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URBAN AGGLOMERATION ECONOMIES

IN

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ABSTRACT

A model of firms' spatial allocation and location in a city is developed by explicit incorporation of urban agglomeration benefits using accessibility measure. In a linear and one-activity city, every firm is assumed to interact each other for face-to-face transactions, and the unit construction cost of office building is considered to be proportional to firm density. It is shown that the market equilibrium distribution is more dispersed than the optimum distribution, and that the equilibrium rent function is concave near the city center and convex near the boundaries.

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1. INTRODUCTION

Urban agglomeration is the most important reason for the existence of cities. With the rapid growth in the service sector, the urban agglomeration economies are essentially attributed to the concentration of office firms in central business districts (CBD). The locational decision of one activity would in general endogenously be determined, ceteris paribus, by accessibility or proximity to other activities. Since most of the literature on macro urban agglomeration [e.g., Moomaw (1983)] often regard a city as a spaceless point, urban agglomeration economies are usually measured by summary statistics of a city like city size. However, if a city were spaceless, transportation costs within a city would become zero while transportation costs between cities would not. In this instance, there is no reason to decentralize economic activities, in the sense that every firm will be concentrated in one point city to realize the maximum agglomeration economies.

If space concept is explicitly taken into account, it is natural to introduce congestion costs as agglomeration diseconomies. Overcongestion of firms would require high construction costs of building, oblige laborers to commute long distance in a crowded train, and presumably raise land rent. Suppose these costs be paid directly or indirectly by firms, then the agglomeration diseconomies might be expressed as a function of a density measure. The location of activities would thus be determined by balancing two kinds of "forces": the force of diversification and the force of unification by Zipf's (1949) terminology.

In the context of the above, a pioneering work is Vaughan (1975) which explictly introduced the notion of continuous spatial interaction between every possible pair of activities. To obtain socially optimum distribution of activities in a linear city, Vaughan minimized a total rent measured by a power function of activity density (the force of diversification) plus a total travel cost between every activity (the force of unification) subject to fixed number of total activities. The functional form of the analytical solution is so complex that Vaughan used empirical values of the power parameters in London and in Sydney for 1966 and found that the density distribution of the activity (population) is inverse S shaped, i.e., concave near the CBD and convex elsewhere, which seems to fit observed distributions in those cities well.

In the past literature, the force of diversification is normally regarded as a function of a commuting cost or density. The commuting cost [Fujita and Ogawa (1982), and Imai (1982)] is out of consideration here due to the assumption of one-activity city as will be mentioned later. The density is therefore the only source of the force of diversification although there are several variations of its interpretations. Beckmann (1976), Odland (1976), and Tabuchi (1982) considered population density as a disutility of residents while Borukhov and Hochman (1977), O'Hara (1977), and Tauchen and Witte (1984) considered firm density to be related to a construction cost of building. Applying the bid rent approach, on the other hand, Fujita and Ogawa (1982) regarded rent as a disutility for households and as a cost for business firms.[1]

The force of unification is due to urban agglomeration economies which is regarded as face-to-face contacts and is usually measured by an average travel cost proportional to distance between all activities with equal probability [Beckmann (1976), Borukhov and Hochman (1977), O'Hara (1977), and Tauchen and Witte (1984)]. On the other side, Tabuchi (1982) and Fujita and Ogawa (1982) considered agglomeration economies as a benefit measured by accessibility to all activities. However, it is quite unlikely for firms to transact or travel to all activities with equal probability. It is well known in the regional science literature that the probability of trip is a distance decay function, which may suggests the introduction of accessibility measure in formulating the urban agglomeration economies.

Following Vaughan (1975), this paper attempts to examine the social optimum and market equilibrium distribution of firm density in a central business district based on the accessibility benefit as urban agglomeration economies and the density cost as a building construction cost. Although this paper is confined to the analysis of office firms, similar discussion on population density distribution would of course be possible assuming a certain suitable utility function of household. For analytical purposes, two simplifications will be made here: a linear rather than two-dimensional city; and one activity (i.e., office firm) rather than two activities.

