of A, which can be checked efficiently based on Prop. 4.1, as explained above. (This is trivially equivalent to the well-established method in electric network theory [9], [10], [13], [27], [28], [29], [30], [32].) If we calculate the determinant of A directly, we obtain

$$\det A = -g_1 - (1-\alpha)(1+\beta)g_2,$$

which is distinct from zero by the algebraic independence of the physical parameters, and hence this network is uniquely solvable under the assumption on generality. |≡|

Time-invariant dynamical systems can also be treated by means of mixed matrices. Consider a control system expressed in the "descriptor form" [19], [20], [37] (sometimes also called "intermediate standard form" [6], [15]):

$$F\, dx/dt = A\, x + B\, u, \tag{4.6}$$

where x and u are the descriptor-vector (standing for internal variables) and the input-vector, respectively, and F, A and B are constant matrices. In case F is nonsingular, (4.6) could be tranformed to Kalman's standard form:

$$dx/dt = F^{-1}A\, x + F^{-1}B\, u, \tag{4.7}$$

but (4.6) is more elementary and hence more suitable for the structural analysis than (4.7) from the combinatorial point of view.

Suppose that F, A and B of (4.6) are matrices over $\mathbb{F}$ ($\supset \mathbb{Q}$) and that their entries which do not belong to $\mathbb{Q}$ are collectively algebraically independent over $\mathbb{Q}$. This is equivalent to saying that the composite matrix $[F|A|B]$ is a mixed matrix with respect to $\mathbb{Q}$:

$$[F|A|B] = [T_F|T_A|T_B] + [Q_F|Q_A|Q_B]. \qquad (4.8)$$

If the system (4.6) is written in the Laplace transform, we have

$$[A-sF \mid B] \binom{x}{u} = 0, \qquad (4.9)$$

where s is a symbol (or an indeterminate over $\mathbb{F}$) standing for the differentiation with respect to time. Then the coefficient matrix

$$D = [A-sF \mid B], \qquad (4.10)$$

regarded as a matrix over $\mathbb{F}(s)$, is again a mixed matrix with respect to $\mathbb{K} = \mathbb{Q}(s)$, since it is expressed as

$$D = T_D + Q_D \qquad (4.11)$$

with

$$T_D = [T_A-sT_F \mid T_B], \quad Q_D = [Q_A-sQ_F \mid Q_B], \qquad (4.12)$$

and the set of the nonvanishing entries of T_D is algebraically independent over $\mathbb{Q}(s)$. The matrix

$$E = A - sF \qquad (4.13)$$

can also be expressed as a mixed matrix with respect to $\mathbb{K} = \mathbb{Q}(s)$ in a similar manner.

Example 4.2 (Mechanical system).

Consider the mechanical system in Fig. 4.2 consisting of two masses m_1 and m_2, two springs k_1 and k_2, and a damper f; u is the force exerted from outside. We may choose $x=(x_1,x_2,x_3,x_4,x_5,x_6)$ as the descriptor-vector, where x_1 (x_2) is the displacement of mass m_1 (m_2) and x_3 (x_4) is its velocity, x_5 is the force by the damper f, and x_6 is the relative velocity of the two masses. Then the system can be expressed in the descriptor form (4.6) with

$$F = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & m_1 & & & \\ & & & m_2 & & \\ & & & & 0 & \\ 1 & -1 & & & & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & & 1 & & & \\ & 0 & & 1 & & \\ -k_1 & & 0 & & -1 & \\ & -k_2 & & 0 & 1 & \\ & & & & -1 & f \\ & & & & & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{4.14}$$

If we regard $\{m_1,m_2,k_1,k_2,f\}$ as being algebraically independent, the matrices T_D and Q_D of (4.12) are given by

$$T_D = [T_E | T_B] = \left[\begin{array}{cccccc|c} 0 & & & & & & 0 \\ & 0 & & & & & 0 \\ -k_1 & & -sm_1 & & & & 0 \\ & -k_2 & & -sm_2 & & & 0 \\ & & & & 0 & f & 0 \\ & & & & & 0 & 0 \end{array}\right] , \qquad (4.15)$$

$$Q_D = [Q_E | Q_B] = \left[\begin{array}{cccccc|c} -s & & 1 & & & & 0 \\ & -s & & 1 & & & 0 \\ & & 0 & & -1 & & 1 \\ & & & 0 & 1 & & 0 \\ & & & & -1 & & 0 \\ -s & s & & & & 1 & 0 \end{array}\right] . \qquad (4.16)$$

4.3. Physically-dimensioned mixed matrices

We have already introduced two concepts on matrices that we encounter in the description of real systems, namely the dimensioned matrix and the mixed matrix. The former is motivated by the dimensional analysis while the latter by the structural analysis. Now these two are combined to yield some useful consequences in the analysis of dynamical systems.

