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Homotopy Base of Acyclic Graphs

---- A Combinatorial Analysis of Commutative Diagrams by Means of Preordered Matroid

by

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Abstract

A diagram is an acyclic graph with (linear) maps associated with arcs. The problem considered is which set of relations characterizes the commutativity of a given diagram and how to find such a set of relations. The dependence among parallel paths of an acyclic graph is analyzed by exploiting their composite algebraic structure of preorder and matroid. The notion of "homotopy base" is introduced; any two bases are shown to be equicardinal, and a base can be found by an efficient dual greedy algorithm, which, starting with a spanning set, deletes dependent elements one by one in an arbitrary order.

Keywords: acyclic graph, commutative diagram, homotopy base,
cyclomatic number, arborescence, preordered matroid,
dual greedy algorithm

1. Introduction --- Commutativity of Diagrams

A diagram, as used in mathematics, is an acyclic graph $G(V,E)$ with (linear) spaces attached to vertices and (linear) maps to arcs. A diagram is said to commute if for each pair of vertices (u,v) $(u,v \in V)$, the product (i.e., the composition) of the maps along any path connecting u to v does not depend on the choice of a path. For example, the diagram in Fig.1 commutes, by definition, iff the following equations hold, where f_i denotes the map associated with the arc e_i and $f \circ g(x)=g(f(x))$:

from v_1 to v_3: $f_2=f_1 \circ f_3$, (1.1)

from v_2 to v_4: $f_4=f_3 \circ f_5$, (1.2)

from v_1 to v_4: $f_1 \circ f_4=f_1 \circ f_3 \circ f_5=f_2 \circ f_5$. (1.3)

It is easy to see here that the relations (1.1) and (1.2) are sufficient for the diagram to commute and the commutativity (1.3) along the paths from v_1 to v_4 are derivable from (1.1) and (1.2). In other words, the commutativity of this diagram is characterized by two relations (1.1) and (1.2), and (1.3) is redundant for it. In this paper, we are concerned with which set of relations characterizes the commutativity of a diagram and how such a set can be found.

More generally, a diagram in this paper will mean a pair of acyclic graph $G(V,E)$ and a mapping $f:E \dashrightarrow F$ from the arc set E to a semi-group F with the associative multiplication denoted by $\circ$. We denote by P_{all} the set of all the directed paths on G. Note that P_{all} is a finite set since G is acyclic. A path p $(\in P_{all})$ can be viewed as a sequence of arcs, say $p=e_1 \dots e_k$, and the mapping f is naturally extended for a path p by $f(p)=f(e_1) \circ \dots \circ f(e_k)$. For a path p $(\in P_{all})$, $\partial^+ p$ (resp. $\partial^- p$) designates the initial (resp. terminal) vertex of p and $\partial p=\{\partial^+ p, \partial^- p\}$ the endvertices. Two distinct paths p and q are called <u>parallel</u> if $\partial p=\partial q$ and the set of all the unordered pairs of parallel paths is denoted by R_{all}, i.e.,

$$R_{all}=\{ \{p,q\} \mid p,q\in P_{all},\ p\neq q,\ \partial p=\partial q \}.$$

We will say that a diagram commutes iff

$$f(p)=f(q) \text{ whenever p and q are parallel.} \qquad (1.4)$$

The following are some of the instances that may be covered by the notion of commutativity as defined above.

(1) In the case where the semi-group F is the set of square matrices of a fixed order and the multiplication $A\circ B$ is defined as the matrix product BA, the commutativity of a diagram in the sense of (1.4) reduces to the usual one for linear maps.

(2) Suppose F is the additive semi-group of nonnegative integers and the acyclic graph G represents the Hasse diagram of a partially ordered set with $f(e)=1$ for all $e\in E$. Then the commutativity of the diagram in the present sense is equivalent to the Jordan-Dedekind condition for the partial order.

(3) As an physical example, consider an electric network with the underlying graph G(V,E), where F is the real numbers with addition as the operation $\circ$ and f(e) stands for the voltage across the arc e. We may assume without loss of generality that G is connected and the arcs are oriented so that there exists a rooted directed spanning tree (an arborescence) on G. Then the diagram commutes in the sense of (1.4) iff the voltages f(e)'s satisfy Kirchhoff's voltage law.

In characterizing the commutativity (1.4) of a diagram, it is evidently sufficient to consider only those parallel paths which have no common intermediate vertices. We will mean by a bilinking such an unordered pair of parallel paths that share no common vertices except their end-vertices, and denote by R_0 ($\subset R_{all}$) the set of all the bilinkings. Note that R_0, as well as R_{all}, is determined by the graph G, independently of the mapping f. The following is obvious.

Proposition 1.1. A diagram (G,f) commutes iff

$$f(p)=f(q) \text{ for any bilinking } \{p,q\}\in R_0.$$

A bilinking $r=\{p,q\}\in R_0$ corresponds to an equation $f(p)=f(q)$, which will be referred to as the elementary relation for r. A set R ($\subset R_0$), or alternatively, the set of elementary relations $f(p)=f(q)$ for $\{p,q\}\in R$, characterizes the commutativity of the diagram iff any elementary relation is derivable from the given set of relations in the semi-group F. If we consider a semi-group in general without taking advantage of any special properties of a particular semi-group, all we have in hand is the associative law; neither the commutative law nor the inverses are available. In such a case, the question of whether an elementary relation is derivable or not from the given set of elementary relations reduces to that of graph-theoretic nature, since it should be determined by some topological relation among bilinkings in G.

In the next section, we will investigate what kind of topological relation in G is relevant to the derivability of a relation from a given set of elementary relations. In particular, we define an equivalence relation, called "homotopy", among parallel paths with respect to a given set of parallel paths and the notion of "homotopy base" is introduced for an acyclic graph, which corresponds to an irredundant set of elementary relations characterizing the commutativity of a diagram. By exploiting the fact that the set R_0 of bilinkings is equipped with a composite algebraic structure of preorder and matroid, it is shown that any two homotopy bases are of equal cardinality and a base can be found by an efficient algorithm of dual greedy type.

2. Homotopy Base of Acyclic Graphs

2.1. Homotopy and derivability

Let G(V,E) be an acyclic graph, and P_{all}, R_{all} and R_0 be, as above, the set of all the paths, the set of all pairs of parallel paths and the set of all bilinkings, respectively. For two paths p and q such that $\partial^- p=\partial^+ q$, we denote by pq ($\in P_{all}$) the concatenation of p and q, where $\partial^+(pq)=\partial^+ p$ and $\partial^-(pq)=\partial^- q$.

Let R be a subset of R_{all}. The following concepts are of central importance.

<u>Definition 2.1.</u> Two paths p, q ($\in P_{all}$) are said to be <u>contiguous</u> with respect to R if

(i) they are parallel (i.e., $\{p,q\}\in R_{all}$),

(ii) there exists $\{p',q'\}\in R$ such that $p=\alpha p'\beta$ and $q=\alpha q'\beta$ for some α, $\beta\in P_{all}$.

<u>Definition 2.2.</u> Two paths p, q ($\in P_{all}$) are said to be <u>homotopic</u> with respect to R (denoted as "$p\simeq q$ (R)") if for some sequence p_i ($\in P_{all}$) ($i=1,\ldots,k$) of paths, p_i is contiguous to p_{i+1} with respect to R for $i=0,\ldots,k$, where $p_0=p$ and $p_{k+1}=q$.

Evidently, the relation $\simeq$ is an equivalence relation on P_{all}. Note that two homotopic paths are necessarily parallel. A comment would be in order here as to the mutual relation between the commutativity of a diagram (G,f) and the homotopy of the paths of G. Suppose that the relations of the form $f(p')=f(q')$ are given for $\{p',q'\}$ belonging to R ($\subset R_{all}$). Then it is easy to see that the relation $f(p)=f(q)$ for a pair of parallel paths $\{p,q\} \in R_{all}$ can be derived from the given set of relations if the paths p and q are contiguous with respect to R. Furthermore, the following is true.

Proposition 2.1. For a pair of parallel paths $\{p,q\}$, the relation $f(p)=f(q)$ is derivable in general from the given set of relations for R iff the paths p and q are homotopic with respect to R.

