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OPTIMAL STOPPING PROBLEM WITH UNCERTAIN RECALL

by

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Abstract

A model of stopping problem is examined in which an offer once passed up is available in the future with a known probability. The main results are as follows. With a finite horizon, the optimal stopping rule is characterized by the twofold reservation values for a present offer, causing the choice among the next three alternatives: 1. Stop the search with accepting the present offer, 2. Pass up it to continue the search, and 3. Stop the search with accepting the past best among offers which still remain available at present. This property, called a double reservation values property, disappears gradually as a time horizon tends to infinity.

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1. Introduction

Different models of an optimal stopping problem have been presented so far in many literatures of economics and management science, and various interesting aspects of this problem have been made clear. For the reason that they represent quite fitly some facets of human's decision behavior, many attempts have been being made to apply these models to examine dynamic natures underlying a class of economic or managerial phenomena which we encounter in the real-world.

Early examples in such applications include an inventory problem by [1], a house selling problem [16], a commodity purchasing problem [12], and so on. Presumably it is to the field of job search problem in labor economics [8],[9], [11],[15] that these models were most extensively applied. In almost all of them it is assumed that an offer once inspected and passed up either becomes immediately and forever unavailable (lost) or is available with certainty later on. The former is referred to as a stopping problem <u>with no recall</u>, the latter a stopping problem <u>with recall</u>. In studying actual economic or managerial decision problems by using these models, however, it will be rather realistic to postulate that an offer once passed up is available in the future with a known probability.

For example, suppose you are searching for a house to live in and have just found a desirable one for sale. Then probably you will avoid giving its owner an immediate answer about whether or not to buy it and will continue the search for another more desirable one. In this case, if it takes a long time to make the decision of buying it and then signify that effect of yours to its owner, then it might have been purchased by any other house searcher. If so and if unfortunately you could not find thereafter a more desirable one than it, it must be said to be a quite regretful end for you. Accordingly it will come to be a crucial problem to decide whether to stop the search with accepting the best among the currently available houses or to continue the search for another one.

In a job search problem, suppose a job searcher had an interview with a personnel manager of a certain company and the manager decided to hire him. If he postpones the decision of joining the company, then the company might hire other applicant before he notifies the intention of his to the manager.

In a research and development, the following problems will arise. Among some prototypes of new products that have been created till now, which and when should be marketed before the similar product might have been placed in the market by one of competitive companies ?

A stopping problem with such future unavailability of offers is called a stopping problem with uncertain recall. The previous works on this subject were made by Landsberger, et al. [7] and Karni et al. [5]. In the former it is assumed that the searcher is only concerned about the past best offer and neglects all the others and that this past best is unavailable at the next time with a known probability. They pointed out that a model in which the probability of future unavailability is defined for each of all past offers will require remarkably complicated and intractable mathematical treatments. The latter literature tackles a model with such difficulties and succeeds in deriving some interesting results but under some severe assumptions, say, that the probability for future availability of offer is strictly decreasing in age, that the marginal cost of search is strictly increasing with the number of searches and reinspections, and so on. The purpose of this paper is to reformulate strictly the model whose mathematical treatment were expected in [7] to become quite difficult and to reveal some interesting properties of its optimal stopping rule.

In Section 2 we present the strict definition of our model. Section 3 provides, using the simplest case, the rough sketch of the structure of the optimal stopping rule. Section 4 which follows summarizes the conclusions that are derived analytically in Section 5. The last section presents some interesting as well as important future studies which should be tackled.

2. Definition of Model

Consider the following version of the standard discrete-time stopping problem with a finite time horizon [2],[14] where points in time are numbered backward from the final time of the horizon, termed time 0. Assume that an offer is made immediately after paying a search cost $c \geq 0$ and that values of successive offers $w, w', w'', \ldots$ are independent identically distributed random variables with a known continuous distribution function $F(w)$ having a finite expectation E. For given real numbers X, Y with $0 < X < Y$, let $F(w) = 0$ for $w < X$, $0 < F(w) < 1$ for $X \leq w \leq Y$ and $F(w) = 1$ for $Y < w$, so $X < E < Y$. Here notice the following. Although an offer w takes in actuality only value on interval $[X,Y]$ owing to the assumption of its distribution function, the domain of the w can be extended, without loss in generality, to $[0,\infty)$ in an analytical treatment of the model. In the case, an offer w with $w < X$ or $Y < w$ should be called a fictitious offer.

Let p_j, $j = 0,1,\ldots$, represent the probability of an offer which was inspected and passed up j periods ago becoming unavailable at the next time, provided that it has been being available till now (so p_0 is the probability that an offer made at present is unavailable at the next time). Here we shall assume that, for any past offer, it is always possible to know free and instantly whether it is currently available or not. Furthermore postulate that there exists a fixed integer $N \geq 1$ such that $0 < p_j < 1$ for $j < N$ and $p_j = 1$ for $j \geq N$. This implies that any offer becomes unavailable with certainty up to N+1 periods after, in other words, every offer has the maximum age of N periods. Let α $(0 < \alpha \leq 1)$ represent a discount factor.

The objective is to maximize the expected gain under the condition that the searcher must accept one offer up to time 0, where the expected gain means the expectation of "the value of an offer accepted less the total search cost incurred up to the termination of the process with its acceptance".

Now suppose the process starts from time t. Then let w_j denote the value of an offer which was made j periods ago, or at time t+j. If the offer w_j has already been unavailable at present, then let $k_j = 0$, otherwise $k_j = w_j$, so always $k_0 = w_0$. For convenience of explanation, we shall use the term offer k_j. Then since the event of $k_j = 0$ implies that the offer w_j has already been withdrawn, such offer k_j should also be called a fictitious offer. Define the vectors $K = (k_0,k_1,\ldots,k_N)$, $R = (k_0,k_1,\ldots,k_{N-1})$ and $G = (k_1,k_2,\ldots,k_N)$, and let the maximum elements in these vectors be denoted by, respectively, k, r, and g. Furthermore let $\bar{K} = \{0,1,\ldots,N\}$, $\bar{R} = \{0,1,\ldots,N-1\}$ and $\bar{G} = \{1,2,\ldots,N\}$, which represent the sets of the subscripts of k_j in these vectors. We shall sometimes write the vector K as (k_0,G).

A state of the decision process can be described by the vector K. Hence the state space is defined by $I = \{(k_0,k_1,k_2,\ldots,k_N) \mid 0 \leq k_j < \infty,\ j \in \bar{K}\}$. Let A_t denote an action space of time t. Then we have $A_0 = \{x_1\}$ and $A_t = \{x_0,x_1\}$ for $t \geq 1$ where x_1 = stop the search with accepting the best offer k and x_0 = continue the search for another offer. The simplest form of a decision rule is a mapping from state space I onto action space A_t. We shall, however, extend the domain of decision rules up to randomized, history-dependent ones. By a decision policy we shall mean the time sequence of decision rules. Then define

$v_t(K)$ = the maximum expected gain over all possible decision policies, starting from time t with offers $K \in I$.

$V_t(G) = E[v_t(k_0,G)]$, the expectation with respect to offer k_0 $(= w_0)$. Let $V_t = V_t(G)$ with $k_j = 0$ for all $j \in \bar{G}$.

From the definition of the model, clearly

$$v_0(K) = k, \quad K \in I \tag{2.1}$$

We shall refer to the decision policy attaining the $v_t(K)$ for all $K \in I$ and all $t \geq 0$ as an optimal decision policy, which is given by a non-randomized, history-independent decision rule [3]. When $t \geq 1$, if action x_1 is taken, then the gain obtained is k, and if action x_0 is taken, then the maximum expected gain is given by $U_{t-1}(R)$, which implies the present value of the maximum expected gain starting from time t-1 less an additional search cost.

Let R* represent the N-vector resulting from replacing part of elements in vector R by value 0, and let an element of R* be denoted by k_j^*, that is, $R^* = (k_0^*,k_1^*,\ldots,k_{N-1}^*)$. There exist 2^N different kinds of R* in all. Now define

$$q_j(1) = 1 - p_j \text{ and } q_j(0) = p_j, \tag{2.2}$$

where $q_j(1)$ $(q_j(0))$ means the probability that offer $k_j \in R$ is available (unavailable) at the next time. Therefore the probability of offers R changing into offers R^* at the next time is provided by the product

(2.3) $P(R^*) = \Pi\, q_j(n_j)$, a product over all $j \in \bar{R}$,

where if offer k_j is available, then let $n_j = 1$, otherwise $n_j = 0$. Then $U_{t-1}(R)$ is expressed as

(2.4) $U_{t-1}(R) = \alpha\, \Sigma\, P(R^*)V_{t-1}(R^*) - c$, a sum over all possible R^*,

where note

(2.5) $U_{t-1}(0) = \alpha V_{t-1} - c$.

From the principle of optimality in dynamic programming, for all K and $t \geq 1$, we have

(2.6) $v_t(K) = \max\{k,\ U_{t-1}(R)\}$.

Now define

(2.7) $Q_{t-1}(K,j) = U_{t-1}(R) - k_j, \quad j \in \bar{K}$.

Suppose k_j is the maximum element in K, that is, $k_j = k$. Then if $Q_{t-1}(K,j) < 0$, it is optimal to stop the search with accepting the best offer k_j, otherwise to continue the search. Define the following sets.