As has been discussed in Section 2, when we describe a physical system in the form of (2.1) with a matrix A over $\mathbb{F}$, we usually know the physical dimensions associated with its rows and columns. Then we can determine the dimensioned matrix $\tilde{A}$ over $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$ that corresponds to A by (2.5) and (2.8) (see (3.1) for the definition of $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$). We call $\tilde{A}$ determined in this way the <u>dimensioned matrix corresponding to</u> A (with the implicit understanding of the given physical dimensions). Conversely, we call A the <u>numerical matrix corresponding to</u> $\tilde{A}$. By Prop. 2.1, this correspondence between numerical matrices over $\mathbb{F}$ and dimensioned matrices over $\mathbb{F}\langle Z_1,\ldots,Z_d\rangle$ are given by

$$\tilde{A} = D_r\ A\ D_c^{-1}, \tag{4.17}$$

where D_r and D_c are the known diagonal matrices of (2.9) and (2.10) representing the physical dimensions of the rows (equations) and the columns (variables).

When A is a mixed matrix of the form (4.2): $A = T_A + Q_A$ with respect to a subfield $\mathbb{K}$ of $\mathbb{F}$, we can express the corresponding dimensioned matrix $\tilde{A}$ of (4.17) as

$$\tilde{A} = T_{\tilde{A}} + Q_{\tilde{A}} \tag{4.18}$$

with

$$T_{\tilde{A}} = \tilde{T}_A = D_r\, T_A\, D_c^{-1}, \tag{4.19}$$

$$Q_{\tilde{A}} = \tilde{Q}_A = D_r\, Q_A\, D_c^{-1}. \tag{4.20}$$

This shows that $\tilde{A}$ is also a mixed matrix, but with respect to (the quotient field of) $\mathbb{K}\langle Z_1,\ldots,Z_d\rangle$. In particular, $\tilde{Q}_A$ is a matrix over $\mathbb{K}\langle Z_1,\ldots,Z_d\rangle$. Note also that the matrices $T_{\tilde{A}}$ and $Q_{\tilde{A}}$ constituting the mixed matrix $\tilde{A}$ coincide with the dimensioned matrices $\tilde{T}_A$ and $\tilde{Q}_A$ corresponding to T_A and Q_A of (4.2), respectively.

The physical observation to be made here concerns with the physical dimensions of the nonvanishing entries of Q_A. When we regard physical parameters, such as $\{g_1,g_2,\alpha,\beta\}$ in Example 4.1 and $\{m_1,m_2,k_1,k_2,f\}$ in Example 4.2, as algebraically independent transcendentals over $\mathbb{Q}$ (assuming GA2), the matrix Q_A then represents various kinds of conservation laws or structural equations, and consists of incidence coefficients such as those induced from the underlying topological/geometrical incidence relations in electric networks and the stoichiometric coefficients in chemical reactions. Thus it is natural to expect that the nonvanishing entries of Q_A (i.e., the accurate numbers) are dimensionless. In fact, this is the case both with the coefficient matrix (4.5) of Example 4.1 and with the matrices F, A and B of (4.14) of Example 4.2.

The above physical observation can be stated in algebraic terms as follows. Let $A = T_A+Q_A$ $(Q_A=(q_{ij}))$ be a mixed matrix with respect to $\mathbb{K}$ and $\tilde{A}$ be the corresponding dimensioned matrix expressed as (2.5) with exponents p_{ijk} of dimensions. The condition that the nonvanishing entries of Q_A are dimensionless is equivalent to:

$$q_{ij} \neq 0 \text{ implies } p_{ijk} = 0 \text{ for } k=1,\ldots,d; \tag{4.21}$$

or alternatively,

$$D_r \, Q_A \, D_c^{-1} = Q_A \tag{4.22}$$

with reference to (4.20). The condition (4.21) or (4.22) does not exclude dimensionless nonvanishing entries from T_A; in Example 4.1, the parameters α and β in T_A are dimensionless.

Now we are in the position to introduce the concept of physical matrix as a mathematical model of the matrices that we encounter in real-world systems. It reflects the dual viewpoint from structural analysis and dimensional analysis. Suppose a matrix A over $\mathbb{F}$ is given along with a pair (D_r, D_c) of diagonal matrices of the forms (2.9) and (2.10), respectively.

Definition 4.2. We will say that A is a <u>physical</u> <u>matrix</u> with respect to $(\mathbb{K}; D_r, D_c)$, where $\mathbb{K} \subset \mathbb{F}$, if

(i) $A = T_A + Q_A$ is a mixed matrix with respect to $\mathbb{K}$, and

(ii) $D_r \, Q_A \, D_c^{-1} = Q_A$.

When (D_r, D_c) is understood, we say simply that A is a physical matrix with respect to $\mathbb{K}$. A dimensioned matrix $\tilde{A}$ that corresponds to a physical matrix A by (4.17) will also be called a physical matrix.

