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Abstract

The notion of homotopy base of an acyclic graph was introduced by K. Murota. Consider an acyclic graph G=(V,A) with a vertex set V and an arc set A. For any directed path in G, p can be translated into another path p' by replacing a subpath p_1 of p by a directed path p_2 such that p_1 and p_2 have the common end-vertices but no common intermediate vertices. Such a pair of paths p_1 and p_2 is called a bilinking and we say p' is obtained by an elementary transformation of p by the bilinking $\{p_1,p_2\}$. A homotopy base of G is a minimal set R (minimal relative to set inclusion) of bilinkings in G with the property that, for any two directed paths p and p' in G with common end-vertices, p' is obtained by repeated elementary transformations of p by bilinkings in R. We give a simple characterization of homotopy bases of G and propose an $O(|V|^2|A|)$ algorithm, with $O(|V|+|A|)$ working space, for finding a homotopy base of G=(V,A).

1. Introduction

Consider an acyclic graph $G=(V,A)$ with a vertex set V and an arc set A. We assume the familiarity with the basic terminology and definitions in graph theory (see e.g. [1], [3]). Throughout the present paper a path means a directed path unless otherwise stated.

For any arc $a \in A$ we denote the initial and the terminal vertices of a by $\partial^{+}a$ and $\partial^{-}a$, respectively. For a finite set S the cardinality of S is denoted by $|S|$. An unordered pair of distinct paths in G with common end-vertices but no common intermediate vertices between the end-vertices is called a <u>bilinking</u> (or bi-linking). Let $r=\{p_1,p_2\}$ be a bilinking consisting of paths p_1 and p_2 in G, and suppose that p is a path in G which contains p_1 as a subpath of p. Then we can obtain a path p' having the same end-vertices as p by replacing the subpath p_1 in p by p_2. We say the path p' is obtained by the <u>elementary transformation</u> of p by $r=\{p_1,p_2\}$, and denote p' by $p \oplus r$. Note that we also have $p=p' \oplus r$.

Given a set R of bilinkings, if for a sequence $r_i \in R$ $(i=1,2,\cdots,k)$ with $k \geqq 0$ we have

$$p' = (\cdots((p \oplus r_1) \oplus r_2) \oplus \cdots) \oplus r_k, \tag{1.1}$$

or equivalently

$$p = ((\cdots(p' \oplus r_k) \oplus \cdots) \oplus r_2) \oplus r_1, \tag{1.2}$$

then two paths p and p' are called <u>homotopic</u> with respect to R, and we write

$$p \simeq_R p'. \tag{1.3}$$

By definition, homotopic p and p' have common end-vertices. A set R of bilinkings in G is called a <u>homotopy base</u> of G if

(1) for every pair of paths p and p' in G with common end-vertices,

we have $p \simeq_R p'$,

and

(2) no proper subset of R enjoys the above property (1).

The notion of homotopy base was first introduced in [2] in an analysis of commutativity of diagrams; a characterization of a homotopy base was given in terms of the preordered matroid associated with the underlying acyclic graph and an $O(|V|^2|A|^4)$ algorithm with $O(|V||A|^2)$ working space for finding a homotopy base was presented as an application of the dual greedy algorithm for preordered matroids. In the present paper we shall give an alternative complete characterization of homotopy bases and show an $O(|V|^2|A|)$ algorithm, with $O(|V|+|A|)$ working space (excluding the space for the storage of the solution), for finding a homotopy base of G=(V,A).

2. A Characterization of Homotopy Bases

For any two vertices $u,v \in V$ an <u>interval</u> $[u,v]$ of G is the subgraph of G induced by the set of all the vertices which lie on paths from v to u in G. The interval $[u,v]$ is empty if there is no path from v to u. The length of a path is the number of arcs on the paths. When $[u,v]$ is nonempty, the <u>length</u> of $[u,v]$ is the maximum length of paths from v to u in G.