(2.8) $S_t(j) = \{K | Q_{t-1}(K,j) < 0\}$ for all $j \in \bar{K}$ and

(2.9) $S_t = \cup\, S_t(j)$, a union over all $j \in \bar{K}$

Now let $k_j = k$. If $K \in S_t$, then $K \in S_t(i)$ for at least one $i \in \bar{K}$, and hence $Q_{t-1}(K,i) < 0$. This yields $Q_{t-1}(K,j) < 0$ due to $k_j \geq k_i$. Therefore it becomes optimal to stop the search with accepting the offer k_j. On the contrary, if $K \notin S_t$, then $K \notin S_t(j)$ for all $j \in \bar{K}$, hence $Q_{t-1}(K,j) \geq 0$ for all $j \in \bar{K}$, which implies that it is optimal to continue the search. Consequently we may refer to S_t as a stop region and to its complement, denoted by C_t, as a continuation region. The current paper will prove that the optimal stopping rule, prescribed by the above sets, has an interesting property such as defined below.

DEFINITION 0. A stopping rule of a certain point in time is said to have a double reservation values property if there exist the following two values ξ, ξ' such as $X < \xi < \xi' < Y$ for at least one fixed $G = (k_1, k_2, \ldots, k_N)$. For a given present offer w_0, if either $w_0 < \xi$ or $\xi' < w_0$, it is optimal to stop the search with accepting the best among presently available offers K, otherwise to continue the search.

That a decision rule has the double reservation values property means that, when w_0 travels from X toward Y, the decision to make changes from stop to continuation at $w_0 = \xi$ and from continuation to stop at $w_0 = \xi'$. The usual definition of a reservation value (price, wage) [4],[8],[13] claims that if, for a given present offer w_0, it is optimal to continue the search, then so also is to continue the search for any offer w_0' with $w_0' < w_0$. This means that there must exist a maximum δ such that, for any offer $w_0 < \delta$, it is optimal to continue the search. The δ is just what is commonly called a reservation value. When there exists such the reservation value, the problem in question is said to have a reservation value property. Accordingly it follows that our problem has not always a reservation value property in a sense of the conventional definition.

Before procceeding to further discussions, we shall here provide in advance one lemma that will be used in the subsequent sections. Define

$$(2.10) \qquad T(x) = E[\max\{w - x, 0\}] = \int_{w>x} (w - x)dF(w), \quad -\infty < x < \infty,$$

$$(2.11) \qquad H(x) = \alpha(x + T(x)) - x - c.$$

which are continuous functions. Let the solution to $H(x) = 0$ (if exists) be denoted by h^*. When there exist more than one solutions, let h^* represent the smallest one of them. The following lemma will be used in the subsequent sections.

LEMMA 0. We have

(a) $T(x)$ is a decreasing convex function, which is strictly decreasing on $x \leq Y$ and equal to $E - x$ on $x \leq X$ and to 0 on $Y \leq x$.

(b) $x + T(x)$ is an increasing convex function, which is strictly increasing on $X \leq x$ and equal to E on $x \leq X$ and to x on $Y \leq x$.

(c) $E[\max\{w,x\}] = x + T(x)$

(d) Suppose $(1-\alpha)^2 + c^2 \neq 0$. Then the solution h^* of $H(x) = 0$ is unique. If $X < \alpha E - c$, then $X < h^* < Y$, and if $\alpha E - c \leq X$, then $h^* = \alpha E - c$ where $H(x) > 0$ for $x < h^*$ and $H(x) < 0$ for $h^* < x$. If $(1-\alpha)^2 + c^2 = 0$, then $h^* = Y$ where $H(x) > 0$ for $x < h^*$ and $H(x) = 0$ for $h^* \leq x$.

(e) If $g(w) < x$ for $X \leq w \leq X + \varepsilon$ with infinitesimal $\varepsilon > 0$, then

$$E[\max\{g(w),x\}] > E[g(w)]. \quad \blacksquare$$

Proof: Easy for (a) to (d). (e). Clear from $\max\{g(w),x\} > g(w)$ and $f(w) > 0$ for $X < w < X + \varepsilon$. $\square$

Note: The inequality $\alpha E - c \leq X$ implies $(1-\alpha)^2 + c^2 \neq 0$ because if not so, then we have the contradiction of $E \leq X$).

3. Simple Cases

It is indicated in section 5 that the optimal stopping rule of our model is characterized by a remarkably complicated structure. Then, in order to facilitate the understanding of a general configuration of the structure, we shall give in the section its rough sketch, using the simplest case with N = 1, i.e., $0 < p_0 < 1$, $p_j = 1$ for $j \geq 1$. In the case, $K = (k_0,k_1)$, $R = (k_0)$, $G = (k_1)$, $\bar{K} = \{0,1\}$, $\bar{R} = \{0\}$, $\bar{G} = \{1\}$, and $k = \max\{k_0,k_1\}$. Now suppose the process starts from time 1 with offer k_0 (= w_0). Then we have

$$v_1(k_0,k_1) = \max\{k, U_0(k_0)\}$$

$$U_0(k_0) = \alpha p_0 E[w] + \alpha(1-p_0)E[\max\{w,k_0\}] - c$$
$$= \alpha p_0 E + \alpha(1-p_0)(k_0 + T(k_0)) - c \qquad \text{(Lemma 0(c))}$$

where E[•] denotes an expectation as to offer w of time 0. Then the stop region S_1 is given by the union of

$$S_1(0) = \{(k_0,k_1)|Q_0((k_0,k_1),0) < 0\}, \quad Q_0((k_0,k_1),0) = U_0(k_0) - k_0$$
$$S_1(1) = \{(k_0,k_1)|Q_1((k_0,k_1),1) < 0\}, \quad Q_1((k_0,k_1),1) = U_0(k_0) - k_1.$$

It can be easily seen by use of Lemma 0(a,b) that the continuation region C_1 is depicted as the domain enclosed by bold lines in Figure 1. Here the curved line $\widehat{ab}$ and the straight line $\overline{cb}$ are the graphs of points (k_0,k_1) satisfying, respectively, $U_0(k_0) - k_1 = 0$ and $U_0(k_0) - k_0 = 0$ where $h_1(1) = U_0(0) = \alpha E - c > 0$ and h_1 is the unique positive solution to $U_0(h) - h = 0$. Then we have $h_1(1) \leq h_1$ because of $h_1(1) = U_0(0) \leq U_0(h_1) = h_1$ owing to $h_1 > 0$. Suppose $h_1(1) < h_1$, and let x be the solution to $U_0(x) = k_1$ for any given k_1 with $h_1(1) < k_1 < h_1$. Then the optimal decision becomes as follows. If $h_1 < k_0$, then stop the search with accepting the offer k_0, if $x \leq k_0 \leq h_1$, then continue the search, and if $k_0 < x$, then stop the search with accepting the offer k_1. Accordingly it follows that the optimal stopping rule of time 1 has a double reservation values property.

It is obvious that this property is lost if $h_1 = h_1(1)$. Then the question arises as to what conditions assure the inequality $h(1) < h_1$ effecting the property. Let $\alpha E - c \leq X$. Then since $U_0(h_1(1)) = U_0(\alpha E - c) = \alpha E - c = h_1(1)$, it must be that $h_1 = h_1(1)$ because h_1 is the unique solution to $U_0(h) = h$. Hence,

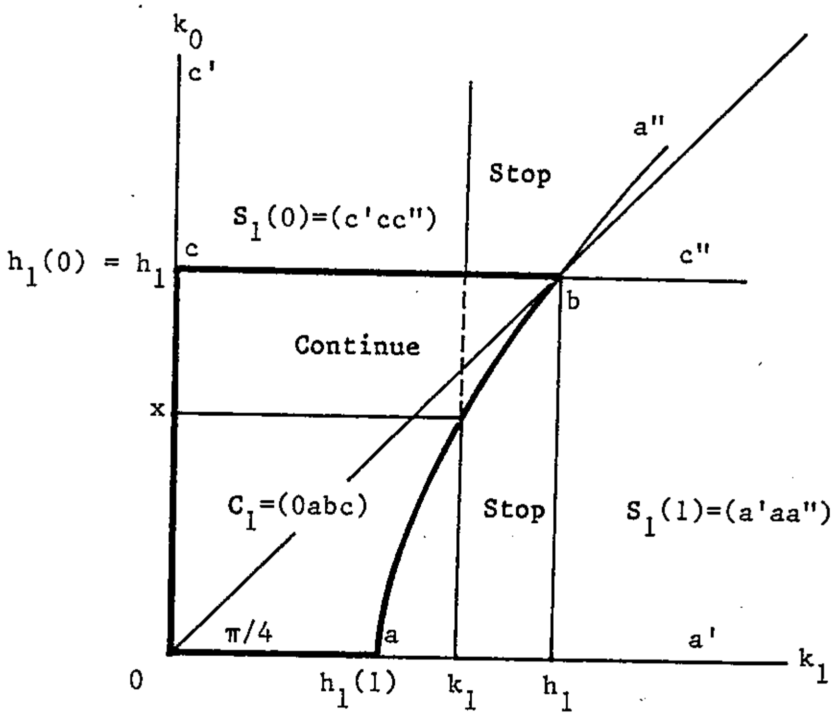

Fig 1: Stop region $S_1 = S_1(0) \cup S_1(1)$ and continuation region C_1 (enclosed by bold lines) of time 1 for case with $0 < p_0 < 1$ and $p_j = 1$ for $j \geq 1$.

if $\alpha E - c \leq X$, then it follows that the property does not appear. Suppose $\alpha E - c > X$. Then we have $h_1 > h_1(1)$ because $h_1 = U_0(h_1) \geq U_0(h_1(1)) = U_0(\alpha E - c) > U_0(X) = \alpha E - c = h_1(1)$. Thus it follows that the inequality $\alpha E - c > X$ provides a necessary and sufficient condition on which the optimal stopping rule of time 1 has the double reservation values property. The present paper will verify that this holds for any $t \geq 2$ not only for this simplest case but also for case with any $N \geq 2$.