4.4. Physical matrices in dynamical systems

We will investigate the Laplace transform (4.9) of a dynamical system by means of the concept of physical matrix introduced above. Consider the matrix $D = [A-sF \mid B]$ of (4.10) and let (D_r, D_c) be the pair of matrices of (2.9) and (2.10) representing the physical dimensions of D, where we may assume that time is chosen as one of the fundamental dimensions, say Z_1. D_c is decomposed in two parts as $D_c = \mathrm{diag}(D_x, D_u)$ with D_x accounting for the dimensions of x and D_u for those of u. Noting that the symbol s should have the dimension of Z_1^{-1} (the inverse of time) since it represents "d/dt" (the differentiation with respect to time), we see that the dimensions associated with F, A and B are given by $(D_r, Z_1^{-1}D_x)$, (D_r, D_x) and (D_r, D_u), respectively. Then the dimensioned matrix $\tilde{D}$ corresponding to D is given by

$$\begin{aligned}\tilde{D} &= D_r\ [A-sF \mid B]\ D_c^{-1} \\ &= [\tilde{A}-sZ_1^{-1}\tilde{F} \mid \tilde{B}]\end{aligned} \qquad (4.23)$$

in terms of the dimensioned matrices corresponding to F, A and B.

Suppose that F, A and B are physical matrices with respect to $(\mathbb{K}; D_r, Z_1^{-1}D_x)$, $(\mathbb{K}; D_r, D_x)$ and $(\mathbb{K}; D_r, D_u)$, respectively, expressed as in Def. 4.2. Accordingly, $\tilde{D}$ of (4.23) is expressed as

$$\tilde{D} = T_{\tilde{D}} + Q_{\tilde{D}},$$

where

$$\begin{aligned}T_{\tilde{D}} &= \tilde{T}_D = [\tilde{T}_A - sZ_1^{-1}\tilde{T}_F \mid \tilde{T}_B], \\ Q_{\tilde{D}} &= \tilde{Q}_D = [\tilde{Q}_A - sZ_1^{-1}\tilde{Q}_F \mid \tilde{Q}_B].\end{aligned} \qquad (4.24)$$

Since F, A and B are physical matrices, we see that $\tilde{Q}_D$ of (4.24) is simplified to

$$\tilde{Q}_D = [Q_A - sZ_1^{-1}Q_F \mid Q_B], \qquad (4.25)$$

which may be regarded as a dimensioned matrix over $\mathbb{K}(s)\langle Z_1\rangle$.

Then it follows from the total unimodularity of $\tilde{Q}_D$ (Theorem 3.2) that every minor of Q_D of (4.12) is a monomial in s over $\mathbb{K}$, as shown below.

Theorem 4.2. Let $Q_D = [Q_A - sQ_F \mid Q_B]$ be defined by (4.11) and (4.12), where F, A and B are physical matrices with respect to $\mathbb{K}$, as above. Then each minor of Q_D is of the form αs^p with $\alpha \in \mathbb{K}$ and p a nonnegative integer. In particular, $Q_D \in U(\mathbb{K}\langle s\rangle)$.

(Proof) By Theorem 3.2 (or by Prop.2.2), each minor of $\tilde{Q}_D$ of (4.25) is of the form $\Delta=\beta Z_1^{-p}$ with $\beta \in \mathbb{K}(s)$ and $p \in \mathbb{Q}$. Since each entry of $\tilde{Q}_D$ is either of the form α ($\in \mathbb{K}$) or of the form $\alpha s Z_1^{-1}$ ($\alpha \in \mathbb{K}$), we immediately see that p is a nonnegative integer and that $\beta = \alpha s^p$ for some $\alpha \in \mathbb{K}$. That is, $\Delta=\alpha s^p Z_1^{-p}$. Noting that Q_D is obtained from $\tilde{Q}_D$ by setting $Z_1=1$ (cf. (4.12) and (4.25)), we see that the corresponding minor of Q_D is equal to αs^p. Q.E.D.

In view of Prop. 4.3 below, this theorem has a significant implication in actual applications in that it means a considerable reduction in the computational complexity of finding the rank of the mixed matrix D of (4.11) (or E of (4.13)) by way of the combinatorial characterization given in Prop. 4.1. See Section 5 for details.

Proposition 4.3. Let Q be a matrix over $\mathbb{K}[s]$ (the polynomial ring in s over a field $\mathbb{K}$) and M(Q) be the associated matroid (cf. Def. 4.1) when Q is regarded as a matrix over $\mathbb{K}(s)$. If each minor of Q is a monomial in s over $\mathbb{K}$, $M(Q)$ is representable over $\mathbb{K}$. Specifically, on expressing as $Q = (\alpha_{ij}s^{p_{ij}})$ with $\alpha_{ij} \in K$ and $p_{ij} \in Z^+$, we have $M(Q)=M(Q_0)$, where $Q_0=(\alpha_{ij})$. |≡|

Example 4.2 (continued).