Definition 2.3. A subset P of P_{all} is said to be connected by R ($\subset R_{all}$) (denoted as "$P \lhd R$" or "$R \rhd P$") if

$$p \simeq q \ (R) \text{ whenever } p,q \in P \text{ and } \partial p = \partial q.$$

Definition 2.4. A subset R of R_{all} is called a homotopy base if $P_{all} \lhd R$ and no proper subset of R satisfies this property.

Not surprisingly, we may confine ourselves to the bilinkings when we consider the homotopy bases, as stated below.

Proposition 2.2. If R ($\subset R_{all}$) is a homotopy base, then $R \subset R_0$.

(Proof) Suppose $r=\{p,q\} \in R$ and that the paths p and q share some vertices other than the end-vertices $\partial p = \partial q$. The symmetric difference of p and q determines k (≥ 1) bilinkings $r_i=\{p_i,q_i\}$ such that p_i and q_i are properly contained in p and q, respectively ($i=1,\ldots,k$). Since R is a homotopy base, we have $p_i \simeq q_i$ (R) for $i=1,\ldots,k$, which is equivalent to the condition that $p_i \simeq q_i$ ($R \backslash r$) for $i=1,\ldots,k$. This implies that $p \simeq q$ ($R \backslash r$), and therefore $P_{all} \lhd (R \backslash r)$, which contradicts the minimality of R. Q.E.D.

It follows from Prop.2.1 and Def.2.3 that the relation $f(p)=f(q)$ holds true for parallel paths $p,q \in P$ if $P \lhd R$ and the relations hold for R ($\subset R_{all}$). Thus, a homotopy base may be viewed as representing an irredundant set of elementary relations that guarantees the commutativity of a diagram. Prop.1.1 can be rephrased as in Prop.2.3 below, and Prop. 2.4 is immediate from the definition.

<u>Proposition 2.3.</u> $P_{all} \lhd R_0$.

<u>Proposition 2.4.</u> Let $R \subset R' \subset R_{all}$ and $P' \subset P \subset P_{all}$. If $P \lhd R$, then $P' \lhd R'$.

Those concepts introduced are illustrated for the graph G shown in Fig.1. By inspection we have

$$R_0 = \{ \{e_2, e_1e_3\}, \{e_4, e_3e_5\}, \{e_1e_4, e_2e_5\} \},$$

$$R_{all} = R_0 \cup \{ \{e_1e_4, e_1e_3e_5\}, \{e_2e_5, e_1e_3e_5\} \}.$$

Let, e.g., $R = \{ \{e_4, e_3e_5\}, \{e_1e_4, e_2e_5\} \}$ and $p_1 = e_2e_5$, $p_2 = e_1e_3e_5$, $p_3 = e_2$, $p_4 = e_1e_3$. Then p_1 and p_2 are homotopic with respect to R, while p_3 and p_4 are not, that is, $\{p_3, p_4\}$ is not connected by R. This corresponds to the fact that the relation (1.1) is not derivable from (1.2) and (1.3). Thus, R is not a homotopy base. The graph G has a unique homotopy base $\{ \{e_2, e_1e_3\}, \{e_4, e_3e_5\} \}$.

2.2. Preordered matroid on bilinkings

This subsection aims at establishing fundamental properties of homotopy bases of an acyclic graph G by taking notice of the algebraic structure of R_0. Specifically, R_0 is a preordered set with the preorder induced from the reachability among vertices of G, and at the same time, R_0 is a matroid with the dependence defined by means of the linear dependence among elementary cycles in G.

First, we define a preorder, i.e., a reflexsive and transitive binary relation $\geq$ on R_0. The notation ∂ is extended for a bilinking $r=\{p,q\}\in R_0$ by the well-defined relation $\partial r=\partial p(=\partial q)$, and similarly for ∂^+ and ∂^-. A vertex u of G is reachable in G to (resp. from) a vertex v if u=v or there exists a path $p\in P_{all}$ with $\partial^+p=u$ and $\partial^-p=v$ (resp. $\partial^-p=u$ and $\partial^+p=v$).

Definition 2.5. For bilinkings $r_1=\{p_1,q_1\}$, $r_2=\{p_2,q_2\}\in R_0$, we define $r_1 \geq r_2$ iff ∂^+r_1 is reachable to ∂^+r_2 and ∂^-r_1 is reachable from ∂^-r_2 in G.

Alternatively, $r_1 \geq r_2$ iff there exists a path p such that $\partial p=\partial r_1$ and p contains p_2. For $r\in R_0$, the set

$$\langle r\rangle = \{s\in R_0 \mid r \geq s\} \tag{2.1}$$

is referred to as the principal ideal defined by r, and the set

$$[r] = \{s\in R_0 \mid r \geq s \text{ and } s \geq r\} \tag{2.2}$$

as the block containing r. Since G is acyclic, $[r_1]=[r_2]$ iff $\partial r_1=\partial r_2$, and therefore each block [r] is identified by a pair $[\partial^+r,\partial^-r]$ of the initial and the terminal vertex of r.

The second algebraic structure, i.e., a matroidal structure [9], is defined on R_0 as follows. A bilinking $r=\{p,q\}$, being a pair of parallel paths, determines an elementary cycle, which is nothing but the collection of arcs contained in either p or q. With this correspondence we can talk of the dependence among bilinkings on the basis of the linear dependence over GF(2) among the associated elementary cycles. The matroid thus defined on R_0 will be denoted by M, which is obviously representable over GF(2). For a subset R of R_0, we denote by cl(R) the closure of R in M, i.e., the set of bilinkings that are dependent on R in M.

To sum up, it is revealed that the matroid M is defined on the ground set $X=R_0$ which is preordered by $\geq$. In general, we will christen such an algebraic structure $\tilde{R}=(X, \geq, M)$ a preordered matroid, in which the "closure function" σ is defined by

$$\sigma(R) = \{r \in X \mid r \in cl(R \cap \langle r \rangle)\}, \qquad (2.3)$$

where $\langle r \rangle$ is the principal ideal defined by (2.1) and cl the closure function of the matroid M. We will denote by $\tilde{R}(G)$ the preordered matroid on the bilinkings R_0 of an acyclic graph G with the preorder and the matroid as above, A preordered matroid reduces to a usual matroid if the preorder is trivial in the sense that $r \geq r'$ for all $r, r' \in X$, namely, $\langle r \rangle = [r] = X$ for any $r \in X$.

The closure function σ of a preordered matroid enjoys the following natural properties.

Proposition 2.5. Let $R, S \subset X$.

(1) $R \subset \sigma(R)$,

(2) $\sigma(R) \subset \sigma(S)$ for $R \subset S$,

(3) $s \in \sigma(R)$ implies $\sigma(R \cup s) = \sigma(R)$,

(4) $S \subset \sigma(R)$ implies $\sigma(R \cup S) = \sigma(R)$,

(5) $\sigma(\sigma(R)) = \sigma(R)$.

(Proof) (1) If $r \in R$, then $r \in R \cap \langle r \rangle$ and therefore $r \in cl(R \cap \langle r \rangle)$.

(2) If $r \in \sigma(R)$, then $r \in cl(R \cap \langle r \rangle) \subset cl(S \cap \langle r \rangle)$.

(3) From (2), we have $\sigma(R \cup s) \supset \sigma(R)$. Let $r \in \sigma(R \cup s)$.

[Case 1: $s \in \langle r \rangle$]: The assumption $s \in \sigma(R)$ implies that $s \in cl(R \cap \langle s \rangle) \subset cl(R \cap \langle r \rangle)$, from which it follows that $cl(R \cap \langle r \rangle) = cl((R \cap \langle r \rangle) \cup s)$, since (3) holds when σ is replaced by cl. On the other hand, $cl((R \cap \langle r \rangle) \cup s) = cl((R \cup s) \cap \langle r \rangle) \ni r$, since $r \in \sigma(R \cup s)$. Hence $r \in cl(R \cap \langle r \rangle)$, i.e., $r \in \sigma(R)$.

