No. 247

Greedy Elimination Algorithm for Closure Systems

by

Kazuo Murota

November 1984

Greedy Elimination Algorithm for Closure Systems

Kazuo Murota

Institute of Socio-Economic Planning, University of Tsukuba

Sakura, Ibaraki 305, Japan

November 1984

Abstract

A kind of greedy algorithm, called the greedy elimination, works for finding an optimal base of a certain class of structures defined in terms of closure functions. Such systems are characterized by the "elimination property", which turns out to be a relaxation of Steinitz's exhange axiom for matroids. The closure function of a preordered matroid is an example.

Keywords: closure system, greedy elimination algorithm, elimination property, matroid

Abbreviated title: Greedy elimination

1. Greedy-type algorithms for closure systems

Let E be a finite set. A function σ: $2^E \to 2^E$ is called a closure function on E iff

(C1) $X \subset \sigma(X)$ for $X \subset E$,

(C2) $X \subset Y \implies \sigma(X) \subset \sigma(Y)$ for $X, Y \subset E$,

(C3) $\sigma(X) = \sigma(\sigma(X))$ for $X \subset E$.

The set of all the closure functions on E will be denoted by C(E). From (C1), (C2) and (C3) follows that

(C4) $X \subset \sigma(Y) \implies \sigma(X \cup Y) = \sigma(Y)$ for $X, Y \subset E$.

By a <u>closure system</u> we mean an ordered pair (E, σ) with $\sigma \in C(E)$, in which the following concepts can be introduced.

(i) e ($\in E$) is said to be <u>dependent</u> on X ($\subset E$) iff $e \in \sigma(X)$,

(ii) X ($\subset E$) is called to be <u>independent</u> iff $x \notin \sigma(X \setminus x)$ for all $x \in X$,

(iii) X ($\subset E$) is called to be <u>spanning</u> iff $\sigma(X) = E$,

(iv) X ($\subset E$) is called a <u>base</u> iff X is independent and spanning.

Any subset of an independent set is independent and any superset of a spanning set is spanning.

We are concerned with greedy-type algorithms for finding an optimal base with respect to a given weight function w: $E \to R$, where the weight w(X) of X ($\subset E$) is defined as $w(X) = \Sigma\{w(x) \mid x \in X\}$. For notational convenience we assume $E = \{e_1, e_2, \ldots, e_n\}$ and $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_n)$.

Two types of algorithms, dual to each other in a sense, are considered here; the one of which is based on the principle of greedy augmentation maintaining independence, whereas the other of which is on the principle of greedy elimination keeping spanning property.

Algorithm A-max (Finding a maximum-weight base B)

```
I:=∅;
for i:=1 to n do                                        (*)
    if e_i ∉ σ(I)  then  I:=I∪e_i;
B:=I
```

Algorithm D-min (Finding a minimum-weight base B*)

```
S:=E;
for i:=1 to n do                                        (*)
    if e_i ∈ σ(S\e_i)  then  S:=S\e_i;
B*:=S
```

(Algorithm A-min and D-max are obtained by replacing (*) in A-max and D-min respectively by

for i:= n downto 1 do)

As is well known [5], the correctness of A-$^{\max}_{\min}$ for any weight w is tantamount to the closure system being a matroid, that is, the greedy augmentation A works iff σ satisfies Steinitz's exchange property:

(C5) $y \notin \sigma(X)$, $y \in \sigma(X \cup z)$ ==> $z \in \sigma(X \cup y)$, for $X \subset E$ and $y, z \in E$.

In this case, algorithms A and D are in some sense equivalent. More precisely, the greedy elimination D-min (D-max, resp.) applied to a matroid (E, σ) is equivalent to the greedy augmentation A-max (A-min, resp.) applied to the matroid dual of (E, σ). Hence the greedy elimination D also works for a matroid.

In this note, we will characterize those closure systems for which the greedy elimination D works. An example of such a closure system, called the preordered matroid, is also given.

2. Closure systems for which greedy elimination works

First we consider the unweighted case where $w(e)=0$ for $e \in E$, i.e., the problem of finding a base of (E,σ). The following is easy to establish.

Prop. 2.1. X ($\subset E$) is a base iff X is a minimal spanning set, i.e., $\sigma(X)=E$ and no proper subset of X satisfies this property.

It follows from Prop. 2.1 that the greedy elimination algorithm D works for any closure system in the case where $w(e)=0$ for $e \in E$.

Prop. 2.2. A base of a closure system can be found by the greedy elimination algorithm D.

Next we turn to the weighted case. The following characterizes those closure systems for which the greedy elimination works.

