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A Bilateral Monopoly and The Nash Solution

by

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Abstract:

This paper considers an economy consisting a seller and a buyer with a commodity and money. We provide an extensive game which is divided into two stages. At the first stage, the quantity of the commodity to be traded from the seller to the buyer is decided, and at the second stage, the price of the commodity is decided. We define a new solution concept, which is a modification of Selten's perfect equilibrium point concept . The idea of this definition is based on the smoothing procedure of Nash [1953]. We show that the new equilibrium point coincides with the Nash's cooperative solution in the economy .

1. Introduction and the Bilateral Economy

Let us consider a bilateral exchange economy, which consists of a seller S and a buyer B. The seller owns $Q > 0$ amount of a commodity , which he wants to sell to the buyer. The buyer owns no amount of the commodity but owns $M > 0$ amount of another commodity, which we call "money" . "money" should be interpreted as the composite commodity of all commodities other than the commodity in question. Of course, the seller may owns a positive amount of money, but since we will consider only the case where the seller never buy any amount of the commodity from the buyer, it is convenient to represent the initial level of money by zero. That is, the initial endowments of the seller and buyer are represented as $w^s = (Q,0)$ and $w^b = (0,M)$. The seller and buyer have von Neumann-Morgenstern utility functions $U(x_1,x_2)$ and $V(x_1,x_2)$ defined on the non-negative orthant E^2_+ of two-dimensinal Euclidean space.

Nash [1950 & 1953] provided a solution concept for two-person cooperative games. If the theory is applied to the above bilateral economy, it selects an allocation $((Q-q,p),(q,M-p))$ such that

$$(1.1) \quad (U(Q-q,p) - U(w^s))(V(q,M-p) - V(w^b)) = \max\ (U(x_1,x_2) - U(w^s))(V(y_1,y_2) - V(w^b))$$

$$\text{subject to} \quad (x_1,x_2) + (y_1,y_2) = (Q,M),\ (x_1,x_2),\ (y_1,y_2) \in E^2_+.$$

We call an allocation satisfying (1.1) a <u>Nash allocation</u>.

Nash provided two complementary approaches to the solution. One is the axiomatic approach, which was established in Nash [1950]. The axiomatic approach focusses it's consideration only on outcomes of a bargaining process instead of the bargaining process itself. Nash [1953] provided another approach to it in which the solution was derived as an equilibrium point of an extensive game. This approach intends to exhibit a structure of bargaining process yielding the Nash solution. As the latter approach provides an explicit structure of bargaining process, it looks like more realistic than the former. But when this approach is applied to the above bilateral economy, the bargaining process never seems to fit for the bilateral economy, i.e., the extensive game given by Nash can not be interpreted in the context of the bilateral economy. The extensive game should be thought of just as an archetype . The purpose of this paper is to modify the latter approach in the case of the bilateral economy to fit for it .

Without loss of generality we can put $U(w^s) = V(w^b) = 0$. We assume the following conditions on the utility functions U and V :

(A): U and V are monotonically increasing, i.e., if $x \geqq y$ and $x \neq y$, then $U(x) > U(y)$ and $V(x) > V(y)$, (B): U and V are functions of C^2 ,i.e., U and V have all second derivatives which are continuous,

(C): U and V are strictly concave functions,

(D): $U(0,M)V(Q,0) < 0$, but there is an m such that $U(0,m),\ V(Q,M-m) > 0$,

(E): for each m with $U(0,m)V(Q,M-m) > 0$, there is a (q,m') such that $U(Q-q,m')V(q,M-m') > U(0,m)V(Q,M-m)$.

Assumptions (A),(B),(D) and (E) impose regularities upon the bilateral economy. Assumption (B) is a mathematical condition. Assumptions (D) and (E) are not familar to us, but they just claim that the Nash solution does not lie on the boundary.

Before we provide a bargaining model, we need to prove the following lemmata.

<u>Lemma 1</u> . For each q ($0 < q \leqq Q$), there is an m such that $U(Q-q,m),\ V(q,M-m) > 0$, but there is no m such that $U(Q-q,m),\ V(q,M-m) \leqq 0$.

<u>Proof</u>. Let $a = q/Q$. Since $a > 0$, it follows from (C) and (D) that for m given in (D),

$$U(Q-q,am) \geqq aU(0,m) + (1-a)U(w^s) = aU(0,m) > 0$$

$$V(q,M-am) \geqq aV(Q,M-m) + (1-a)V(w^b) = aV(Q,M-m) > 0 .$$

It is clear that $V(q,M) > 0$. Since $U(0,M) > 0$ by (D), we have, by (A), $U(Q-q,M) \geqq U(0,M) > 0$. For $\forall m' \in [0,am]$, i.e., $m' = bam$ ($0 \leqq b \leqq 1$), it holds by (C) that

$$V(q,M-bam) \geqq (1-b)V(q,M) + bV(q,M-am) > 0 .$$

For $\forall m' \in [am,M]$, i.e., $m' = bam+(1-b)M$ ($0 \leqq b \leqq 1$), it holds that

$$U(Q-q,bam+(1-b)M) \geqq bU(Q-q,am) + (1-b)U(Q-q,M) > 0 .$$

Q.E.D.

<u>Lemma 2</u>. For each q ($0 < q \leqq Q$), there is a unique $m(q)$ ($0 < m(q) < M$) such that

$$(1.2) \quad U(Q-q,m(q))V(q,M-m(q)) \geqq U(Q-q,m)V(q,M-m)$$

$$\text{for } \forall m \ (0 \leqq m \leqq M) .$$

<u>Proof</u>. For each fixed q, $\log U(Q-q,m) + \log V(q,M-m)$ is defined for some m by Lemma 1. If $m = 0$ or $m = M$, then $U(Q-q,m)V(q,M-m) < 0$ by (A) and (D). Hence $\log U + \log V$ is defined on an open interval (a,b) since U and V are concave functions. Since $U(Q-q,a)V(q,M-a) = U(Q-q,b)V(M-b) = 0$, we have $\lim_{m \to a+0} (\log U + \log V) = \lim_{m \to b-0} (\log U + \log V) = -\infty$. Further since $\log U$

+ log V is a strictly concave function of m , there is a unique m(q) satisfying (1.2). Q.E.D.

Lemma 3. Let q be fixed ($0 < q \leqq Q$). Then m = m(q) if and only if

$$(1.3) \qquad \frac{U_2(Q-q,m)}{U(Q-q,m)} = \frac{V_2(q,M-m)}{V(q,M-m)} \quad .$$

Here we denote $\partial U/\partial x_2$ and $\partial V/\partial x_2$.

Proof. Lemma 2 says that there is a m(q) which maximizes the function log U + log V . Since log U + log V is defined on an open interval (a,b), and since log U + log V is a strictly concave function of m, (1.3) is a necessary and sufficient condition for m to coincide with m(q). Q.E.D.

Lemma 4. (A): m(q) is a differentiable function on (0,Q] . Hence u(q) = U(Q-q,m(q)), v(q) = V(q,M-m(q)) and L(q) = log u(q) + log v(q) are differentiable function on (0,Q] .

(B): L(q) is a strictly concave function.

Proof. Lemma 3 says that the function m(q) is defined by

$$(1.4) \qquad U_2/U - V_2/V = 0 \quad .$$

Let $f(q,m) = U_2(Q-q,m)/U(Q-q,m) - V_2(q,M-m)/V(q,M-m)$. Then, by (B), f has continuous partial derivatives f_1 and f_2. Since U and V are concave and increasing, f is a decreasing function of m. Hence Young's implicit function theorem can be applied to (1.4).[2] Hence $m(q)$ has derivative, which is given as $dm(q)/dq = - f_1/f_2$.

Let q_1, q_2 and a ($0 < a < 1$, $b = 1-a$ and $q_1 \neq q_2$) be arbitrarily chosen. Then we have, by (C) and definition of $m(q)$,

$$aL(q_1) + bL(q_2) \leqq \log(aU(Q-q_1, m(q_1)) + bU(Q-q_2, m(q_2))) + \log(aV(q_1, M-m(q_1)) + bV(q_2, M-m(q_2)))$$

$$< \log U(Q-(aq_1+bq_2), am(q_1)+bm(q_2)) + \log V(aq_1+bq_2, M-(am(q_1)+bm(q_2)))$$

$$\leqq \log U(Q-(aq_1+bq_2), m(aq_1+bq_2)) + \log V(aq_1+bq_2, M-m(aq_1+bq_2))$$

$$= L(aq_1+bq_2) . \quad \text{Q.E.D.}$$

Lemma 5. (A): There is the unique q_N ($0 < q_N < Q$) such that

$$(1.5) \quad L(q_N) \geqq L(q) \quad \text{for } \forall q \in (0, Q] .$$

(B): $q = q_N$ if and only if q satisfies

$$(1.6) \quad u'(q)/u(q) + v'(q)/v(q) = 0 .$$

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2). See Komatsu [1962].

