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by

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1. Introduction

In spite of vast number of literatures on the theoretical model of multi-region economy, a simple neoclassical model of two-region economy was, for the first time, worked out very recently by Carlberg (1981)[1]. Although he took the equivalence between the social optimum and a free market equilibrium almost for granted in his two-region model[2], we need some simple but rigorous assumptions to claim that. Boadway and Flatters (1982) dealt with this problem more carefully and discussed the possible inefficiency of free market (free migration) solution for a two-region economy in a setting of federal nation as well as policy measures to eliminate this inefficiiency. Their inclusion of the third factor of production other than capital and labour (e.g., land, natural resources, etc.) in production functions made their analysis more realistic than cases of linear homogeneous two-factor production functions on the one hand. On the other hand, this made it impossible for them to utilize the convenience of ratio-variable models.

Other cases in which policy intervention is justified were discussed in Borts (1966) very rigorously and in Boadway and Flatters (1981) a little more intuitively[3]. They are cases where we have unemployment in one of the regions owing to several reasons. Existence of migration cost *and* inflexibility of the wage rate in the poor region is one of possible combinations that cause the underemployment equilibrium. This interesting setting is, however, hardly applicable to a country like Japan where the interregional mobility of labour is extremely high, in other words migration cost is negligible.

Another case of possible policy intervention dealt with in Sakashita (1983a) and (1983b) was that in which other social objective than national income maximization was established. In these papers, the objective was a fixed ratio allocation of labour force (population) between two regions. Similar to the Carlberg's paper, equalization of *wage rates* between regions was taken for granted as a condition of market equilibrium without additional assumptions in these papers, and this could be an unrealistic specification in such an economy as we discuss **it in this** paper.

In the present paper, we wish to develop a series of neoclassical models of a two-region economy by which we can examine discrepancy between the market equilibrium and the social optimum under alternative specifications of the pattern of migration and to discuss policy measures to fill the discrepancy. This sort of analysis is, we think, quite relevant to an economy like present-day Japan where the reasons of people's migration among regions are becoming more and more complicated.

In the next section, a simplest model of no public sector goods and no region-specific factors is analyzed. Even in this simplest case, some intervention by the public sector becomes necessary in order to make the market equilibrium equivalent to the social optimum unless a particular pattern of rental income distribution between the two regions exists. An quantitative appraisal of the inefficiency of market equilibrium is also attempted. In section 3, uncontrollable region-specific factors are introduced into the utility functions. Then we see collapse of equivalence among the social optimum, productive efficiency, and the market equilibrium. The market equilibrium becomes only the second-best social optimum even with some intervention in this model. Inefficiency brought by presence of the region-specific factors is also quantitatively appraised. In section 4, public sector goods are introduced into the model instead of or in addition to uncontrollable region-specific factors. Controllability of the supply of own sector goods by the public sector assures the equivalence of the market equilibrium and the social optimum again but it cannot overcome the difficulty caused by the presence of uncontrollable region-specific factors. Section 5 which is followed by concluding remarks of the final section deals with cases of heterogeneous individuals in the sense of taste differentiation. If the public sector has sufficient number of policy instruments which are also feasible from the socio-political viewpoint, there will be no additional difficulty to achieve the social optimum directly or indirectly through the market mechanism. However, socio-political infeasibility of some policy instruments may make such performances impossible.

2. The social optimum for a two-region economy and its equivalence to the market equilibrium under certain assumptions

Consider an economy consisting of two regions which produce a homogenous output by employing two factors of production, i.e. capital and labour with neoclassical, linear homogeous, and well-behaved technology. Each region has a production function which characterizes its production technology and the production function of the one region differs from that of the other region in such a manner that the one function never dominates the other completely. This means that there is at least one common wage-rental ratio under which both production functions are guaranteed to have positive outputs simultaneously. Let us express the per capita production function of region i as $f_i(k_i)$ where k_i is the capital-labour ratio in that region.

We assume that endowments of capital and labour in the nation is fixed so is the national capital-labour ratio $\bar{k}$. If we denote the relative share of national labour to region i by n_i, we have the following two definitional relations among k_i, n_i, i=1, 2, and $\bar{k}$.

$$n_1 + n_2 = 1 \tag{2.1}$$

$$n_1k_1 + n_2k_2 = \bar{k} \tag{2.2}$$

Now we can give a definition of the national per capita output x by

$$x = n_1x_1 + n_2x_2 \tag{2.3}$$

in which x_i is the per capita output in region i, i.e. $x_i = f_i(k_i)$, i = 1, 2, and we can formulate the following maximizing problem:

$$\begin{aligned}&\text{Maximize } x\\ &\text{with respect to } k_i \text{ and } n_i,\ i = 1, 2,\\ &\text{and subject to (2.1) and (2.2).}\end{aligned} \tag{2.4}$$

Assuming the existence of an interior solution, the optimal conditions for (2.4) are easily derived as

$$f_1'(k_1) = f_2'(k_2) \tag{2.5}$$

$$f_1(k_1) - k_1f_1'(k_1) = f_2(k_2) - k_2f_2'(k_2) \tag{2.6}$$

(2.1), (2.2),

in which $f_i'(k_i) = \frac{df_i}{dk_i}$, i = 1, 2. Equations (2.5) and (2.6) are of course the well-known marginal productivity conditions[4]. It is also easy to check whether an interior solution actually holds or not in this case. By equations (2.5) and (2.6) we can obtain the optimal k_1 and k_2 for the case of an interior solution without consulting equations (2.1) and (2.2). If we can obtain the solutions for n_1 and n_2 both of which are positive by equations (2.1) and (2.2) for these optimal values of k_1 and k_2, we can say that we have an interior solution to the problem. In other words, $\bar{k}$ must be a convex combination of the optimal k_1 and k_2 for the existence of an interior solution.