[Case 2: $s \notin \langle r \rangle$]: $r \in \sigma(R \cup s)$ iff $r \in cl((R \cup s) \cap \langle r \rangle) = cl(R \cap \langle r \rangle)$.

(4) By induction using (3).

(5) Immediate from (1) and (4) with $S = \sigma(R)$. Q.E.D.

In parallel with the ordinary matroid, we introduce the following terminology for a preordered matroid with ground set X and closure function σ.

Definition 2.5. r ($\in X$) is said to be dependent on R ($\subset X$) if $r \in \sigma(R)$.

Definition 2.6. R ($\subset X$) is called a spanning set if $\sigma(R) = X$.

Definition 2.7. R ($\subset X$) is called an independent set if $r \notin \sigma(R \setminus r)$ for any $r \in R$.

Definition 2.8. R ($\subset X$) is called a base if R is spanning and independent.

From Prop.2.5 it follows that any superset of a spanning set is spanning; and that any subset of an independent set is independent, i.e., the independence thus defined determines an independence system in the sense of [5].

Stated below are the key observations that link the homotopy of G with the preordered matroid $\tilde{R}(G)$ on the bilinkings of G.

<u>Proposition 2.6.</u> For $r=\{p,q\} \in R_0$ and $R \subset R_0$,

$$p \simeq q \ (R) \quad \text{implies} \quad r \in \sigma(R),$$

where σ is the closure function of $\tilde{R}(G)$.

(Proof) Suppose $p \simeq q$ (R). Then there exists a sequence of paths $p=p_0$, p_1, ..., p_k, $p_{k+1}=q$ such that p_i and p_{i+1} are contiguous with respect to R. Identifying a path with a set of arcs on it, we may write as $p_{i+1} = p_i \oplus r_i$ for $i=0,\ldots,k$, from which it follows that $q=p\oplus r_1\oplus\cdots\oplus r_k$, or $r=r_1\oplus\cdots\oplus r_k$. Since $r_i \in \langle r\rangle$, this implies that $r \in \sigma(\{r_1,\ldots,r_k\}) \subset \sigma(R)$. Q.E.D.

<u>Proposition 2.7.</u> Let $R\subset R_0$. If $\sigma(R)=R_0$ in $\tilde{R}(G)$, then $p\simeq q$ (R) for any $r=\{p,q\}\in R_0$.

(Proof) For any $r=\{p,q\}\in R_0$, there exist $r_i\in R$ $(i=1,\ldots,k)$ such that $r\geq r_i$ and $r=r_1\oplus\cdots\oplus r_k$. We will prove the proposition by induction with respect to the preorder on R_0.

Basis: Suppose [r] is a minimal block, i.e., $[r]=\langle r\rangle$. If we put $p_i = p\oplus r_1\oplus\cdots\oplus r_i$ $(i=0,\ldots,k)$, we have $p_{i-1}\simeq p_i$ (R) for $i=1,\ldots,k$ since $\partial r=\partial r_i$. Hence $p_0\simeq p_k$ (R), i.e., $p\simeq q$ (R).

Induction: Suppose that $p'\simeq q'$ (R) for any $r'=\{p',q'\}\in R_0$ such that $r'\in\langle r\rangle\backslash[r]$. Since $r\geq r_i$, there exist a path α_i from ∂^+r to ∂^+r_i and a path β_i from ∂^-r_i to ∂^-r (including the cases where $\partial^+r=\partial^+r_i$ and/or $\partial^-r=\partial^-r_i$). Put $r_i=\{p_i,q_i\}$.

Since $r=r_1\oplus\cdots\oplus r_k$, we can choose $r_{i(1)}=\{p_{i(1)},q_{i(1)}\}$ such that $p\cap p_{i(1)}\neq\emptyset$. We claim the relation $p \simeq \alpha_{i(1)}p_{i(1)}\beta_{i(1)} \simeq \alpha_{i(1)}q_{i(1)}\beta_{i(1)}$ (R), which is obviously true if $p=p_{i(1)}$, where $\alpha_{i(1)}$

and $\beta_{i(1)}$ are virtually void, and which holds true by the induction hypothesis otherwise.

In a similar manner, it can be shown that, if $r_i \cap r_j \neq \emptyset$, then $\alpha_i p_i \beta_i \simeq \alpha_j p_j \beta_j$ (R). Since $r=r_1 \oplus \ldots \oplus r_k$, we can find a sequence $r_{i(j)}=\{p_{i(j)}, q_{i(j)}\} \in R$ $(j=1,\ldots,m)$ such that $p \simeq \alpha_{i(j)} p_{i(j)} \beta_{i(j)} \simeq \alpha_{i(j)} q_{i(j)} \beta_{i(j)}$ (R), for $j=1,\ldots,m$ and $q \simeq \alpha_{i(m)} q_{i(m)} \beta_{i(m)}$. This means that $p \simeq q$ (R). Q.E.D.

Theorem 2.1. Let $R \subset R_0$. $P_{all} \lhd R$ iff $\sigma(R) = R_0$ in $\tilde{R}(G)$.

(Proof) Immediate from Prop.2.3, Prop.2.6 and Prop.2.7. Q.E.D.

Theorem 2.2. R ($\subset R_0$) is a homotopy base of G iff R is a base of the preordered matroid $\tilde{R}(G)$ on the bilinkings of G.

(Proof) Since R ($\subset R_0$) is independent in the preordered matroid iff $\sigma(R \backslash r) \neq \sigma(R)$ for any $r \in R$, R is a base iff $\sigma(R)=R_0$ and no proper subset of R satisfies this property. The theorem follows from this fact combined with Def.2.4 and Theorem 2.1. Q.E.D.

Theorem 2.2 reveals the algebraic nature of homotopy bases of an acyclic graph in terms of the preordered matroid. Henceforth, we will concentrate on establishing fundamental properties of a preordered matroid. To be specific, the bases of a preordered matroid are shown in Theorem 2.3 to be of the same size. This fact, along with Theorem 2.2, implies that the homotopy bases of an acyclic graph G contain an equal number of bilinkings. The common size of a homotopy base of G will be named the "homotopy rank" of G.

In what follows, $\tilde{R}=(X, \geq, M)$ represents a general preordered matroid with closure function σ defined by (2.3). The following property constitutes the basis of the dual greedy algorithm, given in Section 3, for finding a base of a preordered matroid.

Proposition 2.8. Let $B \subset X$. B is a base iff B is a minimal spanning set, that is, iff B is spanning and $B\backslash r$ is not spanning for any $r \in B$.

(Proof) Suppose B is a base. Since B is independent, $r \notin \sigma(B\backslash r)$ for any $r \in B$. Therefore, $\sigma(B\backslash r) \neq X$ for any $r \in B$.

Conversely suppose that B is a minimal spanning set and not an independent set. Then $r \in \sigma(B\backslash r)$ for some $r \in B$. It follows from Prop.2.5(3) that $\sigma(B\backslash r) = \sigma(B) = X$, which contradicts the minimality of B. Q.E.D.

It should be remarked in connection with Prop.2.8 that a maximal independent set is not necessarily a base, in sharp contrast with ordinary matroids, although a base is certainly a maximal independent set.

For $R \subset X$, $M|R$ denotes the restriction of the matroid M to R, whereas M.R means the contraction of M to R [9]. The following characterize a spanning set and a base of the preordered matroid $\hat{R}$ directly in terms of the bases of minors of M.

Proposition 2.9. Let $R \subset X$. R is spanning in a preordered matroid $\hat{R}=(X, \geq, M)$ iff $R \cap [r]$ is spanning in the matroid minor $M|\langle r\rangle.[r]$ for each $r \in X$.

(Proof) [ONLY IF] Let R be spanning in $\hat{R}$ and fix $r \in X$. Then $R \cap [r]$ is spanning in $M|\langle r\rangle.[r]$ since $s \in cl(R \cap \langle s\rangle) = cl(R \cap \langle r\rangle) \subset cl((R \cap [r]) \cup (\langle r\rangle \backslash [r]))$ for all $s \in [r]$.