The linear city assumption permits an analytical solution. Except a square city model with rectilinear distance [O'Hara (1977), Tauchen and Witte (1984)], Borukhov and Hochman (1977) would be the only paper which derived the optimal and market density distributions in a two-dimensional circular city using polar coordinates. Unfortunately, the average travel cost in their model is not weighted by the density distribution which may be a theoretical flaw in solving endogenous distribution of business firms. Note that Tabuchi (1982) also obtained the optimal density distribution in a two-dimensional plane using quadratic programming, but only in a discrete case.

In regard to the one activity city, one may conceive that neither a multicentric pattern nor a catastrophic structural transition is possible as has taken place in Fujita and Ogawa's (1982) equilibrium model under a fixed lot-size assumption. Since the concern here is office firms, the word, 'city' may be interpreted as CBD throughout this paper. Notice that the distribution of firms would presumably be more decentralized if commuting cost between two activities (firm and residence) were incorporated.

In the next section, the optimal distribution of firms will be solved using the calculus of variations, and the same will be compared with two kinds of market equilibrium solutions obtained in the third section. The final section summarizes the concluding remarks.

2. OPTIMAL DISTRIBUTION OF FIRMS

For the sake of analytical transparency, imagine a linear city locating a fixed number of identical firms N. Firms are infinitesimally divisible and their type of industry is the same. The existence of households and hence their commuting costs are not incorporated here.[2]

Mathematically, one may formulate the firm's net profit $\psi(x)$ consisting of the accessibility benefits to all other firms and the unit construction cost of office building at its location x as

$$\psi(x) = \int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - \beta y(x) , \tag{1}$$

where x and t are the location points at the city, y(x) is the firm density at x, a is the city length divided by 2, α is the "distance friction" or "accessibility" parameter, and β is the "firm density" or "construction cost" parameter.

The first term on the right hand side of (1) is a negative exponential-type accessibility function rather than an inverse power function ($\int_{-a}^{a} |x-t|^{-\alpha} y(t)dt,\ \alpha>0$) utilized in Tabuchi (1982) since this power function is not defined at x=t which gives rise to mathematical difficulties in such a continuous model. Thus, one may infer in this case that it would be optimal to concentrate every firm in a single point because the power function gets infinity.

The second term on the right hand side of (1) shows that the unit construction cost at location x is proportional to firm density there.[3] If there is a nonlinear relationship between the unit construction cost and the firm density, one should add another parameter to y(x) (e.g., $-\beta[y(x)]^{\gamma}$), which is however not conducted here to gain analytical advantages.

Now suppose a city government plans to maximize the net social benefit of all firms (Φ) by summing up the net benefit of individual firms locating at x ($\psi(x)$) under the constraint of a fixed total size of firms. Mathematically, the problem is to:

$$\text{maximize} \quad \Phi = \int_{-a}^{a} \psi(x)y(x)dx$$

$$= \int_{-a}^{a}\int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)y(x)dtdx - \beta\int_{-a}^{a}\{y(x)\}^2dx \tag{2}$$

$$\text{subject to} \quad \int_{-a}^{a} y(x)dx = N, \text{ and } y(x) \geqq 0, \tag{2'}$$

where N is the total number of firms. The first term in (2) is the aggregate accessibility benefits and the second term is the aggregate construction costs. Equation (2') simply states that the total number of firms is fixed in the linear city.

Utilizing the method of Lagrange multiplier, the maximization problem of (2) and (2') are to be rewritten as

$$\Phi = \int_{-a}^{a} [y(x)e^{-\alpha x}\int_{-a}^{x} e^{\alpha t}y(t)dt + y(x)e^{\alpha x}\int_{x}^{a} e^{-\alpha t}y(t)dt - \beta\{y(x)\}^2 + \lambda y(x)]dx - \lambda N, \tag{3}$$

where λ is the Lagrange multiplier.

Equation (3) is ascribed to the problem of the standard calculus of variations with respect to z(x). By use of the Euler's equation $F_y - \frac{d}{dx} F_{y'} = 0$, where F is the functional, the first-order condition for maximum is given by

$$2\int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - 2\beta y(x) + \lambda = 0. \tag{4}$$

This shows that the accessibility is proportional to density in optimum, assuming existence of an interior solution. More generally, if the unit construction cost is a power function of firm density $-\beta[y(x)]^{\gamma}$ and $\gamma > 1$, then one can say that the accessibility is positively related to density.