The physical dimensions associated with $D = T_D+Q_D$ consisting of (4.15) and (4.16) are expressed by the pair (D_r,D_c) given by

$$D_r=\mathrm{diag}(T^{-1}L,T^{-1}L,T^{-2}LM,T^{-2}LM,T^{-2}LM,T^{-1}L), \qquad (4.26)$$

$$D_c=\mathrm{diag}(L,L,T^{-1}L,T^{-1}L,T^{-2}LM,T^{-1}L,T^{-2}LM), \qquad (4.27)$$

where $T=Z_1$=time, $L=Z_2$=length and $M=Z_3$=mass. In this example, any minor of Q_D of (4.16) can easily be identified as a monomial in s over $\mathbb{Q}$. |≡|

5. Structural analysis of dynamical systems

In this section, we deal with some fundamental problems on the dynamical system in the descriptor form (4.6): $F dx/dt = Ax + Bu$. As discussed in Section 4, we may assume that the nonvanishing entries of F, A and B which do not belong to $\mathbb{Q}$ are collectively algebraically independent over $\mathbb{Q}$, so that the matrix $D = [A-sF \mid B]$ of (4.10) can be written as (4.11): $D = T_D + Q_D$, which is a mixed matrix with respect to $\mathbb{Q}(s)$. Furthermore, we may assume that F, A and B are physical matrices with respect to $\mathbb{Q}$; then, by Theorem 4.2, each minor of $Q_D = [Q_A-sQ_F \mid Q_B]$ of (4.12) is a monomial in s over $\mathbb{Q}$.

Thus, we will assume throughout this section that, D of (4.10) is written as $D = T_D + Q_D$, where

(i) the collection $\mathcal{T}$ of the nonvanishing entries of T_D are algebraically independent over $\mathbb{Q}(s)$, (5.1)

and

(ii) every minor of $Q_D = [Q_A-sQ_F \mid Q_B]$ is a monomial in s over $\mathbb{Q}$. (5.2)

5.1. Solvability of a descriptor system

The problem of determining whether the pencil [5] $E = A-sF$ is regular or not, i.e., whether $\det(A-sF)=0$ or not, is of fundamental importance in connection with the unique solvablility [33], [37] (or consistency [6]) of the dynamical system (4.6) or of its discrete counterpart [20].

By (5.1), E is a mixed matrix with respect to $\mathbb{Q}(s)$, as noted in (4.13), i.e.,

$$E = (T_A-sT_F) + (Q_A-sQ_F). \qquad (5.3)$$

The rank of E is then characterized by Prop. 4.1 in terms of a transversal matroid $M(T_A-sT_F) = M(T_A-T_F)$ and a linear matroid $M(Q_A-sQ_F)$ over $\mathbb{Q}(s)$. It is noteworthy that the latter is representable over $\mathbb{Q}$, by (5.2) and Prop. 4.3, as

$$M(Q_A-sQ_F) = M(Q_A-Q_F), \tag{5.4}$$

so that the rank of E can be found using the combinatorial algorithm, to be explained in Section 5.3 in some detail, with arithmetic operations in $\mathbb{Q}$ without involving the indeterminate s. In this way, r(E) is computed with $O(n^4)$ arithmetic operations in $\mathbb{Q}$, where n is the size of E. Note that Theorem 4.2 plays a crucial role in reducing the computational complexity to such a extent that regularity of E can be determined with practicable amount of computation.

5.2. Controllability of a descriptor system

The initiating work [18] on structural controllability has motivated many subsequent refinements (see, e.g., [1], [21], [22], [23], [24], [31] and the references therein). This subsection is a brief summary of the results obtained in [22], [23], giving a combinatorial characterization of the controllability under the present setting of (5.1) and (5.2).

In the following, we assume that F, A and B of (4.6) are real matrices satisfying (5.1) and (5.2), and that F is a nonsingular n by n matrix. In this case, (4.6) can be reduced to the standard form (4.8) and the controllability of (4.6) may be defined for the reduced system in the ordinary sense. Note also that $\det(A-sF)\neq 0$, where s is regarded as an indeterminate. It is well known that the system (4.6) is controllable iff

$$\mathrm{rank}([A-\lambda F \mid B]) = n \tag{5.5}$$

for any complex number $\lambda \in \mathbb{C}$.

By (5.1) and (5.2), we see [22] that any nonzero eigenmode is a transcendental over $\mathbb{Q}$, as stated below.

Proposition 5.1. Assume (5.1) and (5.2), and that F is nonsingular. If $\det(A-\lambda F)=0$, then either $\lambda=0$ or λ is transcendental over $\mathbb{Q}$.

(Proof) Suppose $\lambda\neq 0$ and λ is algebraic. Then $\det(A-\lambda F)$ cannot vanish since it is a nontrivial polynomial (due to nonsingularity of F) in the transcendental numbers in T of (5.1) with algebraic coefficients. Q.E.D.

Based on Prop. 5.1, it can be shown [22], [23] by using the exchange property concerning algebraic independence that (5.5) is equivalent to the conditions:

$$\operatorname{rank}([A \mid B]) = n \tag{5.6}$$

and

$$\operatorname{rank}(D^j) = n \quad \text{for } j=1,\dots,n, \tag{5.7}$$

where D^j is the submatrix of D of (4.10) which is obtained from D by deleting the j-th column. Note that, for $j=1,\dots,n$,

$$D^j = [A^j - sF^j \mid B] \tag{5.8}$$

is a matrix over Q(s), where A^j and F^j are submatrices of A and F, respectively, defined in a similar way.