In case with $N = 2$, i.e., $0 < p_j < 1$ for $j = 0, 1$ and $p_j = 1$ for $j \geq 2$, the continuation region C_t is provided by the curved cube as being depicted in Figure 2, enclosed by three coordinate planes and three curved surfaces. This figure demonstrates that it is the very <u>curvature of a continuation region</u> that causes the double reservation values property. Two points A and B in the figure, at which the straight line LL' intersects with the surfaces, provide twofold critical numbers, ξ and ξ', at which the decision to make changes from stop to continuation and vice versa when k_0 travels from point L to L'.

4. Summery of Results and Considerations

Define the following sets.

(4.1) $\quad O = \{K \mid 0 \leq k_j \leq \alpha E - c,\ j \in \bar{K}\}$

(4.3) $\quad Y = \{K \mid 0 \leq k_j \leq Y,\ j \in \bar{K}\},$

(4.4) $\quad H^* = \{K \mid 0 \leq k_j \leq h^*,\ j \in \bar{K}\},$

(4.5) $\quad H_t = \{K \mid 0 \leq k_j \leq h_t,\ j \in \bar{K}\},$

(4.6) $\quad H_t' = \{K \mid 0 \leq k_j \leq h_t(j),\ j \in \bar{K}\},$

(4.7) $\quad X = \{K \mid 0 \leq k_j \leq X,\ j \in \bar{K}\},$

where $h_t(j)$ is a k_j-intercept of continuation region C_t, and h_t is the coordinates of the intercept of C_t and straight line emerging from origin at angle of $\pi/4$ with each coordinate axis (Figure 2). The main results are:

1. Condition for stop with one search

If $\alpha E - c \leq X$, then $C_t = H^* = O$ for all $t \geq 1$ (Theorem 3(b)) where $h^* = \alpha E - c$. This implies the following. Suppose the process starts from any time t without any offer. Then if once an offer w_0 is made, then it follows that the searcher has offers $K = (w_0,0,0,\ldots,0) \notin C_t$ due to $X < w_0$, and hence that it is optimal to accept the offer w_0 because of $K \in S_t$. In other words, the inequality $\alpha E - c \leq X$ provides the condition on which it is optimal to make a search only one time with accepting an offer from it. This is an intuitively obvious conclusion because $\alpha E - c \leq X$ implies that a net profit gained from a further additional search, $\alpha E - c$, does not make up for passing up the present offer w_0 due to $\alpha E - c < w_0$.

2. Condition for double reservation values property

If $\alpha E - c > X$, then the double reservation values property appears strictly for all $t \geq 1$ (Theorem 6(a)). Why such property will appear in a finite horizon are reasoned as follows, using Figure 3. Consider the vector $G = (k_0,k_1)$ in the figure. Now suppose the present offer k_0 is not so large as to be in the bar L -A in the figure. Then a continuation decision may bring, on the searcher, such risk that the present best offer $k = \max\{k_0,k_1,k_2\}$ becomes unavailable at the next time and unfortunately no any better one than the lost best could be encountered over the remaining finite horizon. Consequently, it will follow presumably in the case that it becomes optimal to stop a search with accepting the present best offer k. If the present offer k_0 is neither so small nor so large to be in the bar A-B, then the risk stated above might be reduced to a certain degree (A degree of the risk for each of the above two cases is of course relative. Consider the following two examples as present available offers: K' = (5,0,20) and K" = (10,0,20). Then although the values of the best ones in offers K' and K" are equal to 20 (= k' = k"), if they will be unavailable at the next time, then the second bests are 5 for K' and 10 for K". Consequently offer K' can be said to be more risky, in a sense that we presented above, than offers K"). Accordingly it might not be so unreasonable that the searcher plans to obtain a better offer than the present best by continuing a search. If the present offer k_0 is enough large to be in the bar B-L', then it is of course that it would be quite reasonable rather to accept the present offer k_0 than to continue a search with such risk.

3. Structure of continuation region

Suppose $\alpha E - c > X$. Then, for any $t \geq 1$, the continuation region C_t is given by (N+1)-dimensional cube, which is enclosed by N+1 coordinate planes and N+1 curved planes (Figures 1,2,3, Theorem 1). The continuation region C_t is increasing in t and has the inclusion relationships of $Y \supset H^* \supset H_t \supset C_t \supset H_t' \supset 0$ (Theorems 3(a), Lemmas 7(c)). The following are true from Lemma 0(d). If $(1-\alpha)^2 + c^2 \neq 0$ and $\alpha E - c > X$, then $Y \supsetneqq H^* \supsetneqq X$, and if $\alpha E - c \leq X$, then $X \supset 0 = C^*$, where h^* is a unique solution of $H(x) = 0$. If $(1-\alpha)^2 + c^2 = 0$, then $Y = h^*$, and hence $Y = C^*$. Furthermore, for all $t \geq 1$, it follows from Theorem 6(b) that, for all $t \geq 1$,

a. If N = 1, then $h_t(1) < h_t$ and $h_t(0) = h_t$,

b. If $N \geq 2$, then $h_t(N) < h_t(j) < h_t$ for all $j \in \bar{R}$, and

c. If $p_i >(=) p_j$, then $h_t(i) <(=) h_t(j)$.

It follows from the above and Theorem 2(b) that it is optimal to

d. Continue a search if $k_j \leq h_t(N)$ for all $j \in \bar{K}$,

e. Stop a search if $k_j > h_t$ for at least one $j \in \bar{K}$,

Furthermore, since $h_t(N)$ is increasing in t (Lemma 5(b)) and $h_t \leq h^*$ (Lemma 4(b)), it is optimal to

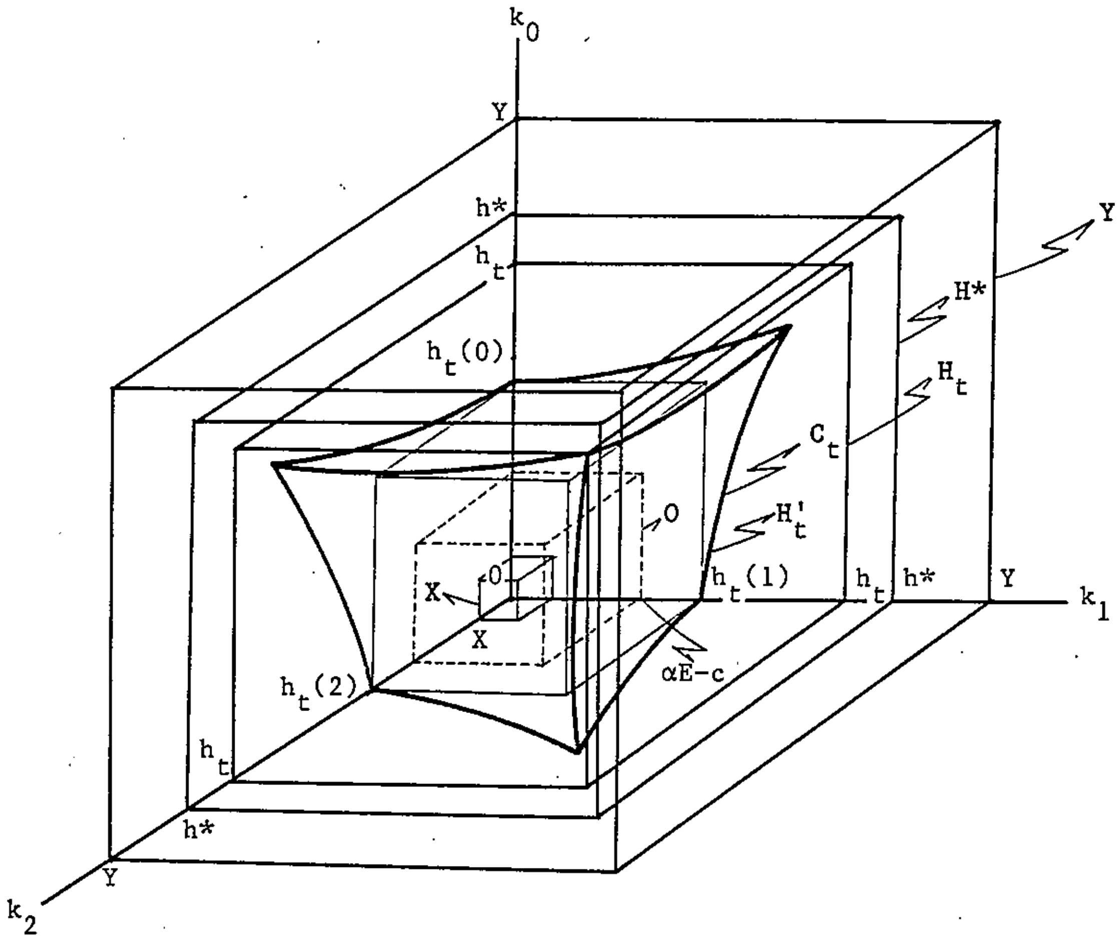

Fig 2: Inclusion relationships, $Y \supset H^* \supset H_t \supset C_t \supset H'_t \supset 0 \supset X$, of continuation region C_t and other regions, given $\alpha E - c > X$

