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Welfare and Work-Incentive Effects of the Redistributive Taxation: A Negative Income Tax Proposal

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*1 This research was motivated from a discussion with Professor J. Trout Rader over his seminal work, privately cooresponded, though the content of the work has been largely changed from and extended over the first motivation. Dr. Yoshiko Nogami, my colleague, helped me to derive the results in Appendix 2.

Abstract

Welfare and work-incentive of the member citizens of an inequal society, characterized by a considerable degree of difference in the productivity of their labor services, are considered in the simple general equilibrium framework formulated by Sheshinski [1972]. Among the others, an implication thereof will be proved precisely: In sofar as the tax revenues hence transfer incomes are expended only upon consumption, the redistributive taxation, as a negative income tax proposal, will induce, on one hand, a rather (the more inequal, the larger) number of citizens, to whom relatively lower productivities endowned, to choose unemployment hence substantially decreasing the "social" productivity, and, on the other hand, induce the government to fix, under the assumed objective, a high (the higher) tax rate, which is more (the more) agreeable to all the voluntarily unemployed (more increased in number).

Introduction and Informal *Presentation of the Results*

In the present analysis, an *inequal* society, in which individual members have identical tastes but differ in the productivity of their labor services such as their skills, will be considered, from the *welfare* viewpoint on one hand, and, from the *work-incentive* viewpoint on the other hand.
To this end, we must relevantly simplfy the framework for analysis.

Consider the general equilibrium framework of *labor* supply under the public policy with taxation, formulated by Mirrlees[1971] who considerd the nonlinear taxation problem and obtained numerical results by using the sum of logarithms of Neumann-Morgenstern utilities, and, reformulated by the followers like Sheshinski[1972], Sadka[1976], Varian[1980], for example among others, who found relatively low tax rates in almost linear that are optimal in terms of some or other social welfare criterion.[*2]

We shall here concern ourselves with an extension of the model formulated by Sheshinski[1972]. There are the continuum of economic agents called *individuals*, who are endowned with labor (income) productivities, measured in efficiency units, and preferences (both being fixed), and an agent called the *government* who takes the income *redistributive taxation* policy under the assumed objective of maximizing the welfare of the individuals, for which the average, for example, is taken. Their endowned productivities are distributed in a *dissimilar (inequal)* way,[*3] but, their preferences are very similar, so that they may be represented simply by a linear homogeneous real-valued function of the Cobb-Douglas type. The welfare is obtained by two goods; one, non-traded, termed "leasure"[*4] and the other, traded at market, termed consumption (or "income", essentially efficient labor in consumption units). A main reason why we keep the first and specify more the second assumption is to bring, as a final analysis, more precise conclusions by providing the basis for actual calculation of tax rates and develope the arguments relevantly as a preclude to fruitful empirical policy studies.

As the final analysis, we shall thus work out the full details of the simple general equilibrium model. We shall there calculate the welfare gain or loss of individuals, as well as the aggregate (average), of the taxation policy, which have not been done in the literature of the previous authors.[*5] In order to have a more explicit understanding of how productivity structure affects the taxation structure, we shall compute the various tax rates of individual after individual, in terms of their welfare, income, optimization. We shall also compute the (incentive-losing) tax rates, at which there will be some individuals who completely lose their incentive for work hence a rational choice to them being unemployment. Of course we shall examine how the tax rates, each being optimal under the assumed objective, vary in diversely given structures of productivities.

It would be most significant to investigate where all those tax rates are located in the unit interval of rates, in particular, relatively to the tax rates, such as of maximizing the aggregate welfare, the welfare of individuals of the lowest productivity, or, trasnfer (the tax revenues) etc., as an alternative policy.the government might choose. *6

Thus, the present analysis will reveal, for each taxation policy and for each degree (somehow defined) of inequality, the diverse structures of individual welfare, income, consumption and (un)employment, hence, the whole economic schedule of taxation (we shall henceforth say the taxation schedule for short), and their structural changes subject to a variation in tax policy and/or the degree of inequality.

This income-redistributive taxation system, in fact, may well be reformulated hence interpreted as a negative income tax proposal, by the introduction of a tax rate in the net sense. The NIT proposal was made originally as an attempt to overwhelm a strongly adverse, psychologically corrosive, effects on the incentive for work of the existing welfare-relief arrangement for support of the unemployed. However it has been argued against; the NIT itself may constitute of a source of psychological corrosion for a much larger number of workers . For *all* individuals now will be able to receive unearned income supplements from the government, even though they are remaining, at any rate, employed. (Hirshleifer[1980, pp.454-459]). We shall see how a *more* inequal productivity structure make the latter argument easier to hold at low tax rates, and, the higher the tax rate, the larger a number of individuals will choose a complete unemployment. We thus reformulate the employment vs welfare controversy in a wider scope of view, covering social welfare and policy, and wish to shed a new light on this moral hazard problem.

A *rather lengthy summary* could be made by presenting the taxation schedule in a largely informal but illustrative way; it rests upon, however, technical analysis that will be available in the later sections.

Inorder to make the exposition easier, it would be a great convenience to define four kinds of hypothetical individuals. The first one is an individual, with the productivity R(t) for each tax rate t, whose labor supply is zero for the tax rate t, hence, a rational choice to him is always a complete unemployment. The intended interpretation for this definition is that, if productivity R^i is larger than this R(t) for tax rate t, then, the individual i of R^i will more or less work at that rate t, whereas, if R^i is equal to or less than R(t), then, this individual won't work at all. Naturally, this productivity R(t) will increase with the tax rate t, with its range interval $[0,\bar{R}]$ where $\bar{R}$ is the highest productivity. In fact, a large class of distributions, including both Pareto and Gibrat distributions, will reveal this property. Let $\underline{R}$ be the lowest productivity, and define the critical rate $\underline{t}$ to be such that $R(\underline{t})=\underline{R}$. Then, $\underline{t} \leq \underline{t}^i$ for any i such that $R(\underline{t}^i)=R^i$. (Subsection 2.1)

The second one is defined to be an individual with the productivity $\tilde{R}(t)$ endowned for each tax rate t, whose labor supply is the aggregate (average) for that tax rate, E(L(t)). $\tilde{\tilde{R}}(t)$ is the average productivity over working individuals at that rate t. This individual is intended to be, for every tax rate t, neither a tax payer nor a recipient of taxes, in the net sense. He is always an earner of the breakeven level of labor income. In view of R(t), this $\tilde{\tilde{R}}(t)$ will stay to be a constant, that is, the average E(R), in so far as every individual is working at any rate, but, will increase with the tax rate as soon as some individuals reserve employment. See Subsection 3.1.

The third one is an individual, to whom the productivity R*(t) would be endowned for each tax rate t, whose welfare would be maximized at that rate t. An interpretation for this is clear. That is, $t^{i}*$ is optimal to individual i if R^{i} is equal to $R*(t^{i}*)$. In so far as the productivity of a working individual is below the average, the higher the productivity, the lower the optimal tax rate of the individual, provided that the optimal rate is unique for a productivity. Thus, we may take R*(t) as a decreasing function of the rate t with its range $[E(R),R(t^{i°}*)]$, where $t^{i°}*$ is both an optimal rate and the smallest incentive losing tax rate to i°. In general, the uniqueness need not be true, on which we shall fully discuss in Subsection 6.3. There we shall see this R*(t) will, after decreasing and attaining its minimum, then increase with tax rate t. See also Subsection 4.1,5.1 & 5.3.