(C): $((Q-q_N, m(q_N)), (q_N, M-m(q_N)))$ is the unique Nash allocation.

Proof. Since $L(q)$ is a strictly concave function and $\lim_{q \to 0} L(q) = -\infty$, there is a q_N satisfying (1.5). By (E) we have $q_N < Q$. The uniqueness follows from the strict concavity of $L(q)$.

Since $L(q)$ is differentiable and q_N is an interior point of $(0,Q]$, (1.6) is a necessary and sufficient condition for q to coincide with q_N.

If there is another allocation $((Q-q^*, m^*), (q^*, M-m^*))$ such that $q^* \neq q_N$ and $L(q_N) \leqq \log U(Q-q^*, m^*) + \log V(q^*, M-m^*)$, then then it holds by Lemma 5.(A) that $\log U(Q-q^*, m^*) + \log V(q^*, M-m^*) > L(q^*)$, which is a contradiction. Q.E.D.

2. The Bargaining Game $G(f_o,g_o)$

The seller and buyer have a negotiation to abide by the rule of the bargaining game $G(f_o,g_o)$. We assume that the bargaining game $G(f_o,g_o)$ is divided into two stages 1 and 2. At the 1st stage, an amount of the commodity to be traded is decided, and at the 2nd stage, an amount of money to be paid to the seller from the buyer is decided. The rule of the bargaining game is as follows:

The 1st stage: The seller S and buyer B select amounts of the commodity that they want to trade from S to B. These decisions are made independently. That is, each must select an amount without the knowledge of the decision of the other. Then they inform each other of the amounts selected. If the decisions are the same, then the negotiation goes on the 2nd stage, but, otherwise, the negotiation is broken off.

The strategy-spaces of S and B at the 1st stage are $(0,Q]$. It should be noted that 0 does not belong to the strategy-space, as it is assumed that they want to trade the commodity. Let $f_o(q_b-q_s)$ be a function defined by

$$(2.1) \quad f_o(q_b-q_s) = \begin{cases} 1 & \text{if } q_b-q_s = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where q_s and q_b are amounts of the commodity selected by S and

B respectively . The rule can be rewritten by this notation as follows . If S and B select strategies q_s and q_b , then the negotiation goes on the 2nd stage with probability $f_o(q_b-q_s)$ and is broken off with probability $1 - f_o(q_b-q_s)$.

The 2nd Stage: Let the negotiation go on the 2nd stage . This case happens if and only if $q_b = q_s = \overline{q}$. But if S and B violate the rule and if they make the negotiation go on the 2nd stage and be concluded in a case of $q_s \neq q_b$, then it is assumed that $(q_s+q_b)/2 = \overline{q}$ amount of the commodity is traded . At this stage, S and B must select prices of the amount of $\overline{q}$ of the commodity . These decisions are also made independently, and they infrom each other of the prices . The negotiation is concluded if and only if prices p_s set by S is not greater than price p_b set by B , and the final price becomes $\overline{p} = (p_s+p_b)/2$. It should be noted that p_s ,p_b or $\overline{p}$ are the total amount of money to be paid from B to S , and that the usual prices are given as $p_s/\overline{q}$,$p_b/\overline{q}$ or $\overline{p}/\overline{q}$ respectively . Let g_o be a function defined by

$$(2.2) \qquad g_o(p_b-p_s) = \begin{cases} 1 & \text{if } p_b-p_s \geqq 0 \\ 0 & \text{otherwise .} \end{cases}$$

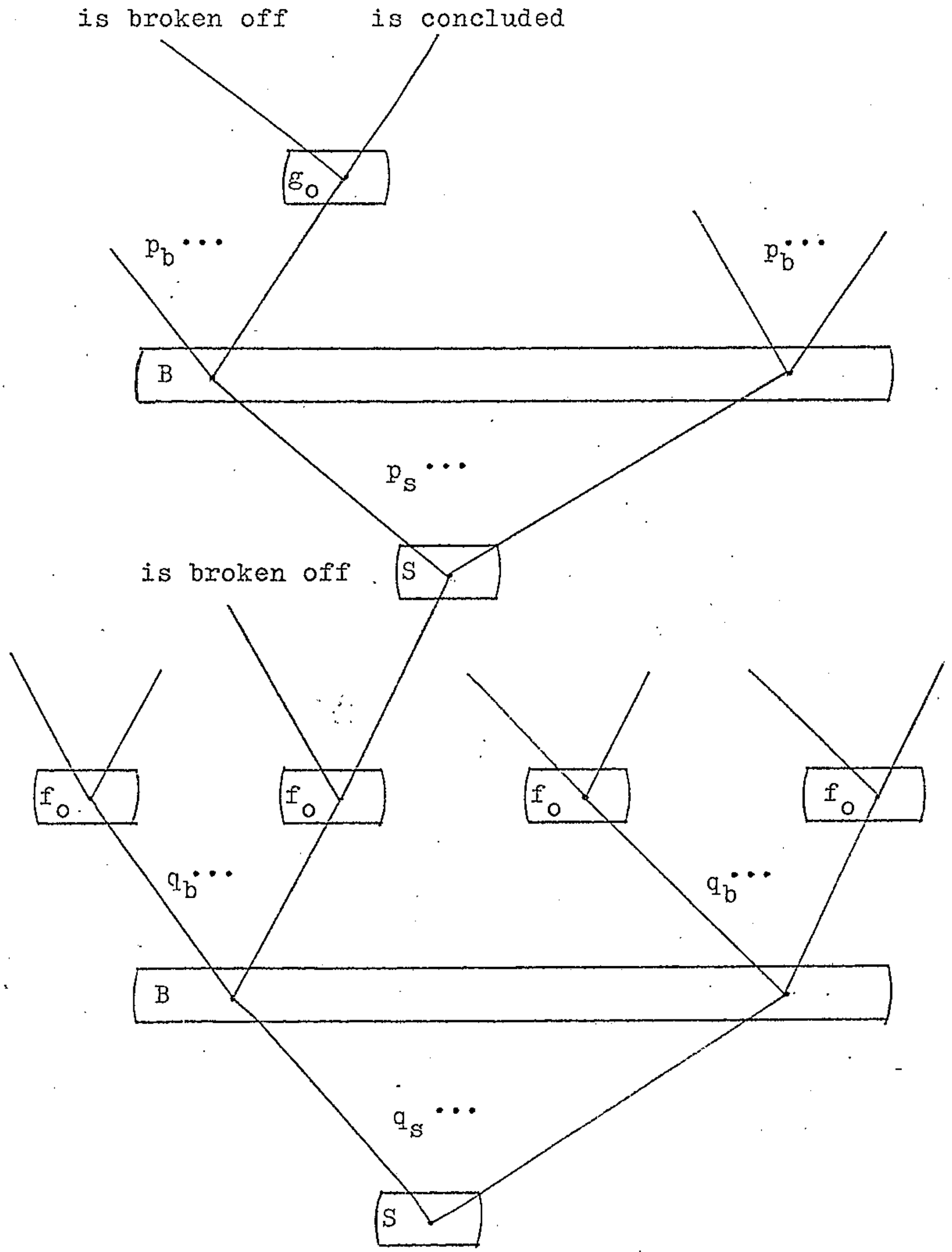
The set of prices to be set by S or B is $(0,M]$. Hence the domain of g_o is the open interval $(-M,M)$. 0-price is omitted from the set of prices by the same reason as that of the omission of 0-quantity . The above rule can be rewritten by g_o as follows . The negotiation is concluded with probability $g_o(p_b-p_s)$ and is broken off with probability $1-g_o(p_b-p_s)$.

The bargaining game $G(f_o,g_o)$ is formulated as an extensive game . The game tree of the game is drawn as Figure 1 . Of course, the order of S's and B's information sets is exchangeable . Even if $q_s \neq q_b$, then the game tree is assumed to have the second stage, though the probability that the negotiation goes on the 2nd stage is zero . A <u>behavior strategy</u> of S is a pair (q_s,t_s) of a point q_s in $(0,Q]$ and a function t_s from $(0,Q]^2 = (0,Q]\times(0,Q]$ to $(0,M]$. A <u>behavior strategy</u> of B is also a pair (q_b,t_b) of a point q_b in $(0,Q]$ and a function t_b from $(0,Q]^2$ to $(0,M]$. The set of all behavior strategies of S is the same as that of all behavior strategies of B , which is denoted by $BS = (0,Q]\times T$. The <u>payoff functions</u> H_s and H_b of the bargaining game $G(f_o,g_o)$ are real valued functions on $BS^2 = BS\times BS$. If S and B use behavior strategies $b_s = (q_s,t_s)$ and $b_b = (q_b,t_b)$ respectively , the payoff functions are represented as

$$(2.3) \quad H_s(b_s,b_b) = U(Q-\overline{q},\overline{p})f_o(q_b-q_s)g_o(p_b-p_s)$$

$$H_b(b_b,b_s) = V(\overline{q},M-\overline{p})f_o(q_b-q_s)g_o(p_b-p_s) ,$$

where $p_s = t_s(q_s,q_b)$, $p_b = t_b(q_s,q_b)$, $\overline{q} = (q_s+q_b)/2$ and $\overline{p}$ =


is broken off
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S
is broken off
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q_b...
B
q_s...
S

Figure 1.