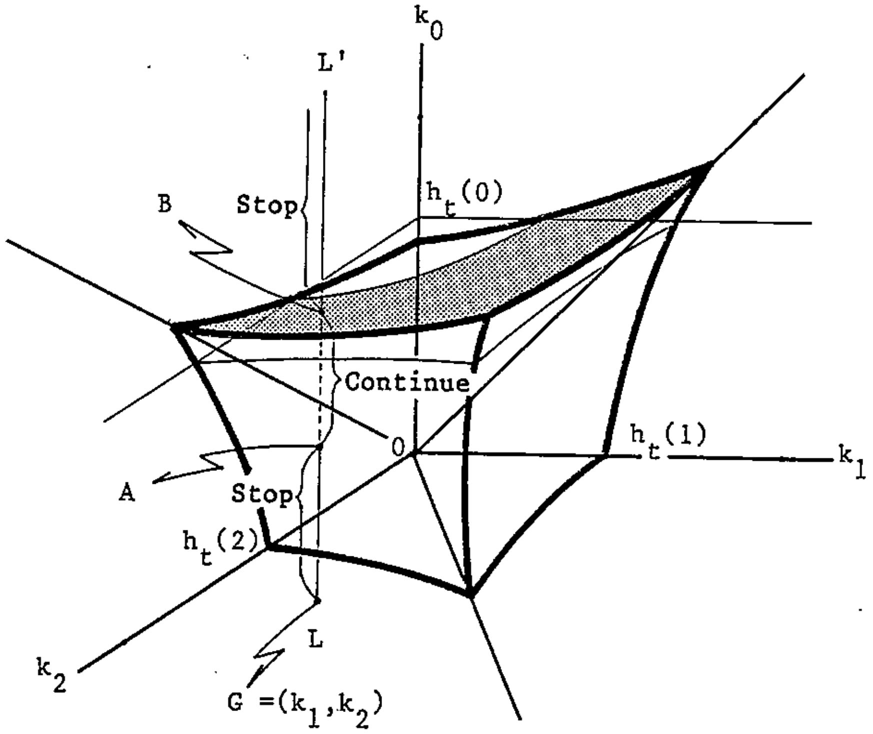

Fig. 3: Double reservation values property caused by a curvature of continuation region C_t in which points A and B provide twofold critical numbers.

f. Continue a search if $k_j \leq h_1(N)$ for all $j \in \bar{K}$,

g. Stop a search if $k_j > h^*$ for at least one $j \in \bar{K}$,

4. Maximum expected gain, search amount, and value realization

Suppose the process starts with offers $K \in H^*$, given a finite horizon. Then the expected gain attained, the search amount (the expected number of searches), and value realization (the expectation of the value of an offer accepted) are not greater than or equal to, respectively, the expected gain, the search amount and the value realization for the standard stopping problem with recall, provided that the optimal stopping rule are followed (Theorem 7(a),8(b)). If $(1-\alpha)^2 + c^2 \neq 0$, then, for the both problems, they converge to the same finite values as a time horizon tends to infinity, provided that K is an inner point of H* (Theorem 7(c),8(b)).

5. Fading double reservation values property as a horizon tends to infinity

A continuation region C_t (curved cube) is increasing in t and converge to H* (a perfect cube) as $t \to \infty$ (Theorems 4(a),5(b)). This implies that the double reservation values property disappears gradually as a time horizon tends to infinity and vanishes in its limit and moreover that the optimal stopping rule with a limiting horizon becomes just the same as the one for the standard stopping problem with recall because h* gives a reservation value of it (the standard stopping problem with recall). This conclusion, derived purely analytically, may be plausible intuitively for the following reason.

Consider again the vector $K = (k_0, G)$ in figure 3. Then suppose that a continuation decision is made, given a sufficiently large horizon, and that the present best offer is lost at the next time. In the case, the risk described in 2. could hardly be incurred because a better offer than the lost best will be almost possibly made sometime during the remaining horizon. Accordingly it follows eventually that the double reservation values property may vanish in the limiting horizon. However, this explanation may not always be persuasive from the following reason. The fact that a cost is incurred every search cannot allow the searcher to continue the search as sufficiently many time as he wants. Accordingly it possibly follows that such risk cannot always be eventually avoided even with a sufficiently large horizon, and hence that the double reservation values property will be preserved even in a limiting horizon. The analytical results obtained in the current paper, however, fully denies this conjecture, and claims that the double reservation values property will fade gradually as a horizon becomes larger and larger and vanishes totally in its limits.

6. Reduction to case with N = 1 in a limiting horizon

Consider case with a limiting horizon. Let $K' = \{k_1, k_2, \ldots, k_{N+1}\}$ be offers of time t+1 (the previous time) and $k' = \max K'$ where clearly $g \leq k'$ ...(*). Now suppose the process continued up to time t+1. Then, since $\lim_{t\to\infty} C_t = H^*$, the optimal stopping rule of time t (the present time) becomes as follows. If $k \leq h^*$, then stop, otherwise continue. In the case, the following results.

If $k \leq h^*$, then $w_0 \leq h^*$ because of $w_0 \leq k$. On the contrary suppose $h^* < k$. Now the assumption that the process had continued up to time t+1 implies $k' \leq h^*$, which yields $g \leq h^*$. Therefore $k = \max\{w_0, g\} \leq \max\{w_0, h^*\}$, from which $h^* < w_0$ follows because $w_0 \leq h^*$ produces the contradiction of $k \leq h^*$. The above things implies that the stopping rule is merely reduced to a comparison of h^* and w_0. In other words, it always suffices to memorize only the current offer w_0 with neglecting all past ones. Eventually it follows that, in a limiting horizon, our model is always reduced to the model with N = 1, i.e., $0 < p_0 < 1$, $p_j = 1$ for $j \geq 1$, and the optimal stopping rule is the same as that for the standard stopping problem with recall.

5. Analysis

5.1. Monotonicity of Optimal Stopping Rule

Throughout this section, a function $f(x)$ is said to be increasing (decreasing) in x if $f(x) \geq (\leq) f(y)$ for any $x > y$ on its domain and strictly increasing (strictly decreasing) in x if $f(x) > (<) f(y)$ for any $x > y$ on its domain. Furthermore a function $f(Z)$ of vector Z is said to be increasing (decreasing) in Z if it is increasing (decreasing) in each element of Z.

LEMMA 1. For all $t \geq 0$,

(a) $V_t(G) \geq E$ and $V_t(G) \geq k_j$ for $j \in \bar{G}$.

(b) $Y \geq V_t(G)$, and $V_t(G)$ and $U_{t-1}(R)$ are increasing as well as convex in R.

(c) $Q_{t-1}(K,j)$ is increasing in k_i with $i \neq j$ and convex in K. ❙

Proof: (a). Obvious from $v_t(k_0,G) \geq k_j$ for $j \in \bar{K}$. (b). Since $v_0(k_0,G) = \max\{k_0,g\} \leq Y$, we have $V_0(G) \leq Y$ and $V_0(G) = g + T(g)$ (Lemma 0(c)). Since g is increasing and convex in G and $g + T(g)$ is increasing and convex in g (Lemma 0(b)), $V_0(G)$ is also increasing as well as convex in G. Suppose $V_{t-1}(R) \leq Y$ for any R and is increasing and convex in R. Then also $V_{t-1}(R^*) \leq Y$ and is increasing and convex in R*.

Remark. Here the elements k_j^* with replacement by 0 are fixed and all others are regarded as variables.

Now since $P(R^*)$ is independent of values of elements in R* and moreover since the sum of $P(R^*)$ over all possible R* equals 1, it follows from the induction hypothesis that $U_{t-1}(R) \leq \alpha Y - c \leq Y$ and is increasing as well as convex in R, and hence also in G. In addition, since $k \leq Y$ and is increasing and convex in G, $v_t(K) = v_t(k_0,G) \leq Y$ and is increasing and convex in G, and hence so also is $V_t(G)$. Thus the induction completes. (c). Immediate from (b). ▯

THEOREM 1. $S_t(j)$ is a convex set for all $j \in \bar{K}$ and all $t \geq 1$. ❙

Proof: Immediate from (2.7), (2.8) and Lemma 1(c). ▯

LEMMA 2. For all $t \geq 1$ and all $j \in \tilde{K}$, we have

(a) $\alpha V_{t-1}(R) - k_j$ is decreasing in k_j and tends, as $k_j \to \infty$, to $-\infty$ if $j = N$, 0 if $j < N$ and $\alpha = 1$, and $-\infty$ if $j < N$ and $\alpha < 1$.

(b) $Q_{t-1}(K,j)$ is strictly decreasing in k_j and tends to $-\infty$ as $k_j \to \infty$.