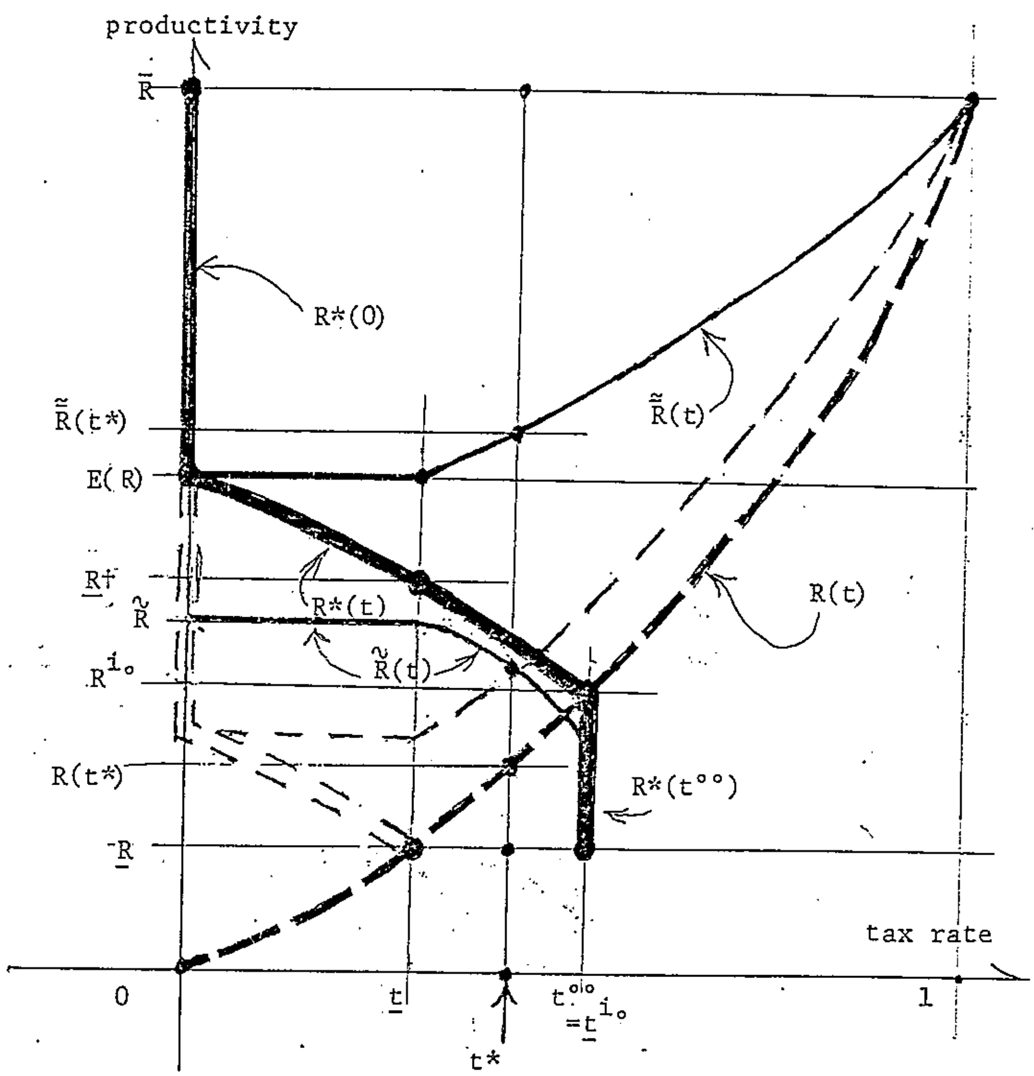
The optimal tax rates of the voluntarily unemployed are[*7] the rate $t^{\circ\circ}$, at which the transfer income hence tax revenues, T, will take its maximum $T(t^{\circ\circ})$. No taxation (t=0) is optimal to those whose productivities are equal to or above the average.[*8] (Subsection 5.2)

The last one is a hypothetical individual, to whom the productivity $\tilde{R}(t)$[*9] would be endowned for each tax rate t, whose welfare is (equal to) the average (aggregate) one. $\tilde{R}(t)$ will stay to be a constant $\tilde{R}$ whenever $t \leq \underline{t}$, while it will decrease as the tax rate t increases when $t > \underline{t}$. Thus, the productivity $\tilde{R}(t)$, producing the aggregate welfare for each t,[*9] will decrease as the number of the voluntarily unemployed increases. Here, $\tilde{R}$ is the ratio of the aggregate of $(R^i)^\alpha$ over the aggregate of $(R^i)^{\alpha-1}$ over individuals. In the large class of distributions relevant here, the ratio $\tilde{R}$ is less than the average productivity $E(R)$. See Subsection 5.3 & 6.1.

With those hypothetical productivities thus far introduced, we may have a diagram below, that is termed taxation schedule and able to depict the whole problem, for each ultimate parametric variables, such as the degree of inequality, the expenditure-income ratio (that is, the marginal propensity to spend on consumption, α).[*10]

Take a cartesian product $[0,1]\times[0,\bar{R}]$ as the taxation-productivity space. We shall take the tax rates along the horizontal line and the productivities along the vertical line. Then, we are able to draw the graphs of $R(.)$, $\tilde{\tilde{R}}(.)$, $R^*(.)$ and $\tilde{R}(.)$ as in the diagram.

Figure 1 Taxation Schedule



Suppose the government takes a taxation policy t* under the assumed objective; see Figure 1. Here the rate t* is set relatively high to any rate which would be lower if there is no unemployment at all (that is, if $\tilde{R}$ be larger than $\underline{R}^{\dagger}$, then t* would be determined so that $R^*(t^*)=\tilde{R}(t^*)=\tilde{R}$). However, in case $\tilde{R}$ is less than $\underline{R}^{\dagger}$[*11] (as in Figure 1), the larger the tax rate t ($> \underline{t}$), the lower (than

$\tilde{R}$) the productivity $\tilde{R}(t)$ for this t. Hence, the rate t* must be determined so that R*(t*) may be larger than $\tilde{R}(t^*)$.[*12]

Any taxation policy will here induce *all* working individuals to choose, more or less, a *less* degree of employment. Not only that but also it will induce $100\int_{\underline{R}}^{R(t^*)} g(R)dR$ % of them to choose a complete unemployment. Their consumption expenditure must be equal to the transfer incomes. $100\int_{R(t^*)}^{\tilde{\tilde{R}}(t^*)} g(R)dR$ % will work to some extent but will be the (net) recipients of income supplements. The rest $100\int_{\tilde{\tilde{R}}(t^*)}^{\bar{R}} g(R)dR$ % will work, being the tax payer in the net sense. Here, the function g(R) is the density function of R. (Subsection 3.1)

The percentage of the voluntarily unemployed would become the largest ($100\int_{\underline{R}}^{R^{i_\circ}} g(R)dR$ %), if the government would take the policy $t^{\circ\circ}$, which maximizes the tax revenues hence the welfare of the unemployed. See Subsection 3.1.

The above argument, however, presumes a certain degree of the inequality in the productivity structure. Let $\underline{R}$† be the productivity of an individual whose welfare will be maiximized at the smallest incentive losing rate $\underline{t}$. Then, how far $\underline{R}$† will be away from $\underline{R}$, depends entirely upon, how much less the ratio of $\underline{R}$ over E(R) will be than a certain value $1/\{1+(\sqrt{1-\alpha})^{-1}\}(<1/2)$. In fact, $\underline{R} = \underline{R}$† if and only if $\underline{R}/E(R) = 1/\{1+(\sqrt{1-\alpha})^{-1}\}$. Therefore, the ratio $E(R)/\underline{R}$ could be an appropriate measure of the inequality of which we can take an expository advantage.[*13] Further, we shall see that the higher the degree of inequality, the larger the number of the unemployed, who think of the rate $t^{\circ\circ}$ as optimal, and the higher the revenues maximizing rate $t^{\circ\circ}$, hence the higher the optimal rate t^* approaching closer to the rate $t^{\circ\circ}$. Thus, the greater extent, to which workers differ in the productivity of their labor services, would induce the larger the number of the voluntarily unemployed, on one hand, and, on the other hand, induce the government to take the higher tax rate t^*, which would become more agreeable to the increased number of the unemployed. See Subsection 5.3, 5.4 & 6.3. for this.

Suppose the degree of inequality be so small that $\underline{R}\dagger$ may be less than $\underline{R}$, then, the government tax rate t^* will be situated within the open interval $(0,\underline{t})$; see the figure. Thus, all individuals will work, and each will choose a *larger* degree of employment. The extreme case of perfect equality, that is, $E(R)/\underline{R} = 1$, would easily show that $t^* = t^{i*} = 0$. See Subsection 5.1.

Propositions that Characterize the Taxation Schedule

We shall prove precisely the propositions listed below in the derivation order in technical analysis.

1. (i) Labor supplies, both individual and the aggregate (average), will monotonely decrease as the tax rate t increases.

(ii) The (work-incentive losing) tax rate, $\underline{t}^i$, at which some individuals including individual i will lose work incentive, follows precisely the order of productivities. The smallest rate, $\underline{t}$, is simply the ratio of $\underline{R}$ over a convex combination of $\underline{R}$ and $E(R)$, that is, $\underline{t} = \underline{R}/\{\alpha\underline{R}+(1-\alpha)E(R)\}$.

2. The average (aggregate) income and the income of an individual i with the productivity R^i higher than a certain multiple of the average ($\alpha E(R)$), will both, in the full income terms, decrease as the tax rate t increases. That of an individual of less productivity will, from the very begining, increase with the tax rate, attaining its maximum at a certain rate, $t^{i\circ}$, and, then decreasing at increasing rates, ending up with 0 at 100% tax rate.