Theorem 2.3. Let (E,σ) be a closure system. The greedy elimination algorithm D works for any weight w iff σ satisfies

(C6) $\sigma(X)=\sigma(Y)=E$, $|X|<|Y|$ $\Longrightarrow$ $\sigma(Y \setminus z)=E$ for some $z \in Y \setminus X$.

(Proof) Let $F=\{X \subset E \mid \sigma(E \setminus X)=E\}$. In view of Prop. 2.1 and the fact that the greedy augmentation principle characterizes matroids, the greedy elimination algorithm D works iff F is the family of independent sets of a certain matroid, i.e., iff

(I1) $\emptyset \in F$,

(I2) $X \in F$ and $Y \subset X$ $\Longrightarrow$ $Y \in F$,

(I3) $X \in F$, $Y \in F$ and $|X|>|Y|$ $\Longrightarrow$ $Y \cup z \in F$ for some $z \in X \setminus Y$.

Obviously (I1) and (I2) follow from (C1) and (C2), whereas (I3) is equivalent to (C6) above. Q.E.D.

We will call (C6) the <u>elimination property</u>. Note that the exchange property (C5) implies (C6), and thus the elimination property is a relaxation of the matroidal property. As a corollary, we obtain the equicardinality of bases of a closure system with the elimination property, by setting $w(e)=1$ $(e \in E)$ in Theorem 2.3.

<u>Prop. 2.4</u>. If a closure system enjoys the elimination property (C6), then $|B_1| = |B_2|$ for two bases B_1 and B_2.

3. An example of a non-matroidal closure system with elimination property

Let (E,cl) be a matroid on E defined in terms of the closure function cl. We assume that a preorder $\geqq$ is also defined on E, independently of the matroid. (A preorder is a reflexive and transitive binary relation.) The triple $(E,\geqq,cl)$ is named the preordered matroid in [3], [4].

For X $(\subset E)$, the order ideal $\langle X \rangle$ determined by X is defined as

$$\langle X \rangle = \{e \in E \mid x \geqq e \ \text{ for some } x \in X\}. \tag{3.1}$$

Note that $\langle\ \rangle$ satisfies (C1), (C2) and (C3), i.e., $\langle\ \rangle \in C(E)$. Thus, two closure functions, cl and $\langle\ \rangle$, are defined in a preordered matroid.

In general, for two closure functions $\sigma, \tau \in C(E)$, we define the <u>localization</u> of σ by τ, denoted as σ/τ, by the identity:

$$(\sigma/\tau)(X) = \{e \in E \mid e \in \sigma(X \cap \tau(e))\} \tag{3.2}$$

for $X \subset E$. It is easy to show that σ/τ is a closure function on E.

<u>Prop. 3.1</u>. $\sigma/\tau \in C(E)$ for $\sigma, \tau \in C(E)$.

The closure function σ of a preordered matroid $(E,\geq,cl)$ is defined [3], [4] as the localization of cl by $\langle\ \rangle$, i.e.,

$$\sigma(X) = (cl/\langle\ \rangle)(X) = \{e\in E \mid e\in cl(X\cap\langle e\rangle)\}. \tag{3.3}$$

By Prop. 3.1, (E,σ) is a closure system, but is not necessarily a matroid. Prop. 3.4 below states that (E,σ) associated with a preordered matroid is an instance of the closure system with the elimination property. To prove it, we need the following propositions, where for $y\in E$,

$$[y] = \{e\in E \mid e\geq y \text{ and } y\geq e\}. \tag{3.4}$$

<u>Prop. 3.2</u>. Let σ be defined by (3.3). If $\sigma(X)=\sigma(Y)=E$, $y\in Y\backslash X$ and $\sigma(Y\backslash y)\neq E$, then there exists $x\in(X\backslash Y)\cap[y]$ such that $\sigma((Y\backslash y)\cup x)=E$.

(Proof) Put $Z=Y\backslash y$. First we claim that

$$x \notin \sigma(Z) \quad \text{for some } x \in X\cap\langle y\rangle. \tag{3.5}$$

Suppose $X\cap\langle y\rangle \subset \sigma(Z)$. Then for any $e \in X\cap\langle y\rangle$, we have $e \in cl(Z\cap\langle e\rangle) \subset cl(Z\cap\langle y\rangle)$, and consequently,

$$X\cap\langle y\rangle \subset cl(Z\cap\langle y\rangle). \tag{3.6}$$

Since $y\in\sigma(X)=E$, i.e., $y\in cl(X\cap\langle y\rangle)$, (3.6) implies $y\in cl(Z\cap\langle y\rangle)$, i.e., $y\in\sigma(Z)$, from which follows that $\sigma(Z)=\sigma(Z\cup y)=\sigma(Y)=E$, a contradiction to the assumption. Thus (3.5) is established.