(c) The equation $Q_{t-1}(K,j) = 0$ with unknown k_j has a unique solution, denoted by $k_{t,j}(K)$. If $\alpha E - c > 0$, then the solution is positive. ▌

Proof: It is obvious from R being independent of k_N that the assertion (a) and (b) holds for $j = N$. Let $j < N$. Now we have

(5.1) $$\alpha V_0(R) - k_j = \alpha(r + T(r)) - k_j = (*).$$

Clearly (*) is decreasing in k_j on $k_j < r$. On $k_j = r$ we have $(*) = \alpha T(k_j) - (1-\alpha)k_j$, which is decreasing in k_j and tends, as $k_j \to \infty$, to 0 if $\alpha = 1$ and $-\infty$ if $\alpha < 1$. Hence (a) holds for $t = 1$. Since the sum of $P(R^*)$ over all possible R^* equals 1, we have

(5.2) $$\alpha v_t(k_0,G) - k_j = \max\{\alpha \cdot \max\{k_0,g\} - k_j,\ \alpha Q_{t-1}(K,j) - (1-\alpha)k_j\},$$

(5.3) $$Q_{t-1}(K,j) = \Sigma\, P(R^*)(\alpha V_{t-1}(R^*) - k_j) - c.$$

Suppose that (a) holds for a given $t \geq 1$. Then $\alpha V_{t-1}(R^*) - k_j$ is decreasing in k_j for all j (Note Remark in the proof of Lemma 1(b)). Hence $Q_{t-1}(K,j)$ is also decreasing in k_j. Consider R^* resulting from replacing all elements of R by value 0. Then, for the null vector R^*, $\alpha V_{t-1}(R^*) - k_j$ for each j is strictly decreasing in k_j and tends to $-\infty$ as $k_j \to \infty$ (Again note Remark). Moreover $P(R^*) = p_0 p_1 \cdots p_{N-1} > 0$ for the R^*. Consequently it follows that, for each j, $Q_{t-1}(K,j)$ is strictly decreasing in k_j and tends to $-\infty$ as $k_j \to \infty$. From the above and the fact that $\alpha \cdot \max\{k_0,g\} - k_j$ for each j "is decreasing in k_j and tends, as $k_j \to \infty$, to 0 if $\alpha = 1$ and $-\infty$ if $\alpha < 1$", it follows that the above assertion put between two double quotation marks is true for the right side of (5.2). Hence the assertion is also true for $\alpha V_t(G) - k_j$ $(= E[v_t(k_0,G) - k_j])$ with $1 \leq j \leq N$. Furthermore it is also true for $j = N+1$ because of it being independent of k_{N+1}. Thus the induction completes. (c). Clear from (b) and the facts that $Q_{t-1}(K,j) \to \infty$ as $k_j \to -\infty$ and $\alpha E - c > 0$ yields $Q_{t-1}(K,j) \geq \alpha E - c > 0$ (Lemma 1(a)). ▯

Let K_i with $1 \leq i \leq N+2$ denote the set of elements in K except <u>the first</u>

i-1 ones, k_0, k_1, ..., k_{i-2}, that is, $K_i = \{k_{i-1}, k_i, \ldots, k_N\}$ where $K_1 = K$ and $K_{N+2} = \phi$.

DEFINITION 1. For any subset $\bar{M}$ of $\bar{K}$ and for any i with $1 \leq i \leq N+2$,

(a) Let $K(\bar{M},i,h)$ be the vector resulting from replacing $k_j \in K_i$ with $j \in \bar{M}$ by a real number h and $k_j \in K_i$ with $j \notin \bar{M}$ by value 0 where $K(\bar{M},N+2,h) = K$.

(b) Let $Q_{t-1}(K(\bar{M},i,h)) = U_{t-1}(R(\bar{M},i,h)) - h$. ∎

For example, let N = 2 and K = (3,7,5) ($\bar{K} = \{0,1,2\}$). If $\bar{M} = \{0,2\}$, then we have $K(\bar{M},1,h) = (h,0,h)$, $K(\bar{M},2,h) = (3,0,h)$, $K(\bar{M},3,h) = (3,7,h)$, $K(\bar{M},4,h) = (3,7,5) = K$. Then we have

(5.4) $$Q_{t-1}(K(\bar{M},i,h)) = \Sigma\, P(R^*(\bar{M},i,h))(\alpha V_{t-1}(R^*(\bar{M},i,h)) - h) - c, \quad 1 \leq i \leq N+2$$

(5.5) $$\alpha v_t(K(\bar{M},i,h)) - h = \max\{s(K(\bar{M},i,h)), \alpha Q_{t-1}(K(\bar{M},i,h)) - (1-\alpha)h\} \quad 1 \leq i \leq N+2$$

where

(5.6) $$s(K(\bar{M},i,h)) = \alpha\cdot\max\{k_0,k_1,\ldots,k_{i-2},\rho\} - h, \quad \rho = 0 \text{ or } h, \quad 1 \leq i \leq N+2,$$

LEMMA 3. For all $\bar{M}$, all $t \geq 1$ and all i with $1 \leq i \leq N+2$

(a) $\alpha V_{t-1}(R(\bar{M},i,h)) - h$ is decreasing in h.

(b) $Q_{t-1}(K(\bar{M},i,h))$ is strictly decreasing in h and tends to $-\infty$ as $h \to \infty$.

(c) $Q_{t-1}(K(\bar{M},i,h)) = 0$ with unknown h has a unique solution. If $\alpha E - c > 0$, then the solution is positive. ∎

Proof: (a,b). First we have

$$\alpha V_0(R(\bar{M},i,h)) - h = \alpha E[\max\{w,k_0,k_1,\ldots,k_{i-2},\rho\}] - h, \quad \rho = 0 \text{ or } h,$$

which is decreasing in h. Thus (a) holds for t = 1. Suppose (a) holds for a given $t \geq 1$. Then $\alpha V_{t-1}(R^*(\bar{M},i,h)) - h$ is also decreasing in h (Note Remark in the proof of Lemma 1(b)). Then, in the same fashion as in the proof of Lemma 2, $Q_{t-1}(K(\bar{M},i,h))$ can be proved to be strictly decreasing in h and tends to $-\infty$ as $h \to \infty$. From this and the fact that (5.6) is decreasing in h, the right side of (5.5), hence its left side also becomes decreasing in h. Let $K' = (k_1, k_2, \ldots, k_{N+1})$ represent offers of time t+1 (the previous time), $R' = (k_1, k_2, \ldots, k_N)$ (= G) and define $\bar{M}'$ to be a subset of $\bar{K}' = \{1,2,\ldots,N+1\}$ (a set of subscripts of elements in K') where note $K = (k_0, G) = (k_0, R')$. Then, for $2 \leq i \leq N+2$, the left side of (5.5) can be expressed as $\alpha v_t(k_0, R'(\bar{M}', i-1, h)) - h$ where $\bar{M}' = \bar{M} - \{0\}$. Accordingly $\alpha V_t(R'(\bar{M}', i-1, h)) - h$ for $2 \leq i \leq N+2$, hence $\alpha V_t(G(\bar{M}', i, h)) - h$ for $1 \leq i \leq N+1$ is increasing in h. Finally we have $\alpha V_t(G(\bar{M}', N+2, h)) - h = \alpha V_t(G) - h$, which is decreasing in h. Therefore it follows that $\alpha V_t(G(\bar{M}', i, h)) - h$ is decreasing in h for $1 \leq i \leq N+1$. Thus the induction completes. (c). Clear from (b) and the facts that $Q_{t-1}(K(\bar{M}, 1, 0)) \to \infty$ as $h \to -\infty$ and $\alpha E - c > 0$ leads to $Q_{t-1}(K(\bar{M}, i, h) \geq \alpha E - c > 0$ (Lemma 1(a)). ▯

DEFINITION 2. Let $h_t(K, \bar{M})$, $h_t(j)$, and h_t denote the unique solutions to, respectively,

(5.7) $$Q_{t-1}(K(\bar{M}, 1, h)) = 0,$$

(5.8) $$Q_{t-1}(K(\{j\}, 1, h)) = 0,$$

(5.9) $$Q_{t-1}(K(\bar{K}, 1, h)) = 0.$$

where note $h_t = h_t(K, \bar{K})$, $h_t(j) = h_t(K, \{j\})$ and $h_t(N) = \alpha V_{t-1} - c$. ▯

THEOREM 2. For all t,

(a) $h_t(j) \leq h_t$ for all $j \in \bar{K}$.

(b) For a given K

1. If $h_t < k_j$ for at least one $j \in \bar{K}$, then $K \in S_t$.
2. If $k_j \leq h_t(j)$ for all $j \in \bar{K}$, then $K \in C_t$. ▯

Proof: (a). Suppose $h_t < h_t(j)$ for a certain $j \in \bar{K}$. Then the following contradiction is derived.

$$\begin{aligned} 0 &= Q_{t-1}(K(\bar{K}, 1, h_t)) \\ &= Q_{t-1}((h_t, h_t, \ldots, h_t)) \\ &= Q_{t-1}((h_t, \ldots, h_t, \ldots, h_t), j) \\ &> Q_{t-1}((h_t, \ldots, h_t(j), \ldots, h_t), j) \quad \text{(Lemma 2(b))} \\ &\geq Q_{t-1}((0, \ldots, 0, h_t(j), 0, \ldots, 0), j) \quad \text{(Lemma 1(c))} \\ &= Q_{t-1}(K(\{j\}, 1, h_t(j))) = 0. \end{aligned}$$

(b1). Consider k_j such as $k_j = k$ where $h_t < k_j$ and $k_i \leq k_j$ for all $i \in \bar{K}$. Then it follows that

$$Q_{t-1}(K,j) = Q_{t-1}((k_0,\ldots,k_j,\ldots,k_N),j)$$
$$\leq Q_{t-1}((k_j,\ldots,k_j,\ldots,k_j),j) \quad \text{(Lemma 1(c))}$$
$$= Q_{t-1}(K(\bar{K},1,k_j)) < Q_{t-1}(K(\bar{K},1,h_t)) = 0 \quad \text{(Lemma 3(b))}$$

Consequently it follows that $K \in S_t(j)$, hence $K \in S_t$. (b2). For all $j \in \bar{K}$, we have

$$Q_{t-1}(K,j) \geq Q_{t-1}((0,\ldots,0,k_j,0,\ldots,0),j) \quad \text{(Lemma 1(c))}$$
$$= Q_{t-1}(K(\{j\},1,k_j))$$
$$\geq Q_{t-1}(K(\{j\},1,h_t(j))) = 0 \quad \text{(Lemma 3(b))}.$$

, which implies $K \notin S_t(j)$. Therefore we have $K \notin S_t$, hence $K \in C_t$. ▯

LEMMA 4. For all $t \geq 0$,

(a) If $K \in H^*$, then $v_t(K) \leq h^*$, and if $K \notin H^*$, then $v_t(K) = k$.