[IF] The converse, i.e., the assertion that $r \in \sigma(R)$ for all $r \in X$, is established by induction with respect to the preorder. For $r \in X$ such that $\langle r\rangle=[r]$, we have $r \in \sigma(R)$ since, by the assumption, $R \cap [r]=R \cap \langle r\rangle$ is spanning in $M|\langle r\rangle.[r]=M|[r]$, i.e., $r\in cl(R\cap\langle r\rangle)$. Next assume that $s\in\sigma(R)$ for $s\in\langle r\rangle\setminus[r]=S$, which implies that $s\in cl(R\cap\langle s\rangle) \subset cl(R\cap\langle r\rangle)$, or that

$$S \subset cl(R\cap\langle r\rangle). \qquad (*)$$

Since $R\cap[r]$ is spanning in $M|\langle r\rangle.[r]$, we have $r \in cl((R\cap[r])\cup S) = cl((R\cap\langle r\rangle)\cup S) = cl(R\cap\langle r\rangle)$, where the last equality follows from the relation (*) above. Q.E.D.

<u>Proposition 2.10.</u> Let $B \subset X$. B is a base of a preordered matroid $\tilde{R}=(X, \geq, M)$ iff $B\cap[r]$ is a base of $M|\langle r\rangle.[r]$ for all $r\in X$.

(Proof) First note that, if $B\cap[r]$ is spanning in $M|\langle r\rangle.[r]$ for each $r\in X$, we have

$$\langle r\rangle\setminus[r] \subset cl(B\cap(\langle r\rangle\setminus[r])) \qquad (*)$$

for each $r\in X$, since $s \in cl((B\cap[s])\cup(\langle s\rangle\setminus[s])) \subset cl(B\cap(\langle r\rangle\setminus[r]))$ for $s\in\langle r\rangle\setminus[r]$.

[IF] By Prop.2.9, it suffices to show that B is independent in $\tilde{R}$. Since $B\cap[r]$ is independent in $M|\langle r\rangle.[r]$ and (*) holds, $r\notin cl((B\cap[r]\setminus r)\cup(\langle r\rangle\setminus[r])) = cl((B\setminus r)\cap\langle r\rangle)$, which is equivalent to $r\notin\sigma(B\setminus r)$. Therefore B is independent in $\tilde{R}$.

[ONLY IF] Let B be a base of $\tilde{R}$. By Prop.2.9, $B\cap[r]$ is spanning in $M|\langle r\rangle.[r]$ for all $r\in X$ and the inclusion (*) above holds. It then follows from (*) and the fact $s\notin\sigma(B\setminus s)$ for $s\in X$ that $cl((B\cap[r]\setminus s)\cup(\langle r\rangle\setminus[r])) = cl((B\setminus s)\cap\langle r\rangle) = cl(B\setminus s\cap\langle s\rangle) \not\ni s$ for all $s\in[r]$. This means that $B\cap[r]$ is independent in $M|\langle r\rangle.[r]$. Q.E.D.

Finally, we obtain the following as an immediate consequence of Prop.2.10.

<u>Theorem 2.3.</u> $|B_1| = |B_2|$ if B_1 and B_2 are bases of a preordered matroid.

(Proof) From Prop.2.10 and the equicardinality of bases of a matroid, we have $|B_1 \cap [r]| = |B_2 \cap [r]|$ for each block $[r]$. Q.E.D.

Theorem 2.3 above suggests that a kind of rank be defined for a preordered matroid just as for a usual matroid. The common size of bases of a preordered maroid $\tilde{R}$ will be called the "rank" of $\tilde{R}$, and denoted by $\rho(\tilde{R})$. Moreover, as a collorary of Theorem 2.3, we obtain the following, where a subset S of R ($\subset X$) is called a base of R if $\sigma(R)=\sigma(S)$ and S is independent.

Proposition 2.11. $|S_1| = |S_2|$ if S_1 and S_2 are bases of R ($\subset X$).
The common size of bases of R ($\subset X$) will be called the rank of R.

Theorem 2.2 and Theorem 2.3 together reveal that any two homotopy bases of an acyclic graph G have the same cardinality, denoted as $\eta(G)$, which we name here the "homotopy rank" of G.

Theorem 2.4. The size $\eta(G)$ of a homotopy base of an acyclic graph G is equal to the rank $\rho(\tilde{R}(G))$ of the associated preordered matroid $\tilde{R}(G)$ on the bilinkings of G.

In relation to our original interest in a commutative diagram, Theorem 2.4 implies, when combined with Prop.2.1, that any irredundant set of elementary relations sufficient for the commutativity of the diagram contains the same number of equations, which is equal to the homotopy rank $\eta(G)$ of the underlying graph G.

The homotopy rank $\eta(G)$ of G(V,E) is no larger than $|V||E|$, as will be stated with more precision in Prop.3.4

2.3. An illustrative example

As an example, consider the acyclic graph G(V,E) depicted in Fig.2 with $V=\{v_1,\ldots,v_7\}$ and $E=\{e_1,\ldots,e_{11}\}$. All the directed paths on G are listed below, where a path p is represented by a sequence of arcs on it.

∂^+p	∂^-p	$p \in P_{all}$
v_1	v_2 :	e_1
v_1	v_3 :	e_2
v_1	v_4 :	e_1e_3, e_2e_6
v_1	v_5 :	e_1e_5, e_2e_4
v_1	v_6 :	$e_1e_3e_7$, $e_2e_4e_8$, $e_1e_5e_8$, $e_2e_6e_7$, e_1e_9
v_1	v_7 :	e_2e_{10}, $e_1e_5e_{11}$, $e_2e_4e_{11}$
v_2	v_4 :	e_3
v_2	v_5 :	e_5
v_2	v_6 :	e_3e_7, e_5e_8, e_9
v_2	v_7 :	e_5e_{11}
v_3	v_4 :	e_6
v_3	v_5 :	e_4
v_3	v_6 :	e_6e_7, e_4e_8
v_3	v_7 :	e_{10}, e_4e_{11}
v_4	v_6 :	e_7
v_5	v_6 :	e_8
v_5	v_7 :	e_{11}

The set R_0 of bilinkings of G consists of 12 elements, r_1 to r_{12}, as below.

$[\partial^+ r, \partial^- r]$		$r \in R_0$
$[v_1, v_4]$	:	$r_1=\{e_1e_3, e_2e_6\}$
$[v_1, v_5]$	:	$r_2=\{e_1e_5, e_2e_4\}$
$[v_1, v_6]$	:	$r_3=\{e_1e_3e_7, e_2e_4e_8\}$, $r_4=\{e_1e_5e_8, e_2e_6e_7\}$, $r_5=\{e_1e_9, e_2e_4e_8\}$, $r_6=\{e_1e_9, e_2e_6e_7\}$
$[v_1, v_7]$	:	$r_7=\{e_2e_{10}, e_1e_5e_{11}\}$
$[v_2, v_6]$	:	$r_8=\{e_3e_7, e_5e_8\}$, $r_9=\{e_3e_7, e_9\}$, $r_{10}=\{e_5e_8, e_9\}$
$[v_3, v_6]$	:	$r_{11}=\{e_6e_7, e_4e_8\}$
$[v_3, v_7]$	:	$r_{12}=\{e_{10}, e_4e_{11}\}$

The preorder ($\geq$) of R_0 is shown in Fig.3, where it should be remembered that each block $[r]$ is identified by a pair $[\partial^+ r, \partial^- r]$ of the initial and the terminal vertex.

As for the matroid M, the bilinking r_2, for instance, is dependent on the set $\{r_3, r_8\}$ in M, i.e., $r_2 \in cl(\{r_3, r_8\})$, since the cycle corresponding to r_2 can be expressed as the sum of those corresponding to r_3 and r_8, i.e., $r_2 = r_3 \oplus r_8$. However, r_2 is not dependent on $\{r_3, r_8\}$ in the preordered matroid $\tilde{R}(G)$, i.e., $r_2 \notin \sigma(\{r_3, r_8\})$, since $\{r_3, r_8\} \cap \langle r_2 \rangle = \emptyset$. On the other hand, the bilinking r_7 is dependent on the set $\{r_2, r_{12}\}$ in $\tilde{R}(G)$, i.e., $r_7 \in \sigma(\{r_2, r_{12}\})$, since $\langle r_7 \rangle = \{r_2, r_7, r_{12}\}$ and the cycle corresponding to r_7 is expressed as $r_7 = r_2 \oplus r_{12}$.