$(p_s+p_b)/2$.

The second stage of the bargaining game $G(f_o,g_o)$ is subgames of $G(f_o,g_o)$.[3)] When q_s and q_b are selected at the 1st stage, the subgame of the 2nd stage is denoted by $SG(q_s,q_b,g_o)$. $SG(q_s,q_b,g_o)$ is characterized as a game as follows . The set of strategies of S or B is $(0,M]$, which is the set of values of behavior strategies $t_s(q_s,q_b)$ or $t_b(q_s,q_b)$, i.e., $(0,M] = \{t(q_s,q_b) \mid t \in T\}$. The payoff functions h_s and h_b of $SG(q_s,q_b,g_o)$ are given as

$$(2.4) \quad h_s(p_s,p_b) = U(Q-\overline{q},\overline{p})g_o(p_b-p_s)$$

$$h_b(p_s,p_b) = V(\overline{q},M-\overline{p})g_o(p_b-p_s) ,$$

where $\overline{q} = (q_s+q_b)/2$ and $\overline{p} = (p_s+p_b)/2$. We call h_s and h_b the <u>induced payoff functions</u> of H_s and H_b on $SG(q_s,q_b,g_o)$. We call $p_s = t^*_s(q_s,q_b)$ and $p_b = t^*_b(q_s,q_b)$ the <u>induced strategies</u> of $b^*_s = (q^*_s,t^*_s)$ and $b^*_b = (q^*_b,t^*_b)$ on $SG(q_s,q_b,g_o)$.

When strategies t_s and t_b in T at the 2nd stage are fixed, $G(f_o,g_o)$ is reduced to a game having one stage . We call the game a <u>reduced game</u> , which is denoted by $R(f_o,g_o,t_s,t_b)$. The sets of all strategies are $(0,Q]$, and the payoff functions

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3). For definition of a subgame, see Selten [1973 or 1975] .

r_s of S and r_b of B are represented as

(2.5) $r_s(q_s,q_b) = H_S((q_s,t_s),(q_b,t_b))$

$r_b(q_s,q_b) = H_b((q_s,t_s),(q_b,t_b))$.

Let (X_1,X_2,k_1,k_2) be a two-person game, where X_1 and X_2 are the sets of strategies , and where k_1 and k_2 are the payoff functions . Then we call a pair (x_1^*,x_2^*) of strategies an <u>equilibrium</u> <u>point</u> if

(2.6) $k_1(x_1^*,x_2^*) \geqq k_1(x_1,x_2^*)$ for $\forall x_1 \in X_1$

$k_2(x_1^*,x_2^*) \geqq k_2(x_1^*,x_2)$ for $\forall x_2 \in X_2$.

We call a pair $(b_s^*,b_b^*) = ((q_s^*,t_s^*),(q_b^*,t_b^*))$ of behavior strategies of $G(f_o,g_o)$ an <u>subgame</u> <u>perfect</u> <u>equilibrium</u> <u>point</u> if (b_s^*,b_b^*) satisfies the following (2.7) and (2.8);

(2.7) For each $(q_s,q_b) \in (0,Q]^2$, the pair (p_s,p_b) of the induced strategies of b_s^* and b_b^* on $SG(q_s,q_b,g_o)$ is an equilibrium point of $SG(q_s,q_b,g_o)$.

(2.8) (q_s^*,q_b^*) is an equilibrium point of the reduced game $R(f_o,g_o,t_s^*,t_b^*)$.

This solution concept is defined by R.Selten , and the basic idea is explained very precisely in Selten [1973 or 1975] .

In the bargaining game $G(f_o, g_o)$ there are a lot of sub-game perfect equilibrium points . For example , if a pair $(b_s^*, b_b^*) = ((q_s^*, t_s^*), (q_b^*, t_b^*))$ satisfies

$$(2.9) \quad q_s^* = q_b^* \quad \text{and} \quad t_s^*(q_s, q_b) = t_b^*(q_s, q_b) \quad \text{for} \ \forall (q_s, q_b) \in (0, Q]^2 ,$$

then it is a subgame perfect equilibrium point . Hence we wuold like to select an equilibrium point from the set of these equilibrium points . In the next section, we shall define a new solution concept for this purpose .

3. The (F,G)-Perfect Equilibrium Point of $G(f_o,g_o)$

For the purpose of a selection of an subgame perfect equilibrium point, we shall employ the basic idea of the smoothing procedure provided by Nash [1953] . In our case it is that the bargaining game $G(f_o,g_o)$ is thought of as a limit case of bargaining games where the probability that S and B violate the rule represented by f_o and g_o is taken into consideration. Selten [1975] reconsidered the concept of subgame perfect equilibrium point in extensive games due to the similar idea and defined a solution concept called "perfect equilibrium point", but it is not necessary to refer to it in this paper .

We shall approximate f_o and g_o by classes of smooth functions. Let F be the set of all continuously differentiable functions f on $(-Q,Q)$ which satisfy

(3.1) $f(w)$ is increasing on $(-Q,0)$ and is decreasing on $(0,Q)$, and $f(w) > 0$ for $\forall w \in (-Q,Q)$ with $f(0) = 1$,

(3.2) there are unique $d_1(f)$ in $(-Q,0)$ and unique $d_2(f)$ in $(0,Q)$ such that $f(w)$ is convex on $(-Q,d_1(f))$ and $(d_2(f),0)$ and is concave on $(d_1(f),d_2(f))$.

d_1 and d_2 are functions on F to $(-Q,0)$ and $(0,Q)$ respectively. Let G be the set of all continuously differentiable functions g which satisfy

(3.3) $g(w)$ is increasing on $(-M,0)$ and $g(w) > 0$ for $\forall w \in (-M,0)$ and $g(w) = 1$ for $\forall w \in (-M,0)$,

(3.4) for each $g \in G$, there is an unique $d_3(g)$ in $(-Q,0)$ such that g is convex on $(-M,d_3(g))$ and is concave on $(d_3(g),M)$.

Here d_3 is a function on G to $(-M,0)$.

A function f in F assigns a probability that the negotiation goes on the 2nd stage . When q_s is not equal to q_b , it is the probability that S and B violate the rule and make the negotiation go on the 2nd stage . (3.1) means that the probability becomes smaller as the disparity between q_s and q_b becomes larger . (3.2) is a regularity condition . A function g in G assigns a probability that the negotiation is concluded, assuming that it goes on the 2nd stage . When p_s is greater than p_b , it is the probability that S and B violate the rule and make the negotiation be concluded . (3.4) is a regularity condition .

We shall think of the rules of f_o and g_o as limit cases of functions in F and G . Hence it is necessary to define approximating sequences of f_o and g_o . We call a sequence $\{f_n\}$ in F an <u>approximating sequence</u> of f_o if

(3.5) $\{f_n\}$ converges pointwise to f_o , i.e., $\lim_{n\to\infty} f_n(w) = f_o(w)$ for $\forall w \in (-Q,Q)$, (3.6) $\lim_{n\to\infty} f_n(d_1(f_n)) = \lim_{n\to\infty} f_n(d_2(f_n)) = 0$,

(3.7) $\{f_n'/f_n\}$ converges uniformly to 0 on $(-Q,w_1)$ and on (w_2,Q) for $\forall w_1 \in (-Q,0)$ and $\forall w_2 \in (0,Q)$,

where $f_n' = df_n/dw$. We call a sequence $\{g_n\}$ in G an <u>approximating sequence</u> of g_o if

(3.8) g_n converges pointwise to g_o ,

(3.9) $\lim_{n\to\infty} g_n(d_3(g_n)) = 0$,

(3.10) $\{g_n'/g_n\}$ converges unifromly to 0 on $(-Q,w)$ for $\forall w \in (-Q,0)$.

It is easily verified that $\lim_{n\to\infty} d_1(f_n) = \lim_{n\to\infty} d_2(f_n) = 0$ and $\lim_{n\to\infty} d_3(g_n) = 0$.

In the following, we shall consider the games $SG(q_s,q_b,g)$ and $R(f,g_o,t_s,t_b)$ in which f_o and g_o are substituted by $f \in F$ and $g \in G$. The definitions of games $SG(q_s,q_b,g)$ and $R(f,g_o,t_s,t_b)$ are clear and unmistakable .