A free market equilibrium for this two-region economy produces the same conditions as (2.5) and (2.6) provided that the following two assumptions are met. First, the capital moves freely between the regions corresponding to the rental (return) differential. This assumption which assures (2.5) can be taken as an indispensable characteristic of a free market economy beyond question. Secondly, the labour also moves freely between regions corresponding to the *wage differential*. This second assumption which corresponds to (2.6) can be justified only under certain institutional settings because the distributive pattern of rental income is not yet specified in our model.

One situation that justifies the second assumption is the case in which each labour unit of the nation receive an equal share of the national rental income and therefore equalization of per capita incomes y_i, i = 1, 2, of the two regions, distinct from per capita outputs x_i, i = 1, 2, becomes equivalent to that of wages. We may call this case "uniform national dividend scheme".

If there is any regional bias in the distribution of rental income, equation (2.6) ceases to be a market equilibrium condition. In the strongest case in which there is no interregional flow of rental income, we have the following expression for the regional per capita income.

$$y_i = w_i + rk_i \ , \quad i = 1,\ 2 \qquad (2.7)$$

In equation (2.7), $r = f_1' = f_2'$ and $w_i = f_i - k_i f_i'$, i =

1, 2, which correspond to the real rental and the real wage respectively. It is easily shown that $y_i = x_i = f_i(k_i)$, $i = 1, 2$, in this case so that the corresponding market equilibrium conditions will be as follows[5].

$$f_1'(k_1) = f_2'(k_2) \tag{2.8}$$

$$f_1(k_1) = f_2(k_2) \tag{2.9}$$

(2.1), (2.2)

Assuming that both of the systems expressed by equations (2.1), (2.2), (2.5), and (2.6) and by equations (2.1), (2.2), (2.8), and (2.9) (system I and system II) have interior solutions, although we need rather strong conditions for it in the case of system II[6], firstly we are interested in the degree of inefficiency involved in system II. In order to evaluate this we start with the following general system (system III) with equations (2.1) and (2.2)[7].

$$f_1'(k_1) = f_2'(k_2) = r \tag{2.10}$$

$$f_1(k_1)-\alpha k_1 f_1'(k_1) = f_2(k_2)-\alpha k_2 f_2'(k_2), \quad 0 \leqq \alpha \leqq 1 \tag{2.11}$$

If $\alpha = 1$ in system III we have system I and if $\alpha = 0$ system II follows. Actually system III implies a situation in which 100α percent of the regional rental income in each region is poured into the uniform national dividend but $100(1-\alpha)$ percent is retained within the region.

Applying a comparative static technique to system III, we obtain the following[8]:

$$\frac{dk_1}{d\alpha} = -\frac{1}{A} r(k_2 - k_1)f_2'' \tag{2.12}$$

$$\frac{dk_2}{d\alpha} = -\frac{1}{A} \; r(k_2 - k_1)f_1'' \qquad (2.13)$$

in which

$$A = (1-\alpha)(f_1'f_2''-f_2'f_1'') + \alpha f_1''f_2''(k_2 - k_1) \qquad (2.14),$$

and then

$$\frac{dn_1}{d\alpha} = \frac{1}{k_2-k_1}\left\{ n_1 \frac{dk_1}{d\alpha} + (1-n_1) \frac{dk_2}{d\alpha} \right\} \qquad (2.15)$$

$$\frac{dx}{d\alpha} = -\frac{1-\alpha}{A} \, r^2(k_2-k_1)\{n_1 f_2'' + (1-n_1)f_1''\} \qquad (2.16).$$

Without loss of generality we can assume $k_2 > k_1$ in the relevant range of solution to system III, and under this assumption we may further assume that

$$\left|\frac{f_1''}{f_1'}\right| > \left|\frac{f_2''}{f_2'}\right| \qquad (2.17)$$

again in the relevant range[9]. Under these assumption we can show that $A > 0$ and therefore

$$\frac{dx}{d\alpha} \geqq 0 \quad \text{for all } \alpha, \quad 0 \leqq \alpha \leqq 1 \qquad (2.18).$$

Now we can evaluate the difference between x in system I (x_I) and the same in system II (x_{II}) using a theorem of mean values as follows:

$$x_{II} - x_I = \int_0^1 \left(\frac{dx}{d\alpha}\right) d\alpha \qquad (2.19)$$

$$= \phi(\hat{\alpha}), \qquad \hat{\alpha} = \text{some } \alpha \text{ between 0 and 1,}$$

in which $\phi(\alpha)$ is the right-hand-side of equation (2.16). Equation (2.16) also implies that x attains its maximum when $\alpha = 1$ so that a regional cross transfer of income implied by a more-than-unity value of α generates an inefficient equilibrium as well as a partially-retained-rental-income equilibrium with $\alpha < 1$.