3. The tax revenues hence transfer income will, in the begining, increase with the tax rate t. They will increase at decreasing rates, attaining their largest value at a certain rate $t^{\circ\circ}$, then decreasing at increasing rates ending up with 0 at 100% tax rate. The revenues maximizing rate $t^{\circ\circ}$ is larger than $1/\{1+\sqrt{1-\alpha}\}$ where $t^{\circ\circ}=1/\{1+\sqrt{(1-\alpha)\int_R g(R)\,dR}\}$, $\int_R = \int_{R(t^{\circ\circ})}^{\bar{R}} = 1-\int_{\underline{R}}^{R(t^{\circ\circ})}$, if there is a number of the unemployed. Otherwise $t^{\circ\circ}=1/\{1+\sqrt{1-\alpha}\}$.

4. Every individual, to whom the average or higher income productivity is endowned, will lose, at any positive tax rate, parts of the welfare, more or less, that would be gained at 0 tax rate.

With the productivity endowned lower than the average, an individual will gain, more or less, an extra welfare relatively to the welfare gained at no taxation.*14 The lower the income productivity, the more the extra welfare at certain positive tax rates.

5. With the lower productivity endowned, the higher tax rate an individual will think of as his (smallest) optimum. Thus, the (smallest) welfare maximizing rates, t^{i*}'s, are ordered inversely with the order of their productivities. See Section 5, Subsection 6.3.

6. The aggregate (average) welfare maximizing tax rate t^* is less than or equal to the smallest incentive losing rate $\underline{t}$ if and only if the productivity $\tilde{R}$ is (less than the average and) larger than the critical productivity $\underline{R}\dagger$.

7. The optimal rate t^* is larger than the rate $\underline{t}$ if and only if the productivity $\tilde{R}$ is less than the productivity $\underline{R}\dagger$ and larger than the smallest productivity $\underline{R}$. The latter is equivalent to that the social productivity $\tilde{R}(t^*)$ is less than the productivity $\underline{R}\dagger$ (but not necessarily larger than $\underline{R}$).

8. The propositions 6 & 7 hold for the set (t^{i*},R^{i}) in stead of the set $(t^*,\tilde{R}(t^*))$, provided that the individually optimal rate t^{i*} is unique for each R^{i}.

9. The condition of 7 holds if and only if the ratio of the the average over the lowest productivity; $E(R)/\underline{R}$, the inequality measure, is larger than $1+(\sqrt{1-\alpha})^{-1}$, which is also equivalent to that the largest optimal rate $\underline{t}^*$ of individuals of the lowest productivity $\underline{R}$ is higher than their incentive losing rate $\underline{t}$. Hence it is also equivalent to the condition of 8 if the uniqueness is true. This condition is equivalent also to that the revenues maximizing rate $t^{\circ\circ}$ is larger than the incentive losing tax rate $\underline{t}$.

10. Suppose further a Pareto distribution for the given productivities. Then, the Pareto index β, the inverse of which can measure the inequality degree, is less than a certain small index $\beta\dagger$(less than $1+\sqrt{1-\alpha}$) if and only if the optimal rate t^* will be so large (larger than the rate $\underline{t}$) that there will be a number of the unemployed.

11. There exists a certain $\beta^{\sim}$, smaller than $\underline{\beta}^{\dagger}$, such that (i) β is not larger than $\beta^{\sim}$ if and only if the rate t^* is equal to the rate $t^{\circ\circ}$ and (ii) β is larger than $\beta^{\sim}$ if and only if the rate t^* is less than the rate $t^{\circ\circ}$.

12. The Pareto index β is not smaller than $2-\alpha$, if and only if $R^*(t)$ is decreasing on an interval $[\underline{t}, t^{\circ\circ})$ where $t^{\circ\circ}=1/\beta$. Consequently, β is smaller than $2-\alpha$, if and only if $R^*(t)$ is not monotone on that interval, where $R^*(t)$ decreases, attains its minimum (larger than $\underline{R}$), then, increases as t increases from $\underline{t}$ to $t^{\circ\circ}$.

13. There exists a smaller β° such that the index β is equal to or smaller than β° if and only if the (largest) productivity (= $R(1/\beta)$), individuals of which will voluntarily choose unemployment at $t^{\circ\circ}$, is equal to or larger than the average $E(R)$.

Part II Technical Analysis

1. A Simple General Equilibrium Model of Taxation

1.0 The (Income) Productivity Distribution and Identical Tastes Representable by an Linear Homogenious Cobb-Douglas Function

Let R^i be (labor) productivity of individual i [0,1]. The productivity distribution of R^i, $i \epsilon [0,1]$, is externally given and well known to the central authority which makes a decision on the public policy. For a later, specific analysis, we shall make use of Pareto and lognormal distributions. Let g be the given density function. Let L^i be a voluntary labor supply of i, both R^i and L^i measured in terms of consumption good. The consumption good bought by i is given by y^i. Let r^i measure leisure, and let us define $r^i = (R^i - L^i)/R^i$.

Assume all individuals possess the same taste, which is represented by an identical utility function defined on two dimensional space of consumption good and leisure. This utility function is further assumed to take the form of linear homogenious Cobb-Douglas; that is, $u^i = u(y^i, r^i) = (y^i)^{\alpha}(r^i)^{1-\alpha}$. A main reason for these assumptions is, we can avoid a difficulty often involved in the aggregation problem of (different) preferences and productivities.

1.1 Linear Taxation Policy, Ex-Ante Transfer and Ex-Post Tax Revenues of the Government, and Individual Decision

The taxation structure is specified by a linear tax rate, designated t and $0 \leq t \leq 1$. We shall simply assume that the tax revenues are to be paid out to all indiyiduals in equal quantities. However, the tax revenues, to be ex-post collected by the authority through taxation on the realized labor supplies, need not be the transfer to be ex-ante announced to deliver to every individual.

Let the density function of ex-post labor supply distribution be given and known to the authority, designated $f(L^i)$ where L^i is supplied by $f(L^i)$ individuals and $\int_0^\infty f(L^i)\,dL^i=1$. The (ex-post) tax $t\int_0^\infty L^i f(L^i)\,dL^i$ are thus ex-post determined with L^i realized. However, a budget constraint, on which the realized L^i depends, is determined, for each t, only with the transfer ex-ante planned, T, by the authority. The planned transfer, whatever it may be, is feasible only if it does not exceed the revenues. Let L be the value expected ex-ante by the authority on the basis of information elicited from individuals of productivities, as well as preferneces. Then, $T = t\int_0^\infty Lf(L)\,dL \leqq t\int_0^\infty L^i f(L^i)\,dL^i$. Given the policy $(t,\ t\int_0^\infty Lf(L)\,dL)$, the budget constraint is in terms of full income,

(0) $y^i+(1-t)R^i r^i \leqq (1-t)R^i+t\int_0^\infty Lf(L)\,dL,$

where $(1-t)R^i$ corresponds to the after tax wage of individual i.

With this ex-ante (full) income constraint, each individual maximizes his utility u^i with respect to $(y^i,\ r^i)$. A solution must satisfy the necessary condition for (y^i, r^i) to be individually optimal;

(1) $y^i(t,\ t\int_0^\infty Lf(L)\,dL) = \alpha I^i,$

(2) $(1-t)(R^i-L^i)=(1-\alpha)I^i,$

as an interior solution, or,

(1') $y^i(t,\ t\int_0^\infty Lf(L)\,dL) = t\int_0^\infty Lf(L)\,dL,$

as a corner solution.

Here, $I^i=(1-t)R^i+t\int Lf(L)dL$ and $L^i=L(t, t\int Lf(L)dL)$.

The latter $L(t,t\int Lf(L)dL)$ is equivalent to:

(3) $L^i(t)=\alpha R^i-(1-\alpha)t\int Lf(L)dL/(1-t)$,

which can be negative but taken to be zero in that case.