By Prop. 4.1, the first condition (5.6) is equivalent to

$$\operatorname{rank}(M([T_A|T_B]) \vee M([Q_A|Q_B])) = 2n. \tag{5.9}$$

The second condition (5.7) can also be put in a more convenient form as follows. First note that, for a matrix G in general, we have

$$M(G^j) = M(G)\backslash\{j\}, \tag{5.10}$$

where the left-hand side is the matroid determined by Def. 4.1 by the submatrix of G with the j-th column deleted, and the right-hand side is the matroid minor of M(G) obtained from it by deleting the element j (corresponding to the j-th column). By Prop. 4.1, (5.10) and the fact that the operations of union and deletion commute, we have for $j=1,\dots,n$:

$$\begin{aligned} r(D^j) &= \operatorname{rank}(\ M(T_D{}^j) \vee M(Q_D{}^j)\ ) - n \\ &= \operatorname{rank}(\ (M(T_D)\backslash\{j\}) \vee (M(Q_D)\backslash\{j\})\ ) - n \\ &= \operatorname{rank}(\ (M(T_D) \vee M(Q_D))\backslash\{j\}\ ) - n \\ &= \operatorname{rank}(\ (M([T_A - sT_F|T_B]) \vee M([Q_A - sQ_F|Q_B]))\backslash\{j\}\ ) - n. \end{aligned} \tag{5.11}$$

Furthermore, it follows from (5.2) (cf. (5.4)) that

$$r(D^j) = \operatorname{rank}(\ (M([T_A - T_F|T_B]) \vee M([Q_A - Q_F|Q_B]))\backslash\{j\}\ ) - n \tag{5.12}$$

for j=1,...,n. On substituting (5.12) into (5.7), we have a combinatorial restatement of (5.7):

$$\text{rank}(\ (M([T_A-T_F|T_B]) \vee M([Q_A-Q_F|Q_B])) \setminus \{j\}\ ) = 2n \quad \text{for } j=1,\ldots,n, \qquad (5.13)$$

which says that none of the columns corresponding to the descriptor variables x is a coloop [35] in the union matroid.

The controllability condition is now given.

Theorem 5.2. Assume (5.1) and (5.2), and that F is nonsingular. Then, the descriptor system (4.6) is controllable iff

(5.9): $\text{rank}(M([T_A|T_B]) \vee M([Q_A|Q_B])) = 2n$

and

(5.13): $\text{rank}((M([T_A-T_F|T_B]) \vee M([Q_A-Q_F|Q_B])) \setminus \{j\}) = 2n$ for j=1,...,n. |≡|

The first condition (5.9), as well as the nonsingularity of F, can be verified with $O(n^4)$ arithmetic operations in $\mathbb{Q}$ by a combinatorial algorithm based on Prop. 4.1. If the second condition (5.13) is checked for each j separately, it can be done with $O(n^4(n+r))$ arithmetic operations in $\mathbb{Q}$, where r is the number of inputs, i.e., the number of columns of B. It has been pointed out, however [22, Theorem 25.2], [23, Theorem 3], that (5.13) can be rephrased into a condition with reference to the principal partition [9], [11] with respect to the pair of matroids so that it can be verified with $O(n^3(n+r))$ operations in $\mathbb{Q}$.

5.3. Dynamical degree of a descriptor system

Suppose that the descriptor system (4.6) is solvable (cf. Section 5.1), i.e.,

$$\det (A-sF) \neq 0. \tag{5.14}$$

The degree of det(A-sF), to be denoted as $\delta(\det(A-sF))$, is one of the fundamental characteristics to the dynamical behavior of the system, since it represents the number of independent state-variables, or the dynamical degrees of freedom [6], [33]. Following [6], we call it the <u>dynamical degree</u> of (4.6) and denote it by dd(F,A), i.e.,

$$dd(F,A) = \delta(\det(A-sF)). \tag{5.15}$$

The obvious relation

$$dd(F,A) \leq \text{rank } F \leq n \tag{5.16}$$

may be noted, where n is the size of F. We are mainly interested in a singular F, since otherwise dd(F,A) is trivially equal to n.

In the special, but important, case of electric networks, the dynamical degree of (5.15) agrees with what is known as the order of complexity in network theory [6], [10], [13], [27], [28], [29]. The problem of determinimg the order of complexity has been settled [10], [12], [13] for its most general form with mutual couplings, under the generality assumption that the element characteristics are inaccurate numbers which are algebraically independent; it is formulated as a combinatorial optimization problem of independent assignment, and a practical matroid-theoretic algorithm for it is known.

This subsection formulates the problem of determining the dynamical degree dd(F,A) by means of the independent-linkage problem [4], [8], [11], which is a generalization of the independent-assignment problem.

The present assumption (5.2) on the accurate numbers are trivially met by the standard descriptions (with voltages and currents as variables) of the electric networks treated in e.g. [10], [12], [13], [27], [28], [29], [30], since Q_D represents the structural equations (Kirchhoff's laws) being free from the symbol s (cf. (4.5) of Example 4.1). Thus the subsequent result may be regarded as a direct extension of the previously known results on the order of complexity of networks.

First notice the following, where the assumption (5.2) is not needed.