Second, we wish to explore a tax-subsidy policy which realizes the same production equilibrium as system I under the setting of system III. Denoting the optimal values of k_1, k_2, and n_1 in system I as k_1^*, k_2^*, and n_1^*, a self-financing tax-subsidy policy of this type will be expressed as follows:

$$(1+s)\{f_1(k_1^*)-\alpha k_1^* f_1'(k_1^*)\}=(1-t)\{f_2(k_2^*)-\alpha k_2^* f_2'(k_2^*)\} \tag{2.20}$$

$$sn_1^*\{f_1(k_1^*)-\alpha k_1^* f_1'(k_1^*)\}=t(1-n_1^*)\{f_2(k_2^*)-\alpha k_2^* f_2'(k_2^*)\} \tag{2.21}$$

in which s is the rate of income subsidy to region 1 and t is the rate of income tax to region 2. (Recall that $k_2^* > k_1^*$ and therefore $f_1(k_1^*) < f_2(k_2^*)$ but $f_1(k_1^*)-k_1^* f_1'(k_1^*)=f_2(k_2^*)-k_2^* f_2'(k_2^*)$.) From equations (2.20) and (2.21) we have

$$t = \frac{n_1^*}{f_2^*-\alpha k_2^* f_2'^*}\{(f_2^*-f_1^*)-\alpha r^*(k_2^*-k_1^*)\} \tag{2.22}$$

$$s = \frac{1-n_1^*}{f_1^*-\alpha k_1^* f_1'^*}\{(f_2^*-f_1^*)-\alpha r^*(k_2^*-k_1^*)\} \tag{2.23}$$

in which $f_i^* = f_i(k_i^*)$ and so on[10]. For system II with $\alpha = 0$ these equations become

$$t(0) = \frac{n_1^*}{f_2^*}(f_2^* - f_1^*) \tag{2.24}$$

$$s(0) = \frac{1-n_1^*}{f_1^*}(f_2^* - f_1^*) \tag{2.25}$$

which imply a simple scheme of regional income redistribution.

In this section we have examined the degree of inefficiency involved in regional income redistribution schemes other than the uniform national dividend scheme and also have discussed a possible tax-subsidy policy usable to restore the efficient equilibrium[11]. So far the cause of market distortion has been an institutional one, and therefore necessary form of intervention can also be a simple institutional measure of correction. Now we turn to the case in which region-specific factors enter as causes of distortion in the next section.

3. Existence of region-specific factors and discrepancy between the social optimum and the market equilibrium

In the previous section, we have been implicitly assuming that the level of utility of an individual and that of per capita income are practically equivalent to each other in both of the two regions. However, even if the utility functions are basically the same in the two regions they can be different as functions of per capita incomes between the regions because of the existence of region-specific factors $\bar{q}_i$, i = 1, 2, which affects the level of utility. Such factors could be regional amenities or attractive landscape influence of which is common to all residents in a specific region as a sort of local public goods.

In such a case, the utility of a resident in a specific region, u_i, is expressed by

$$u_i = u(y_i, \bar{q}_i) = u_i(y_i), \ i = 1, 2 \qquad (3.1)$$

in which y_i is the level of per capita income and $\bar{q}_1 \neq \bar{q}_2$.

Firstly we describe the social optimum in this setting. The maximizing problem takes the following form.

$$\text{Maximize } \{W=n_1u(y_1,\bar{q}_1)+n_2u(y_2, \bar{q}_2)\} \tag{3.2}$$

with respect to y_i, k_i, and n_i, $i=1, 2$,

and subject to (2.1), (2.2), and

$$n_1y_1+n_2y_2=n_1f_1(k_1)+n_2f_2(k_2) \tag{3.3}.$$

Again assuming the existence of an interior solution we have the following set of optimal conditions:

$$u_1' = u_2' = \mu \tag{3.4}$$

$$f_1' = f_2' = \frac{\lambda}{\mu} = r \tag{3.5}$$

$$u_1+\mu\{(f_1-k_1f_1')-y_1\}=u_2+\mu\{(f_2-k_2f_2')-y_2\} \tag{3.6}$$

(2.1), (2.2), (3.3),

in which λ and μ are Lagrangean multipliers corresponding to constraints (2.2) and (3.3). We call this system of social optimum system RS-I.

It should be noticed that we have equalization of marginal utilities of income between the two regions (see equation (3.4)) but not equalization of utilities in this social optimum. As an example let us assume that the utility function (3.1) is separable with respect to y and $\bar{q}$. Then certainly we will have $y_1 = y_2$ at the optimum by equation (3.4), but then it is obvious that $u_1 \neq u_2$ because $\bar{q}_1 \neq \bar{q}_2$ by assumption. Inequality of the utilities in this case means a difference between the residents of region 1 and region 2 in their *abilities* of utility enjoyment owing to the difference in regional amenities for instance. In the present case of separable utility, this utility difference is compensated by the opposite difference of marginal productivities of labour between the two regions (see equation (3.6)). The latter means that there is no equalization of these marginal productivities so that no maximization of per capita national output in system RS-I.

Equation (3.6) in general implies that the net social benefit produced by moving one labour unit (resident) from region 1 to region 2 or vice versa is exactly equal to zero as an intuitively clear condition of the optimum. For the capital which has no ability of enjoying utility by itself, we observe an ordinary condition of equalized marginal productivities between the two regions ((3.5)). Therefore we can say that the failure of marginal productivities equalization for the labour comes from its dual character as a factor of production and as a resident of the particular region enjoying utility. This duality also makes the comparison between the social optimum and the market equilibrium more complicated for this sort of model.