We call a pair $(b_s^*,b_b^*) = ((q_s^*,t_s^*),(q_b^*,t_b^*))$ of behavior strategies an <u>(F,G)-perfect equilibrium point</u> if it satisfies (3.11),(3.12) and (3.13);

(3.11) (b_s^*, b_b^*) is an subgame perfect equilibrium point of $G(f_o, g_o)$.

(3.12) Let $\{g_n\}$ be an arbitrary approximating sequence , and let (q_s, q_b) be arbitrary in $(0,Q]^2$. Let $\{E_n\}$ be the sequence such that each E_n is the set of equilibrium points of $SG(q_s, q_b, g_n)$. Then the sequence $\{E_n\}$ converges to $(\overline{p}_s, \overline{p}_b)$ which is the induced strategy pair of (b_s^*, b_b^*) on $SG(q_s, q_b, g_o)$.[4)]

(3.13) Let $\{f_n\}$ be an arbitrary approximating sequence of f_o . Let $\{Q_n\}$ be the sequence such that each Q_n is the set of equilibrium points of $R(f_n, g_o, t_s^*, t_b^*)$. Then the sequence $\{Q_n\}$ converges to (q_s^*, q_b^*) .

This definition is based on the intuitive idea that a reasonable equilibrium point should be stable against arbitrary small imperfections of rationality . Approximating sequences $\{f_n\}$ and $\{g_n\}$ represent arbitrary small imperfections of

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4) Let a sequence $\{E_n\}$ of sets be given . We say that the sequence $\{E_n\}$ <u>converges to</u> e if there is a converging sequence $\{e_n\}$ to e with $e_n \in E_n$ for $\forall n$ and if all sequence $\{e_n\}$ with $e_n \in E_n$ for $\forall n$ converge to e . Here it is admitted that a fnite number of e_n in E_n is not defined . Hence a finite number of E_n may be empty .

rationality . Hence the definition requires an equilibrium point to have a property that for any approximating sequence it is the limit of the sequence of the sets of all equilibrium points of the corresponding games . We impose this requirement upon every subgame $SG(q_s, q_b, g_o)$ and the reduced game $R(f_o, g_o, t_s^*, t_b^*)$.

In particular , it is noted that every f_n or g_n is a continuous function . When a deviation from the rule of game occurs with a positive probability , it is natural to assume that the probability changes continuously with respect to strategies used . This continuity assumption is essential in our definition , but the defferentiability of f_n or g_n is just a technical condition .

It is clear by the definition that there is at most one (F,G)-perfect equilibrium point . The reason for this is that the sequences of the equilibrium sets must converges to the equilibrium point .

Nash [1953] surely considered this type of equilibrium point and showed that the equilibrium point coincides with the Nash's cooperative solution in two-person cooperative games . Selten [1975] used a very similar idea to redefine a concept of "perfect equilibrium point" , and made a very exact explanation of it .

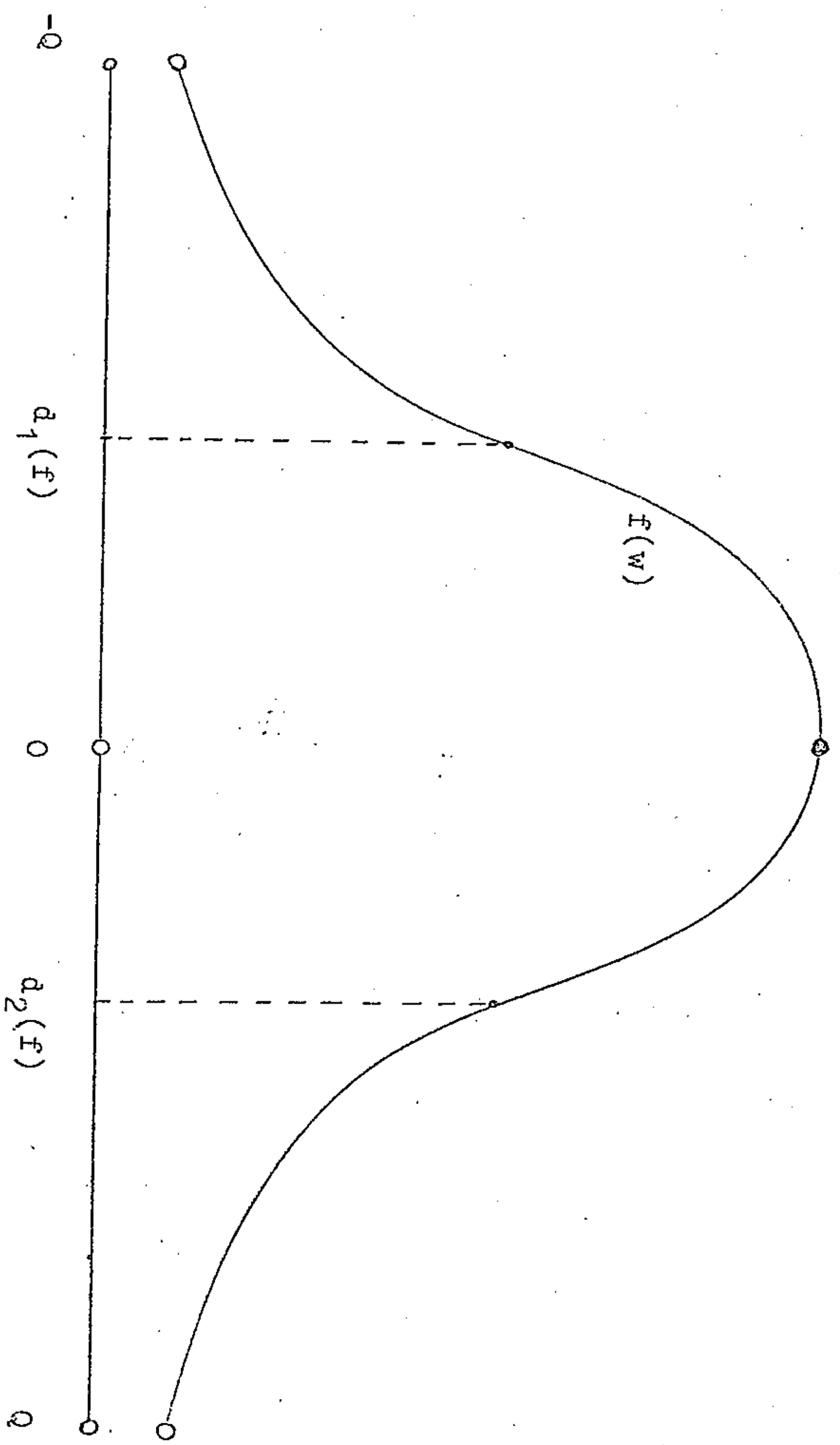

Figure 2.