The set $B_0=\{r_1, r_2, r_{11}, r_{12}\}$ is included in any base of $\tilde{R}(G)$. A base of $\tilde{R}(G)$ is obtained from B_0 by adding any two of $\{r_8, r_9, r_{10}\}$ to B_0. Thus the rank $\rho(\tilde{R}(G))$ of $\tilde{R}(G)$ is equal to $|B|=6$ and, by Theorem 2.4, the homotopy rank $\eta(G)$ of G is also equal to 6.

3. Algorithm for Finding a Homotopy Base

This section describes a polynomial-time algorithm for finding a homotopy base of an acyclic graph G as a special case of the general procedure, given in Section 3.1, for finding a base of a preordered matroid by deleting dependent elements one by one from a spanning set. Section 3.2 is devoted to the construction of a spanning set, which is small in size, of the preordered matroid $\tilde{R}(G)$ of bilinkings of an acyclic graph G by repeated applications of depth-first search on G. Then in Section 3.3 it is shown that a homotopy base of G can be found in time polynomial in the size of G. An example is also given for illustration.

3.1. Dual greedy algorithm for a base of a preordered matroid

In this subsection, we deal with a general preordered matroid $\tilde{R}=(X, \geq, M)$, the closure function of which is denoted by σ as usual. It is shown below that a base of $\tilde{R}$ can be found by an algorithm of dual greedy type which starts with a spanning set and deletes dependent elements one by one.

The algorithm, labelled as Algorithm B, is described as follows.

Algorithm B [Finding a base of a preordered matroid]

Step 1: Find a spanning set $R_s=\{r_1, \ldots, r_K\}$.

Step 2: $R:=R_s$;

<u>for</u> $i:=1$ <u>to</u> K <u>do</u>

<u>if</u> $r_i \in \sigma(R \setminus r_i)$ <u>then</u> $R:=R \setminus r_i$;

$B:=R$

The ordering of the elements of R_s is not relevant and R decreases monotone; in this sense, this algorithm is greedy or myopic. It is easy to see, by Prop.2.5(3), that the relation $\sigma(R)=\sigma(R_s)=X$ is maintained throughout the iteration, and therefore the set B obtained is spanning. B is also independent, since, otherwise $r_j \in \sigma(B \backslash r_j)$ for some $r_j \in B$, which contradicts the fact that r_j has not been deleted. Thus, the set B is a base of the preordered matroid $\tilde{R}$. The complexity of Algorithm B depends on the size $K=|R_s|$ of the initial spanning set R_s and, of course, on the cost for checking the dependence of r_i at each iteration.

Suppose a weight function $w:X\text{-->}R^+$ is given and a base B of minimum weight is to be found. The Algorithm B is extended to the weighted version as follows.

Algorithm WB [Finding a base of minimum weight]

Step 1: Order all the elements of $X=\{r_1, \ldots, r_N\}$ in such a way that $i \leq j$ iff $w(r_i) \geq w(r_j)$.

Step 2: R:=X;

<u>for</u> i:=1 <u>to</u> N <u>do</u>

<u>if</u> $r_i \in \sigma(R \backslash r_i)$ <u>then</u> $R:=R \backslash r_i$;

B:=R

A base of the maximum weight can be found similarly by ordering the elements of X in an ascending order with respect to their weight. The reason Algorithm WB works lies in the fact that the family F of the complements of spanning sets, $F = \{R \subset X \mid \sigma(X \backslash R)=X\}$, forms the independent sets of a matroid. Prop.2.10 shows that $R \in F$ iff $R \cap [r]$ is independent in the dual of $M|\langle r\rangle.[r]$ for each $r\in X$. Thus F coincides with the family of the independent sets of the direct sum of $(M|\langle r\rangle.[r])^*$, where [r] runs over all the blocks with respect to the prescribed preorder.

3.2. Finding a small spanning set of bilinkings

A homotopy base of an acyclic graph G(V,E) could be found by Algorithm B described above with the obvious choice of the totality R_0 of the bilinkings of G as the initial spanning set R_s. This naive choice of R_s, however, does not lead to a polynomial-time algorithm, since the size of R_0 can be exponentially large; e.g., for the graph G(V,E) of Fig.4, which is a cascade of m complete graphs $K_{2,2}$, we have $|V|=2m+2$ and $|E|=4m$, whereas $|R_0|>2^m$.

In the following, we construct a spanning set R_s ($\subset R_0$) with $|R_s|\le|V||E|$. For each $u\in V$, we denote by V(u) those vertices of G which are reachable from u, by G(u) the vertex-induced subgraph of G on V(u), and by E(u) the arc set of G(u). The cyclomatic number [2] (or the nullity) of G(u) will be designated by $\nu(u)=\nu(G(u))$, i.e.,

$$\nu(u) = |E(u)| - |V(u)| + 1. \tag{3.1}$$

Let $u\in V$ be fixed for the moment. It is well known that there exists an arborescence, or a rooted directed tree, of G(u) with root u; such an arborescence T(u) can easily be found, e.g., by the depth-first search from u on G [1]. Those arcs of G(u) which are not contained in T(u) will be called the cotree arcs of G(u) with respect to T(u). Then $\nu(u)$ is equal to the number of cotree arcs of G(u).

Each cotree arc e=(v,w) (directed from v to w) determines an elementary cycle, i.e., the unique cycle composed of the arc e and some of the arcs of $T(u)$. As a consequence of the fact that $T(u)$ is an arborescence, such a cycle, in turn, determines a bilinking as follows.

In the case (cf. Fig.5(a)) where v is an ancestor of w with respect to $T(u)$, i.e., where there is a path $p=e_1 \ldots e_k \in P_{all}$ on $T(u)$ such that $\partial^+ p=v$ and $\partial^- p=w$, the corresponding elementary cycle, consisting of $e_1,\ldots,e_k$ and e, determines a bilinking $r=\{e_1\ldots e_k, e\} \in R_0$. Otherwise (cf. Fig.5(b)), there is a common ancestor $x \in V$ of v and w, and therefore two paths $p=e_1\ldots e_k$ and $q=f_1\ldots f_l$ exist on $T(u)$ such that $\partial^+ p=\partial^+ q=x$, $\partial^- p=w$ and $\partial^- q=v$; such paths p and q are uniquely determined if they are further required to be arc-disjoint and the bilinking corresponding to e is given by $r=\{e_1\ldots e_k, f_1\ldots f_l e\} \in R_0$. Obviously, distinct cotree arcs determine distinct bilinkings. Note that $\partial r=\partial e$ in the former case, whereas $\partial^- r=\partial^- e$ but $\partial^+ r \neq \partial^+ e$ in the latter.

For each $u \in V$, we denote by $R[u]$ $(\subset R_0)$ the collection of the bilinkings that correspond in the above-mentioned manner to the cotree arcs with respect to a fixed arborescence $T(u)$. We obtain the following.

Proposition 3.1. $|R[u]| = \nu(u)$ $(u \in V)$.

Proposition 3.2. For each $u \in V$, $R[u]$ is independent in the preordered matroid $\mathring{R}(G)$.

(Proof) By the well-known fact that the elementary cycles corresponding to cotree arcs are linearly independent. Q.E.D.

We now claim the key lemma, where $P(u)$ is defined by

$$P(u) = \{p \in P_{all} \mid \partial^+ p=u\} \qquad (3.2)$$

and the notation $\lhd$ is given in Def.2.3.

Proposition 3.3. $P(u) \lhd R[u] \quad (u \in V)$.

(Proof) Take a path $p=e_1 \ldots e_k \in P(u)$. Since $T(u)$ is an arborescence, there exists a unique path q on $T(u)$ such that $\partial p = \partial q$. It suffices to show that p and q are homotopic with respect to $R[u]$, i.e., $p \simeq q \; (R[u])$.

