No. 25

78-4

GENERALIZED DOMINATION AND FUZZY DOMINATION IN PREFERENCE STRUCTURE

March 1978

Hajime Eto

University of Tsukuba

GENERALIZED DOMINATION AND FUZZY DOMINATION IN PREFERENCE STRUCTURE

Hajime Eto*

ABSTRACT

Domination which had been defined only on the conventional inequalities was generalized into domination structure. However its merit was neither fully recognized nor exploited yet to develop a realistic solution concept in a multi-objective optimization problem. This paper shows that the generalized domination structure can yield various irrationalities pertinent with domination and preference. It further shows that fuzzy domination is included in the generalized domination

1. INTRODUCTION

Domination is usually defined upon the conventional inequalities in the real vector space as in Def. 1.1. However, there is no ground to confine domination to this. It can equivalently be defined in terms of domination cone of the non-negative orthant in the real space as in Def. 1.2 and can further be defined in terms of domination cone in general as in Def. 1.3. However, there is no need to confine domination to cone. In fact Yu's definition of domination structure [3] as in Def. 1.4 allows cases where domination is not expressed in terms of cone. Despite this general property of the definition itself, its realistic applicability has not been explored enough

* The University of Tsukuba, Institute of Sócio-Economić Planning Ibaraki, Japan

To the best knowledge of the author it was touched only in the course of discussion on stability of non-dominated solutions [2]. This paper shows that this is general enough to encompass various irrationalities and fuzziness.

Our discussion will be as a rule concerned with the n-dimensional real space R^n. R^n_+ and R^n_{++} denote its non-negative orthant and the interior of R^n_+ respectively. Similarly R^n_- denotes the non-positive orthant of R^n and R^n_{--} denotes the interior of R^n_-. In discussing domination, a, b, c and d are elements in R^n, and their equality and inequality are accordant to the conventional vector equality and inequality. When discussion is concerned with sets, & and ϕ denote the intersection of sets and the empty set respectively.

Hereafter "a dom_j b" means that a dominates b according to Def. 1.j. Subscript j may be omitted if unnecessary.

Def. 1.1 (conventional definition)

a dom_1 b if and only if $a > b$.

Def. 1.2 (non-negative orthant cone definiton)

a dom_2 b if and only if there exists d in R^n_{++} such that $a = b + d$.

Remark that a dom_1 b if and only if a dom_2 b and that R^n_+ is a convex cone which may be termed the non-negative orthant cone whose interior is R^n_{++}.

Def. 1.3 (general cone definition)

a dom_3 b if and only if there exists d inside of a given cone C in R^n such that $a = b + d$.

When necessary, let C be specified as follows; $C_+(q)$ means that C is the translation of R^n_+ so that its apex is at q in R^n_+, $C_{+-}(q)$ means that $C \& R^n_+ \neq \phi$ and $C \& R^n_- \neq \phi$ and its apex is at q, $C^o(q)$ means that

C contains 0 aside from its apex at q.

Note that $C_{+-}(q)$ is neither necessarily $C^o(q)$ nor vice versa.

Def. 1.4 (general region definition)

a dom_4 b if and only if there exists d inside of a given region D in R^n such that $a = b + d$.

This region D will be called domination region and, when necessary, be specified as follows;

D_+ : if D is in R^n_+,

D_{+-} : if $D \,\&\, R^n_+ \neq \phi$ and $D \,\&\, R^n_- \neq \phi$

D^p : if D contains point p when reference to p is needed.

According to this notation, R^n_+ is an example of D^o_+ and a hyper-ball around the origin is an example of D^o_{+-}.

Note that Def. 1.1 - Def. 1.4 are concerned with strict domination. To extend Def. 1.1 to weak domination, the strict inequality is replaced with the weak inequality. Similarly for Def. 1.2, R^n_{++} is replaced with "R^n_+ except the origin", and for Def. 1.3 - 1.4, "inside of" is replaced with "in". "a weakly dominates b according to Def. 1.j" will be abbreviated as "a wdom_j b".

Domination cone and region are constant in Def. 1.1 - 1.4. They can be variable in association with Def. 1.3 and 1.4.

Var D.1 (dominator variable)

Domination is variable in a.

Var D.2 (dominatee variable)

Domination is variable in b.

Var D.3 (distance variable)

Domination is variable in $a - b$.

Variable domination will be referred to Var D.j when necessary. When no specific mention will be made, "domination" will denote constant domination.