To solve this integral equation with respect to y(x), first of all, differentiate (4) with respect to x,

$$-2\alpha\int_{-a}^{x} e^{-\alpha|x-t|}\, y(t)dt + 2\alpha\int_{x}^{a} e^{-\alpha|x-t|}\, y(t)dt - 2\beta y'(x) = 0 . \tag{4'}$$

Again differentiate (4') with respect to x,

$$2\alpha^2\int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - 4\alpha y(x) - 2\beta y''(x) = 0 . \tag{4''}$$

Eliminating the integral part in (4) and (4"), one can obtain

$$\beta y''(x) + (2\alpha-\alpha^2\beta)y(x) = -\frac{\alpha^2\lambda}{2}. \tag{5}$$

Solving the differential equation (5), the optimal distribution of the firm density is then obtained as

$$y(x) = C_1 e^{\sqrt{\alpha^2-2\alpha/\beta}\ x} + C_2 e^{-\sqrt{\alpha^2-2\alpha/\beta}\ x} + \frac{\lambda\alpha}{2(\alpha\beta-2)} . \qquad (6)$$

Now, to eliminate three undetermined constants C_1, C_2 and λ, we need three equations. Firstly, as we are considering a linear city symmetrical about the midpoint x=0, the optimal solution should have the property,

$$y(x) = y(-x), \quad \text{for all } x. \qquad (7)$$

Secondly, if the city length were allowed to vary so that the density at both of the city edges is equal to that of outside the city, i.e., zero, then,

$$y(\pm a) = 0 . \qquad (8)$$

Note that $y(\pm a) > 0$ is possible if the city length 2a is fixed due to institutional constraints like zoning or geographical constraints such as mountains and seas. To reduce the number of parameters, however, it is set zero. Finally, the total number of firms is fixed which is equation (2').

Thus, by use of (2'), (7) and (8), the firm density y(x) can be expressed by N, a, α, β, and x as

$$y(x) = \frac{N}{2a}\,\frac{\left(e^{\sqrt{\alpha^2-2\alpha/\beta}\ a} + e^{-\sqrt{\alpha^2-2\alpha/\beta}\ a}\right) - \left(e^{\sqrt{\alpha^2-2\alpha/\beta}\ x} + e^{-\sqrt{\alpha^2-2\alpha/\beta}\ x}\right)}{\left(e^{\sqrt{\alpha^2-2\alpha/\beta}\ a} + e^{-\sqrt{\alpha^2-2\alpha/\beta}\ a}\right) - \dfrac{\left(e^{\sqrt{\alpha^2-2\alpha/\beta}\ a} - e^{-\sqrt{\alpha^2-2\alpha/\beta}\ a}\right)}{\sqrt{\alpha^2-2\alpha/\beta}\ a}} \qquad (9)$$

In accordance with the value of the square root $\sqrt{\alpha^2-2\alpha/\beta}$, equation (9) is then classified into the following five cases:

i) $\alpha = 0$

$$y(x) = \frac{N}{2a} \tag{9a}$$

ii) $\beta = 0$

concentration on one point

iii) $0 < \alpha\beta < 2$

$$y(x) = \frac{N}{2a}\,\frac{\cos(\sqrt{2\alpha/\beta-\alpha^2}\ x)/\cos(\sqrt{2\alpha/\beta-\alpha^2}\ a) - 1}{\tan(\sqrt{2\alpha/\beta-\alpha^2}\ a)/(\sqrt{2\alpha/\beta-\alpha^2}\ a) - 1} \quad \text{for } \sqrt{2\alpha/\beta-\alpha^2}\ a \leqq \pi \tag{9b}$$

iv) $\alpha\beta = 2$

$$y(x) = \frac{N}{2a}\left[\frac{3}{2} - \frac{3}{2}\left(\frac{x}{a}\right)^2\right] \tag{9c}$$

v) $2 < \alpha\beta$

$$y(x) = \frac{N}{2a}\,\frac{\cosh(\sqrt{\alpha^2-2\alpha/\beta}\ x)/\cosh(\sqrt{\alpha^2-2\alpha/\beta}\ a) - 1}{\tanh(\sqrt{\alpha^2-2\alpha/\beta}\ a)/(\sqrt{\alpha^2-2\alpha/\beta}\ a) - 1} \tag{9d}$$

Notice that the inequality in (iii) is necessary to ensure the smoothness of $y(x)$ for $-a \leq x \leq a$.