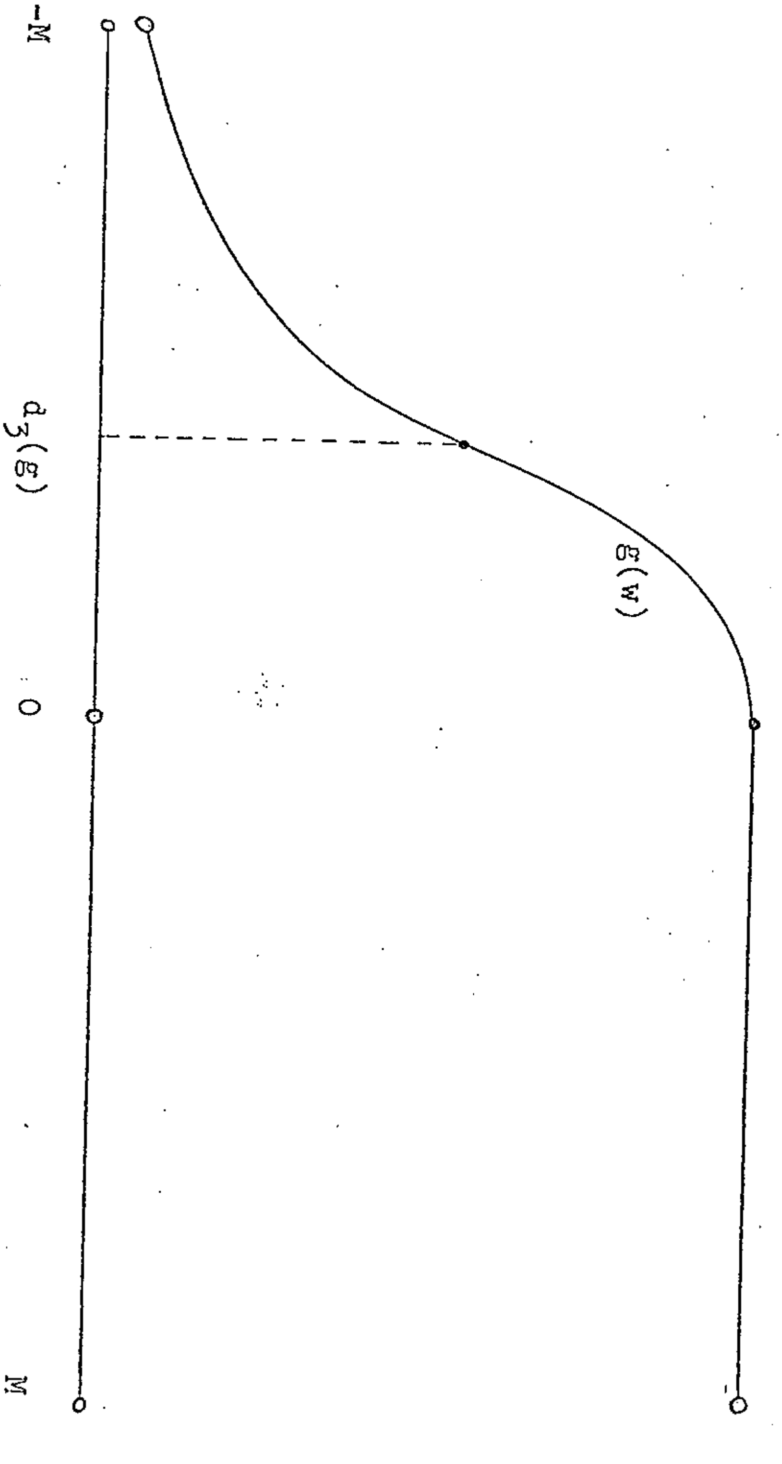

Figure 3.

4. The (F,G)-Perfect Equilibrium Point at the 2nd Stage

The purpose of this section is to prove the following theorem.

Theorem A. Let $(b_s^*, b_b^*) = ((q_s^*, t_s^*), (q_b^*, t_b^*))$ be an (F,G)-perfect equilibrium point. Then it holds that

$$(4.1)\quad t_s^*(q_s, q_b) = t_b^*(q_s, q_b) = m((q_s+q_b)/2) \quad \text{for } \forall (q_s, q_b) \in (0, Q]^2$$

The following lemmata are necessary to prove this theorem. In the following, let $(q_s, q_b) \in (0, Q]^2$ be arbitrarily fixed and let $\bar{q} = (q_s+q_b)/2$. Let $\{g_n\}$ be an arbitrary approximating sequence of g_o .

Lemma 6. (i). If $p_s \leqq p_b$, then (p_s, p_b) can not be an equilibrium point of $SG(q_s, q_b, g_n)$ for $\forall n \geqq 1$.

(ii). If $V(q, M-(p_s+p_b)/2) \leqq 0$, then (p_s, p_b) can not be an equilibrium point of $SG(q_s, q_b, g_n)$ for $\forall n \geqq 1$.

(iii). If $p_s = M$, then there is an integer N such that (p_s, p_b) can not be an equilibrium point of $SG(q_s, q_b, g_n)$ for $\forall n \geqq N$.

(iv). If $U(Q-\bar{q}, (p_s+p_b)/2) \leqq 0$, then there is an integer N such that (p_s, p_b) can not be an equilibrium point of $SG(q_s, q_b, g_n)$ for $\forall n \geqq N$.

<u>Proof</u>. (i): Let $p_s \leqq p_b$. Then $g_n(p_b-p_s) = 1$ and $g_n'(p_b-p_s) = 0$ for $\forall n \geqq 1$. Hence we have

$$\partial h_b^n(p_s,p_b)/\partial p_b = - V_2(\overline{q},M-(p_s+p_b)/2)/2 < 0 \quad \text{for } \forall n \geqq 1 .$$

Here $h_s^n(p_s,p_b)$ and $h_s^n(p_s,p_b)$ are the payoff functions of $SG(q_s, q_b,g_n)$ ($\forall n \geqq 1$). Hence (p_s,p_b) can never become an equilibrium point of $SG(q_s,q_b,g_n)$ for $\forall n \geqq 1$.

(ii): Let $V(\overline{q},M-(p_s+p_b)/2) \leqq 0$. Then any $p_b' < p_b$ in $(0,M]$ satisfies

$$h_b^n(p_s,p_b') = V(\overline{q},M-(p_s+p_b')/2)g_n(p_b'-p_s)$$

$$> V(\overline{q},M-(p_s+p_b)/2)g_n(p_b-p_s) = h_b^n(p_s,p_b) ,$$

because $g_n(p_b'-p_s) \leqq g_n(p_b-p_s)$ and $V(\overline{q},M-(p_s+p_b')/2) > V(\overline{q},M-(p_s+p_b)/2)$. Hence (p_s,p_b) can not be an equilibrium point of $SG(q_s,q_b,g_n)$ for $\forall n \geqq 1$.

(iii): Let $p_s = M$. We need to consider the case where $V(\overline{q},M-(p_s+p_b)/2) > 0$. Since $V(\overline{q},0) \leqq V(Q,0) < 0$ by assumption (D), there is a $\overline{p}_b < M$ such that $V(\overline{q},M-(p_s+\overline{p}_b)/2) = 0$. When p_s is fixed, $V_2(\overline{q},M-(p_s+p_b)/2)/V(\overline{q},M-(p_s+p_b)/2)$ is an increasing function of p_b on $(0,\overline{p}_b)$. Let $c = V_2(\overline{q},M/2)/V(\overline{q},M/2) = \inf_{p_b} V_2(\overline{q},M-(p_s+p_b)/2)/V(\overline{q},M-(p_s+p_b)/2)$. Since $\{g_n'/g_n\}$ converges uniformly to 0 on $(-M,\overline{p}_b-M)$, we can select an N such that for $\forall n \geqq N$, $c > g_n'(w)/g_n(w)$ for $\forall w \in (-M,\overline{p}_b-M)$. Hence it holds that

$$V_2(\overline{q},M-(p_s+p_b)/2)/V(\overline{q},M-(p_s+p_b)/2) \geqq c >$$

$$2g_n'(p_b-p_s)/g_n(p_b-p_s) \qquad \text{for } \forall n \geqq N ,$$

which is equivalent to $\partial h_b^n(p_s,p_b)/\partial p_b < 0$ for $\forall n \geqq N$. Hence (p_s,p_b) can not be an equilibrium point of $SG(q_s,q_b,g_n)$ for $\forall n \geqq N$.

(iv): Let $U(Q-\overline{q},(p_s+p_b)/2) \leqq 0$. We need to consider the case where $p_s < M$. There is an $p_s' > p_s$ such that $h_s^n(p_s',p_b) = U(Q-\overline{q},(p_s'+p_b)/2)g_n(p_b-p_s') > U(Q-\overline{q},(p_s+p_b)/2)g_n(p_b-p_s) = h_s^n(p_s,p_b)$, because $U(Q-\overline{q},(p_s'+p_b)/2) > U(Q-\overline{q},(p_s+p_b)/2)$ and $g_n(p_b-p_s') \leqq g_n(p_b-p_s)$. Hence (p_s,p_b) can not be an equilibirum point of $SG(q_s,q_b,g_n)$ for $\forall n \geqq N$, where N is given in (iii) .

Q.E.D.

Lemma 7. There is an integer N such that for $\forall n \geqq N$, (p_s^n, p_b^n) is an equilibrium point of $SG(q_s, q_b, g_n)$ if and only if

$$(4.2) \quad \frac{U_2(Q-\overline{q}, (p_s^n+p_b^n)/2)}{U(Q-\overline{q}, (p_s^n+p_b^n)/2)} = \frac{V_2(\overline{q}, M-(p_s^n+p_b^n)/2)}{V(\overline{q}, M-(p_s^n+p_b^n)/2)}$$

$$= 2\,\frac{g_n'(p_b^n-p_s^n)}{g_n(p_b^n-p_s^n)} ,$$

$$(4.3) \quad \left|p_b^n - p_s^n\right| \leqq \left|d_3(g_n)\right| .$$

Here $|a|$ is the absolute value of a .

Proof. It is easily verified that if at least one of the assumptions of Lemma 6: (i),(ii),(iii) and (iv) is true for (p_s, p_b), then (p_s, p_b) can not satisfy (4.2) for $\forall n \geqq$ some N. Hence we need to consider the case where no assumption of Lemma 6 is true .