An aggregation of both hand-sides of (3) (by the authority) will bring out the revenues which is not always equal to the transfer already announced or delivered. *For the present analysis, we assume;*

(4) $L^i(t) = L$, or, $\int_0^\infty L^i(t)f(L^i(t))dL^i(t) = \int_0^\infty Lf(L)dL$. *15

Integrating (3) over the individuals, the authority has

(5) $$\int_0^\infty L(t)f(L(t))dL(t) = \int_R^\infty Rg(R)dR - \{(1-\alpha)t\int_0^\infty Lf(L)dL/(1-t)\}\int_R^\infty g(R)dR$$
$$\geqq \int_0^\infty Rg(R)dR-(1-\alpha)t\int_0^\infty Lf(L)dL/(1-t),$$

where $L(t)=L^i(t)\geqq 0$, $f(0)=\int_{\underline{R}}^{R(t)}g(R)dR$, $f(L(t))=g(R)$ if $L(t)>0$, and R of $\int_R^\infty$ is $R(t)$, such that $L(t)=0$ (to be defined precisely later).

Under the consistency condition (4), (5) may imply (6), and the authority knows the aggregate labor for each t;

(6) $$\int_0^\infty L(t)f(L(t))dL(t) = [\alpha(1-t)/\{(1-t)+(1-\alpha)t\int_R^\infty g(R)dR\}]\int_R^\infty Rg(R)dR$$
$$\geqq \{\alpha(1-t)/(1-\alpha t)\}\int_0^\infty Rg(R)dR,$$

and also the aggregate (and average) income;

(7) $$\int_0^\infty I(t)h(I(t))dI(t) = (1-t)\int_0^\infty Rg(R)dR+t[\alpha(1-t)/\{(1-t)+(1-\alpha)t\int_R^\infty g(R)dR\}$$
$$]\int_R^\infty Rg(R)dR \quad \geqq [(1-t)+t\alpha(1-t)/(1-\alpha t)]\int_0^\infty Rg(R)dR,$$

where the strict equalities hold only when $L^i>0$ for all i.

The authority exactly knows labor supply of each individual i, in terms of tax and productivity (t, R^i); from (3) and (6),

(3') $L^i(t) = \alpha R^i - [\alpha(1-\alpha)t/\{(1-t)+t(1-\alpha)\int_R g(R)dR\}]\int_R Rg(R)dR$

$\geqq \alpha R^i - \{\alpha(1-\alpha)t/(1-\alpha t)\}\int_0 Rg(R)dR.$

Also from (6) and the definition of $I^i(t)$ follows

(8) $I^i(t) = (1-t)R^i + [\alpha t(1-t)/\{(1-t)+(1-\alpha)\int_R g(R)dR\}]\int_R Rg(R)dR$

$\geqq (1-t)R^i + t\alpha(1-t)/(1-\alpha t)\int_0 Rg(R)dR.$

Here the strict equalities hold only when $L^i(t)>0$ for all i.

1.2 <u>The Walras Law and Market Equilibrium</u>

Lastly, for each t, the Walras Law is obtained by integrating (0) when (1) and (2) hold, so that from (6)

(9) $\int_0^\infty y(y)j(y(t)dy(t) = \int_0^\infty L(t)f(L(t))dL(t),$

where $j(y)=f(L)$, hence, the market is always in equilibrium for an arbirary price and supply of labor always creates its demand.

1.3 <u>Labor Incentive and Tax Rate</u>

We shall see first, from the "incentive" view point, how the levels of individual labor supplied are related with tax rates.

From (3'),

(10) $L^i(t)>0 \leftrightarrow t< R^i/[(1-\alpha)\int_R Rg(R)dR + \{1-(1-\alpha)\int_R g(R)dR\}R^i].$

Suppose R^i is not less than the average over working individuals;

$R^i \geqq \int_R Rg(R)dR/\int_R g(R)dR,$

then, it follows that, for any t< 1,

$L^i > 0$.

Otherwise, the right hand side of the inequality (10) is less than 1, hence,

(11) $L^i(t) > 0$

if $t < \min [1, R^i/[(1-\alpha)\int_R Rg(R)dR+\{1-(1-\alpha)\int_R g(R)dR\}R^i]]$.

In view of (3), let us define $R(t)$ to be such that

(12) $R(t) = (1-\alpha)t\int_R Rg(R)dR/\{(1-t)+(1-\alpha)t\int_R g(R)dR$.

Note that $R(t)=R^i \leftrightarrow L^i(t)=0$.

For each t, define such $R(t)$, then, $\int_R = \int_{R(t)}$, and, we shall see later this function is an increasing function in t. Define also a work-incentive losing rate $\underline{t}^i$ to be such that $R(\underline{t}^i)=R^i$. The intended interpretation is that $\underline{t}^i$ is the critical tax rate, at which or at any rate higher than which, the individual i won't work at all.

The role of $R(t)$ will be important in the following analysis. From (6) (7) (8) (3') etc., it immediately follows that,

(6') $t\int_0^\infty L(t)f(L(t))dL(t) = \alpha(1-t)R(t)/(1-\alpha)$,

(7') $\int_0^\infty I(t)h(I(t))dI(t) = (1-t)(1-\alpha)^{-1}\{(1-\alpha)\int_0^\infty Rg(R)dR+\alpha R(t)\}$

(8') $I^i(t) = (1-t)(1-\alpha)^{-1}\{(1-\alpha)R^i+\alpha R(t)\} = (1-t)(1-\alpha)^{-1}\{R^i-L^i(t)\}$

(3'') $L^i(t) = \alpha\{R^i-R(t)\}$.

2. Variation of Labor Supplies, Incomes and Tax Revenues, Subject to Tax Variation

2.1. We shall first examine how this $R(t)$ behaves as rate t varies. It may be without loss of generality if we assume the existence of supremum and infimum of R^i. Let $\underline{R} = \inf_i R^i$ and $\bar{R} = \sup_i R^i$. Behaviour of $R(t)$ in response to changes in rate t will be investigated firstly when $R(t) \leq \underline{R}$.

Suppose

(13) $\underline{R} \geqq R(t)$,

then, since in (12), $\int_{\underline{R}}^{\infty} Rg(R)dR = \int_0^{\infty} Rg(R)dR$, and $\int_{\underline{R}}^{\infty} g(R)dR = \int_0^{\infty} g(R)dR = 1$,

(14) $R(t) = (1-\alpha)tE(R)/(1-\alpha t)$

where $E(R) = \int_0^{\infty} RgdR$, and, $0<t<1$,

(15) $R(t)^{-1}\partial R(t)/\partial t = \{(1-\alpha t)t\}^{-1} > 0$, $\partial R(t)/\partial t = (1-\alpha)E(R)/(1-\alpha t)^2 > 0$,

Let $\underline{t}$ be such that $R(\underline{t}) = \underline{R}$. Then,

(16) $\underline{t} = \underline{R}/\{\alpha\underline{R}+(1-\alpha)E(R)\}$,

which is less than or equal to $R^i/\{\alpha R^i+(1-\alpha)E(R)\}$.

From (3") & (13), for any $t \leqq \underline{t}$,

(17) $L^i(t) = \alpha\{R^i - R(t)\} \geqq 0$, for all i,

with the equality only when $R^i = \underline{R}$ and $t = \underline{t}$.

Suppose, instead,

(18) $\underline{R} \leqq R(t) \leqq \bar{R}$.

Then, from (3') and (12),

(19) $\int_{R(t)} RgdR/\int_{R(t)} gdR \geqq R(t) > (1-\alpha)tE(R)/(1-\alpha t)$, $\underline{t} \leqq t < 1$.

Let $\bar{t}$ be such that $R(\bar{t}) = \bar{R}$, then, $L^i(\bar{t}) = 0$ and $\int L(\bar{t})f(L(\bar{t}))dL(\bar{t}) = 0$, and,

(20) $\bar{t} = \bar{R}/\{\bar{R}+(1-\alpha)\{\int_{\bar{R}} RgdR - \bar{R}\int_{\bar{R}} gdR\}$

which is less than or equal to 1, since from (19)

(21) $\bar{R}-\varepsilon \leqq \int_{\bar{R}-\varepsilon}^{\bar{R}} RgdR/\int_{\bar{R}-\varepsilon}^{\bar{R}} gdR$ $\varepsilon>0$,

By definition, $\bar{R} \geqq \int_{R(t)} RgdR/\int_{R(t)} gdR$, $\underline{t} \leqq t < 1$, hence

(22) $\bar{t} = 1$,

or equivalently,

(23) $R(\bar{t}) = R(1) = \bar{R}$.

Thus, 100% rate is the smallest tax rate at which individuals with the highest productivity $\bar{R}$ will not work, whatever his productivity $\bar{R}$ may be. No one will work if and only if the tax rate is 100%.