For the model with region-specific factors in utility functions, the market equilibrium under free migration and the uniform national dividend assumption takes the following form.

$$u_1(y_1) = u_2(y_2) \qquad (3.7)$$

$$f_1' = f_2' = r \qquad (3.8)$$

$$y_i = (f_i - k_i f_i') + r\,\bar{k}, \quad i = 1, 2 \qquad (3.9)$$

(2.1), (2.2)

Constraint (3.3) can be derived from equations (3.9) and

(2.2) so that it is not an independent constraint to the model. We call this type of market equilibrium system RS-II.

Apparently the market equilibrium described by system RS-II does not coincide with the social optimum of system RS-I. This situation is usually known as the inefficiency of free migration[12]. Since $\bar{q}_1 \neq \bar{q}_2$ it is obvious that $y_1 \neq y_2$ in general by equation (3.7), and it follows that the marginal productivities are not equalized between the two regions (see equations (3.9)) but by a different reason from in the case of system RS-I. In addition it is very important to notice that system RS-I cannot be derived from system RS-II using ordinary fiscal measure of the income transfer because the latter system cannot get rid of equalization of utilities in any case which does not exist in the former system.

However, whether the market equilibrium without intervention is also different from the possible second-best optimum in the sense of the best situation with an equal utility constraint or not is a different question. In order to see this point, we formulate the following sub-maximizing problem with an additional constraint of utility equalization.

Maximize $\{W=n_1u_1(y_1) + n_2u_2(y_2)\}$

with respect to y_i, k_i, and n_i, i=1, 2,and (3.10)

subject to (2.1), (2.2), (3.3), and

$u_1(y_1) = u_2(y_2)$ (3.11)

The set of optimal conditions will be:

$$\left(\frac{n_1 + \nu}{n_1}\right)u_1' = \left(\frac{n_2 - \nu}{n_2}\right)u_2' = \mu \quad (3.12)$$

$$f_1' = f_2' = r \quad (3.13)$$

$$(f_1 - k_1 f_1') - y_1 = (f_2 - k_2 f_2') - y_2 \quad (3.14)$$

(2.1), (2.2), (3.3), (3.11),

in which ν is a new Lagrangean multiplier corresponding to constraint (3.11). We may call this set of equations system RS-III but it is easily shown that system RS-II with equations (3.7), (3.8), (3.9), (2.1), and (2.2) is exactly equivalent to system RS-III with equations (3.11), (3.13), (3.14), (3.3), (2.1), and (2.2).[13] There is, therefore, no real distinct RS-III.

Now it has become clear that the market equilibrium with the assumption of uniform national dividend is just equal to the second-best social optimum in system RSs. Even in the case in which the above assumption is not met, we can find some regional income transfer scheme by which the market equilibrium is made equivalent to the send-best social optimum similar to the case of previous section. We are, however, more interested in the question how much will the per capita national output be increased when the two regions tend to have similar amenity conditions for instance, i.e. $\bar{q}_2$ approaches $\bar{q}_1$. In order to examine this matter, we start with the following summary of market equilibrium.

$$u\{f_1 - f_1'(k_1 - \bar{k}), \bar{q}_1\} = u\{f_2 - f_2'(k_2 - \bar{k}), \beta\bar{q}_1\}, \quad 0 < \beta \leqq 1 \quad (3.15)$$

$$f_1' = f_2' = r \quad (3.16)$$

For this system we have

$$\frac{dk_1}{d\beta} = \frac{1}{B} u_q \bar{q}_1 f_2'' \qquad (3.17)$$

$$\frac{dk_2}{d\beta} = \frac{1}{B} u_q \bar{q}_1 f_1'' \qquad (3.18)$$

in which

$$B = -f_1''f_2''(k_2-k_1)\{n_1 u_y(y_2,\beta\bar{q}_1)-(1-n_1)u_y(y_1,\bar{q}_1)\} \qquad (3.19)$$

and $u_y = \frac{\partial u}{\partial y}$ and $u_q = \frac{\partial u}{\partial q}$ (y_2,q_2) both of them being assumed to be positive. Referring to equation (2.15) (replace α by β), finally we have the following expression.

$$\frac{dx}{d\beta} = \left(r- \frac{f_2-f_1}{k_2-k_1}\right)\left(\frac{u_q\bar{q}_1}{B}\right)\{n_1 f_2'' + (1-n_1)f_1''\} \qquad (3.20)$$

Firstly let us assume $\beta < 1$, then it must be that $y_2 > y_1$ because both of u_y and u_q are positive. This implies $r < (f_2-f_1)/(k_2-k_1)$ under the assumption of $k_2 > k_1$ which does not harm generality so that the first bracket in the right-hand-side of (3.20) is negative. Second, since $y_2 > y_1$ it might be $u_y(y_1) > u_y(y_2)$. If the production technologies and the national capital-labour ratio ($\bar{k}$) are not so biased in favour of region 1, i.e. n_1 is not much bigger than $(1-n_1)$, we usually have $n_1 u_y(y_2)-(1-n_1)u_y(y_1) < 0$ and $B > 0$ subsequently. Under the circumstances described above, we will have $\frac{dx}{d\beta} > 0$ at a particular value of β which is smaller than unity. In addition it is highly likely for us to have the persistence of positive $\frac{dx}{d\beta}$ throughout $0 < \beta \leq 1$ because $\frac{dn_1}{d\beta} < 0$ when $B > 0$ particularly if $n_1 < 1-n_1$ when $\beta = 1$[14]. If we have all of these favourable conditions we can say that

$$\frac{dx}{d\beta} > 0 \qquad \text{for} \quad 0 < \beta \leqq 1 \tag{3.21}$$

and then we can make a similar quantitative evaluation of continuous change in β as was done by equation (2.19). At least we can say that the market equilibrium, welfare maximization, and productive efficiency are mutually compatible with each other only when $\beta = 1$ therefore $y_1 = y_2$ and $\frac{dx}{d\beta} = 0$.