Let (4.2) be not true for (q_s^n, q_b^n) . Then we have

$$U_2(Q-\overline{q}, (p_s^n+p_b^n)/2)/U(Q-\overline{q}, (p_s^n+p_b^n)/2) \neq 2g_n'(p_b^n-p_s^n)/g_n(p_b^n-p_s^n)$$

or $$V_2(\overline{q}, M-(p_s^n+p_b^n)/2)/V(\overline{q}, M-(p_s^n+p_b^n)/2) \neq 2g_n'(p_b^n-p_s^n)/g_n(p_b^n-p_s^n),$$

which is equivalent to $\partial h_s^n(p_s^n, p_b^n)/\partial p_s \neq 0$ or $\partial h_b^n(p_s^n, p_b^n)/\partial p_b \neq 0$. Hence (p_s^n, p_b^n) can not be an equilibrium point of

$SG(q_s, q_b, g_n)$.

In the following, suppose that (p_s^n, p_b^n) satisfies (4.2) for $\forall n \geqq 1$. Then we have $(p_s^n + p_b^n) = m(\overline{q})$ for $\forall n \geqq 1$ by Lemma 3 . Let ε and δ be positive real numbers such that if $|p_s - m(\overline{q})| \leqq \delta$ and $|p_b - m(\overline{q})| \leqq \delta$, then

$$(4.4) \quad U(Q-\overline{q}, M)\varepsilon < U(Q-\overline{q}, p_b) \quad \text{and} \quad V(\overline{q}, M)\varepsilon < V(\overline{q}, M-p_s) .$$

Since U and V are continuous functions, this section of such ε and δ is possible .

Let (p_s^n, p_b^n) not satisfy (4.3) for any $n \geqq 1$. Since $\lim_{n \to \infty} g_n(d_3(g_n)) = 0$, there is an integer N_1 such that $g_n(d_3(g_n)) \leqq \varepsilon$ for $\forall n \geqq N_1$. Since $\{g_n'/g_n\}$ converges unifromly to 0 on $(-M, w)$ for $\forall w \in (-M, 0)$, there is an integer N_2 such that $|p_b^n - p_s^n| \leqq \delta$ for $\forall n \geqq N_2$. Hence it holds for $\forall n \geqq \max(N_1, N_2)$ that

$$h_s^n(p_s^n, p_b^n) = U(Q-\overline{q}, (p_s^n + p_b^n)/2) g_n(p_b^n - p_s^n) \leqq U(Q-\overline{q}, M)\varepsilon$$

$$< U(Q-\overline{q}, p_b^n) = h_s^n(p_b^n, p_b^n) .$$

Hence (p_s^n, p_b^n) can not be an equilibrium point of $SG(q_s, q_b, g_n)$

for $\forall n \geqq \max(N_1, N_2)$.

Let (p_s^n, p_b^n) satisfy (4.3) for $\forall n \geqq 1$. We can select an integer N_3 such that $g_n(d_3(g_n)) \leqq \varepsilon$ and $-d_3(g_n) \leqq \delta$ for $\forall n \geqq N_3$, because $\lim_{n \to \infty} g_n(d_3(g_n)) = \lim_{n \to \infty} d_3(g_n) = 0$. Since $\log h_s^n(p_s, p_b^n)$ and $\log h_b^n(p_s^n, p_b)$ are concave functions of p_s on $(0, p_b^n - d_3(g_n))$ and of p_b on $(p_s^n + d_3(g_n), M]$ respectively, and since $\partial \log h_s^n(p_s^n, p_b^n)/\partial p_s = \partial \log h_b^n(p_s^n, p_b^n)/\partial p_b = 0$ by (4.2), it holds that

$$\log h_s^n(p_s^n, p_b^n) \geqq \log h_s^n(p_s, p_b^n) \qquad \text{for } \forall p_s \in (0, p_b^n - d_3(g_n))$$

$$\log h_b^n(p_s^n, p_b^n) \geqq \log h_b^n(p_s^n, p_b) \qquad \text{for } \forall p_b \in (p_s^n + d_3(g_n), M]$$

that is,

$$h_s^n(p_s^n, p_b^n) \geqq h_s^n(p_s, p_b^n) \quad \text{for } \forall p_s \in (0, p_b^n - d_3(g_n))$$

$$h_b^n(p_s^n, p_b^n) \geqq h_b^n(p_s^n, p_b) \quad \text{for } \forall p_b \in (p_s^n + d_3(g_n), M] .$$

These inequalities hold for $\forall n \geqq 1$. By (4.4) we have, for $\forall n \geqq N_3$,

$$h_s^n(p_s, p_b^n) = U(Q-\bar{q}, (p_s + p_b^n)/2) g_n(p_b^n - p_s) \leqq U(Q-\bar{q}, M)\,\varepsilon$$

$$< U(Q-\bar{q}, p_b^n) = h_s^n(p_b^n, p_b^n) \leqq h_s^n(p_s^n, p_b^n) \quad \text{for } \forall p_s \in [p_b^n - d_3(g_n), M]$$

and

$$h_b^n(p_s^n, p_b) = V(\overline{q}, M-(p_s^n+p_b)/2)g_n(p_b-p_s^n) \leqq V(\overline{q}, M)\varepsilon$$

$$< V(\overline{q}, M-p_s^n) = h_b^n(p_s^n, p_s^n) \leqq h_b^n(p_s^n, p_b^n) \quad \text{for } \forall p_b \in (0, p_s^n+d_3(g_n)] .$$

Hence (p_s^n, p_b^n) satisfying (4.2) and (4.3) is an equilibrium point of $SG(q_s, q_b, g_n)$ for $\forall n \geqq N_3$.

Let N be a greater integer than the integers selected in the above. Then for $\forall n \geqq$ this N, (p_s^n, p_b^n) is an equilibrium point of $SG(q_s, q_b, g_n)$ if and only if it satisfies (4.2) and (4.3).

Q.E.D.

<u>Lemma 8</u>. There is an integer N^* such that for $\forall n \geqq N^*$, there is a unique equilibrium point (p_s^n, p_b^n) of $SG(q_s, q_b, g_n)$.

<u>Proof</u>. Since $g_n'(0) = 0$ for $\forall n \geqq 1$, we have $g_n'(0)/g_n(0) = 0$ for $\forall n \geqq 1$. Since $g_n(w)$ is concave on $(d_3(g_n), 0)$, $g_n'(w) \geqq g_n'(d_3(g_n))$ for $\forall w \in (d_3(g_n), 0)$. Since $\lim_{n \to \infty} g_n(d_3(g_n)) = \lim_{n \to \infty} d_3(g_n) = 0$, we have $\sup_n g_n'(d_3(g_n)) = +\infty$. Hence there is an integer N_1 such that

$$U_2(Q-\overline{q}, m(\overline{q}))/U(Q-q, m(\overline{q})) = V_2(\overline{q}, M-m(\overline{q}))/V(\overline{q}, M-m(\overline{q}))$$

$$< 2g_n'(d_3(g_n))/g_n(d_3(g_n)) \quad \text{for } \forall n \geqq N_1 .$$

Since $g_n'(w)/g_n(w)$ is a continuous function of w, there is a w^n in $(d_3(g_n),0)$ such that

$$2g_n'(w_n)/g_n(w_n) = U_2(Q-\overline{q},m(\overline{q}))/U(Q-\overline{q},m(\overline{q})).$$

Since $g_n(w)$ is increasing on $(d_3(g_n),0)$ and $g_n'(w)$ is nonincreasing on $(d_3(g_n),0)$, $g_n'(w)/g_n(w)$ is decreasing on $(d_3(g_n),0)$. Hence the above w^n is uniquely determined for each n. We put $p_s^n = m(\overline{q}) - w^n/2$ and $p_b^n = m(\overline{q}) + w^n/2$ for $\forall n \geqq N_1$. These (p_s^n,p_b^n) satisfy (4.2) and (4.3) of Lemma 7. Let $N^* = \max(N,N_1)$, where N is an integer given in the previous lemma. For this N^*, this lemma holds.

Q.E.D.

<u>Proof of Theorem A</u>. By Lemma 8, there is a sequence $\{(p_s^n,p_b^n)\}$ such that (p_s^n,p_b^n) is an equilibrium point of $SG(q_s,q_b,g_n)$ for $\forall n \geqq N^*$. By Lemma 7, it holds that

$$(p_s^n+p_b^n)/2 = m(\overline{q}) \quad \text{and} \quad |p_b^n - p_s^n| \leqq |d_3(g_n)| \quad \text{for } \forall n \geqq N^* .$$

Hence $\{(p_s^n,p_b^n)\}$ converges to $(m(\overline{q}),m(\overline{q}))$. Q.E.D.

5. The (F,G)-Perfect Equilibrium Point at the 1st Stage

In the previous section we showed that the (F,G)-perfect equilibrium point at the 2nd stage was the Nash's cooperative solutions of the subgames . The purpose of this section is to show that the (F,G)-perfect equilibrium point at the 1st stage coincides with the Nash's cooperative solution of the reduced game, which implies that the Nash allocation is derived as the (F,G)-perfect equilibrium point of the bargaining game $G(f_o, g_o)$. To prove it , however , we need to assume an additional condition .