(9a) is constant density, (9b) is a cosine curve, (9c) is a quadratic curve, and (9d) is a cosine hyperbolic curve or sum of positive and negative exponential curves. For illustrative purposes, they are drawn together in Figure 1 by setting $N = 100$, $a = 4$, $\beta = 2$, and several values of α. One can observe that the density distribution becomes flatter as α increases for $0.5 \leq \alpha$. For $0 < \alpha < 0.5$, however, the density distribution becomes sharper as α increases. It thus follows that the degree of firm concentration is not a monotonic function of α. To see this, define the second-order moment of x as the degree of firm decentralization:

$$m = \int_{-a}^{a} x^2 y(x)/N \, dx \ , \tag{10}$$

where the division by N is just to standardize the density function $y(x)$. Needless to say, smaller the m, the more concentrated to the center of the linear city. Substituting $y(x)$ of (9) into (10),

$$m = \frac{a^2 \, [(1/3 + 2/A^2)(e^A + e^{-A}) - (1 + 2/A^2)(e^A - e^{-A})/A]}{(e^A + e^{-A}) - (e^A - e^{-A})/A} \ , \tag{10'}$$

where $A = \sqrt{\alpha^2 - 2\alpha/\beta}\; a$. Equation (10') is valid whether A is real or imaginary.

Figure 2 describes a contour map of the moment value (10') on the two-dimensional plane of (α,β) for $a = 4$. The value of m is smaller than

$a^2/5$ [= 3.2] if $0 < \alpha\beta < 2$ (cosine curves), equal to $a^2/5$ if $\alpha\beta = 2$ (a quadratic curve), and greater than $a^2/5$ if $2 < \alpha\beta$ (cosh curves). The hatched area ($\sqrt{2\alpha/\beta-\alpha^2}\ a > \pi$, which is the opposite to (9b)) does not yield a smooth curve at optimum in the range of $-a \leqq x \leqq a$ because Jacobi's necessary condition does not hold there [Gelfand and Fomin (1963)]. The optimal solution in that hatched area would look like Figure 3, in which $y(x) = 0$ for $-a^* \leqq |x| \leqq a^*$. The switching points $\pm a^*$ should be calculated by optimal control theory.

Observing Figure 2, one would find that the moment function is an increasing function of β in most of the cases. In other words, firms tend to concentrate as the unit construction cost rate β becomes relatively cheaper.

As expected from Figure 1, the optimal density $y(x)$ is a non-increasing function of $|x|$, viz.,

$$\frac{dy}{d|x|} \leqq 0,$$

where the equality holds when $\alpha\beta = 0$ or when $x = 0$. The proof is obvious due to the functional characteristics of the cosine curve and the cosine hyperbolic curve. The former ($\alpha\beta = 0$) is trivial because the optimal density is flat everywhere. The latter ($x = 0$) implies that <u>the optimal density is always smooth at the center of the city irrespective of the parameter values of α and β</u>. This is the inherent consequence of the calculus of variations where optimal solutions are necessarily continuous and smooth. This finding is in striking contrast to the classical urban economics models where the spatial interaction is limited to access to the exogenous CBD.

The convexity or concavity can be ascertained by computing the second derivative of y(x) with respect to $|x|$. As can be inferred from Figure 1, the optimal density is concave when $\alpha\beta \geqq 2$. When $0 \leqq \alpha\beta < 2$, on the other hand, there is a possibility that two inflection points exist. After some manipulation, one gets

$$\frac{d^2y}{d|x|^2} = -2C_1''\cos(\sqrt{2\alpha/\beta-\alpha^2}\,|x|).$$

Thus, if $\pi > \sqrt{2\alpha/\beta-\alpha^2}\ a \geqq \pi/2$, there exist inflection points at $x = \pm\ \pi/2\sqrt{2\alpha/\beta-\alpha^2}$. That is, the optimal density distribution is concave near the CBD and convex apart from the CBD, whose result is similar to Vaughan (1975). On the other hand, if $\sqrt{2\alpha/\beta-\alpha^2}\ a < \pi/2$, the above second derivative becomes non-positive for all x, which means that the optimal density is concave everywhere. It thus follows that in most of cases the optimal distribution of firm density is concave whereas the classical CBD model often yields a convex density function, such as negative exponential.