For any nonempty interval $[u,v]$ of G with $u \neq v$, we call a set of subgraphs $G_i=(V_i,B_i)$ $(i \in I)$ a <u>decomposition</u> of $[u,v]$ if

(i) for each $i,i' \in I$ we have $V_i \cap V_{i'}=\{u,v\}$,

(ii) $\{B_i | i \in I\}$ is a partition of the arc set of the interval $[u,v]$.

Moreover, if there is no decomposition $G'_j=(V'_j,B'_j)$ $(j \in J)$ of $[u,v]$ such that the partition $\{B'_j | j \in J\}$ is strictly finer than $\{B_i | i \in I\}$, then we call the decomposition $G_i=(V_i,B_i)$ $(i \in I)$ a <u>minimal</u> decomposition of $[u,v]$. (Here, the partition $\{B'_j | j \in J\}$ is <u>strictly finer</u> than $\{B_i | i \in I\}$ if $\{B'_j | j \in J\} \neq \{B_i | i \in I\}$ and for each $i \in I$ there exists $j \in J$ such that $B'_j \subseteq B_i$.)

We can easily show

Lemma 2.1: For each nonempty interval $[u,v]$ with $u \neq v$, there exists a unique minimal decomposition $G_i=(V_i,B_i)$ $(i \in I)$ of $[u,v]$. Moreover, for each $i \in I$, either

(a) $V_i=\{u,v\}$ and $B_i=\{a_i\}$ with $\partial^- a_i=u$ and $\partial^+ a_i=v$,

or

(b) $|V_i| \geq 3$, there is no arc $a \in B_i$ with $\partial^- a=u$ and $\partial^+ a=v$, and by deleting from G_i the vertices u and v (together with arcs incident to u or v), we have a connected graph.

We omit the proof. The minimal decomposition of an interval is closely related to the structural theory of two-connected graphs (cf. [3], [4]). An example of a minimal decomposition is shown in Fig.2.1.

Given the minimal decomposition G_i ($i \in I$) of an interval $[u,v]$, we define the <u>degree</u> of the interval $[u,v]$ by the cadinality $|I|$, and denote it by $d([u,v])$. Furthermore, suppose that R is a set of bilinkings in G. We define an undirected graph $\tilde{G}([u,v],R) = (\tilde{V},\tilde{E})$ with a vertex set

$$\tilde{V} = \{\tilde{v}_i \mid i \in I\} \qquad (2.1)$$

and an edge set $\tilde{E}$ as follows: Let R_{uv} be the set of bilinkings $r=\{p,p'\}$ in R such that p and p' are paths from v to u. The edge set $\tilde{E}$ is defined by

$$\tilde{E} = \{\tilde{e}_r \mid r \in R_{uv}\}, \qquad (2.2)$$

where for each $r=\{p,p'\} \in R_{uv}$ with p and p' contained in G_i and $G_{i'}$, respectively, the edge $\tilde{e}_r$ connects vertices $\tilde{v}_i$ and $\tilde{v}_{i'}$ in $G([u,v],R)$. Here, note that G_i and $\tilde{v}_i \in \tilde{V}$ have the same index $i \in I$. We call $\tilde{G}([u,v],R)$ the <u>graph associated with</u> $[u,v]$ and R.

Now, the following theorem gives a complete characterization of homotopy bases of G.

Theorem 2.2: A set R of bilinkings in G is a homotopy base of G if and only if for each nonempty interval $[u,v]$ of G with $u \neq v$ the graph $\tilde{G}([u,v],R)=(\tilde{V},\tilde{E})$ associated with $[u,v]$ and R is a spanning tree, i.e., $\tilde{G}([u,v],R)$ is connected and $|\tilde{E}| = |\tilde{V}|-1$.