(F): log u(q) and log v(q) are concave functions of q on $(0,Q]$, where u(q) and v(q) are defined in Lemma 4 .

If U and V satisfy assumptions (A),(B),(C),(D),(E) but not (F) , then they would be irregular functions , since L(q) = log u(q) + log v(q) is a strictly concave function as it was proved in Lemma 4 . Hence assumption (F) would be a weak condition .

The purpose of this section is to prove the following theorem .

Theorem B. Let $(b_s^*, b_b^*) = ((q_s^*, t_s^*), (q_b^*, t_b^*))$ be an (F,G)-perfect equilibrium point. Then it holds that

$$(5.1) \quad q_s^* = q_b^* = q_N \quad \text{and} \quad t_s^*(q_s, q_b) = t_b^*(q_s, q_b)$$
$$= m((q_s+q_b)/2) \quad \text{for } \forall (q_s, q_b) \in (0,Q]^2 .$$

As the latter part of (5.1) was proved in Theorem A, it is sufficient to show that $q_s^* = q_b^* = q_N$. The following lemmata are necessary to prove it. In the following, let $\{f_n\}$ be an arbitrary approximating sequence of f_o.

Lemma 9. If $q_s = Q$ or $q_b = Q$, then there is an integer N such that (q_s, q_b) can not be an equilibrium point of the reduced game $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq N$.

Proof. Since $L(q)$ is a strictly concave function of q and has a maximum at an interior point in $(0,Q]$, we have $L'(Q) < 0$. Then it must hold that $u'(Q)/u(Q) < 0$ or $v'(Q)/v(Q) < 0$. Let δ be a positive real number such that $L'(Q-\delta) < 0$. Since $L'(q)$ is a continuous function, it is possible to select such δ.

Suppose that $q_s = Q$ and $q_b \leqq Q - \delta$. Then $\lim_{n \to \infty} f_n'(q_b - q_s)/f_n(q_b - q_s) = 0$ but $f_n'(q_b - q_s)/f_n(q_b - q_s) \neq 0$ for $\forall n \geqq 1$.

Hence if $v'((q_s+q_b)/2)/v((q_s+q_b)/2) \neq 0$, there is an N_1 such that

$$\left|v'((q_s+q_b)/2)/v((q_s+q_b)/2)\right| > \left|2f'_n(q_b-q_s)/f_n(q_b-q_s)\right| \quad \text{for } \forall n \geqq N_1 ,$$

which implies that $\partial r^n_b(q_s,q_b)/\partial q_b \neq 0$ for $\forall n \geqq N_1$. Here $r^n_s(q_s,q_b)$ and $r^n_b(q_s,q_b)$ are the S's and B's payoff functions of the reduced game $R(f_n,g_o,t^*_s,t^*_b)$ ($\forall n \geqq 1$) . Since q_b is an interior point of $(0,Q]$, (q_s,q_b) can not be an equilibrium point of $R(f_n,g_o,t^*_s,t^*_b)$ for $\forall n \geqq N_1$. If $v'((q_s+q_b)/2)/v((q_s+q_b)/2) = 0$, then it holds that $\partial \log r^n_b(q_s,q_b)/\partial q_b = f'_n(q_b-q_s)/f_n(q_b-q_s) \neq 0$ for $\forall n \geqq 1$. Hence (q_s,q_b) can not be an equilibrium point of $R(f_n,g_o,t^*_s,t^*_b)$ for $\forall n \geqq 1$. In the case where $q_s \leqq Q-\delta$ and $q_b = Q$, we can prove similarly that (q_s,q_b) is not an equilibrium point of $R(f_n,g_o,t^*_s,t^*_b)$ for $\forall n \geqq$ some N_1 .

Suppose that q_s and q_b are in $(Q-\delta, Q]$. Let us consider the case where $u'((q_s+q_b)/2)/u((q_s+q_b)/2) \leqq v'((q_s+q_b)/2)/v((q_s+q_b)/2) \leqq 0$. Of course, it holds that $u'((q_s+q_b)/2)/u((q_s+q_b)/2) < 0$. If $u'((q_s+q_b)/2)/u((q_s+q_b)/2) < 2f_n(q_b-q_s)/f_n(q_b-q_s)$, then $\partial r^n_s(q_s,q_b)/\partial q_s < 0$, which implies that (q_s,q_b) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$. If $u'((q_s+q_b)/2)/u((q_s+q_b)/2) \geqq 2f_n'(q_b-q_s)/f_n(q_b-q_s)$, then $v'((q_s+q_b)/2)/v((q_s+q_b)/2) < -2f_n'(q_b-q_s)/f_n(q_b-q_s)$, which implies that $\partial r_b^n(q_s,q_b)/\partial q_b < 0$. Hence (q_s,q_b) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq 1$. In the case where $v'((q_s+q_b)/2)/v((q_s+q_b)/2) \leqq u'((q_s+q_b)/2)/u((q_s+q_b)/2) \leqq 0$, we can prove similarly that (q_s,q_b) is not an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq 1$.

Let us consider the case where $u'((q_s+q_b)/2)u((q_s+q_b)/2) < 0$ and $v'((q_s+q_b)/2)/v((q_s+q_b)/2) > 0$. If $u'((q_s+q_b)/2)/u((q_s+q_b)/2) < 2f_n'(q_b-q_s)/f_n(q_b-q_s)$, then $\partial r_s^n(q_s,q_b)/\partial q_s < 0$. Hence (q_s,q_b) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$. If $u'((q_s+q_b)/2)/u((q_s+q_b)/2) \geqq 2f_n'(q_b-q_s)/f_n(q_b-q_s)$, then $v'((q_s+q_b)/2)/v((q_s+q_b)/2) < -2f_n'(q_b-q_s)/f_n(q_b-q_s)$, because $L'((q_s+q_b)/2) = u'((q_s+q_b)/2)/u((q_s+q_b)/2) + v'((q_s+q_b)/2)/v((q_s+q_b)/2) < 0$. Hence we get $\partial r_b^n(q_s,q_b)/\partial q_b < 0$. Hence we have shown that (q_s,q_b) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq 1$. In the case where $u'((q_s+q_b)/2)/u((q_s+q_b)/2) > 0$ and $v'((q_s+q_b)/2)/v((q_s+q_b)/2) < 0$, we can prove similarly that (q_s,q_b) is not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq 1$. Q.E.D.

<u>Lemma</u> <u>10</u>. There is an integer N such that for $\forall n \geqq N$, (p_s^n, p_b^n) is an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ if and only if it satisfies

$$(5.2) \quad u'((q_s^n+q_b^n)/2)/u((q_s^n+q_b^n)/2)$$

$$= - v'((q_s^n+q_b^n)/2)/v((q_s^n+q_b^n)/2) = 2f_n'(q_b^n-q_s^n)/f_n(q_b^n-q_s^n) ,$$

$$(5.3) \quad |q_b^n - q_s^n| \leqq \max(-d_1(g_n), d_2(g_n)) .$$

<u>Proof</u>. When $q_s = Q$ or $q_b = Q$, it is shown that (q_s, q_b) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq$ some N , and it is easily verified that (5.2) is not true for $\forall n \geqq$ some N . Hence we need to consider the case where q_s and q_b are in $(0,Q)$.

If (5.2) does not hold for (q_s^n, q_b^n) , then $\partial r_s^n(q_s^n, q_b^n)/\partial q_s \neq 0$ or $\partial r_b^n(q_s^n, q_b^n)/\partial q_b \neq 0$. Hence (q_s^n, q_b^n) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$.

In the following, suppose that (5.2) holds for (q_s^n, q_b^n) ($\forall n \geqq 1$) . Then we have, by Lemma 5 , $(q_s^n+q_b^n)/2 = q_N$. Let ε and δ be positive real numbers such that if $|q_s - q_N| \leqq \delta$ and $|q_b - q_N| \leqq \delta$, then

$$u(q_b) > \varepsilon \sup_q u(q) \quad \text{and} \quad v(q_s) > \varepsilon \sup_q v(q) .$$

Since u and v are continuous functions and $u(q_N)$, $v(q_N) > 0$, it is possible to select such ε and δ.

Suppose that (5.3) does not hold. Since $\lim_{n \to \infty} f_n(d_1(f_n)) = \lim_{n \to \infty} f_n(d_2(f_n)) = 0$, there is an integer N_1 such that $f_n(d_1(f_n)) \leqq \varepsilon$ and $f_n(d_2(f_n)) \leqq \varepsilon$ for $\forall n \geqq N_1$. Since $\{f_n'/f_n\}$ converges uniformly to 0 on $(-Q, w_1)$ and (w_2, Q) for $\forall w_1 \in (-Q, 0)$ and $\forall w_2 \in (0, Q)$, $\{(q_s^n, q_b^n)\}$ satisfying (5.2) has a property that $\{|q_b^n - q_s^n|\}$ converges to 0. Then there is an integer N_2 such that $|q_b^n - q_s^n| \leqq \delta$ for $\forall n \geqq N_2$. It holds for $\forall n \geqq \max(N_1, N_2)$ that

$$r_s^n(q_s^n, q_b^n) = u((q_s^n + q_b^n)/2) f_n(q_b^n - q_s^n) \leqq \varepsilon \sup_q u(q)$$

$$< u(q_b^n) = r_s^n(q_b^n, q_b^n) .$$

Hence (q_s^n, q_b^n) can not be an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq \max(N_1, N_2)$.