2. IRRATIONALITY WITH GENERALIZED DOMINATION

Conventional domination has some reasonable properties such as existence of the non-dominated solutions and the transistivity relation of domination for Def. 1.1 or Def. 1.2. Some strange properties will be shown which are incurred by drastic generalization of domination structure.

Hereafter Y is convex and bounded in R^n and is interpreted as the set of objective values of multi-objective mathematical programming with n objectives. N denotes the set of non-dominated solutions for given definition of domination. Y is assumed to have interior.

2.1 Reflexivity

Rationally domination structure is required irreflexive. Namely, "a dom a" is excluded. However, Def. 1.3 incurs the reflexivity in some cases. Let domination cone be of type $C^o(q)$ for arbitrary q, then a dom a because $a = a + 0$ and 0 is in $C^o(q)$. The same argument applies to D^o in Def. 1.4.

Reflexivity results in self-domination which, in turn, excludes existence of non-dominated solutions.

2.2 Symmetricity

Rationally domination structure is required asymmetric. Symmetricity that a dom b and b dom a occurs under reflexivity if $a = b$. Set aside from reflexivity, $C_{+-}(q)$ for arbitrary q in Def. 1.3 and D_{+-} in Def. 1.4 provide examples that a dom b and b dom a. In fact suppose $a > b$ then there exist $d_1 > 0$ and $d_2 < 0$ such that $a = b + d_1$, $b = a + d_2$ and d_1 and d_2 are in $C_{+-}(q)$ or D_{+-}. In general, this situation can occur when domination cone or region meets the interior of plural orthants so as to contain pairs of points symmetric around the origin. Hence even a cone with apex at the origin can yield symmetricity when its angle around the apex is obtuse.

Symmetricity causes self-dominance and therethrough excludes non-dominant solutions from existence.

2.3 Intransitivity

Transitivity is often referred to as the most crucial relation in domination. Intransivity is, however, easily deduced from Def. 1.4.

Suppose domination region $D = B \& R^n_{++}$ where B denotes the unit hyper-ball around the origin. Also suppose Y is a unit hyper-ball in R^n_{++}. Then there exist three points a, b and c in Y such that $a = b + d_1$ and $b = c + d_2$ for $d_1, d_2 < 1$ (i.e., d_1 d_2 in D) and $a = c + d_3$ for $d_3 > 1$ (i.e., d_3 not in D). This means that a dom b, b dom c and c dom a. That is, intransitivity occurs here.

Intransitivity causes self-dominance and hence non-existence of non-dominated solutions.

2.4 Non-existence of Non-dominant Solutions

Reflexivity, symmetricity and intransitivity exclude non-dominated solutions from existence. However, non-dominated solutions can be nonexistent without them as the following example shows.

For simplicity let Y be a hyper-ball with radius r and let domination region D by Def. 1.4 be as follows,

$$D = C - \{0\}$$

where C is a cone obtained by translating R^n_+ so that its apex is at $q < -2r$. Hence every point in Y is dominated by the all other points in Y. Hence no non-dominant solution exists without self-dominance.

2.5 Unconnectedness

The conventional definition of domination (Def. 1.1 - 1.2) yields that the set of non-dominated solutions is connected in Y. Variable domination cone can cause its unconnectedness.

First take constant cone over Y and let A in Y be a subset of N which cut off the remaining set N - A to separated subsets. Then take cone $C^o(q)$ for arbitrary q as domination cone on A, resulting in self-dominance on A. Hence A ceases to be non-dominated, yielding the unconnectedness of the non-dominated solutions.

2.6 Interiority

The conventional definition of domination (Def. 1.1 - 1.2) yields that N is located on the boundary of Y. General domination cone $C_+(q)$ for $q > 0$ in Def. 1.3 yields an example that N contains interior of Y. In fact the interior inside from the frontier boundary with not more than depth q is non-dominated.

Furthermore the non-dominated solutions can be located in the center of Y unconnectedly from the boundary for variable domination region.

Let Y be, for simplicity, a unit hyper-cube in R^n_+ with a vertex on the origin and B be a hyper-ball with radius r ($r < 1/2$) in the center of Y (namely, both centers are on the same point, (1/2)). Then take variable domination region D by Var D.2 as follows.

$$D = C_+(q) + \{ 0 \} \quad \text{for } y \text{ in } Y - B$$
$$D = C_+(q) \quad \text{for } y \text{ in } B$$

where $q > n^{1/2}$.

Y − B is self-dominated while B is non-dominated because q is large enough for size of Y. Hence only the center is non-dominated unconnectedly from the boundary.