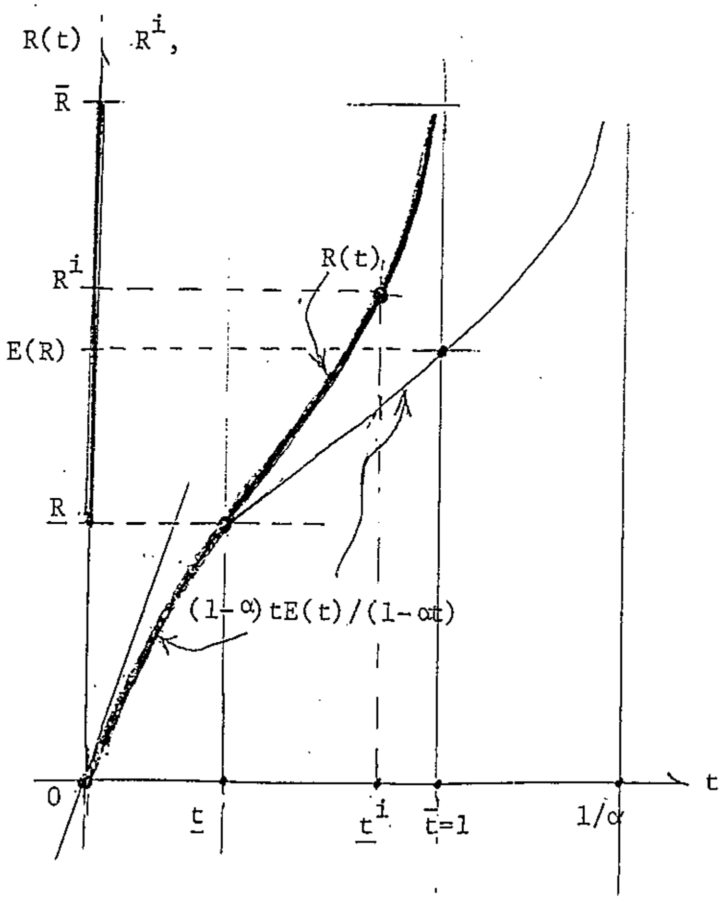
Differentiating (12) with respect to t and substituting from (12), we can have,

(24) $\partial R(t)/\partial t/R(t) = 1/t + \{1-(1-\alpha)\int_{R(t)}^{\bar{R}} g(R)dR\}R(t)/(1-\alpha)t\int_{R(t)} Rg(R)dR$

which is positive for any t, and may be reduced to (15) for $t \leq \underline{t}$.

Now, by collecting the results thus obtained above through (13)-(24), we are able to illustrate how R(t) varies with the tax rate t. Figure 2 below shows a one-to-one correspondence of the tax rate $\underline{t}^i$ and the productivity R^i.

Figure 2 Variation of R(t) in t.



2.2 How Labor Supplies and Incomes, Both Individual and Aggregate (Average), and Tax Revenues Change in Tax Rate ?

Now, we can examine how $L^i(t)$, $\int L(t)f(L(t))dL(t)$, $I^i(t)$, $\int I(t)h(I(t))dI(t)$, and, $t.\int L(t)f(L(t))dL(t)$, will vary as t varies.

2.2.1 Labor Supplies, Both Individual and Aggregate, Will Decrease as Tax Rate t increases.

Differentiate both hand sides of (6')-(8') and (3"), then, we can have;

(25) $\partial L^i(t)/\partial t/L^i(t) = -\partial R(t)/\partial t/\{R^i-R(t)\}$,

where $L^i(0)=\alpha R^i$, $L^i(\underline{t}^i)=L^i(t)=0$ for any $t \geq \underline{t}^i$, $\partial L^i(t)/\partial t< 0$ for every $t \leq \underline{t}^i$, and $\partial L^i(0)/\partial t=-(1-\alpha)R^i<0$ and $\partial L^i(\underline{t}^i)/\partial t/L^i(\underline{t}^i)=-\infty$.

Let $E(L(t))=\int_0^\infty L(t)f(L(t))dL$, then, from (6'), we can have;

(26) $\partial E(L(t))/\partial t/E(L(t)) =\partial R(t)/\partial t/R(t)- t(1-t)^{-1}$,

where note $E(L(0))=\alpha E(R)$, $E(L(1))=0$ from (23), $\partial E(L(0))/\partial t =-\alpha(1-\alpha)E(R)<0$. In view of (25), in which $\partial L^i(t)/\partial t/L^i(t)< 0$ for all $t< \underline{t}^i$, there is a good reason why $\partial E(L(t))/\partial t/E(L(t))< 0$ for all $t \leq 1$.

Intuition would tell, moreover, that $E(L(t))$ will decrease just as $L^i(t)$ decreases for every i for $t \leq \underline{t}^i$.

2.2.2 The (Full) Incomes of Individuals of Low Productivities Will Increase with Tax Rate, Attaining Their Highest Values at Different Positive Rates.

From (8'), we may have,

(27) $\partial I^i(t)/\partial t/I^i(t) = -(1-t)^{-1}+\{\partial R(t)/\partial t/R(t)\}\alpha R(t)/\{(1-\alpha)R^i+\alpha R(t)\}$,

where $I^i(0)=R^i$, $I^i(1)=0$, and,

(28) $\partial I^i(0)/\partial t = -\{R^i-\alpha E(R)\} \lesseqgtr 0$ as $R^i \gtreqless E(R)$.

Full income $I^i(t)$ of indibidual i with $R^i > E(R)$, thus, will in the begining decrease as t increases. In fact, we can see $\partial I^i(t)/\partial t \neq 0$ for any $t \leq \underline{t}$, hence it will continue to decrease until t reaches $\underline{t}$. On the other hand, the income of individual of $R^i < E(R)$ will increase in the begining, then reach a maximum value, then it will eventually decrease to zero.

The existence of income-maximizng tax rate:

We confirm the existence of a tax rate $t^{i\circ}$, at which the income $I^i(t)$ will be maximized. We call this rate the income-maximizing rate of individual i.

Let t† be the largest rate such that $0 \leq t† \leq 1$, $\partial I^i(t†)/\partial t<0$. In fact, such t† exists, because $I^i(0)=R^i$ and $I^i(1)=0$ may imply the existence of $\tilde{t}$ such that $\partial I^i(\tilde{t})/\partial t =-\{I^i(0)-I^i(1)\}< 0$, $0<\tilde{t}<\underline{1}$, hence we can choose $t†=\max \tilde{t}$. Suppose that $R^i<\alpha E(R)$, then, from (28) $\partial I^i(0)/\partial t>0$ and $\partial I^i(t†)/\partial t<0$ implies the existence of $t^{i\circ}$ such that $I^i(t^{i\circ}) \geqq I^i(t)$ for any t such that $0<t<1$, $0<t^{i\circ}<t† \leq 1$.

Let $E(I(t))=\int Ih(I)dI$. We may have

(29) $\partial E(I(t))/\partial t/E(I(t)) = -(1-t)^{-1}+\alpha\partial R(t)/\partial t/\{(1-\alpha)E(R)+\alpha R(t)\}$,

where $E(I(0))=E(R)$, $E(I(1))=0$,

(30) $\partial E(I(0))/\partial t = -(1-\alpha)E(R)<0$,

hence, $E(I(t))$ will decrease as a small t increases.

2.2.3 The Tax Revenues Are Concave and Takes its Maximum at a Certain Positive Rate $t^{\circ\circ}$

Lastly, we shall examine how the tax revenues $tE(L(t))$ will change as t increases. Let $T(t)=tE(L(t))$. Then, $T(0)=T(1)=0$, and,

(31) $\partial T(t)/\partial t/T(t) = -(1-t)^{-1}+\partial R(t)/\partial t/R(t)$,

where for $t\leqq \underline{t}$ so that $R(t)=(1-\alpha)tE(R)/(1-\alpha t)$, $\partial T(0)/\partial t = \alpha E(R)>0$.

The last inequality may imply the existence of $t^{\circ\circ}$ such that $\partial T(t^{\circ\circ})/\partial t=0$ and $T(t^{\circ\circ})\geqq T(t)$. From (12)(24) and (31) we can derive;

(32) $t^{\circ\circ} = 1/\{1+\sqrt{(1-\alpha)\int_R g(R)dR}$,

hence this relation implies;

(32') $1/\{1+\sqrt{1-\alpha}\}\leqq t^{\circ\circ}< 1$.

Observe that $t^{\circ\circ}=1/\{1+\sqrt{1-\alpha}\}$if $t^{\circ\circ}\leqq \underline{t}$.