In this section we have analyzed the difference between the social optimum and the market equilibrium in the presence of region-specific factors in the utility functions of residents. It has been shown that there is no way to fill the gap between the two systems by fiscal measures in this case and also that both systems have to sacrifice some of their productive efficiency in order to achieve the first-best or the second-best social optimum respectively.

4. Public sector goods as region-specific factors or in addition to region-specifice factors

In system RS-II, impossibility of bringing the market equilibrium to the social optimum came from the uncontrollability of region-specific factors. How will the picture change when these factors are manipulated by the public sector? To examine this problem let us replace $\bar{q}_i$ in the utility function (3.1) by public sector goods p_i supplied by that sector. Here we confine ourselves to a case in which the public sector goods have no externality, i.e. they are pure private goods but supplied by the public sector. The education voucher may be a good example.

Now the maximizing problem for the social optimum will take the following form.

Maximize $\{W=n_1 u(y_1, p_1) + (1-n_1)u(y_2, p_2)\}$ (4.1)

with respect to y_i, p_i, k_i, $i=1, 2$, and n_1, and subject to

$\bar{k}-n_1 k_1-(1-n_1)k_2=0$, (Lagrangean multiplier: λ) (4.2)

$n_1 f_1+(1-n_1)f_2-n_1 y_1-(1-n_1)y_2-n_1 p_1-(1-n_1)p_2=0$,

(Lagrangean multiplier: μ) (4.3)

Optimal conditions are as follows (system PS-I).

$$f_1' = f_2' = \frac{\lambda}{\mu} = r \quad (4.4)$$

$$u_y(y_1,p_1)=u_y(y_2,p_2)=u_p(y_1,p_1)=u_p(y_2,p_2)=\mu,$$

$$u_y = \frac{\partial u}{\partial y},\ u_p = \frac{\partial u}{\partial p} \quad (4.5)$$

$$u(y_1,p_1)-u(y_2,p_2)+\mu\{r(k_2-k_1)+(f_1-f_2)-(y_1+p_1)+(y_2+p_2)\} = 0 \quad (4.6)$$

(4.2), (4.3)

If we assume strict concavity of the utility function, a set of equations in (4.5) gives a solution of $y_1 = y_2$ and $p_1 = p_2$. Therefore $u(y_1, p_1) = u(y_2, p_2)$ and equation (4.6) is reduced to the usual condition on marginal productivities of labour, i.e.

$$f_1 - k_1 f_1' = f_2 - k_2 f_2' \quad (4.7).$$

Thus there is no contradiction between the social optimum and productive efficiency in this case of controllable region-specific factors and identical individuals. In addition this social optimum is attainable from a market equilibrium with the uniform national dividend. Let the solution to system PS-I be $y_1^* = y_2^*$, $p_1^* = p_2^*$, k_1^*, k_2^*, n_1^* and r^*. Then a common tax rate t^* derived from

$$y_i^* = (1-t^*)\{(f_i^* - k_i^* f_i'^*) + r^*\bar{k}\}, \ i=1,\ 2 \qquad (4.8)$$

and a corresponding common value of p_i^* given by

$$t^*\{(f_i^* - k_i^* f_i'^*) + r^*\bar{k}\} = p_i^*\ , \quad i=1,\ 2 \qquad (4.9)$$

assures the optimum-equivalent market equilibrium because conditions of marginal utilities and marginal productivities are already satisfied by equations (4.4), (4.5), and (4.7) as well as the equal utility requirement, and equation (4.9) is derived from equations (4.8) and (4.3).

Even if we do not have the uniform national dividend assumption, the social optimum can be still achieved by an additional lump-sum income transfer between the regions. Therefore, the region-specific factors do not cause any difficulties regarding the equivalence of the market equilibrium to the social optimum as far as they are controllable by the public sector. If there are, however, still uncontrollable region-specific factors as the third arguments of the utility functions, we encounter the difficulty again. In this new case there are two different utility functions, $u_1(y_1, p_1)$ and $u_2(y_2, p_2)$ practically, and equations (4.5) are changed into

$$u_{1y}(y_1, p_1)=u_{2y}(y_2, p_2)=u_{1p}(y_1, p_1)=u_{2p}(y_2, p_2)=\mu,$$

$$u_{iy} = \frac{\partial u_i}{\partial y_i}\ ,\ u_{ip} = \frac{\partial u_i}{\partial p_i}\ ,\ i=1,\ 2 \qquad (4.10)$$

for which we have $y_1 \neq y_2$, $p_1 \neq p_2$, and $u_1(y_1, p_1) \neq$

$u_2(y_2, p_2)$ in general. We can, therefore, derive neither the condition of productive efficiency nor the market equilibrium condition from the set of optimal conditions in this system with public sector goods and uncontrollable region-specific factors (system PS-II).