3. EQUILIBRIUM DISTRIBUTION OF FIRMS

Provided that each firm be free to move its location without any relocation costs in accordance with its profit maximization principle, the firm's net profit may be written:

$$\psi_e(x) = \int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - \beta y(x) - r(x)/y(x), \tag{11}$$

where r(x) is the land rent at location x. Since there are y(x) firms, one firm has to pay r(x)/y(x) for land in addition to the unit construction cost of office building $\beta y(x)$. If the third term on the right hand side of (11) is omitted, (11) reduces to (1).

Suppose now that the firm chooses density so as to maximize its profit. The condition for profit maximization is obtained from the following first-order condition by differentiating (11) with respect to y(x),

$$-\beta + r(x)/[y(x)]^2 = 0. \tag{12}$$

The sufficient condition for the maximum is easily confirmed by the negativity of the second-order differentiation of the left hand side of (12). Putting (12) into (11),

$$\psi_e = \int_{-a}^{a} e^{-\alpha|x-t|}\, y(t)dt - 2\beta y(x), \tag{13}$$

which should be identical everywhere in the linear city ($-a \leqq x \leqq a$) in equilibrium (and hence, (x) is dropped on the left hand side), otherwise there will exist an incentive for firms to move.

Differentiating (13) twice with respect to x, and eliminating the integral part, one can get

$$\beta y''(x) + (\alpha-\alpha^2\beta)y(x) = \alpha^2\psi_e/2 \; . \tag{14}$$

Solving the differential equation (14), the equilibrium distribution of the firm density is then given by

$$y(x) = C_3 e^{\sqrt{\alpha^2-\alpha/\beta}\; x} + C_4 e^{-\sqrt{\alpha^2-\alpha/\beta}\; x} + C_5 \; , \tag{15}$$

where the constants C_3, C_4 and C_5 are to be determined by the boundary conditions and the constraint of total number of firms (2') as before.

The crucial difference between the optimum solution (6) and equilibrium solution (15) is the values of the exponents: $\pm\sqrt{\alpha^2-\alpha/\beta}\; x$ in equilibrium while $\pm\sqrt{\alpha^2-2\alpha/\beta}\; x$ in optimum. One can say that by comparing with the optimum city, the equilibrium city tends to be too dispersed since the weight of the construction cost β in equilibrium affects the value of the exponent twice as much as that in optimum. Recall that the degree of the firm decentralization measured by m (equation (11) in the previous section) is an increasing function of β in most cases. This conclusion is in agreement with Borukhov and Hochman (1977) and Imai (1982) although their model assumptions differ.

In this situation, government intervention is indispensable; for instance, a locational subsidy in accordance with firm density, or possibly a proper zoning ordinance to prevent from decentralization. The conclusion obtained here is opposed to the decentralization policy which many city governments adopt. However, if costs of commuting from outside the city is taken into account, or if air pollution as a negative externality is caused owing to commuting congestion, then it would be desirable to decentralize firms compared to the optimal distribution.

It should be noted that (13) can be obtained by the Alonso's (1964) bid rent approach in residential location. Namely, rearranging (11) and maximizing r(x) with respect to y(x), one arrives at the same condition as (13). If, on the other hand, the aggregate land rent of the city be maximized holding the net profit of the firm $\psi_e(x)$ constant at every location with given N, then one arrives at the optimum condition (4) in Section 2.

Next, consider the characteristics of the equilibrium rent function. Equation (12) indicates that the unit construction cost of office building $\beta y(x)$ is equal to the unit land rent $r(x)/y(x)$ throughout the city. The rent function itself is in proportion to the square of the firm density in market equilibrium. Hence, from the characteristics of the firms' density function as examined in the previous section, one is able to draw the several relationships as follows:

$$r(\pm a) = 0, \tag{16}$$

$$\frac{dr(x)}{d|x|} \leq 0, \text{ in which the equality holds when } x=\pm a \text{ or } 0, \tag{17}$$

$$\frac{d^2r(x)}{d|x|^2} < 0, \text{ near the midpoint, and} \tag{18a}$$

$$\frac{d^2 r(x)}{d|x|^2} \geqq 0, \text{ elsewhere.} \tag{18b}$$

The rent function is therefore "bell-shaped" regardless of positive values of α and β. That is to say: from (16), the rent function becomes zero at the city boundaries; from (17), the rent gradient is zero at the city center and boundaries, and the rent decreases with distance from the city center; and from (18a) and (18b), the rent function is concave near the center and convex near the boundaries. In other words, the rent function always has two inflection points between the center and two boundaries for any α, $\beta > 0$.