Suppose that (5.3) holds for (q_s^n, q_b^n) ($\forall n \geqq 1$). Then $\partial r_s^n(q_s^n, q_b^n)/\partial q_s = \partial r_b^n(q_s^n, q_b^n)/\partial q_b = 0$. Since $\log r_s^n(q_s, q_b^n)$ and $\log r_b^n(q_s^n, q_b)$ are concave functions of q_s on $(q_b^n - d_2(f_n), q_b^n - d_1(f_n))$ and of q_b on $(q_s^n + d_1(f_n), q_b^n + d_2(f_n))$, we have

$$\log r_s^n(q_s^n,q_b^n) \geqq \log r_s^n(q_s,q_b^n) \quad \text{for } \forall q_s \in (q_b^n-d_2(f_n),\, q_b^n-d_1(f_n)),$$

$$\log r_b^n(q_s^n,q_b^n) \geqq \log r_b^n(q_s^n,q_b) \quad \text{for } \forall q_b \in (q_s^n+d_1(f_n),\, q_s^n+d_2(f_n)),$$

that is ,

$$r_s^n(q_s^n,q_b^n) \geqq r_s^n(q_s,q_b^n) \quad \text{for } \forall q_s \in (q_b^n-d_2(f_n),\, q_b^n-d_1(f_n))$$

$$r_b^n(q_s^n,q_b^n) \geqq r_b^n(q_s^n,q_b) \quad \text{for } \forall q_b \in (q_s^n+d_1(f_n),\, q_s^n+d_2(f_n)) \;.$$

Since $\lim_{n\to\infty} d_1(f_n) = \lim_{n\to\infty} d_2(f_n) = 0$ and $\lim_{n\to\infty} f_n(d_1(f_n)) = \lim_{n\to\infty} f_n(d_2(f_n)) = 0$, there is an integer N_3 such that $f_n(d_1(f_n)) \leqq \varepsilon$ and $f_n(d_2(f_n)) \leqq \varepsilon$ for $\forall n \geqq N_3$ and $-d_1(f_n) \leqq \delta$ and $d_2(f_n) \leqq \delta$ for $\forall n \geqq N_3$. Hence we have, for $\forall n \geqq N_3$,

$$r_s^n(q_s,q_b^n) = u((q_s+q_b^n)/2)f_n(q_b^n-q_s) \leqq \varepsilon \sup_q u(q)$$

$$< u(q_b^n) = r_s^n(q_b^n,q_b^n) \leqq r_s^n(q_s^n,q_b^n)$$

$$\text{for } \forall q_s \in (0,\, q_b^n-d_2(f_n)] \cup [q_b^n-d_1(f_n),\, Q] \;,$$

and

$$r_b^n(q_s^n,q_b) = v((q_s^n+q_b)/2)f_n(q_b-q_s^n) \leqq \varepsilon \sup_q v(q)$$

$$< v(q_s^n) = r_b^n(q_s^n,q_s^n) \leqq r_b^n(q_s^n,q_b^n)$$

$$\text{for } \forall q_b \in (0,\, q_s^n+d_1(f_n)] \cup [q_s^n+d_2(f_n),\, Q] \;.$$

Hence we have shown that (q_s^n, q_b^n) is an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq N_3$.

Let N be a greater integer than the integers selected in the above . Then for $\forall n \geqq$ this N , (q_s^n, q_b^n) is an equilibrium point of $R(f_n, g_o, t_s^*, t_b^*)$ if and only if it satisfies (5.2) and (5.3). Q.E.D.

<u>Lemma 11</u> . There is an integer N^* such that for $\forall n \geqq N^*$, there is a unique equilibrium point (q_s^n, q_b^n) of $R(f_n, g_o, t_s^*, t_b^*)$.

<u>Proof</u>. Since $f_n(w)$ is concave on $(d_1(f_n), d_2(f_n))$ and since $f_n(w)$ is increasing on $(-Q, 0)$ and is decreasing on $(0, Q)$, $\log f_n(w)$ is a strictly concave function on $(d_1(f_n), d_2(f_n))$. Hence $f_n'(w)/f_n(w)$ is a decreasing function on $(d_1(f_n), d_2(f_n))$. Since $\lim_{n \to \infty} f_n(d_1(f_n)) = \lim_{n \to \infty} f_n(d_2(f_n)) = 0$ and $\lim_{n \to \infty} d_1(f_n) = \lim_{n \to \infty} d_2(f_n) = 0$, it must hold that $\sup_n f_n'(d_1(f_n)) = +\infty$ and $\sup_n f_n'(d_2(f_n)) = -\infty$, which implies $\sup_n f_n'(d_1(f_n))/f_n(d_1(f_n)) = +\infty$ and $\sup_n f_n'(d_2(f_n))/f_n(d_2(f_n)) = -\infty$.

Hence there is an integer N_1 such that for $\forall n \geqq N_1$,

$$2f_n'(d_2(d_2(f_n))/f_n(d_2(f_n)) < u'(q_N)/u(q_N)$$

$$< 2f_n'(d_1(f_n))/f_n(d_1(f_n)) \ .$$

Since $f_n'(w)/f_n(w)$ is a continuous function for each n , there is a w^n in $(d_1(f_n), d_2(f_n))$ for each $n \geqq N_1$ such that $u'(q_N)/u(q_N) = 2f_n'(w^n)/f_n(w^n)$. We put $q_s^n = q_N + w^n/2$ and $q_b^n = q_N - w^n/2$ if $u'(q_N)/u(q_N) \geqq 0$, and $q_s^n = q_N - w^n/2$ and $q_b^n = q_N$ $w^n/2$ if $u'(q_N)/u(q_N) < 0$. This (q_s^n, q_b^n) satisfies (5.2) and (5.3) for $\forall n \geqq N_1$. Hence this lemma holds for $\forall n \geqq N^* = \max(N, N_1)$, where N is the integer given in Lemma 10 .

Q.E.D.

<u>Proof of Theorem B.</u> By lemma 11 , there is a sequence $\{(q_s^n, q_b^n)\}$ such that (q_s^n, q_b^n) is an equilibrium point of $K(f_n, g_o, t_s^*, t_b^*)$ for $\forall n \geqq N^*$. By Lemma 10 , it holds that

$$(q_s^n + q_b^n)/2 = q_N \quad \text{and} \quad |q_b^n - q_s^n| \leqq \max(-d_1(f_n), d_2(f_n))$$
$$\text{for } \forall n \geqq N^* .$$

Hence $\{(q_s^n, q_b^n)\}$ converges to q_N . Q.E.D.

6. Conclusion

In the prvevious two sections, we have shown the following theorem .

Theorem C. There exists a unique (F,G)-perfect equilibrium point of the bargaining game $G(f_o,g_o)$. It is given by (5.1) .

We have shown that Nash's cooperative solution is derived as an (F,G)-perfect equilibrium point of the bargaining game $G(f_o,g_o)$. This result strengthens the point of Harsanyi [1961] that Nash's theory should be interpreted not as a normative theory but as a positive one . This never contradicts the point of Kaneko and Nakamura [1977] that Nash's theory can be interpreted as a normative theory , because Kaneko and Nakamura assert that if the threat point is set as the worst state , the hell , which we can imagine , then the Nash's bargaining game can become a unique social welfare function .

The bargaining game $G(f_o,g_o)$ is one of models which we can consider . For example , we can formulate other models in which a quontity of the commodity and a price of it are decided simulataneously and in which a price is decided before a quontity is done ,etc. . In such cases, our basic approach of smoothing procedure may be useful . It should be, however, noted that the Nash's cooperative solution is not necessarily derived . The reason for the variety of models is that the bilateral economy considered has two decision-variables . If an economy with two bargainers has a unique decision-variable, then a bargaining model of the economy may be the same as the game of the 1st stage or that of the 2nd stage of the bargaining game $G(f_o,g_o)$. For example , when a commodity in question has a very large and indivisible unit, e.g., a house , a tanker, a petrochemical plant, etc. , the negotiation model would have the same structure as that of the 2nd stage . When bargainers are not in a situation of a perfect conflict of interests , the game of the 1st stage may fit for such a situation . Thus it is thought that our basic approach may be applied to wide fields .

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