(b) $h_t(K,\bar{M}) \leq h^*$ for all $\bar{M}$, and hence $h_t \leq h^*$ and $h_t(j) \leq h^*$ for all $j \in \bar{K}$. ▮

Proof: (a). Since $v_0(K) = k$, the assertion is true for $t = 0$. Suppose it is true for t-1. Notice $V_{t-1}(R)$ is expressed as

$$V_{t-1}(R) = E[v_{t-1}(w,R)I(w>h^*) + v_{t-1}(w,R)I(h^*\geq w)].$$

A. Let $K \in H^*$. Then $k \leq h^*$ and $r \leq h^*$. If $w > h^*$, then $(w,R) \notin C^*$ and $\max\{w,R\} = w$. If $h^* \geq w$, then $(w,R) \in H^*$. Consequently from the induction hypothesis, we have $V_{t-1}(R) \leq E[wI(w>h^*) + h^*I(h^*\geq w)] = h^* + T(h^*)$. Therefore since also $V_{t-1}(R^*) \leq h^* + T(h^*)$, we get $U_{t-1}(R) \leq \alpha(h^* + T(h^*)) - c = H(h^*) + h^* = h^*$ ...(*), which produces $v_t(K) \leq \max\{k,h^*\} = h^*$.

B. Let $K \notin C^*$. Then $k > h^*$.

B1. Suppose $k_N \leq h^*$. Then since $k_j > h^*$ for at least one $j \in \bar{R}$, we have $(w,R) \notin H^*$ for any w, hence $v_{t-1}(w,R) = \max\{w,r\} = \max\{w,k\}$, from which we have $V_{t-1}(R) = k + T(k)$, and hence $V_{t-1}(R^*) \leq V_{t-1}(R)$ (Lemma 1(b)). Consequently $U_{t-1}(R) \leq \alpha(k + T(k)) - c = H(k) + k < k$ because of $H(k) < 0$ owing to $k > h^*$ (Lemma 0(d,e)), and hence it follows that $v_t(K) = k$.

B2. Suppose $k_N > h^*$. Then if $k_j > h^*$ for at least one $j \in \bar{R}$, then $(w,R) \notin H^*$ for all w, hence $v_{t-1}(w,R) = \max\{w,r\} \leq \max\{w,k\}$ because $r \leq k$ in the case, hence $V_{t-1}(R) \leq k + T(k)$. Accordingly the same discussion as in B1 produces $v_t(K) = k$. On the contrary, suppose $k_j < h^*$ for all $j \in \bar{R}$. Then applying the same discussions as in A yields also $U_{t-1}(R) \leq h^*$, and hence $V_{t-1}(R) \leq k$. Therefore we have $v_t(K) = k$

(b). If $h \leq h^*$, then $K(\bar{M},1,h) \in H^*$. Noticing this and (*), we have $Q_{t-1}(K(\bar{M},1,h)) \leq h^* - h$. Thus $Q_{t-1}(K(\bar{M},1,h^*)) \leq 0$, which means $h_t(K,\bar{M}) \leq h^*$ from Lemma 3(b,c). ▯

THEOREM 3. For all $t \geq 1$,

(a) $Y \supset H^* \supset H_t \supset C_t \supset H_t'$.

(b) If $(1-\alpha)^2 + c^2 \neq 0$ and $\alpha E - c \leq X$, then $C_t = H^* = 0$ for all t, and hence the double reservation values property does not appear at all. ▮

Proof: (a). $Y \supset H^*$ and $H^* \supset C_t$ are clear from, respectively, Lemma 0(d) and Lemma 4(b). Consider any $K \in C_t$, for which $Q_{t-1}(K,j) \geq 0$ for all $j \in \bar{K}$. Here if $K \notin H_t$, then since $h_t < k_i$ for $k_i = k$, we have the contradiction of

$$\begin{aligned}
0 &\leq Q_{t-1}(K,i) \\
&= Q_{t-1}((k_0,\ldots,k_i,\ldots,k_N),i) \\
&\leq Q_{t-1}((k_i,k_i,\ldots,k_i),i) \quad \text{(Lemma 1(c))} \\
&= Q_{t-1}(K(\bar{K},1,k_i)) \\
&< Q_{t-1}(K(\bar{K},1,h_t)) = 0 \qquad \text{(Lemma 3(b))},
\end{aligned}$$

Therefore it must be that $K \in H_t$, hence $H_t \supset C_t$. $C_t \supset H_t'$ is clear from Theorem 2(b2). (b). $h^* = \alpha E - c$ from Lemma 0(d). For all $j \in \bar{K}$, $Q_0(K(\{j\},1,h)) = \alpha(1-p_j)(h + T(h)) + \alpha p_j E - c - h$. Therefore since $Q_0(K(\{j\},1,\alpha E - c)) = 0$ from Lemma 0(b), we have $h_1(j) = \alpha E - c$ from Lemma 3(c). Hence we have $C_1 = H^* = 0$, which implies $C_t = H^* = 0$ for all t because C_t is increasing in t and converges to H^*. ▯

LEMMA 5. For all K, j and $\bar{M}$,

(a) $v_t(K)$ and $Q_{t-1}(K,j)$ is increasing in t.

(b) $h_t(K,\bar{M})$ is increasing in t. ▮

Proof: (a). Easily proved by induction, starting with $v_1(K) \geq k = v_0(K)$ for any K. (b). Clear from (a) and Lemma 3(b,c). ▯

THEOREM 4. We have

(a) S_t is decreasing in t, and C_t is increasing in t.

(b) H_t and H'_t are increasing in t. ▯

Proof: (a). Since $Q_t(K,j) < 0$ leads to $Q_{t-1}(K,j) < 0$ from Lemma 4(a), we have $S_{t+1}(j) \subset S_t(j)$ for all j, which implies $S_{t+1} \subset S_t$, hence $C_{t+1} \supset C_t$.
(b). Clear from Lemma 4b. ▯

5.2. Behavior of Optimal Stopping Rule near a Limiting Horizon

We shall investigate the behaviors of the optimal stopping rule as a time horizon tends to infinity. To do this, some results that have been derived for two types of the standard stopping problems: One is of with-no-recall, the other is of with-recall. To begin with, we shall summarize them that are used in this section [2],[11].

In case with no recall, all past offers are unavailable with certainty, and hence G is always null vector. Therefore it suffices to define only $v_t(k_0,0,0,\ldots)$. Then let this and its expectation as to a present offer k_0 ($=w_0$) be denoted by, respectively, $v_t(k_0,-|0)$ and $V_t(-|0)$. As being well-known, we have

$$(5.10)\quad v_t(k_0,-|0) = \max\{k_0,\ \alpha V_{t-1}(-|0) - c\},\quad t \geq 1.$$

with $v_0(k_0,-|0) = k_0$ and $V_0(-|0) = E$ where $\alpha V_{t-1}(-|0) - c$ provides a reservation value of time t. In case with recall, the maximum expected gain attained depends only on the best offer k so far. Then we shall denote the maximum expected gain by $v_t(k|1)$, and define $V_{t-1}(k|1) = E[v_{t-1}(\max\{w,k\}|1)]$, the expectation as to an offer w of time t-1 (the next time). Then we have the well-known equation

$$(5.11)\quad v_t(k|1) = \max\{k,\ \alpha V_{t-1}(k|1) - c\},\quad t > 0$$

where $v_0(k|1) = k$ and $V_0(k|1) = k + T(k)$ and where $\alpha V_{t-1}(k|1)$ give a reservation values of time t.

LEMMA 6. We have

(a) In case with no recall,

1. $v_t(k_0,-|0)$ and $V_{t-1}(-|0)$ are increasing and converges in t,
2. Their limits are given by $v(k_0,-|0) = \max\{k_0,h^*\}$ and $V(-|0) = (h^*+c)/\alpha$,
3. The limiting reservation value is provided by h^*.

(b) In case with recall,

1. $v_t(k|1)$ and $V_{t-1}(k|1)$ are increasing and converges in t.
2. Their limits in t are provided by $v(k|1) = \max\{k,h^*\}$ and $V(k|1) = \max\{k + T(k),\ h^* + T(h^*)\}$,
3. The reservation values for all $t \geq 1$ are given by h^* (This time-independency of the reservation value is commonly referred to as a myopic property). ∎

Proof: Refer to [2],[11]. ∎

LEMMA 7. For all $t \geq 0$,

(a) $v_t(k|1) \geq v_t(K) \geq v_t(k_0,-|0)$

(b) $V_t(g|1) \geq V_t(G) \geq V_t(-|0)$

(c) $\alpha V_{t-1}(-|0) - c \leq h_t(K,\bar{M})$ for all $\bar{M}$, and hence $\alpha E - c \leq h_t(K,\bar{M})$, which implies $H_t' \supset 0$. ∎

Proof: (a,b). It is clear that $v_0(k|1) = k = v_0(K) = v_0(k_0,G) \geq k_0 = v_0(k_0,-|0)$, from which $V_0(G) \geq V_0(-|0)$. Since $v_0(k|1) = v_0(\max\{k_0,g\}|1) = \max\{k_0,g\} = v_0(k_0,G)$, we have $V_0(g|1) = V_0(G)$. Thus the assertions (a,b) are true for $t = 0$.