Proposition 5.3. Let $[F|A] = [T_F|T_A] + [Q_F|Q_A]$ be a mixed matrix (with respect to a field $\mathbb{K}$), and put $E=A-sF$, $T_E=T_A-sT_F$ and $Q_E=Q_A-sQ_F$. Then

$$\delta(\det(E)) = \max\{\delta(\det(T_E(R\backslash I,J)))+\delta(\det(Q_E(I,C\backslash J))) \mid I\subset R, J\subset C\}, \quad (5.17)$$

where $\delta(\cdot)$ is the degree of a polynomial in s and $\delta(0)=-\infty$.

(Proof) As noted in (4.13) (cf. also (5.3)), $E = T_E + Q_E$ is a mixed matrix with respect to $\mathbb{K}(s)$. Then (5.17) is easy to establish from the proof (cf. [25]) of Prop. 4.1. Q.E.D.

Next we will mold Prop. 5.3 into a more tractable combinatorial form using (5.2). The first term of (5.17)

$$\delta_T(I,J) = \delta(\det(T_E(R\backslash I,J))) \quad (5.18)$$

can be formulated in terms of the assignment problem or the weighted matching problem [17] defined as follows. Consider the bipartite graph G_T associated with the matrix $T_E=T_A-sT_F$; G_T has vertices corresponding to the rows R and the columns C of T_E, and edges corresponding to the nonvanishing entries of T_E. By the algebraic independence of the nonvanishing entries, $T_E(R\backslash I,J)$ is nonsingular iff there exists a complete matching (of size equal to $|R\backslash I|=|J|$) on $G_T|((R\backslash I)\cup J)$, the subgraph of G_T induced by the vertices of $(R\backslash I)\cup J$; it should also be noted that $T_E(R\backslash I,J)$ is nonsingular iff $I\cup J$ is a base of $M(T_E)$. We will assign to each edge (i,j) ($i\in R$, $j\in C$) of G_T a weight w_{ij} defined by

$$w_{ij} = \begin{cases} 1 & \text{if } (T_F)_{ij} \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.19}$$

Then we see that

$$\delta_T(I,J) = \text{maximum weight of a complete matching on } G_T|((R\backslash I)\cup J). \tag{5.20}$$

To handle the second term of (5.17):

$$\delta_Q(I,J) = \delta(\det(Q_E(I,C\backslash J))), \tag{5.21}$$

the property (5.2) is essential, which implies that $Q_E \in U(\mathbb{Q}\langle s\rangle)$ (cf. Theorem 4.2). Then, by Prop. 3.3, there exist such integers r_i ($i=1,\ldots,n$) and c_j ($j=1,\ldots,n$) associated with the rows and columns of Q_E that

$$(Q_E)_{ij} = (Q_{AF})_{ij}\, s^{r_i - c_j} \tag{5.22}$$

for $i,j=1,\ldots,n$, where

$$Q_{AF} = Q_A - Q_F. \tag{5.23}$$

Then, $\delta_Q(I,J)$ is expressed as

$$\delta_Q(I,J) = \sum_{i\in I} r_i - \sum_{j\in C\backslash J} c_j, \tag{5.24}$$

so long as the submatrix $Q_{AF}(I,C\backslash J)$ is nonsingular. In other words,

$$\delta_Q(I,J) = \sum_{i\in I} r_i + \sum_{j\in J} c_j - c_0 \tag{5.25}$$

if $I \cup J$ is a base of $M(Q_A - Q_F)^*$, and $\delta_Q(I,J) = -\infty$ if not, where

$$c_0 = \sum_{j \in C} c_j. \tag{5.26}$$

Note that the integers r_i's and c_j's of (5.22) can easily be found with $O(n)$ integer operations (additions and subtractions), as mentioned in Section 4.1 (see also the proof of Prop. 3.3).

With the observation above, $dd(F,A)$ can be found by solving a weighted independent-linkage problem defined as follows. Let R_T and R_Q (C_T and C_Q) be disjoint copies of the row-set R (the column-set C) of the matrix E. For each $i \in R$, the corresponding elements in R_T and R_Q are denoted by i_T ($\in R_T$) and i_Q ($\in R_Q$), respectively. This convention will apply to column indices $j \in C$, as well as to subsets of R and C; e.g., for $I \subset R$, $I_T = \{i_T \mid i \in I\} \subset R_T$. The underlying graph G has the vertex-set $R_T \cup C_T \cup R_Q \cup C_Q$ and the arc-set $A_T \cup A_R \cup A_C$, where

$$A_T = \{(i_T, j_T) \mid i \in R,\ j \in C,\ (T_E)_{ij} \neq 0\}, \tag{5.27}$$

$$A_R = \{(i_T, i_Q) \mid i \in R\}, \tag{5.28}$$

$$A_C = \{(j_T, j_Q) \mid j \in C\}, \tag{5.29}$$

The weight $w(a)$ of an arc a is defined as

$$w(a) = \begin{cases} 1 & \text{if } a \in A_T,\ a = (i_T, j_T),\ (T_F)_{ij} \neq 0, \\ 0 & \text{if } a \in A_T,\ a = (i_T, j_T),\ (T_F)_{ij} = 0, \\ r_i & \text{if } a \in A_R,\ a = (i_T, i_Q), \\ c_j & \text{if } a \in A_C,\ a = (j_T, j_Q). \end{cases} \tag{5.30}$$

Note that the weights for $a \in A_T$ are consistent with (5.19).