(Proof) [ONLY IF part] Let $[u,v]$ be a nonempty interval of G with $u \neq v$, and $G_i=(V_i,B_i)$ $(i \in I)$ be the minimal decomposition of $[u,v]$. The graph $\tilde{G}([u,v],R)=(\tilde{V},\tilde{E})$ is defined by (2.1) and (2.2). Suppose that $\tilde{G}([u,v],R)$ is not connected and that vertices $\tilde{v}_i, \tilde{v}_{i'} \in \tilde{V}$ for some $i,i' \in I$ belong to different connected components of $\tilde{G}([u,v],R)$. Let p and p' be any paths from v to u in G_i and $G_{i'}$, respectively. (Here, note that $\tilde{v}_i$ and $\tilde{v}_{i'}$ correspond to G_i and $G_{i'}$, respectively, in the definition of $\tilde{G}([u,v],R)$.) Then we can easily see that p' cannot be obtained from p by repeated elementary transformations by elements of R, i.e., p and p' are not homotopic with respect to R. This is a contradiction. So, $\tilde{G}([u,v],R)$ must be connected.

Now, if $|\tilde{E}| \geq |\tilde{V}|$, then $\tilde{G}([u,v],R)$ contains a cycle, expressed by a sequence of vertices $\tilde{v}_{i_0}, \tilde{v}_{i_1}, \cdots, \tilde{v}_{i_k}, \tilde{v}_{i_0}$, say. For each $j=0,1,\cdots,k$ there exists a bilinking $r_j=\{p_j^{(1)},p_{j+1}^{(2)}\}$ in R such that $p_j^{(1)}$ and $p_{j+1}^{(2)}$ are, respectively, paths from v to u in G_{i_j} and $G_{i_{j+1}}$, where $i_{k+1}=i_0$. We define $p_0^{(2)}=p_{k+1}^{(2)}$. For each $j=0,1,\cdots,k$, if $p_j^{(1)} \neq p_j^{(2)}$, then there are bilinkings $\hat{r}_l=\{\hat{p}_l,\hat{q}_l\}$ $(l=1,2,\cdots,m;\ m \geq 1)$ with paths $\hat{p}_l$, $\hat{q}_l$ $(l=1,2,\cdots,m)$ in G_{i_j} such that $p_j^{(1)}$ and $p_j^{(2)}$ are homotopic with respect to $\{\hat{r}_l \mid l=1,2,\cdots,m\}$ and that none of the paths $\hat{p}_l$, $\hat{q}_l$ $(l=1,2,\cdots,m)$ in G_{i_j} connect u and v (due to Lemma 2.1(b)). Hence, $\hat{p}_l$ and $\hat{q}_l$ $(l=1,2,\cdots,m)$ must be homotopic with respect to $R' \equiv R \setminus \{r_k\}$. Consequently, $p_j^{(1)} \simeq_{R'} p_j^{(2)}$ $(j=0,1,\cdots,k)$. We thus have

$$p_{k+1}^{(2)} \simeq_{R'} p_0^{(1)} \simeq_{R'} p_1^{(2)} \simeq_{R'} p_1^{(1)} \simeq_{R'} \cdots \simeq_{R'} p_k^{(2)} \simeq_{R'} p_k^{(1)}. \qquad (2.3)$$

This implies that $r_k=\{p_k^{(1)},p_{k+1}^{(2)}\}$ is redundant, which contradicts the assumption that R is a homotopy base. Therefore, we must have $|\tilde{E}| = |\tilde{V}|-1$ and $\tilde{G}([u,v],R)=(\tilde{V},\tilde{E})$ is connected, i.e., $\tilde{G}([u,v],R)$ is a spanning tree.

[IF part] Suppose that for each nonempty interval $[u,v]$ with $u \neq v$, $\tilde{G}([u,v],R)$ is a spanning tree. We show the "IF" part by induction.

For each interval $[u,v]$ of length 1, $[u,v]$ is composed of parallel arcs from v to u. Therefore, we can easily see from the connectedness of $\tilde{G}([u,v],R)$ that any two paths, each composed of a single arc, from v to u are homotopic with respect to R.

Next, for some $k \geq 1$ let us suppose that for each interval $[u,v]$ of length $k' \leq k$ any two paths from v to u are homotopic with respect to R. Let $[u',v']$ be any interval of length k+1, and p_1 and p_2 be any paths from v' to u'. Then, because of the induction hypothesis and the connectedness of $\tilde{G}([u,v],R)$ we have $p_1 \simeq_R p_2$ by the same argument as in the proof of the "ONLY IF" part.