Suppose $V_{t-1}(R) \geq V_{t-1}(-|0)$ for all R. Then since $V_{t-1}(R^*) \geq V_{t-1}(-|0)$, we have $U_{t-1}(R) \geq \alpha V_{t-1}(-|0) - c$, which yields $v_t(K) = v_t(k_0,G) \geq \max\{k_0, \alpha V_{t-1}(-|0) - c\} = v_t(k_0,-|0)$, and hence $V_t(G) \geq V_t(-|0)$. Suppose $V_{t-1}(R) \leq V_{t-1}(r|1)$ for any R. Then since $V_{t-1}(R^*) \leq V_{t-1}(r|1) \leq V_{t-1}(k|1)$ because $V_{t-1}(k|1)$ is increasing in k (easily proved by induction), we have $U_{t-1}(R) \leq \alpha V_{t-1}(k|1) - c$. Therefore we have $v_t(K) \leq \max\{k, \alpha V_{t-1}(k|1) - c\} = v_t(k|1)$, and hence $v_t(k_0,G) \leq v_t(\max\{k_0,g\}|1)$, which yields $V_t(G) \leq V_t(g|1)$. Thus the induction completes. (c). Since $V_{t-1}(R) \geq V_{t-1}(-|0)$ for all R from (b), we have $Q_{t-1}(K(\bar{M},1,h)) \geq \alpha V_{t-1}(-|0) - c - h$, the both sides of which are strictly decreasing h. Accordingly the solution to $Q_{t-1}(K(\bar{M},1,h)) = 0$, or $h_t(K,\bar{M})$, must be greater than or equal to that for $\alpha V_{t-1}(-|0) - c - h = 0$, or $\alpha V_{t-1}(-|0) - c$. Since $V_{t-1}(-|0)$ is increasing in t, $\alpha E - c = \alpha V_0(-|0) - c \leq h_t(K,\bar{M})$. ∎

THEOREM 5. We have

(a) If $K \in H^*$, then both $v_t(K)$ and $h_t(K,\bar{M})$ converges to h^* as $t \to \infty$.

(b) H_t, C_t, H_t' converge to H^* as $t \to \infty$. ∎

Proof: (a). Suppose $K \in H^*$. Then $k \leq h^*$ and $k_0 \leq h^*$. In the case, from Lemma 6(a2,b2), the first and the third terms of the inequalities in Lemma 7(a) tends to h^* as $t \to \infty$. Hence $v_t(K) \to h^*$ as $t \to \infty$. Now since $\alpha V_{t-1}(-|0) - c \to h^*$ as $t \to \infty$ from Lemma 6(a2), it follows from Lemmas 4(b) and 7(c) that $h_t(K,\bar{M}) \to h^*$ as $t \to \infty$. (b). From (a), since both h_t and $h_t(j)$ converge to h^* for all $j \in \bar{K}$, both H_t and H_t' converge to H^*, and hence C_t must also converge to H^* from Theorem 3. ∎

5.3. Double Reservation Values Property

Figure 3 claims that the double reservation values property will appear if and only if $h_t(j) < h_t$ for at least one $j \in \bar{G}$. The following theorem guarantees that the conjecture is right in general.

LEMMA 8. Suppose $\alpha E - c > X$. For a given $t \geq 1$, if $h_t(j) < h_t$ for at least one $j \in \bar{G}$, then the optimal stopping rule of time t has a double reservation values property. I

Proof: Suppose $h_t(j) < h_t$ for a certain $j \in \bar{G}$. Then consider an infinitesimal $\varepsilon > 0$ such as $h_t(j) + \varepsilon < h_t$. Now for a fixed $x > h^* - h_t$ (≥ 0), let $\lambda = h^*/(x+h_t)$ where $0 < \lambda < 1$ by noticing $h^* > 0$ and $h_t > 0$ from Lemmas 0(d) and 3(c). Using the λ, define the following (N+1)-vector

$$K(x) = (1-\lambda)(0,\ldots,0,h_t(j)+\varepsilon,0,\ldots,0) + \lambda(h_t+x,h_t,\ldots,h_t)$$

$$= (\lambda(h_t+x),\lambda h_t,\ldots,\lambda h_t,L,\lambda h_t,\ldots,\lambda h_t),$$

$$K_\delta = (\delta,\lambda h_t,\ldots,\lambda h_t,L,\lambda h_t,\ldots,\lambda h_t).$$

where $L = (1-\lambda)(h_t(j)+\varepsilon) + \lambda h_t$, $\lambda h_t < L < h_t$ ($\leq h^*$), and

$$K(0) = (1-\lambda)K(\{j\},1,h_t(j)+\varepsilon) + \lambda K(\bar{K},1,h_t)$$

$$= (\lambda h_t,\ldots,\lambda h_t,L,\lambda h_t,\ldots,\lambda h_t).$$

Consider any given $x' > x$. Then we have $\lambda(x'+h_t) > h^*$, which implies $K(x') \notin H^*$, and hence $K(x') \in S_t$ because $K(x') \notin C_t$ from Theorem 3(a). Therefore, given the present offers $K(x')$ (= K_δ with $\delta = \lambda(x'+h_t)$), it is optimal to stop the search, that is, $Q_{t-1}(K_\delta,0) < 0$. Here notice $\lambda(x'+h_t) \geq h^* > L$. Accordingly it follows that there exists a $\delta \geq L$ such as $Q_{t-1}(K_\delta,0) < 0$. Then let $\xi' = \min\{\delta | Q_{t-1}(K_\delta,0) < 0, L \leq \delta\}$. Next, from Lemmas 1(c) and 3(b), we have $Q(K(0),j) \leq (1-\lambda)Q(K(\{j\},1,h_t(j)+\varepsilon) + \lambda Q(K(\bar{K},1,h_t)) < (1-\lambda)Q(K(\{j\},1,h_t(j)) = 0$, implying $K(0) = K_\delta \in S_t(j)$ where $\delta = \lambda h_t < L$. Hence if the K_δ are present offers, then it is optimal to stop the search. Accordingly it follows that there exists a $\delta < L$ such as $Q_{t-1}(K_\delta,j) < 0$. Then let $\xi = \max\{\delta | Q_{t-1}(K_\delta,j) < 0, \delta < L\}$. It is of course that $\xi < \xi'$. Here notice that $Q_{t-1}(K_\delta,0)$ is decreasing in δ (Lemma 2(b)) and $Q_{t-1}(K_\delta,j)$ is increasing in δ (Lemma 1(c)). Then the optimal decision for a present offers K_δ becomes as follows. If either $\delta \leq \xi$ or $\xi' \leq \delta$, then stop the search, if $L \leq \delta < \xi'$, then continue the search because $Q_{t-1}(K_\delta, 0) \geq 0$ and $\delta \geq \max\{L, \lambda h_t\}$, and if $\xi < \delta < L$, then continue the search because $Q_{t-1}(K_\delta, j) \geq 0$ and $L \geq \max\{\delta, \lambda h_t\}$. ▮

Below, for $0 \leq j \leq N-1$, let such an N-vector $(x,y,\ldots,z)$ that the $(j+1)$-th element from the left is h be denoted by $(x,y,\ldots,z)_j$.

LEMMA 9. Suppose $\alpha E - c > X$. Then for all $t \geq 1$, if $h_t(N) < h$, then

$$V_t(0,\ldots,0,h,0,\ldots,0)_j > V_t, \quad 0 \leq j \leq N-1. \quad ▮$$

Proof: First we shall prove case of $j = N-1$. It is easily proved by induction that $U_{t-1}(0,\ldots,0) = U_{t-1}(X,\ldots,X)$ and $U_{t-1}(w,0,\ldots,0) = U_{t-1}(w,X,\ldots,X)$ for $X \leq w$. Here note $U_{t-1}(w,X,\ldots,X) \geq \alpha V_{t-1} - c = h_t(N)$ and $U_{t-1}(X,X,\ldots,X) = \alpha V_{t-1} - c = h_t(N)$ where $X < \alpha E - c \leq h_t(N)$ (Lemma 7(c)). From this and the fact that $U_{t-1}(\cdot)$ is a continuous function, for w such as $X \leq w \leq X + \varepsilon < h$ with an infinitesimal $\varepsilon > 0$, we have $h_t(N) \leq U_{t-1}(w,0,\ldots,0)) < h$, hence $\max\{w, U_{t-1}(w,0,\ldots,0)\} < h$. Therefore

$$\begin{aligned} V_t(0,\ldots,0,h) &= E[\max\{w,h,U_{t-1}(w,0,\ldots,0)\}] \\ &> E[\max\{w,U_{t-1}(w,0,\ldots,0)\}] \quad \text{(Lemma 0(e))} \\ &= V_t. \end{aligned}$$

Next for $0 \leq j \leq N-2$

$$\begin{aligned} V_t(0,\ldots,0,h,0,\ldots,0)_j &= E[\max\{w,h,U_{t-1}(w,0,\ldots,0,h,0,\ldots,0)_{j+1}\}] \\ &\geq E[\max\{w,h,U_{t-1}(w,0,\ldots,0)\}] \quad \text{(Lemma 1(b))} \\ &> V_t. \quad ▮ \end{aligned}$$

THEOREM 6. Suppose $\alpha E - c > X$. Then

(a) The optimal stopping rule of any time $t \geq 1$ has a double reservation values properties.