The set $R_Q \cup C_Q$ is regarded as being endowed with the matroid structure $M(Q_A - Q_F)^*$ with the obvious correspondence of the ground set. We also attach the free matroid to the set R_T. An <u>independent linkage</u> on

G with the entrance R_T and the exit $R_Q \cup C_Q$ will mean in this paper a set of vertex-disjoint directed paths from vertices of R_T to vertices of $R_Q \cup C_Q$ such that the initial (terminal) vertices of its paths form an independent set of the matroid attached to the entrance (exit) set. Here the matroidal constraint on the entrance set R_T is virtually void, since the free matroid is attached to it. By the size of an independent linkage is meant the number of paths in it, or equivalently, the number of the initial vertices. The weight w(L) of an independent linkage L is the sum of the weights (defined by (5.30)) of the arcs contained in the paths in L.

The observation made in [22], [25] is that

$$r(E) = \text{maximum size of an independent linkage on G}, \quad (5.31)$$

which is a restatement of Prop. 4.1. Thus, the nonsingularity (5.14) of E is expressed as:

there exists an independent linkage of size n. (5.32)

The main result of the present subsection is given below.

Theorem 5.4. Assume (5.1), (5.2) and $\det(A-sF)\neq 0$. Let the graph G be defined as above with the weight (5.30) and the matroids on the entrance and the exit. Then

$$dd(F,A)=\max\{w(L) \mid L: \text{independent linkage on G of size } n\}-c_0, \quad (5.33)$$

where c_0 is defined by (5.26).

(Proof) First note that (5.32) guarantees the existence of such an L. The identity (5.33) follows from Prop. 5.3, (5.20) and (5.25), as well as from the fact that both $T_E(R\setminus I,J)$ and $Q_E(I,C\setminus J)$ are nonsingular iff $I \cup J$ is a common base of $M(T_A-T_F)$ and $M(Q_A-Q_F)^*$. Q.E.D.

This characterization of the dynamical degree enables us to determine it by the efficient combinatorial algorithms [4], [8] for the optimal independent-linkage problem. Since $M(Q_A-Q_F)^*$ is represented over Q, the amount of arithmetic computations in Q and graph manipulations is bounded by $O(n^4)$.

By finding a minimum-weight independent linkage on G, we can also determine the lowest degree of the nonvanishing term in det(A-sF).

Example 5.1.

Recall the mechanical system of Example 4.2, which is described by the matrices (4.15) and (4.16). From (4.16) we may choose $r_1=r_2=r_6=1$, $r_3=r_4=r_5=2$, and $c_1=c_2=0$, $c_3=c_4=c_6=1$, $c_5=2$, to satisfy (5.22); in fact these values are the dimensions of the inverse of time given by (4.26) and (4.27). Fig. 5.1 shows the graph G for this problem, in which the row indices are given with asterisks and the weights of arcs are in parentheses. The maximum-weight independent linkage L, indicated by bold lines, has the weight w(L)=9, from which we obtain by Theorem 5.4: $dd(F,A)=w(L)-c_0=9-5=4$. The existence of an independent linkage of size n=6 also implies that $\det(A-sF)\neq 0$. The terminal vertices of L reveal that I={1*,2*,5*,6*} and J={3,4} attain the maximum in (5.17). |≡|

6. Conclusion

It has been demonstrated that the concept of physical dimensions can play useful roles in the structural analysis of systems. The notion of "column structured matrix" introduced in [36] seems to have a close connection with the approach developed in this paper.

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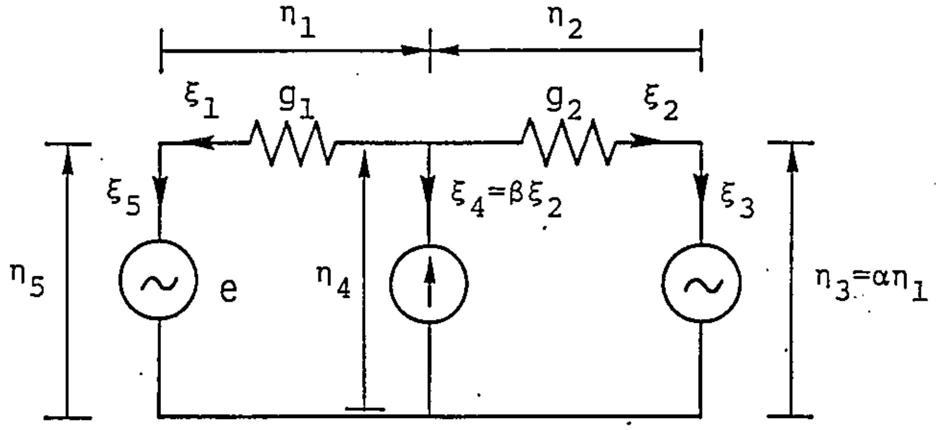