In this section the relation between controllability/uncotrollability of region-specific factors and the equivalence of the market equilibrium and the social optimum has been discussed by a model with an active public sector.

5. Heterogenous individuals

In this semi-final section we introduce heterogenous groups of individuals in the sense of different tastes into system PS-I analyzed in the previous section[15]. Now we have two groups of individuals by a fixed ratio of $\bar{n}_a$ and $(1-\bar{n}_a)$ having utility functions of $u^a(y, p)$ and $u^b(y, p)$. Letting the share of type x individuals in region i be n_{xi}, we can formulate the following maximizing problem[16].

$$\text{Maximize } \{W=n_{a1}\ u^a(y_{a1}, p_{a1})+n_{b1}\ u^b(y_{b1}, p_{b1}) +(\bar{n}_a-n_{a1})u^a(y_{a2},p_{a2})+(1-\bar{n}_a-n_{b1})u^b(y_{b2},p_{b2})\} \quad (5.1)$$

with respect to y_{ai}, y_{bi}, p_{ai}, p_{bi}, k_i, $i=1, 2$, n_{a1}, and n_{b1}, and subject to

$$\bar{k}-(n_{a1}+n_{b1})k_1-(1-n_{a1}-n_{b1})k_2=0, \text{ (Lagrangean multiplier: } \lambda) \quad (5.2)$$

$$(n_{a1}+n_{b1})f_1+(1-n_{a1}-n_{b1})f_2-n_{a1}(y_{a1}+p_{a1})-n_{b1}(y_{b1}+p_{b1}) -(\bar{n}_a-n_{a1})(y_{a2}+p_{a2})-(1-\bar{n}_a-n_{b1})(y_{b2}+p_{b2})$$

$= 0,$ (Lagrangean multiplier: μ) (5.3)

In this formulation we are assuming strong power of discrimination for the public sector in the sense that it can assign different per capita incomes as well as different amounts of per capita public sector goods to the individuals in different groups and in different region. With this strong discrimination, optimal conditions become as follows (system PSH-I).

$$f_1' = f_2' = \frac{\lambda}{\mu} \quad (5.4)$$

$$u_y^a(y_{a_1}, p_{a_1}) = u_y^a(y_{a_2}, p_{a_2}) = u_y^b(y_{b_1}, p_{b_1}) = u_y^b(y_{b_2}, p_{b_2})$$

$$= u_p^a(y_{a_1}, p_{a_1}) = u_p^a(y_{a_2}, p_{a_2}) = u_p^b(y_{b_1}, p_{b_1}) = u_p^b(y_{b_2}, p_{b_2}) = \mu,$$

$$u_y^a = \frac{\partial u^a}{\partial y},\ u_y^b = \frac{\partial u^b}{\partial y},\ u_p^a = \frac{\partial u^a}{\partial p},\ u_p^b = \frac{\partial u^b}{\partial p} \quad (5.5)$$

$$u^a(y_{a_1}, p_{a_1}) - u^a(y_{a_2}, p_{a_2}) + \mu\{r(k_2 - k_1) + (f_1 - f_2) - (y_{a_1} + p_{a_1}) + (y_{a_2} + p_{a_2})\} = 0 \quad (5.6)$$

$$u^b(y_{b_1}, p_{b_1}) - u^b(y_{b_2}, p_{b_2}) + \mu\{r(k_2 - k_1) + (f_1 - f_2) - (y_{b_1} + p_{b_1}) + (y_{b_2} + p_{b_2})\} = 0 \quad (5.7)$$

(5.2), (5.3)

Careful examination of equation (5.5) reveals that we will have $y_{a_1} = y_{a_2}$, $p_{a_1} = p_{a_2}$, $y_{b_1} = y_{b_2}$, $p_{b_1} = p_{b_2}$, and therefore $u^a(y_{a_1}, p_{a_1}) = u^a(y_{a_2}, p_{a_2})$ and $u^b(y_{b_1}, p_{b_1}) = u^b(y_{b_2}, p_{b_2})$ under the assumption of strict concavity of the utility functions. Then we again have the marginal productivities condition of equation (4.7) in addition to equation (5.4), and another expression of equation (5.2), i.e. $\bar{k} - n_1 k_1 - (1-n_1)k_2 = 0$ in which $n_1 = n_{a_1} + n_{b_1}$. These three equations suffice to determine the unique optimal values of k_1, k_2, and n_1 but there is no uniqueness regarding the optimal values of n_{a1} and n_{b1} in this case. But the productive efficiency still does not contradict the social optimum.

There is no fundamental contradiction between the social optimum and the market equilibrium for system PSH-I provided that the public sector can use discriminatory income tax rates as well as discriminatory amounts of the public sector goods supplied corresponding to difference in tastes of individuals. Such discrimination seems, however, to be very difficult in the actual circumstance of an economic society. Even in the planned economy of system PSH-I it might be impossible to discriminate the individuals on the basis of different tastes not on the basis of different residence.