4. CONCLUSION

This paper has investigated the implications of the accessibility benefits as urban agglomeration economies in a linear city of fixed length. It is assumed that each firm chooses locations to maximize net profit comprizing the accessibility benefits to all other firms and the office construction cost measured by firm density.

Applying the calculus of variations, it is derived that the optimal distribution of firms' density is a constant, cosine, quadratic, or cosine hyperbolic curve dependent upon the parameter range as illustrated in Figure 1. It is found that in the greater part of the parameter range, both the rate of the distance friction costs (α) and the rate of the construction cost (β) are negatively related to the degree of firm concentration as depicted in Figure 2. It is also found that the optimal density is always smooth at the center of the city irrespective of the parameter values, which is in contrast to the classical urban economics models. The concavity of the optimal density function is shown to hold in most of the cases, which is contrary to the well-known negative exponential density function.

The market equilibrium distribution of firm density is also obtained supposing the net benefits are identical at every location under the same constraints. It is revealed that the optimal distribution of firm density is more concentrated than the market distribution, which is exhibited by the differences in the parameter values of the exponents in equations (6) and (15). In this instance, the city is in need of a pertinent government intervention, such as a locational subsidy or zoning restriction, in order to prevent from dispersion. The equilibrium rent function is shown to be concave near the city center and convex near the city borders.

FOOTNOTES

* The author has benefited from useful comments by Noboru Sakashita, Yoshitsugu Kanemoto, Katsumi Nishina, and Muttur R. Narayana.

1. Also the force of repelling between firms like Hotelling's spatial competition is not taken into consideration in this paper.

2. If households are incorporated in this model, one would conceive that due to the effect of commuting costs, the firm distribution is more decentralized as demonstrated by Fujita and Ogawa(1982) in their market equilibrium model. However, such a discussion is beyond the scope of this paper because the major objective of this paper is to explore urban agglomeration economies of firms' interaction.

3. See Tabuchi (1982) for justification of this assumption of proportionality.

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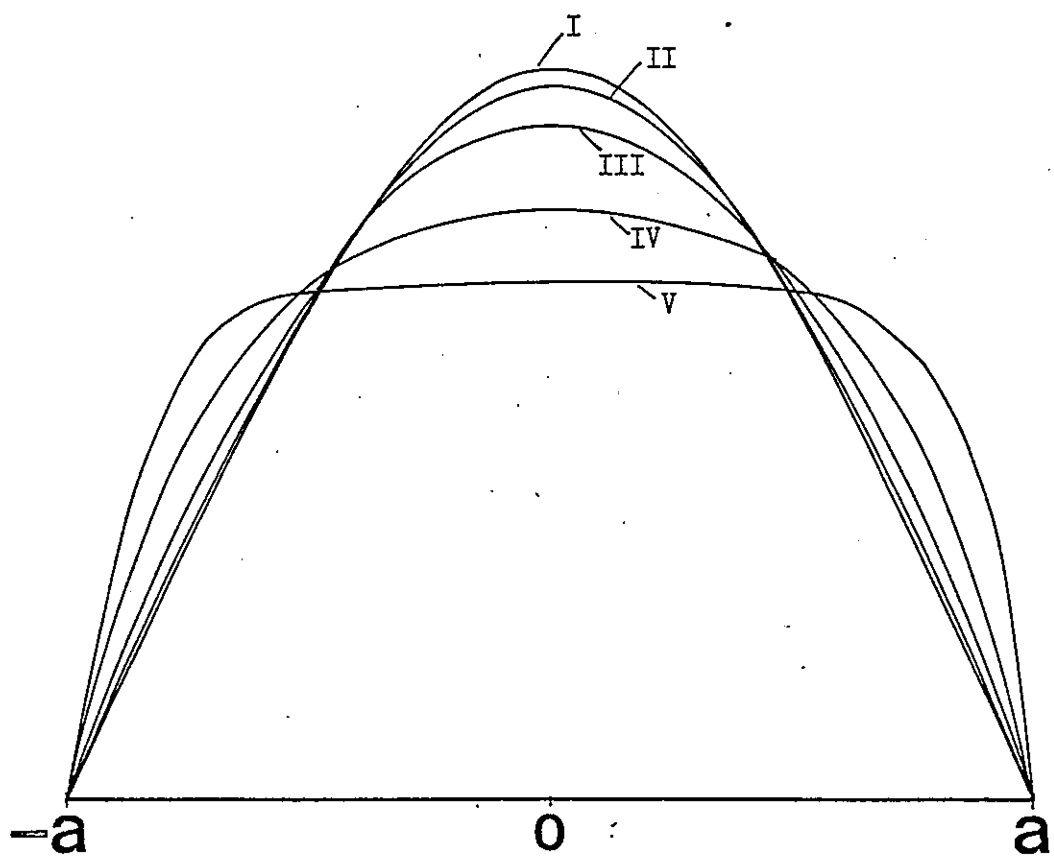