Put $p_j = e_1 \ldots e_j \in P(u)$ for $j=1,\ldots,k$ and let q_j be the unique path on $T(u)$ such that $\partial^+ q_j = u$, $\partial^- q_j = \partial^- p_j = \partial^- e_j$. We will show by induction with respect to j that

$$p_j \simeq q_j \; (R[u]) \qquad (*)$$

for $j=1,\ldots,k$. For $j=1$, (*) obviously holds. Assume that $p_{j-1} \simeq q_{j-1}$ $(R[u])$, which implies

$$p_j \simeq q_{j-1} e_j \; (R[u]) \qquad (**)$$

since $p_j = p_{j-1} e_j$.

If e_j is contained in $T(u)$, we have $q_j = q_{j-1} e_j$, from which (*) follows by (**). Otherwise, e_j is a cotree arc, and it is easy to see that q_j is contiguous to $q_{j-1} e_j$ with respect to the bilinking determined by e_j, and hence $q_j \simeq q_{j-1} e_j \; (R[u])$. This implies (*) since (**) is assumed. Q.E.D.

The union of $R[u]$ $(u \in V)$:

$$R_s = \cup_{u \in V} R[u] \qquad (3.3)$$

is qualified as the initial spanning set in Algorithm B, as stated below. Note that $R[u]$'s for distinct u are not necessarily disjoint.

Theorem 3.1. (1) $|R_s| \leq \Sigma_{u \in V} \nu(u) \leq |V||E|$,

(2) $P_{all} \lhd R_s$, that is, $\sigma(R_s) = R_0$.

(Proof) (1) Obvious from Prop.3.1 and (3.3).

(2) For p, $q \in P_{all}$ with $\partial p = \partial q$, we have from Prop.3.3 that $p \simeq q \; (R[u])$, where $u = \partial^+ p = \partial^+ q$, and hence $p \simeq q \; (R_s)$. The equivalence of the two statements is due to Theorem 2.1. Q.E.D.

The homotopy rank $\eta(G)$ of G is bounded in terms of the cyclomatic number of the subgraphs G(u).

<u>Proposition 3.4.</u> $\max\{\nu(u)|u\in V\} \le \eta(G) \le \Sigma_{u\in V}\nu(u)$.

(Proof) Immediate from Prop.3.2 and Theorem 3.1. Q.E.D.

The example in Fig.6 demonstrates that the homotopy rank $\eta(G)$ of G(V,E) can be as large as $\Theta(|V||E|)$; the graph G(V,E) is a cascade of three complete bipartite graphs, $K_{1,m}$, $K_{m,m}$ and another $K_{m,m}$, and $\eta(G)=m^3-m$, while $|V|=3m+1$ and $|E|=2m^2+m$. Note also that $\max \nu(u)=\nu(s)=2m^2-2m$ and $\Sigma\nu(u)=m^3+m-2$, the upper bound in Prop.3.4 being asymptotically tight for m large.

This subsection is concluded with the consideration on the computational complexity. For each $u\in V$, the directed tree T(u) can be found in $O(|E(u)|)$ time by the straightforward application of the depth-first search rooted at u. For each cotree arc e, the corresponding bilinking can be constructed in $O(|V(u)|)$ time. Since R[u] contains $\nu(u)$ bilinkings, it can be constructed in $O(|E(u)|+|V(u)|\nu(u))$ time. Therefore the spanning set R_s can be found in $O(\Sigma_{u\in V}(|E(u)|+|V(u)|\nu(u)))$, or roughly in $O(|V|^2|E|)$ time.

When the commutativity of a diagram (G(V,E),f) is in question, it can be deduced, on the basis of Prop.2.1 and Theorem 2.1, from the fact that $f(p)=f(q)$ for all $\{p,q\}\in R_s$. In other words, the complexity of testing the commutativity of a diagram is not large, as stated below.

<u>Theorem 3.2.</u> The commutativity of a diagram (G(V,E),f), where f:E-->F, can be checked in $O(c_F|V|^2|E|)$ time if one operation $\circ$ in the semi-group F can be done in a constant time c_F.

The procedure above will be illustrated by means of the example of Fig.2 described in Section 2.3. A possible choice of rooted trees T(u) of G(u) are shown in Fig.7, in which the trees are drawn in bold lines and the cotree arcs in broken lines. The cotree arcs and the corresponding bilinkings are listed below, where r_1 to $r_{12} \in R_0$ are defined in Section 2.3.

root u	$\nu(u)$	cotree arc	bilinking
v_1	4	e_4	r_2
		e_6	r_1
		e_7	r_8
		e_9	r_{10}
		e_{11}	r_7
v_2	2	e_8	r_8
		e_9	r_9
v_3	2	e_8	r_{11}
		e_{11}	r_{12}
v_4	0	-	-
v_5	0	-	-
v_6	0	-	-
v_7	0	-	-

Thus we obtain $R[v_1]=\{r_1,r_2,r_7,r_8,r_{10}\}$, $R[v_2]=\{r_8,r_9\}$, $R[v_3]=\{r_{11},r_{12}\}$ and $R[v_i]=\emptyset$ for i=4,...,7. Notice that $R[v_1]\cap R[v_2]=\{r_8\}\neq\emptyset$ and $R_s=\cup R[u]$ is certainly spanning. Also notice that Prop.3.4 holds with $\eta(G)=6$, $\max_{u\in V}\nu(u)=5$ and $\Sigma_{u\in V}\nu(u)=8$.

3.3. Finding a homotopy base

A homotopy base of an acyclic graph G(V,E) can be found in a polynomial time by applying Algorithm B of Section 3.1 with the initial spanning set R_s ($|R_s| \leq |V||E|$) constructed in Section 3.2, since the dependence of r_i on $R \backslash r_i$ can be tested in $O(|R_s||E|^2)$ time as follows.

First recall that $r_i \in \sigma(R \backslash r_i)$ is equivalent to $r_i \in cl(R \cap \langle r_i \rangle \backslash r_i)$. We may assume that, for each r_i, the pair $[\partial^+ r_i, \partial^- r_i]$ is stored in memory. The set $R \cap \langle r_i \rangle$ can be determined in $O(|R||E|)$ time if Def.2.5 is checked directly on G for each $r \in R$. (If $O(|V|^2)$ space is afforded for listing explicitly the reachability relation among the vertices of G, the set $R \cap \langle r_i \rangle$ is determined in $O(|R|)$ time.)

Suppose that each bilinking $r \in R_s$ is represented by an incidence vector of size $|E|$ that expresses which arcs are contained in r. Then the dependence $r_i \in cl(R \cap \langle r_i \rangle \backslash r_i)$ reduces to the linear dependence of the corresponding 0-1 vectors over GF(2), and can be determined in $O(|R_s||E|^2)$ time by the usual Gaussian elimination over GF(2). (A bilinking can also be represented by a vector of size $\nu(G)$, if we take advantage of the fact that it corresponds to a cycle and can be expanded with respect to a cycle basis of G.)

Since the Step 1 of Algorithm B can be done in $O(|V|^2|E|)$ time and $|R_s| \leq |V||E|$, the whole algorithm runs in $O(|V|^2|E| + |R_s|^2|E|^2) = O(|V|^2|E|^4)$ time to find a homotopy base of G. Thus we have the following.

<u>Theorem 3.3.</u> A homotopy base of an acyclic graph G(V,E) can be found by Algorithm B in $O(|V|^2|E|^4)$ time and in $O(|V||E|^2)$ space.

It is pointed out in [7] that a homotopy base of G(V,E) can be found in $O(|V|^2|E|)$ time and in $O(|E|)$ working space by a straightforward graph algorithm without involving the notion of preordered matroid.

4. Concluding Remarks

It has been shown that a base of a preordered matroid $\tilde{M}=(X, \geq, M)$ can be found by Algorithm B of Section 3.1 which eliminates dependent elements from a spanning set. It is likewise possible to construct an algorithm which augments elements, maintaining independence, to get a base; more precisely, the following augmenting algorithm also finds a base of $\tilde{M}$.