In that case, the public sector will have additional constraints of $y_{a1} = y_{b1} = y_1$ and $p_{a1} = p_{b1} = y_2$ in system PSH-I, and for optimal conditions equation (5.5) will be replalced by

$$\frac{n_{a1}u_y{}^a(y_1,p_1)+n_{b1}u_y{}^b(y_1,p_1)}{n_{a1}+n_{b1}} = \frac{n_{a1}u_p{}^a(y_1,p_1)+n_{b1}u_p{}^b(y_1,p_1)}{n_{a1}+n_{b1}}$$

$$= \frac{(\bar{n}_a-n_{a1})u_y{}^a(y_2,p_2) + (1-\bar{n}_a-n_{b1})u_y{}^b(y_1,p_1)}{1-n_{a1}-n_{b1}}$$

$$= \frac{(\bar{n}_a-n_{a1})u_p{}^a(y_2,p_2) + (1-\bar{n}_a-n_{b1})u_p{}^b(y_2,p_2)}{1-n_{a1}-n_{b1}} = \mu \qquad (5.8)$$

if there is an interior solution (although it is dubious).

Since equations (5.4), (5.6), and (5.7) remain the same in this new system with y_1, y_2, p_1, and p_2 (system PSH-II), the marginal productivities condition ceases to be satisfied. Also there is no way to establish equivalence between the market equilibrium and the social optimum in this case because it is clear that in general $u^a(y_1, p_1) \neq u^a(y_2, p_2)$ and $u^b(y_1, p_1) \neq u^b(y_2, p_2)$ at the social optimum.

There might be a strong tendency in system PSH-II or in its market counterpart to regionally separate the individuals belonging to the different groups by having $n_{a1} = 0$ or $n_{b1} = 0$. This is a case of corner solution[17]. Complete separation of the two groups is, however, impossible except for a very special case in which $\bar{n}_a$ accidentally coincides with n_1^* or n_2^* which is a part of the solution to equations (2.1), (2.2), (2.5), and (2.6). So it is almost completely unlikely for system PSH-II or its market counterpart to have compatibility with productive efficiency.

In this section we have examined some consequences of introducing heterogeneity among individuals. It has been shown that attainability of the first-best social optimum and equivalence between the social optimum and the market equilibrium are dependent on availability or feasibility of multiple policy instruments for the public sector.

6. Concluding Remarks

In this paper we analyzed several cases of correspondence between the social optimum and the market equilibrium in a neo-classical two-region economy. There are three possibilities. The first is the case in which equivalence of the two systems is established without any policy intervention. Secondly, there are cases in which some appropriate policy intervention can bring the equivalence. Finally in some cases it is impossible to modify the market equilibrium to be equivalent to the social optimum with any policy instruments other than a compulsory planning with prohibition of free migration. Differentiation among the cases is dependent on the presence of uncontrollable region-specific factors and/or availability of sufficient number of policy instruments.

The present paper excluded the possibility of a federal nation with different levels of governments by assuming one consolidated public sector[18]. Also we excluded the analysis of public goods or quasi-public goods supplied by the public sector[19]. Finally heterogeneity of the individuals can be extended to include different labour productivities among them. This sort of simplicity in our models itself suggests possible directions of extending the analysis.

Footnotes

1. However, there was an argument concerning the exactly same model of two-region economy as Carlberg's in Sakashita (1980).
2. Carlberg (1981) pp.193-194.
3. Also see Sakashita (1970).
4. These conditions already appeared in Sakashita (1980) p.604, Carlberg (1981) p.194, and Sakashita (1983a) p.1176.
5. It is unlikely for us to have an interior solution to the system of equations (2.1), (2.2), (2.8), and (2.9) if $\bar{k} > k°$ in which $k°$ is the solution to $f_1(k°) = f_2(k°)$. See Mathematical Appendix for the detailed discussion.
6. See Mathematical Appendix again.
7. Notice that system III may have an interior solution even if system II fails to do so when α is close to 1.
8. This technique of using α was suggested by Mr. Jun Nishimura for which the present author wishes to thank him.
9. See Mathematical Appendix for a related discussion.
10. Both of the values of t and s naturally become zeros when $\alpha = 1$ because $r^* = f_1'^* = f_2'^* = (f_2^* - f_1^*)/(k_2^* - k_1^*)$ in system I.
11. A similar model to system II was already discussed in Mera (1975) but from a different viewpoint of trade-off between equity and efficiency.

12. See Boadway and Flatters (1982) particularly pp.620-623. Also see Hartwick (1980) in which region-specific factors appear in the production functions.

13. Equation (3.14) is easily derived from equations (3.9), and the reverse derivation of (3.9) from (3.14) etc. can be done as follows.

$$y_i-(f_i-k_if_i')=n_1\{y_1-(f_1-k_1f_1')\}+n_2\{y_2-(f_2-k_2f_2')\}$$
$$=n_1y_1+n_2y_2-n_1f_1-n_2f_2+(n_1k_1+n_2k_2)r=r\bar{k}, \quad i=1,\ 2.$$

14. Actually we can show that $\frac{dn_1}{d\beta} = \frac{u_q\bar{q}_1}{B(k_2-k_1)}\{n_1f_2'' + (1-n_1)f_1''\}$ in this case.

15. In Boadway and Flatters (1982), however, the difference in income-earning abilities was meant by the heterogenous labour. See pp.628-630 of their paper.