The second derivative of T will be reduced to

(33) $\partial^2T(t)/\partial t^2/T(t) = -2\partial R(t)/\partial t/(1-t) +\partial^2R(t)/\partial t^2/R(t)$,

hence, for t such that $t\leqq \underline{t}$,

(33') $\partial^2T(t)/\partial t^2/T(t) = -(1-\alpha)/\{t(1-\alpha t)(1-t)\}< 0$.

We assume this is negative for any $t> \underline{t}$. In fact, this holds for a Pareto distribution.

Thus, the tax revenues will increase in t at decreasing rates, attain its maximum, then decreasing at increasing rates.

We shall see this rate $t^{\circ\circ}$ is the largest possible that the government might take as its policy and is optimal to all the voluntarily unemployed.

3. A Negative Income Tax System : An Interpretation

The government policy provides the equal transfer, $T(t)$, for each tax rate t. The average labor income (hence consumption) $E(L(t))$ is established as the breakeven level of labor income (consumption). We shall here reconsider the taxation policy as a whole, as a negative income tax proposal. Figure 3 will depict the NIT proposal for a fixed tax rate t.

3.1 Net Tax Rate and A NIT Proposal

Let τ^i be a net tax rate of individual i. Then, for each t,

$$(34)\quad (1-\tau)L^i(t) = (1-t)L^i(t)+tE(L(t))$$

or, from (6') and (3"),

$$(35)\quad \tau^i = \tau(R^i,t,\alpha) = t - (1-t)/(1-\alpha)\{R^i/R(t)-1\},$$

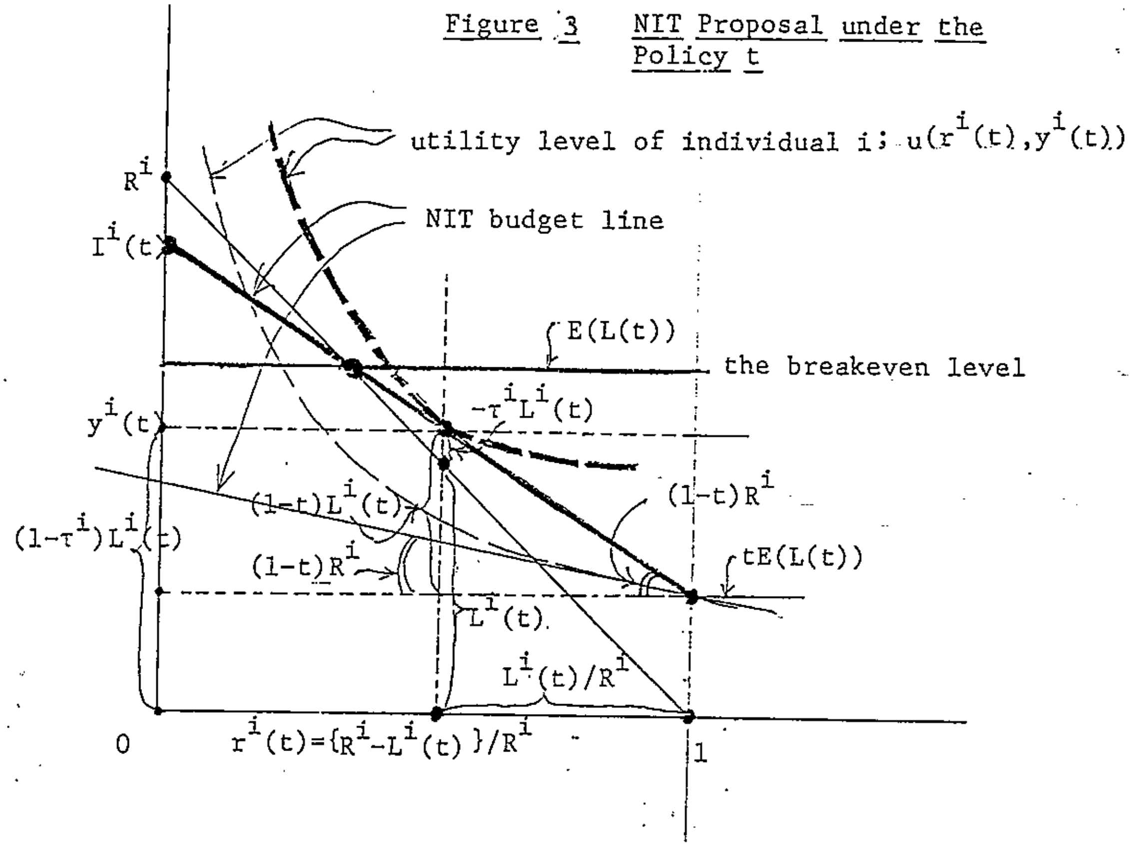
Individual i is a payer of (positive) taxes if τ^i is positive and is a recipient if τ^i is negative. The net rate τ will change together with productivity R^i; $\partial\tau^i/\partial R^i = (1-t)/R(t)\{R^i/R(t)-1\}^2>0$.

Suppose $t \leqq \underline{t}$, that is, $R(t) \leqq R^i$ for any i. Then, from (14), $R(t)=(1-\alpha)tE(R)/(1-\alpha t)$, hence,

$$(36)\quad \tau^i = (1-\alpha t)\{R^i-E(R)\}/(1-\alpha)\{R^i/R(t)-1\}t^2 \gtreqless 0 \text{ as } R^i \gtreqless E(R).$$

Thus, whenever all individuals are working, whether individual i will be a tax payer or a recipient, depends entirely upon whether his productivity R^i is larger or less than the average $E(R)$. Then, what about the case in which $t > \underline{t}$, so that there are some individuals who are not working at all?

Figure 3 NIT Proposal under the Policy t



The RHS of (35) can be reduced generally to the following:

$$(37)\quad \tau^i = t/\{R^i-R(t)\}\{\, R^i-(1-\alpha t)R(t)/(1-\alpha)t \,\}$$
$$=\{I^i(t)/(R^i-R(t)\}\{t - T(t)/y^i(t)\},$$

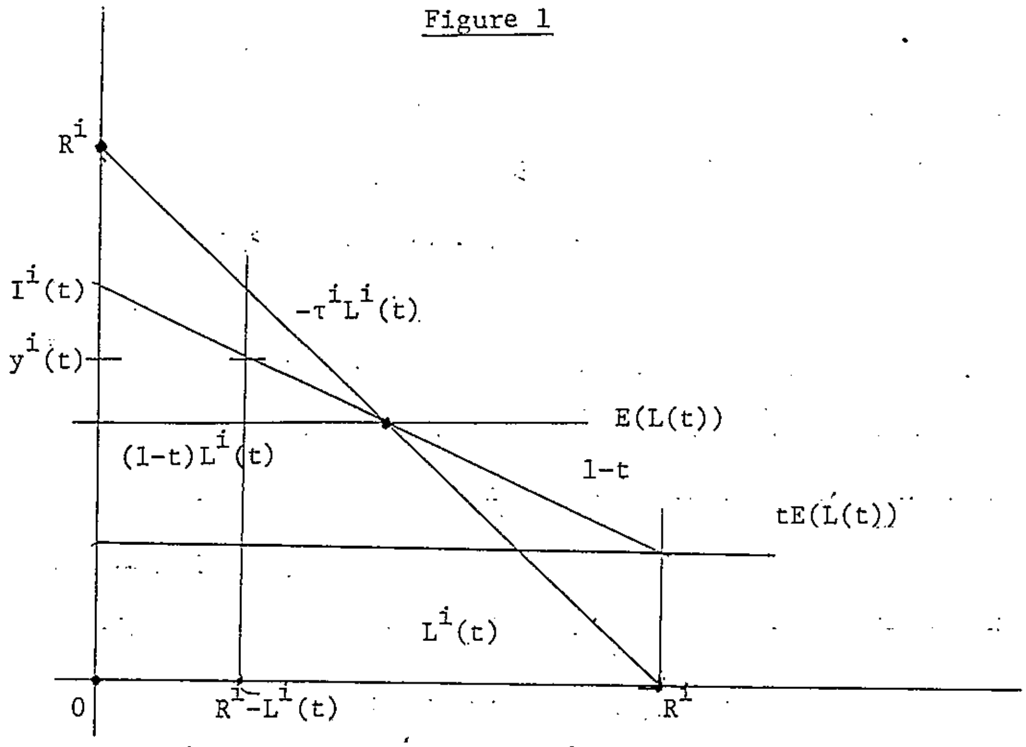
where $T(t)/y^i(t)$ is the ratio of the transfer and consumption of individual i.[*16]

The first one says that τ^i is positive if R^i is larger than $(1-\alpha t)R(t)/(1-\alpha)t$, negative if it is less, and it is zero if they are equal. Note $(1-\alpha t)R(t)/(1-\alpha)t = E(R)$ for $t \leq \underline{t}$, and $(1-\alpha t)R(t)/(1-\alpha)t > E(R)$ for $t > \underline{t}$. Also from (12),

$$(38)\quad E(R) \leq (1-\alpha t)R(t)/(1-\alpha)t \leq \int_{R(t)} Rg(R)dR/\int_{R(t)} g(R)dR,$$

with the equality only when $t \leq \underline{t}$. That is, the LHS of (38) is less than the average productivity of individuals who work at tax rate t, whenever $t > \underline{t}$.