Algorithm A [Finding a base of a preordered matroid]

Step 1: Find a spanning set $R_s=\{r_1,\ldots,r_K\}$ with the ordering being consistent with the preorder $\geq$, i.e., such that

$$r_i \in \langle r_j \rangle \setminus [r_j] \implies i<j \;;$$

Step 2: $R:=\emptyset$;

<u>for</u> i:=1 <u>to</u> K <u>do</u>

<u>if</u> $r_i \notin \sigma(R)$ <u>then</u> $R:=R \cup r_i$;

B:=R

In contrast with Algorithm B, the consistent ordering of the elements of R_s in Step 1 of Algorithm A is indispensable. The validity of Algorithm A can be easily verified by Prop.2.10. The above algorithm can readily be adapted to the weighted case.

The concept of the preordered matroid introduced in this paper can be useful in other contexts. For example, the equisignum decomposition of real vectors [6] and, in particular, the decomposition of flows into conformal simple flows [3] can be treated in terms of the preordered matroid as follows.

Let E be a finite set and $X \subset \mathbf{R}^E$, and for $r=(r(e) \mid e \in E) \in \mathbf{R}^E$ define $car^+ r=\{e \in E \mid r(e)>0\}$ and $car^- r=\{e \in E \mid r(e)<0\}$. A vector s ($\in \mathbf{R}^E$) is said to be r-equisignum iff $car^+ s \subset car^+ r$ and $car^- s \subset car^- r$. We will say here that $B=\{r_1,\ldots,r_m\}$ ($\subset X$) is an <u>equisignum basis</u> of X if any $r \in X$ can be expressed as

$$r = \Sigma_{i=1}^{m} \alpha_i r_i \quad (\alpha_i \geq 0),$$

where r_i is r-equisignum if $\alpha_i > 0$.

For r, $s \in X$, we can define the preorder $\geq$ by

$$r \geq s \text{ iff } s \text{ is r-equisignum.}$$

Thus we can recognize a preordered matroid $\tilde{R}=(X,\geq,M)$ on X with the matroid M defined by the usual linear dependence. An equisignum basis of X is then nothing but a base of $\tilde{R}$, and the arguments for general preordered matroids apply to this problem; e.g., any two equisignum bases are equicardinal (by Theorem 2.3) and an optimum equisignum basis with respect to a given weight can be found by a dual greedy algorithm (i.e., Algorithm WB of Section 3.1).

The equisignum decomposition has been treated (e.g. [3], [6]) in terms of elementary vectors [8] (or minimal vectors [6]) in the particular case where X is a linear subspace of $\mathbf{R}^E$. Then the structure of equisignum bases is known to be quite simple since an equisignum base B can be expressed as

$$B = \{\alpha_i r_i | i \in I\} \cup \{-\beta_i r_i | i \in I\},$$

where $\alpha_i > 0$, $\beta_i > 0$ and $\{r_i | i \in I\}$ denotes a collection of all the noncollinear elementary vectors of X.

An algebraic structure, named the geometry on a poset, involving partial order and matroid has been investigated in [4], but does not seem to have a direct connection with the concept of preordered matroid.

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References

[1] A.V. Aho, J.E. Hopcroft and J.D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, 1974.

[2] C. Berge, Graphes et Hypergraphes, Dunod, Paris, 1971.

[3] G. Busacker and T.L. Saaty, Finite Graphs and Networks, McGraw Hill, 1965.

[4] U. Faigle, Geometries on partially ordered sets, J. Combinatorial Theory B, Vol.28 (1980), 26-51.

[5] D. Hausmann, B. Korte and T.A. Jenkyns, Worst case analysis of greedy type algorithms for independence systems, Math. Prog. Study, Vol.12 (1980), 120-131.

[6] M. Iri, Network Flow, Transportation and Scheduling, Academic Press, New York, 1969.

[7] K. Murota and S. Fujishige, Finding a homotopy base for directed paths in an acyclic graph, submitted for publication.

[8] R.T. Rockafellar, Convex Analysis, Princeton Univ. Press, Princeton, 1970.

[9] D.J.A. Welsh, Matroid Theory, Academic Press, London, 1976.

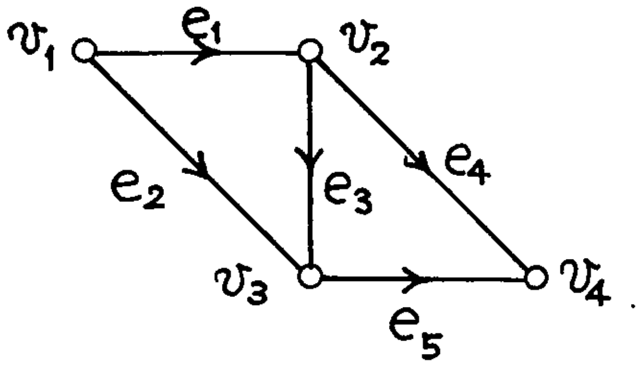

Fig.1. A simple diagram

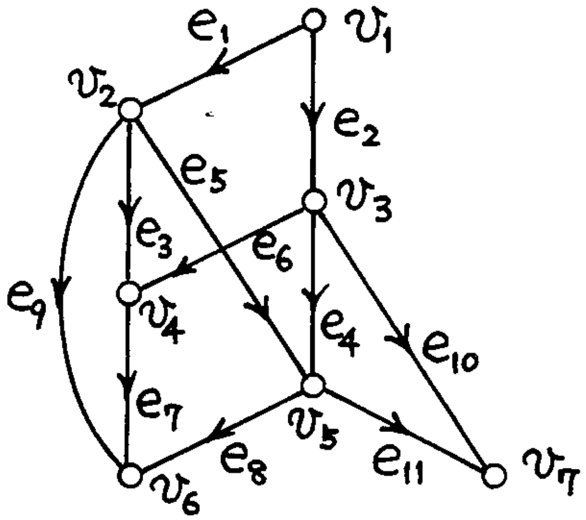

Fig.2. An acyclic graph G

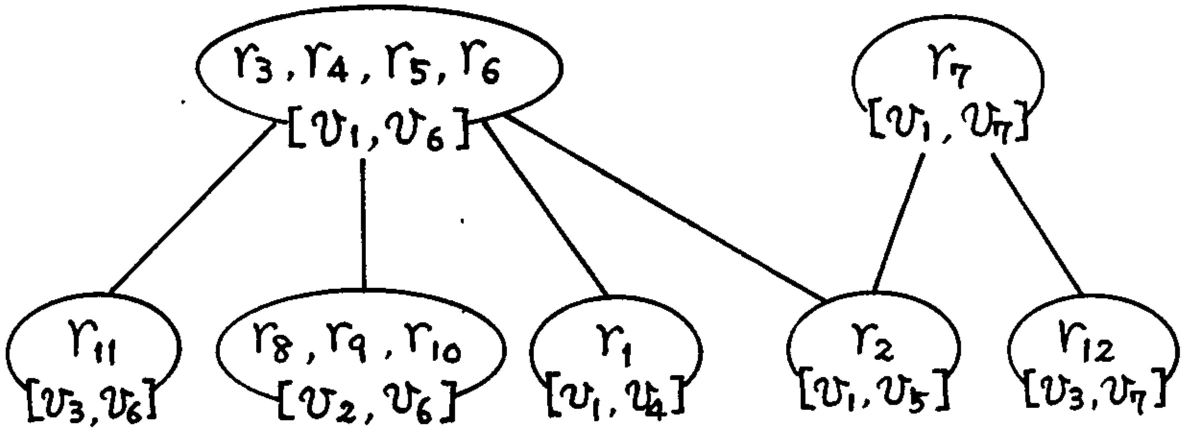

Fig.3. Preorder of the bilinkings of G in Fig.2.