2.7 Whole Boundariness

The conventional definition of domination (Def. 1.1 - 1.2) yields that only the frontier part of the boundary is non-dominated. Variable domination cone yields a case where the whole boundary, front and back is non-dominated.

For simplicity let Y be a hyper-ball around the origin. Let O_j be the j-th orthant in R^n. For example $O_1 = R^n_+$. Note that O_j is a cone with apex at the origin. Take variable domination cone for y in Y by Var D.2 as follows.

$$C = O_j \quad \text{for } y \text{ in } O_j$$

Then for each orthant, the boundary of Y in this orthant is non-dominated. Hence the whole boundary, and no other point in Y, is non-dominated.

2.8 Wholeness

The whole region Y can all be non-dominated. A very simple example is as follows. Let Y be a hyper-ball with radius r and domination cone by Def. 1.3 be $C_+(q)$ for $q > 2r$. Then no point in Y is dominated by other points in Y. Hence the whole region of Y is non-dominated.

3. FUZZY THEORETIC INTERPRETATION OF GENERALIZED DOMINATION

The foregoing disucssion shows that the general domination structure yields many examples of irrationality. But its interest is only in showing them without mentioning their realistic applicability.

Hereafter it will be shown that the general domination structure is rooted in some realistic problems and is applicable.

It has long been recognized that preference is somewhat fuzzy. Among various possible ways of introducing the fuzziness into domination, Roy [1] set up a framework of thresholds between which preference is fuzzy. Def. 3.1 and Val. 1 below may reflect his concept.

Def. 3.1 (Roy's fuzzy domination region definition)

a fdom b if and only if there exists a vector d such that $d > q(b) > 0$, $d \not> s(b)$ for $s(b) > q(b)$ and $a = b + d$.

Val. 1. Membership function $\mu_b(d)$ is valuated as follows.

$$\mu_b(d) = \prod_{i=1}^{n} \min (1, ((d_i - q_i(b))/(s_i(b) - q_i(b)))).$$

Here $q(b)$ and $s(b)$ are termed thresholds by Roy. The fuzzy weak domination can be defined by replacing the strict inequalities with the instrict inequalities in Def. 3.1 (yielding "a fwdom b") as in the aforementioned extension of Def. 1.1 - 1.4.

Interpreted in domination cone, Def. 3.1 yields the following properties.

(i) the boundary of domination cone is fuzzy,

(ii) the cone is derived from the non-negative orthant cone by translating its apex from the origin into R^n_{++} and is contained in R^n_{++}.

(iii) the cone is not constant but variable in b, (Var D.2)

(iv) a point inner the cone takes on the greater value of membership function under the monotone increase valuation (Val. 1).

The concept of fuzzy domination can be extended to include non-fuzzy domination as a special case. Thereby the domination region becomes unbounded to form a cone.

Def. 3.2 (Roy's fuzzy domination cone definition)

a fdom^E b if and only if there exists a vector d such that $d > q(b) > 0$ and $a = b + d$.

Membership function is valuated by Val. 2 in addition to Val. 1.

Val. 2.

$$\mu_b(d) = 1 \quad \text{for} \quad d > s(b) \quad \text{and} \quad \mu_b(d) = 0 \quad \text{unless} \quad d > q(b).$$

The fuzzy weak domination extended can be defined by proper modification of Def. 1.3, introducing a new notation fwdom^E.

For a more sophisticated domination, the positivity condition that $q(b)$, $s(b) > 0$ can be relaxed. To include a case where fuzzy domination cone is in R^n_-, it is natural to relax also the condition that $s(b) > q(b)$. Another variation is obtained by taking other kinds of variable domination like Var D.1 or 3.

Thereby some irrationalities occur in a natural way.

Reflexivity: In Def. 3.1 let $q < 0$ or in Def. 3.2 let $s < 0$. Both can happen in a fuzzy case.

Symmetricity: Domination cone can have so fuzzy boundary that its angle around the apex is obtuse.

Intransitivity: In Var D.1 let $q(a)$ and $s(a)$ be so quickly increasing in a that, when a increases, a does not dominate no other points in Y. It is quite natural that the fuzziness increases as the objective value increases.

For closer connection with mathematical programming, let X be considered as the feasible region, convex and bounded, of mathematical programming with linear multi-objective functions. (Note that Y is not necessarily considered as a linear map of X. X is always associated with Y and is considered as a linear inverse map of Y meanwhile Y can be considered in no association with X.)

Let the set of non-dominated points be defined relative to definition of domination. Whenever necessary, it is referred to which definition is relevant.