Figure 1



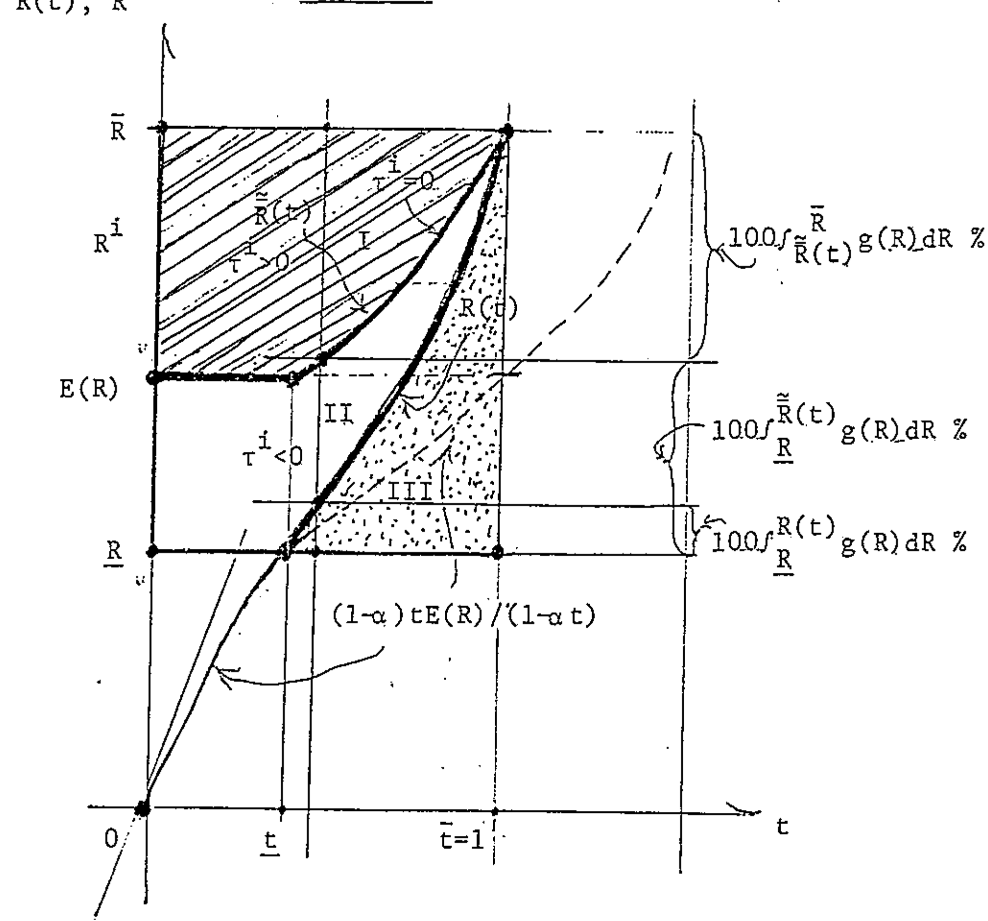
Since from (37) $\tau^i = 0 \leftrightarrow R^i=(1-\alpha t)R(t)/(1-\alpha)t$, we may say $(1-\alpha t)R(t)/(1-\alpha)t$ is the breakeven level of productivity; hereinafter we shall designate this by $\tilde{\tilde{R}}(t)$. Then,

(39) $\tilde{\tilde{R}}(t) > R(t), \quad 0<t<1,$

where $\partial\tilde{\tilde{R}}(t)/\partial t/\tilde{\tilde{R}}(t) = \partial R(t)/\partial t/R(t) - 1/t(1-\alpha t)$, $\tilde{\tilde{R}}(t)=E(R)$ for $t \leq \underline{t}$ and $\tilde{\tilde{R}}(t)=R(1)$.

An economic meaning of $\tilde{\tilde{R}}(t)$ is the productivity of a hypothetical individual whose labor supply is always the average (aggregate) supply, that is, $E(L(t))$. This individual is always neither a taxpayer nor a recipient. Figure 3 below illustrates the NIT interpretation.

Figure 4



In Figure 4, the two solid curves, $R(t)$ and $\tilde{\tilde{R}}(t)$, divides the rectangular area into 3 parts labelled by I, II and III. Individual i of the coordinate (t,R^i) inside Part I will be a tax payer when the policy $(t,T(t))$ is taken. An individual on the $\tilde{\tilde{R}}(t)$ will be neither a tax payer nor a recipient. Individual i of (t, R^i) in Part II will work at any rate but he will be a recipient, whereas, individual i in Part III or on the $R(t)$ curve will be a recipient but won't work at all.

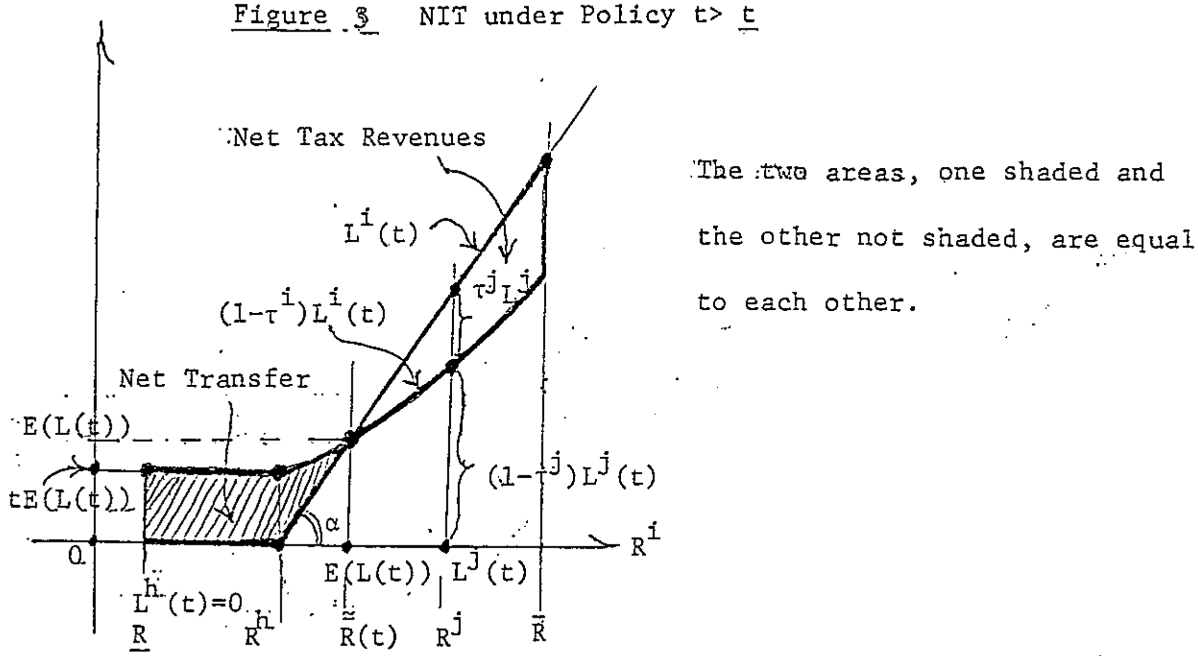
3.2 A Moral Hazard Problem

Under each policy $(t,T(t))$, the lower the productivity endowed to a working individual, the higher the marginal tax rate to the individual, so more likely to choose a less and less degree of employment. In a more inequal society, the taxation policy itself may constitute a source of psychological corrosion for a larger number of workers; see Figure 3.

The higher the tax rate, the larger the number of working individuals will choose a less and less degree of employment. If the tax rate exceeds the critical rate $\underline{t}$, then, $100\int_{\underline{R}}^{R(t)} g(R)dR$ % of workers who are endowed with the lower productivities, will be able to choose "unemployment", because of the transfer income from the government; see Figure 4.