Fig. 4.1. An electric network of Example 4.1

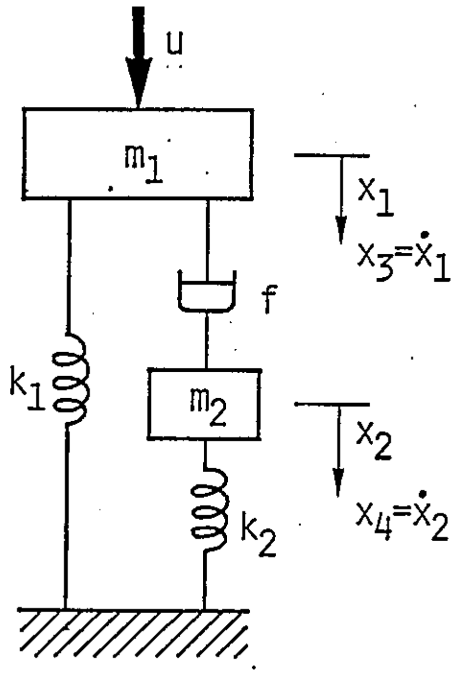

Fig. 4.2. A mechanical system of Example 4.2

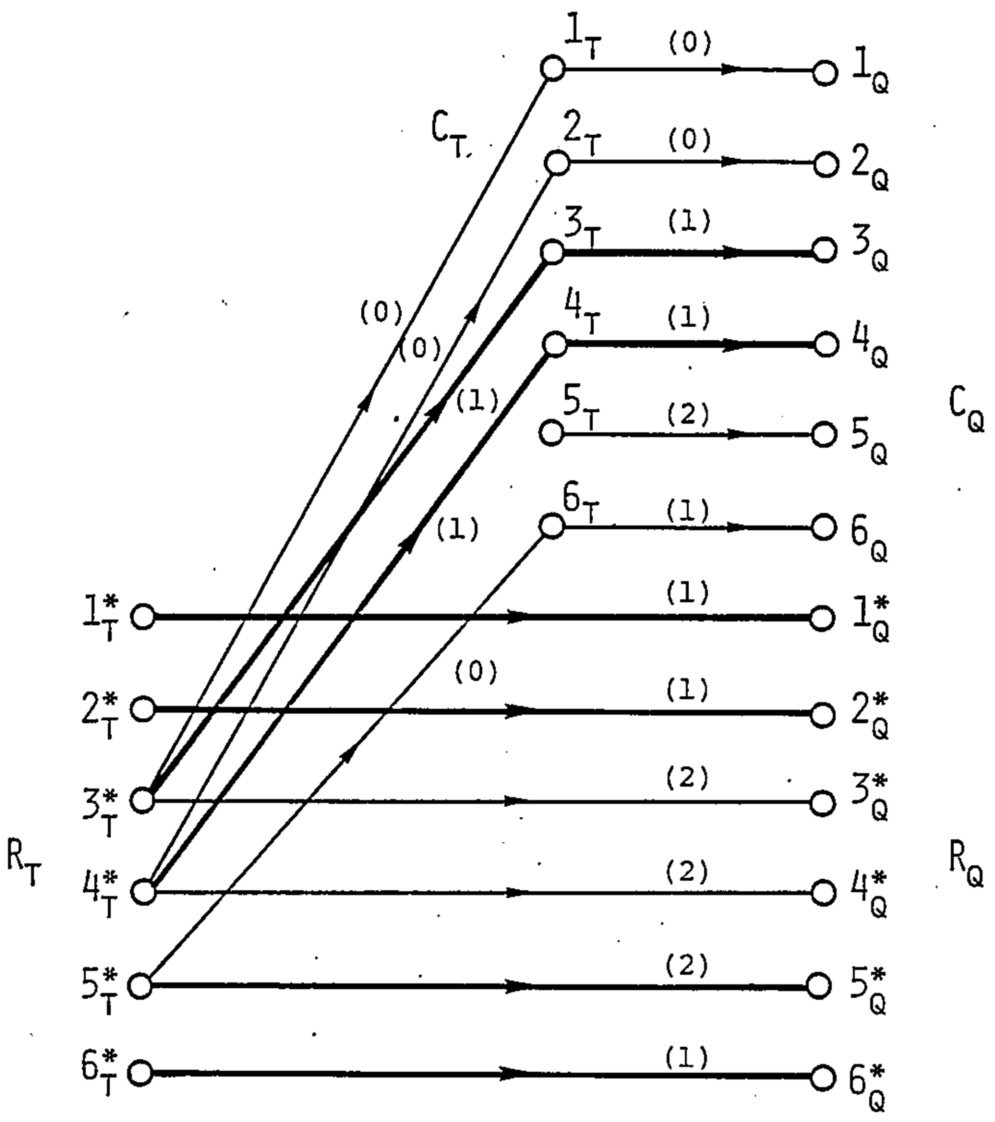

Fig. 5.1. Weighted independent-linkage problem for determining the dynamical degree of Example 4.2

(The weights are given in parentheses and the maximum-weight independent linkage is indicated by bold lines.

$I \cup J = \{1^*, 2^*, 5^*, 6^*\} \cup \{3, 4\}$ is a common base of $M(T_A - T_F)$ and $M(Q_A - Q_F)$.)