16. We are assuming equal social significance of all individuals regardless their tastes.

17. This case has some resemblance to the model in Tiebout (1956).

18. See Boadway and Flatters (1982) for a discussion of the federal economy.

19. See Hartwick (1980) in this respect.

Mathematical Appendix On the existence of a solution to system II

Let us specify an independent variable z at the level of which $f_1(k_1)$ and $f_2(k_2)$ are equalized (see figure A), i.e.

$$z = f_1(k_1) = f_2(k_2) \qquad (A.1).$$

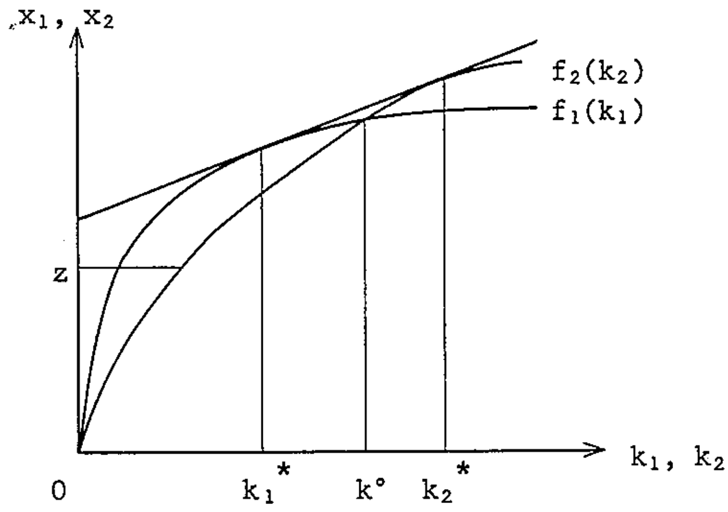

Figure A

Then let us define a function $\phi(z)$ as

$$\phi(z) = f_1'\{k_1(z)\} - f_2'\{k_2(z)\} \qquad (A.2),$$

in which $k_1(z)$ and $k_2(z)$ are the solutions to equation (A.1). By appropriately general specifications of $f_1(k_1)$ and $f_2(k_2)$, we can assume that

$$\lim_{z \to +0} \phi(z) > 0 \qquad (A.3)$$

and

$$\phi\{f_i(k°)\} < 0 , \quad i=1 \text{ or } 2 \qquad (A.4)$$

in which $k°$ is the solution to $f_1(k°) = f_2(k°)$ (see figure A).

Since we can safely assume the continuity of $\phi(z)$ there will certainly be a value of $z(=\hat{z})$ between 0 and $f_i(k°)$ which satisfies

$$\phi(\hat{z}) = f_1'\{k_1(\hat{z})\} - f_2'\{k_2(\hat{z})\} = 0 \qquad (A.5)$$

as well as equation (A.1). If we can get the national capital-labour ratio $\bar{k}$ as a convex combination of $k_1(\hat{z})$ and $k_2(\hat{z})$ both of which are smaller than k°, this is the case in which we have an interior solution to system II.

Since

$$\frac{d\phi}{dz} = \frac{f_1''\{k_1(z)\}}{f_1'\{k_1(z)\}} - \frac{f_2''\{k_2(z)\}}{f_2'\{k_2(z)\}} \qquad (A.6)$$

and it is most likely that $\frac{d\phi}{dz}$ is always negative in the case of (A.3) and (A.4), i.e. f_1 may have stronger curvature than f_2 does for the same value of z, there will be no solution to system II if $\bar{k} > k^\circ$. Even if we have an interior solution to system II, the possibility that we have $k_1^* < \bar{k} < k_2^*$ (k_1^* and k_2^* are solutions to equations (2.5) and (2.6)) at the same time is very limited. Let us see this point by an example of Cobb-Douglas functions such as

$$f_1(k_1) = Ak_1^{\alpha},\ f_2(k_2) = k_2^{\beta},\ A > 1,\ \beta > \alpha \qquad (A.7),$$

(see Sakashita (1983a) for a similar example).

For this example the solutions to equations (2.5) and (2.6) will be

$$k_1^* = \{A\ (\frac{\alpha}{\beta})^{\beta}\ (\frac{1-\alpha}{1-\beta})^{1-\beta}\}^{\frac{1}{\beta-\alpha}} \qquad (A.8)$$

$$k_2^* = \{A\ (\frac{\alpha}{\beta})^{\alpha}(\frac{1-\alpha}{1-\beta})^{1-\alpha}\}^{\frac{1}{\beta-\alpha}} \qquad (A.9)$$

and those to equations (A.1) and (A.5) will be

$$\hat{k}_1 = \{A\ (\frac{\alpha}{\beta})^{\beta}\}^{\frac{1}{\beta-\alpha}} \qquad (A.10)$$

$$\hat{k}_2 = \{A\ (\frac{\alpha}{\beta})^{\alpha}\}^{\frac{1}{\beta-\alpha}} \qquad (A.11).$$

A necessary condition for both of system I and system II to have interior solutions is $\hat{k}_2 > k_1^*$ and this requires

$$\left(\frac{\beta}{\alpha}\right)^{\beta-\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\beta} > 1 \qquad \text{(A.12).}$$

Relation (A.12) may be satisfied only when the gap between β and α is rather big that means great heterogeneity between the two regions' technology (e.g. $\beta=0.9$ and $\alpha=0.1$). Also notice that definitely $\hat{k}_1 < \hat{k}_2 < k^\circ$ $(=A^{\frac{1}{\beta-\alpha}})$ in this example.

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