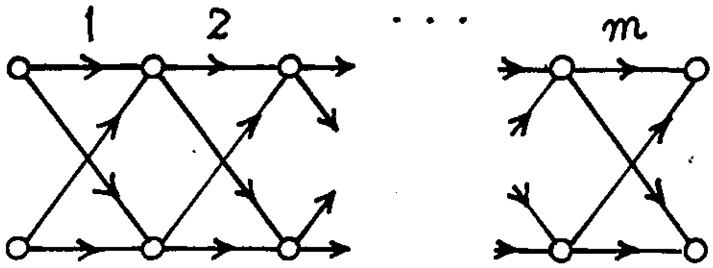

Fig.4. A graph with exponentially many bilinkings

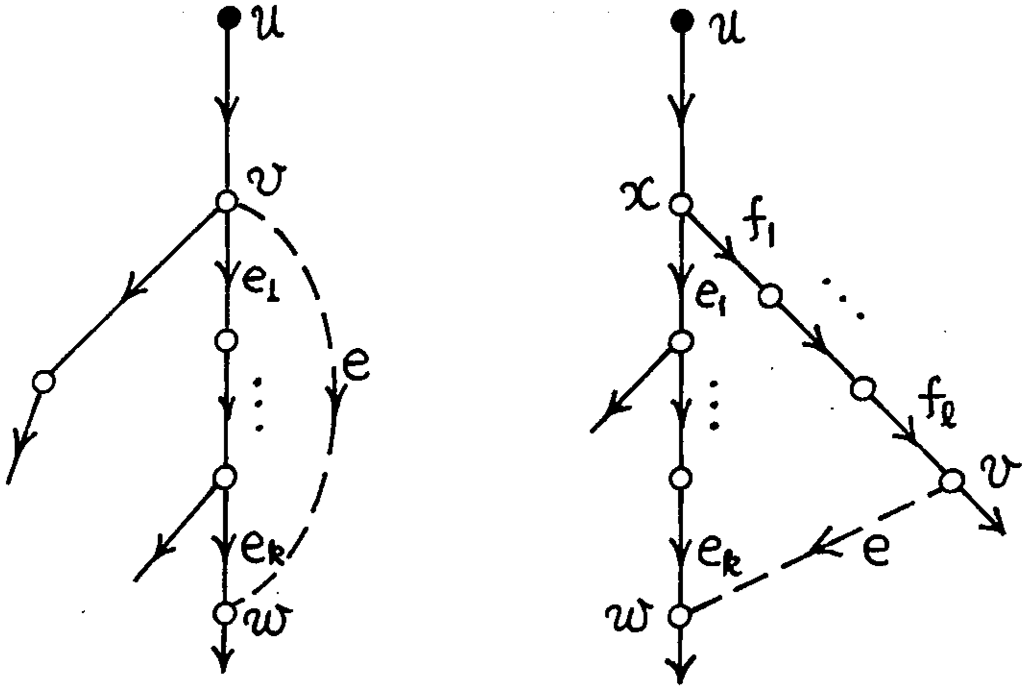

Fig.5. The elementary cycle determined by a cotree arc

(a) $r=\{e_1 \cdots e_k, e\}$ (b) $r=\{e_1 \cdots e_k, f_1 \cdots f_l e\}$

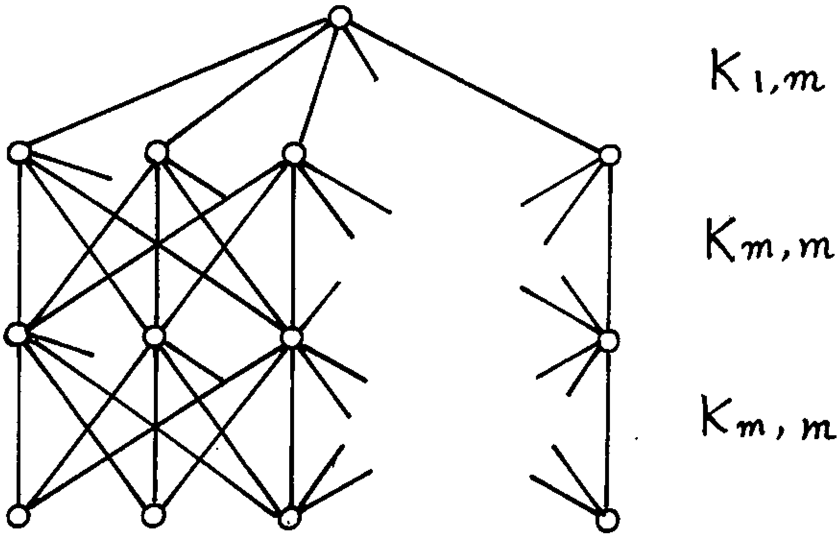

Fig.6. A graph $G(V,E)$ with $\eta(G)$ as large as $\Theta(|V||E|)$

(The arcs are directed downwards.)

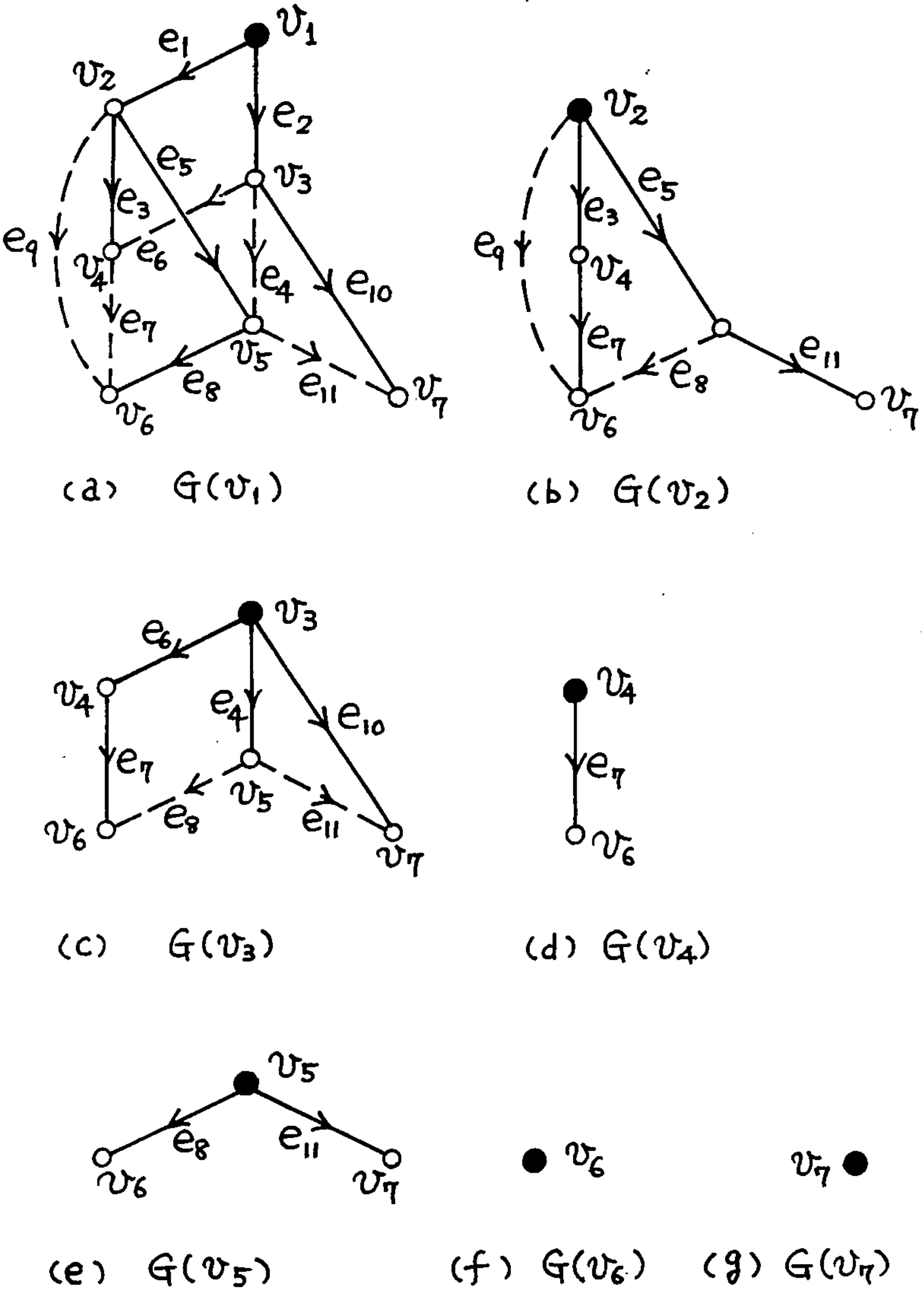

Fig.7. The arborescence T(u) of the subgraph G(u) for G in Fig.2.