Figure 1. Optimal distributions of firm density for α=0.5 (I), α=0.75 or 0.25 (II), α=1 (III), α=1.5 (IV), α=2.5 (V), where β=2 (fixed)

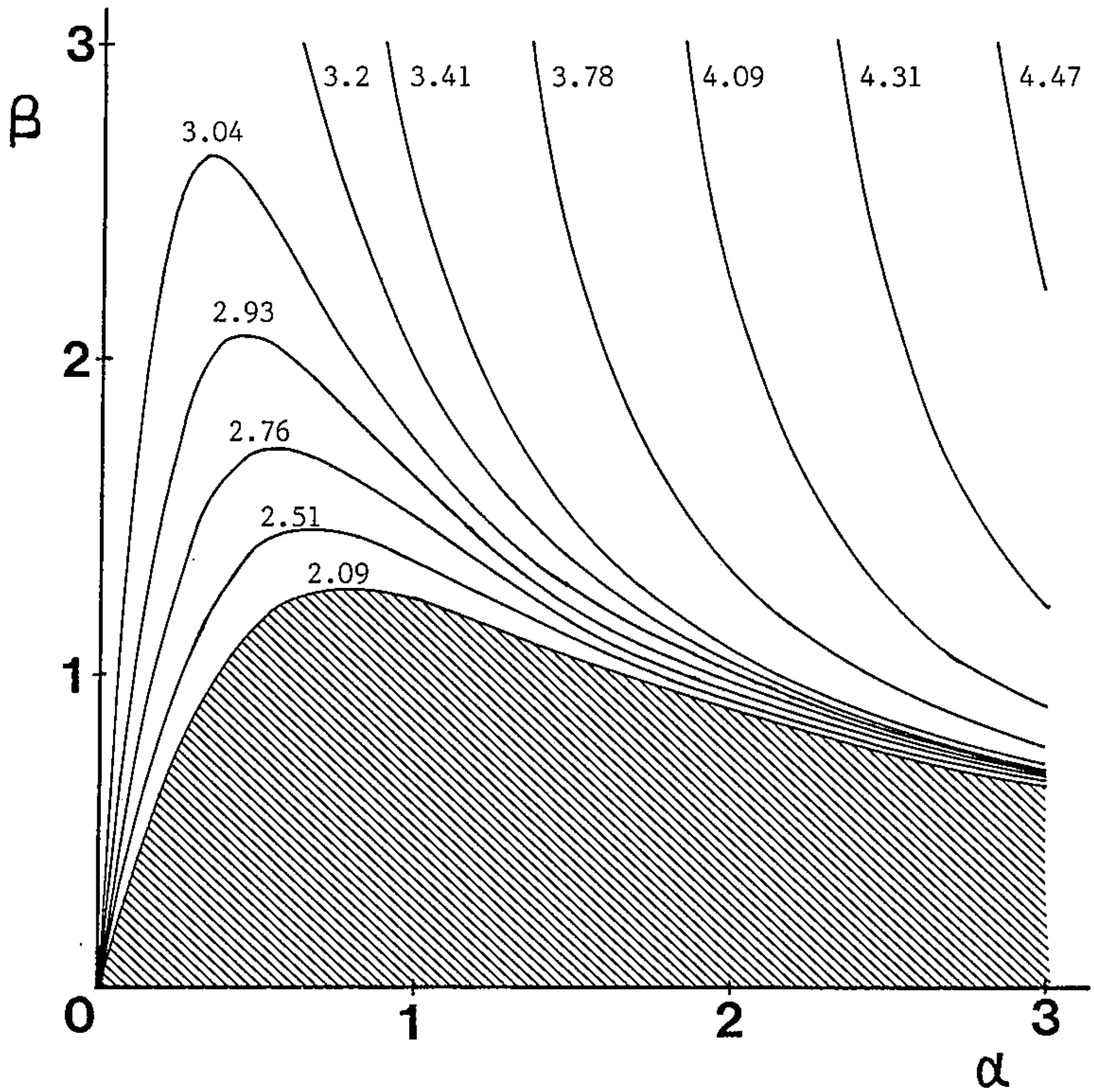