Consequently, for any two paths p_1, p_2 with common end-vertices, we have $p_1 \simeq_R p_2$ by induction.

To complete the proof of the "IF" part, we must show the minimality of R. For any $r=\{p,p'\} \in R$ let p and p' be paths from v to u with $u \neq v$. Then $\tilde{G}([u,v],R \backslash \{r\})$ defined by $R \backslash \{r\}$ instead of R is not connected. By the same argument as in the proof of the "ONLY IF" part, p and p' are not homotopic with respect to $R \backslash \{r\}$. This completes the proof. Q.E.D.

It follows from Theorem 2.2 that all the homotopy bases have the same cardinality, which is the homotopy rank $\eta(G)$ of G [2].

Furthermore, from Theorem 2.2, we have

Corollary 2.3: The homotopy rank $\eta(G)$ is expressed as

$$\eta(G) = \Sigma\{d([u,v])-1 \mid [u,v]\text{:a nonempty interval of } G \text{ with } u\neq v\}, \quad (2.4)$$

where $d([u,v])$ is the degree of the interval $[u,v]$.

Corollary 2.4 (cf. Proposition 3.4 of [2]): Define $W=\{v \mid v\in V,\ \delta^+v\neq\emptyset\}$, where $\delta^+v=\{a \mid a\in A,\ \partial^+a=v\}$. Then we have $\eta(G) \leqq |V|(|A|-|W|)$.

(Proof) Since $d([u,v])$ is not larger than the out-degree $|\delta^+v|$ of v, it follows from Corollary 2.3 that

$$\eta(G) \leqq \sum_{u\in V} \sum_{v\in W} (|\delta^+v|-1) = |V|(|A|-|W|).$$

Q.E.D.

3. An Algorithm for Finding a Homotopy Base

Based on Theorem 2.2, an algorithm for finding a homotopy base R of $G=(V,A)$ is furnished as follows.

An Algorithm

(1) Put $R:=\emptyset$.

(2) For each distinct vertices u, v such that the interval $[u,v]$ is nonempty, do the following:

(a) Find the minimal decomposition $G_i=(V_i,B_i)$ $(i=1,2,\cdots,k)$ of the interval $[u,v]$.

(b) If $k\geqq 2$, then find a path p_i in G_i from v to u for each $i=1,2,\cdots,k$, and put $R:=R\cup\{\{p_i,p_{i+1}\} \mid i=1,2,\cdots,k-1\}$.

(End of the algorithm)

The validity of the algorithm follows from Theorem 2.2. By the use of Lemma 2.1 we can carry out (a) in Step (2) in $O(|A|)$ running time for each interval [u,v]. Therefore, the total running time is $O(|V|^2|A|)$, and the required working space (not for storing the homotopy base) is $O(|V|+|A|)$.

Based on Theorem 2.2, we can generate all the homotopy bases by slightly modifying the above algorithm.

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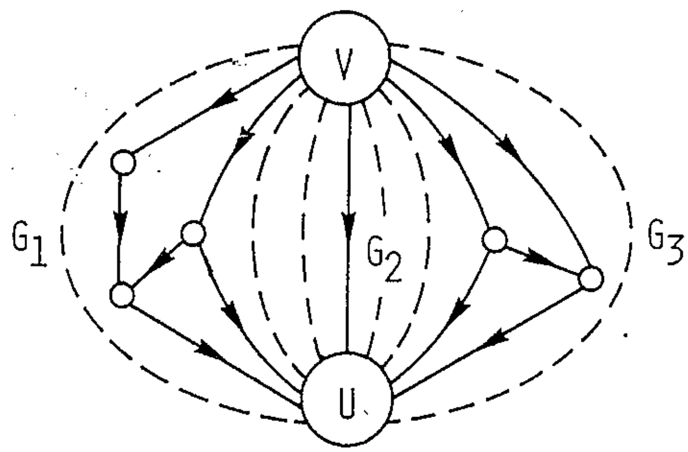

Fig.2.1. An interval [u,v] and its minimal decomposition G_i (i=1,2,3)