(b) If $N = 1$, then $h_t = h_t(0)$ and $h_t > h_t(1)$. If $N \geq 2$, then $h_t > h_t(j) > h_t(N)$ for $j \in \bar{R}$. $h_t(N) = \alpha V_{t-1} - c$ for all $t \geq 1$, $N \geq 1$.

(c) If $p_j >(=) p_i$ for $i, j \in \bar{R}$, then $h_t(j) <(=) h_t(i)$ for all $t \geq 1$. If $p_j = p$ for $j \in \bar{R}$, then $h_t(j)$ is constant on $j \in \bar{R}$. ▮

Proof: If (b) is true, then (a) is immediate from Lemma 8. Then first let us prove (b). Suppose $N = 1$. Then, for all $t \geq 1$, since $Q_{t-1}((h,h)) = Q_{t-1}((h,0))$, we have $h_t = h_t(0)$. Now $Q_{t-1}((h,h)) = \alpha(p_0V_{t-1} + (1-p_0)V_{t-1}(h)) - c - h = (*)$. From Lemma 9, $(*) > \alpha V_{t-1} - c - h = Q_{t-1}((0,h))$. Accordingly $h_t > h_t(1)$ from Lemma 3(b,c). Next let $N \geq 2$. For $0 \leq j \leq N-1$, let $(0,\ldots,0,h,0,\ldots,0)_j$ be N-vector such that the (j+1)-th element from the left is $h > 0$ and all the others are 0. From the definition, we have immediately

$$Q_{t-1}(K(\{N\},1,h)) = \alpha V_{t-1} - c - h$$

$$\begin{aligned} Q_{t-1}(K(\{j\},1,h)) &= \alpha(1-p_j)V_{t-1}(0,\ldots,0,h,0,\ldots,0)_{j+1} + \alpha p_jV_{t-1} - c - h \\ &= \alpha(1-p_j)(V_{t-1}(0,\ldots,0,h,0,\ldots,0)_{j+1} - V_{t-1}) + \alpha V_{t-1} - c - h \qquad \ldots(1^*) \end{aligned}$$

for $0 \leq j \leq N-1$. Suppose $h_t(N) < h$. Then from Lemma 9, we have $Q_{t-1}(K(\{j\},1,h)) > Q_{t-1}(K(\{N\},1,h))$, which yields $h_t(j) > h_t(N)$ from Lemma 3(b,c). Since $U_{t-1}(K(\bar{K},1,h)) \geq U_{t-1}(K(\{j\},1,h))$ from Lemma 1(b), $Q_{t-1}(K(\bar{K},1,h)) \geq Q_{t-1}(K(\{j\},1,h))$ ...(2*) Consider the following corresponding terms in both sides of the inequality (2*)

$$(1^*) = p_j\Pi_{i\neq j}(1-p_i)V_{t-1}(h,\ldots,h,0,h,\ldots,h) \text{ and}$$
$$(2^*) = p_j\Pi_{i\neq j}(1-p_i)V_{t-1}.$$

Since $V_{t-1}(h,\ldots,h,0,h,\ldots,h) \geq V_{t-1}(h,0,\ldots,0) > V_{t-1}$ from Lemmas 1(b) and 9, we have $(1^*) > (2^*)$, hence $Q_{t-1}(K(\bar{K},1,h)) > Q_{t-1}(K(\{j\},1,h))$. Accordingly it follows that $h_t > h_t(j)$. (c). Clear from the fact that $Q_{t-1}(K(\{j\},1,h))$ is strictly decreasing in p_j from (1*). ▯

5.4. Search Amount and Value Realization

Let $n_t(K)$ and $E_t(K)$ denote, respectively, the search amount and the value realization, provided that the process starts from time t with offers K. The approach employed in the section is a generalization of that in [7]. From the definition, $n_0(K) = 0$ for all K and $n_t(K) = 0$ if $K \notin C_t$. Then we have

$$(5.12) \qquad n_t(K) = 1 + \Sigma\, P(R^*)E[n_{t-1}(w,R^*)I((w,R^*)\in H^*)], \quad K \in C_t \ (\subset H^*)$$

where E[•] represents the expectation as to an offer w of time t-1 and where the sum is over all possible R*. In case <u>with recall</u>, if the process starts with the maximum offer $k \leq h^*$, then the search amount is independent of k. Then let it be denoted by n_t. The n_t satisfies $n_t = 1 + n_{t-1}F(h^*)$ for all $t > 0$ with $n_0 = 0$ and increases in t, and furthermore if $(1-\alpha)^2 + c^2 \neq 0$ and $X < \alpha E - c$, then n_t converges to $(1 - F(h^*))^{-1}$ as $t \to \infty$ due to $F(h^*) < 1$ because $h^* < Y$ in the case ([7], Lemma 0(d)).

THEOREM 7. We have

(a) $n_t(K) \leq n_t$ for all t, K.

(b) $n_t(K)$ is increasing in t for all K.

(c) If $(1-\alpha)^2 + c^2 \neq 0$ and $X < \alpha E - c$, then $n_t(K)$ converges to $(1 - F(h^*))^{-1}$ for any inner point K of H*. ∎

Proof: (a). Clear for t = 0. Suppose the assertion is true for a given t-1. If $K \notin C_t$, then $n_t(K) = 0 \leq n_t$. If $K \in C_t$, then we have, from (5.16)

$$\begin{aligned} n_t(K) &\leq 1 + n_{t-1}\ \Sigma\, P(R^*)E[I(w\leq h^*)] \\ &= 1 + n_{t-1}E[I(w\leq h^*)] \\ &= 1 + n_{t-1}F(h^*) = n_t. \end{aligned}$$

(b). Easy. (c). Since $n_t(K) \leq n_t \leq (1-F(h^*))^{-1}$ for all t, K (, or upper bounded), the limit of $n_t(K)$ exists, denoted by n(K). For any given inner point K of H*, we have $K \in C_t$ for all $t \geq T$ with a sufficiently large T. Hence, for such K, (5.16) holds for all $t \geq T$. Then t approaching ∞ produces

$$n(K) = 1 + \Sigma\, P(R^*)E[n(w,R^*)I((w,R^*)\in H^*)] \quad \ldots(*).$$

Suppose there exist two different solutions for the above equation, m(K) and

$n(K)$, and let $\Delta = \sup_K |m(K) - n(K)| > 0$. Then from (*) we have immediately $\Delta \leq \Delta \cdot F(h^*)$, which yields the contradiction of $1 \leq F(h^*) < 1$ (Lemma 0(b)). Thus the solution must be unique. It is easy to check that $(1 - F(h^*))^{-1}$ satisfies the equation. Accordingly $n_t(K)$ must converges to $(1 - F(h^*))^{-1}$. ▯

Value realization for case with recall does not depend on the best offer k on $k \leq h^*$. Then let it be represented by $E_t(-|1)$. Clearly we have

$$(5.13) \qquad v_t(K) = E_t(K) - cn_t(K), \quad v_t(k|1) = E_t(-|1) - cn_t,$$

Let the limits of $E_t(K)$ and $E_t(-|1)$ be denoted by, respectively, $E(K)$ and $E(k|1)$.

THEOREM 8. We have

(a) $E_t(K) \leq E_t(-|1)$ for all t, K.

(b) If $(1-\alpha)^2 + c^2 \neq 0$ and $X < \alpha E - c$, then $E(K) = E(k|1)$ for any inner point K of H*. ▯

Proof: (a). From (5.12) we have $E_t(K) - E_t(-|1) = v_t(K) - v_t(k|1) + c(n_t(K) - n_t)$, which is non-positive from Lemma 7(a) and Theorem 7(a). Hence $E_t(K) \leq E_t(-|1)$. (b). Immediate from Lemma 6(b2) and Theorems 5(a), 7(c). ▯

6. Future studies

The final section presents some interesting future studies for this model, every subject of which is expected to be very difficult to explore but is very challenging as well as worth intensive investigations.

1. In general, an offer with relatively high value will be also preferable for any other searcher. This implies that the higher the value of an offer, the larger the probability of future unavailability of it. Such consideration can be incorporated in our model by assuming the probabilities p_j to be a function of w, that is, $p_j(w)$. This extention tells us the possibility of a game theoretic variation of our model. Consider a searcher Γ, and suppose each of other searchers employs a randamised strategy for accepting a present best offer. Then, for the searcher Γ, future unavailability for each of his present offers K will become stochastic, and hence the probabilities of unavailability for each offer will be to be defined as a function of its value, which are constructed through integrating other searchers' randamized strategies by use of simple probabilty computations. The introduction of such w-dependency, however, will make a mathematical treatment of the model formidably intractable. 2. The introduction of a baysian up-to-date for the distribution function of w with some unknown parameters is another interesting subject not only from the practical viewpoint but also from the theoretical viewpoint. In the case, however, we should note that this case may often cause a non-existence of the reservation value [13]. 3. Relationship between model parameters and a degree to which the double reservation values property appears. The degree can be measured by, for instance, $(h^*-h_t(N))/h^*$, showing the extent of the curvature of the continuation region which is the very reason effecting the property. Which parameters is the most contributive to its appearance ?

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