Obviously, $x\notin Y$ in (3.5), since $x\notin\sigma(Y\backslash y)$ and $y\in Y\backslash X$. Also notice that $y\in\langle x\rangle$ holds in (3.5), since $x\notin\sigma(Z)$ and $x\in\sigma(Z\cup y)$, i.e., $x\notin cl(Z\cap\langle x\rangle)$ and $x\in cl((Z\cup y)\cap\langle x\rangle)$. Hence, (3.5) is strengthened to

$$x \notin \sigma(Z) \quad \text{for some } x \in (X\backslash Y)\cap[y]. \tag{3.7}$$

From (3.7) it follows that $x \notin cl(Z\cap\langle x\rangle)=cl(Z\cap\langle y\rangle)$, whereas $x \in cl(Y\cap\langle x\rangle)=cl((Z\cap\langle y\rangle)\cup y)$ since $\sigma(Y)=E$. The exchange property (C5) of cl implies that $y \in cl((Z\cap\langle y\rangle)\cup x)=cl((Z\cup x)\cap\langle y\rangle)$, i.e., $y\in\sigma(Z\cup x)$, which means $\sigma(Z\cup x)=\sigma((Y\backslash y)\cup x)=E$ when combined with $\sigma(Z\cup y)=E$. Q.E.D.

Prop. 3.3. Let σ be defined by (3.3). If $\sigma(X)=\sigma(Y)=E$ and $\sigma(Y\backslash y)\neq E$ for all $y\in Y\backslash X$, then for any $y\in Y\backslash X$ there exists $x\in X\backslash Y$ such that $Z=(Y\backslash y)\cup x$ satisfies $\sigma(Z)=E$ and $\sigma(Z\backslash z)\neq E$ for all $z\in Z\backslash X$.

(Proof) Prop. 3.2 guarantees that, for any $y\in Y\backslash X$, there exists $x\in(X\backslash Y)\cap[y]$ such that $\sigma(Z)=E$, where $Z=(Y\backslash y)\cup x$. For any $z\in Z\backslash X$ we have $y\notin\sigma(W)$, i.e.,

$$y\notin cl(W\cap\langle y\rangle), \tag{3.8}$$

where $W=Y\backslash\{y,z\}$.

Suppose $\sigma(Z\backslash z)=E$ for some $z\in Z\backslash X$. Then we have, in particular, $y\in\sigma(Z\backslash z)$, i.e.,

$$y \in cl((W\cap\langle y\rangle)\cup x). \tag{3.9}$$

By the exchange property of cl, it follows from (3.8), (3.9) and $\langle x\rangle=\langle y\rangle$ that $x \in cl((W\cap\langle y\rangle)\cup y)=cl((W\cup y)\cap\langle x\rangle)$, i.e., $x\in\sigma(W\cup y)$, which, along with $\sigma(W\cup x)=\sigma(Z\backslash z)=E$, implies $\sigma(W\cup y)=\sigma(Y\backslash z)=E$, a contradiction. Q.E.D.

Prop. 3.4. The closure function of a preordered matroid defined by (3.3) satisfies the elimination property (C6).

(Proof) Suppose (C6) fails and put $Y_0=Y$. Then by Prop. 3.3, there exists Y_1 such that $|Y_1|=|Y_0|$, $|X\cap Y_1|=|X\cap Y_0|+1$ and $\sigma(Y_1\backslash z)\neq E$ for all $z\in Y_1\backslash X$. Repeated application of Prop. 3.3 leads to a contradiction to Prop. 3.2, since $|Y\backslash X| > |X\backslash Y|$. Q.E.D.

4. Conclusion

It has been pointed out that a kind of greedy algorithm, called here the greedy elimination algorithm D, works for closure systems with elimination property (C6). This does not seem to fall in the category considered in [1] or [2].

References

[1] U. Faigle: The Greedy Algorithm for Partially Ordered Sets. Discrete Mathematics, Vol.28 (1979), pp.153-159.

[2] B. Korte and L. Lovasz: Greedoids --- A Structural Framework for the Greedy Algorithm. In "Progress in Combinatorial Optimization (W. R. Pulleyblank, ed.)", Academic Press, 1984, pp.221-243.

[3] K. Murota: Homotopy Base of Acyclic Graphs --- A Combinatorial Analysis of Commutative Diagrams by Means of Preordered Matroid. Discussion Paper Series No.238, Institute of Socio-Economic Planning, University of Tsukuba, 1984.

[4] K. Murota: "Preordered Matroid" --- Matroidal Dependence with Preorder Constraint (in Japanese), Proc. 1984 Fall Conference of OR Society of Japan, 2-D-8, pp.165-166.

[5] D. J. A. Welsh: Matroid Theory. Academic Press, 1976.