$Y_N = \{y \in Y$: there exists no y' in Y such that y' dom $y\}$

Y_{WN}, Y_{FN}, Y_{FWN}, Y^E_{FN} and Y^E_{FWN} can analogously be defined by replacing dom with wdom, fdom, fwdom, fdom^E and fwdom^E respectively. in definition of Y_N.

$Y_D = Y - Y_N$

Y_{WD}, Y_{FD}, Y_{FWD}, Y^E_{FD} and Y^E_{FWD} can analogously be defined by replacing Y_N with Y_{WN}, Y_{FN}, Y_{FWN}, Y^E_{FN} and Y^E_{FWN} respectively in definition of Y_D.

Similarly, X_N, X_{WN}, X_{FN}, X_{FWN}, X^E_{FN}, X^E_{FWN}, X_D, X_{WD}, X_{FD}, X_{WFD}, X^E_{FD}

and X^E_{FWD} can be defined with X. For example, X_N = { x ε X: there exists no x' in X such that y' dom y} where y and y' are objective values of x and x' respectively.

To restrict consideration to a particular set of values of membership function, let the following definition be introduced for a given set of values M in [0, 1]. Y^M_{FN} = { y ε Y: there exists no y' in Y such that y' fdom y for domination factor d with $\mu_y(d) \in M$ }. Y^M_{FWN}, X^M_{FN}, X^M_{FWN} etc. can be defined similarly.

Whenever fuzzy domination in any sense is compared with non-fuzzy domination, definitions of domination must be correspondent. That is, Def. 3.1 corresponds to Def. 1.4, and Def. 3.2 does to Def. 1.3 unless otherwise stated.

Let Z be a set variable which assumes X and Y as values and "$Z_1 > Z_2$" and "$Z_1 \geq Z_2$" mean that Z_1 includes Z_2 as a proper subset and as a subset respectively. For constant domination the following set properties are derived from definitions, where terms in the same formula are based on the corresponding definition of domination.

SP1 $\quad Z_N \leqq Z_{WN}$

SP2 $\quad Z_D \geqq Z_{WD}$

SP3 $\quad Z_{FN} \leqq Z_{FWN}$

SP4 $\quad Z_{FD} \geqq Z_{FWD}$

SP5 $\quad Z^E_{FN} = Z_N + Z_{FN}$

SP6 $\quad Z^E_{FWN} = Z_{WN} + Z_{FWN}$

SP7	$Z_{FN} + Z_{FD} = Z_D$	
SP8	$Z_{FD}^{E} = Z_{FD}$	
SP9	$Z_{FN}^{E} \geqq Z_{FN}$	
SP9'	$Z_{FN}^{E} = Z_{FN}$	if $Z_N = \phi$
SP9"	$Z_{FN}^{E} > Z_{FN}$	unless $Z_N = \phi$
SP10	$Z_{FN}^{\{1\}} \& Z_{WN} \neq \phi$	unless $Z_{WN} = \phi$ for Val. 1 and 2.
SP11	$Z_{FN}^{\{1\}} \& Z_D \neq \phi$	for Val. 1 and 2.
SP12	$Z_{FN}^{\{1\}} \& Z_N = \phi$	for Val. 1.
SP13	$Z_{FN}^{(0,1)} \leqq Z_D$	
SP13'	$Z_{FN}^{(0,1)} < Z_D$	unless $Z_{FD} = \phi$
SP14	$Z_{FN}^{(0,m)} < Z_{FN}^{(0,m')}$	if $m < m'$
SP15	$Z_{FN}^{(m,1)} < Z_{FN}^{(m',1)}$	if $m > m'$
SP16	$Z_{FN}^{\{1\}} = Z_{FN}^{E}$	for Val. 2.

Theorem 3.1 (Separation Theorem)

Z_{FN} separates Z_N from Z_{FD}.

Proof. Clear from SP7 and 12 in view that $Z_{FN} \& Z_{FD} = \phi$.

4. REFERENCES

1. B. Roy, A Conceptual Framework for a Prescriptive Theory of "Decision Aid", TIMS Studies in the Management Sciences 6, 179 - 210, 1977, North-Holland Pub., Amsterdam

2. T. Tanino and Y. Sawaragi, Stability in Multicriteria Decision Making,

presented at 3rd System Symposium, Tokyo, 1977

3. P. L. Yu and M. Zeleny, The Set of All Nondominated Solutions in Linear Cases and a Multicriteria Simplex Method, J. Math. Anal. Appl. 49, 430 - 468, 1975