Figure 2. Degree of Decentralization m for (α,β)

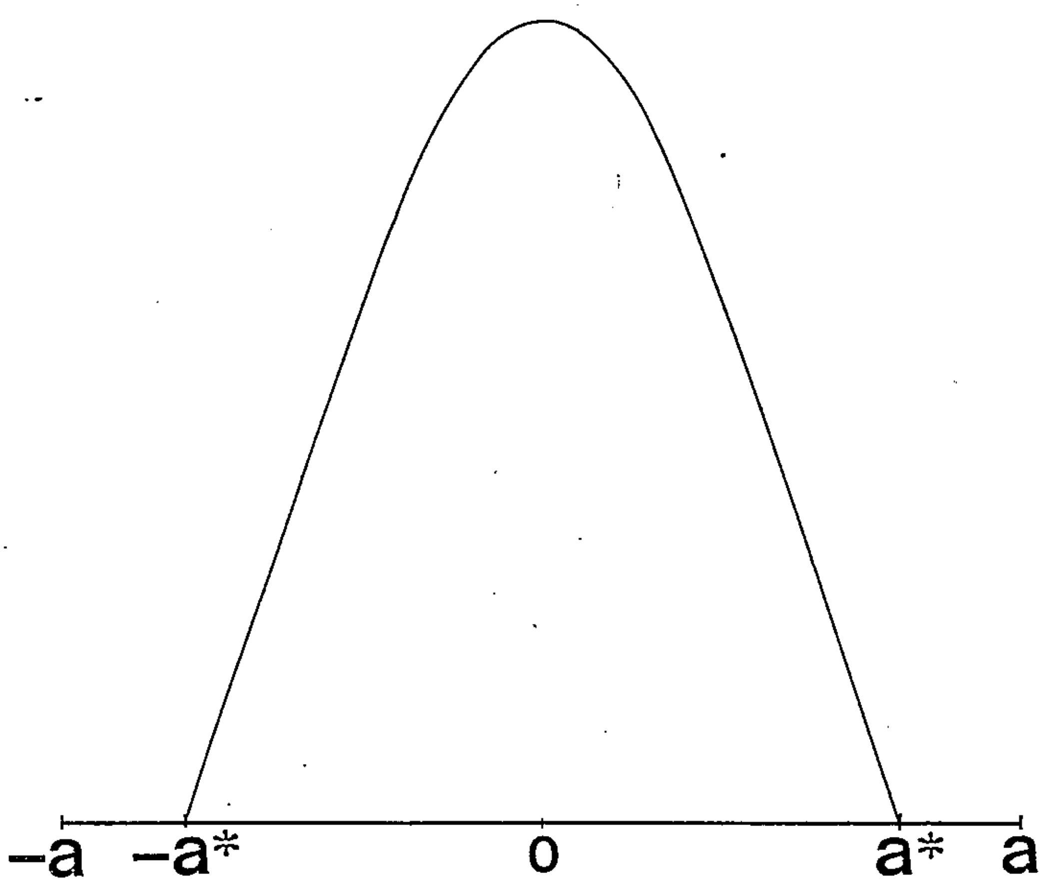

Figure 3. Optimal distribution of firm density when $\sqrt{2\alpha/\beta-\alpha^2}$