This moral hazard will be aggravated if the government takes the aggregate welfare maximizing policy. We shall examine later this problem in detail.

Figure 3 NIT under Policy t> t



We shall add, before closing this subsection, the following: From (3")&(35), for each t, $\partial L^i(t)/\partial R^i=\alpha$, $\partial(1-\tau^i)L^i(t)/\partial R^i = (1-\tau^i)\alpha+(1-t)L^i(t)/(1-\alpha)R(t)> 0$, $\partial^2(1-\tau^i)L^i(t)/\partial R^{i2}=\alpha(1-t)/(1-\alpha)R(t)> 0$. Hence, the net tax rate $\tau^i(R^i,t,\alpha)$ can be illustrated with $L^i(t)$ as in Figure .

4. Individual and the Aggregate Relative Welfares

Let us compute utility u^i of individual i in terms of his productivity R^i and tax rate t. From (1) (2) and (8'), we can have

$$(40)\quad u(R^i,t) = (1-t)^{\alpha}\alpha^{\alpha}(1-\alpha)^{-\alpha}(R^i)^{\alpha-1}\{(1-\alpha)R^i+\alpha R(t)\} \quad \text{if} \quad R^i \geqq R(t)$$
$$= \{T(t)\}^{\alpha} \qquad\qquad \text{if} \quad R^i \leqq R(t).$$

Define a relative welfare of individual i as the ratio of utilitties $u(R^i,t)$ and $u(R^i,0)$, and denote it by $V(R^i,t)$. Then, from (8') and (40), we may have

$$(41)\quad V^i = V(R^i,t) = (1-t)^{\alpha}\{1+\alpha R(t)/(1-\alpha)R^i\} \quad \text{if} \quad R^i > R(t),$$
$$=\{(1-t)R(t)/R^i\}^{\alpha}/(1-\alpha) \qquad \text{if} \quad R^i \leqq R(t).$$

Note that $V(R^i,0)=1$, $I^i(0) = R^i$ and $V(R^i,1) = 1$, $I^i(1) = 0$.

In view of (40), define the relative welfare V(t) as the ratio of two aggregate utilities, E(u(R,t)) and E(u(R,0)); that is, $E(u(R,t))=\int_{R(t)}u(R,t)g(R)dR + \int_{\underline{R}}^{R(t)}T(t)^{\alpha}g(R)dR$, and, $E(u(R,0))=\int_0^{\infty}u(R,0)g(R)dR$. Then, we can obtain,

$$(42)\quad V(t) = (1-t)^{\alpha}(1-\alpha)^{-1}\{(1-\alpha)\tilde{R}(t)+\alpha R(t)\}/\tilde{R},$$

where

$$(43)\quad \tilde{R}(t) = \tilde{R}[1+\int_{\underline{R}}^{R(t)}\{R(t)^{\alpha}-R^{\alpha}\}g(R)dR/E(R^{\alpha})$$
$$+\{\alpha R(t)/(1-\alpha)\}\int_{\underline{R}}^{R(t)}\{R(t)^{\alpha-1}-R^{\alpha-1}\}g(R)dR/E(R^{\alpha})\ ].$$

Observe that $V(0)=1$, $V(1)=0$, and $\tilde{R}(\underline{t})=\tilde{R}$ $(=E(R^{\alpha})/E(R^{\alpha-1}))$, also,

$$(43')\quad \tilde{R}(t) = \tilde{R}, \quad t<\underline{t}.$$

4.1 Individual Welfare and the Existence of Its Optimal Tax Rate

We shall first confirm how V^i takes its maximum at a certain rate t^{i*}, which we shall term the optimal tax rate of individual i.

Differentiating V^i w.r.t. t, we can obtain

(44) $\partial V^i(t)/\partial t/V^i(t) = \partial T(t)/\partial t/T(t)$ if $t \geqq \underline{t}^i (\leftrightarrow R^i \leqq R(t))$,

and otherwise,

(44') $\partial V^i/\partial t/V^i = (1-\alpha)(1-t)^{-1} + \partial I^i(t)/\partial t/I^i(t)$

$= -\alpha/(1-t) + \{\alpha\partial R(t)/\partial t\}/\{(1-\alpha)R^i + \alpha R(t)\}$

Note that the utility of i will take its maximum at a tax rate equal to or higher than the rate at which the income of i will take its maximum. This is also true for the aggregate.

From (14) and (44'), and since $\underline{t}^i \geqq \underline{t} > 0$,

(45) $\partial V^i(0)/\partial t = -\alpha\{R^i - E(R)\}/R^i \gtreqless 0$ as $R^i \lesseqgtr E(R)$.

Thus, the previous argument, extended in Subsection 2.2.2-3, may apply here to show the existence of t^{i*} in particular in case $R^i < E(R)$. There should be a good reason for V^i to take its maximum at $t^{i*}=0$ in case $R^i \geqq E(R)$, but we would like to specify this argument.

4.2 <u>The Existence of a Tax Rate Maximizing the Aggregate Welfare</u>

A differentiation of V(t) with respect to t will give,

(46) $\partial V(t)/\partial t/V(t) = -\alpha/(1-t)$

$+\{(1-\alpha)\partial\tilde{R}(t)/\partial t + \alpha\partial R(t)/\partial t\} / \{(1-\alpha)\tilde{R}(t) + \alpha R(t)\}$,

where

(47) $\partial\tilde{R}(t)/\partial t = -\{\alpha\tilde{R}/(1-\alpha)\}\{\partial R(t)/\partial t/E(R^\alpha)\}\int_{\underline{R}}^{R(t)}\{R^{\alpha-1} - R(t)^{\alpha-1}\}g(R)dR$

which is negative if $R(t) > R^i$ for some i and zero otherwise. Hence,

(47') $\tilde{R}(t) < \tilde{R}$ $t > \underline{t}$.

We are able to show, if $\tilde{R} < E(R)$ and together with (29), that $\partial V(t)/\partial t > 0$ for such a t that $\partial E(I(t)/\partial t = 0$.

Since,

(49) $\partial V(0)/\partial t = -\alpha\{1-E(R)/\tilde{R}\} \gtreqless 0$ as $\tilde{R} \lesseqgtr E(R)$,

and from (43), the existence of the tax rate t*, at which $\partial V(t^*)/\partial t = 0$, will be confirmed if the argument made in section 2.2-3 applies.

Thus, we assume throughout the present analysis,

(50) $\tilde{R} < E(R)$.

In fact, this holds in either case of Pareto and Gibrat distribution.*17

5. <u>Inequality of Productivities and the Various Tax Rates</u>

We shall here examine how closely the various rates, such as t^*, t^{i*}, $t^{i\circ}$ and $t^{\circ\circ}$ etc., are related with a certain extent of the inequality, which will be described in (58). Eventually we shall characterize those rates in terms of the given productivities and the relative degree of their inequality.

In the large class of distributions, we shall first confirm the intuition is correct, that the orders of t^{i*}'s in the unit interval of tax rates inversely follow that of productivities, R^i's, in the interval $[\underline{R}, \bar{R}]$. First, note that $t^{i*}=0$ if $R^i \geqq E(R)$ and $t^{i*}=t^{\circ\circ}$ if $R^i \leqq R(t^{\circ\circ})$. Both are supported by the inverse (unique) relationship between t^{i*} and R^i.

From (44) for the case $R^i > \underline{R}(t)$, immediately, $\partial V^i(t^{i*})/\partial t = 0$ $\leftrightarrow 1/(1-t^{i*}) = \partial R(t^{i*})/\partial t/\{(1-\alpha)R^i + \alpha R(t^{i*})\}$, hence, we can have, $\partial t^{i*}/\partial R^i < 0 \leftrightarrow (1-t^{i*})\partial^2 R(t^{i*})/\partial t^2 < (1+\alpha)\partial R(t^{i*})/\partial t$, after a calculation. Similarly for $t^{i\circ}$ from (27) to obtain

$\partial t^{i\circ}/\partial R^i < 0 \leftrightarrow (1-t^{i\circ})\partial^2 R(t^{i\circ})/\partial t^2 < 2\partial R(t^{i\circ})/\partial t$.

In case $t \leq \underline{t}$, from (14), $R(t)=(1-\alpha)tE(R)/(1-\alpha t)$, hence, $\partial^2R(t)/\partial t^2 / \partial R(t)/\partial t = 2\alpha/(1-\alpha t) < (1+\alpha)/(1-t)$. Thus, the condition for the inverse correspondence holds for this case. So does for $t^{i\circ}$. For the case in which $t > \underline{t}$, however, that does not always hold. For a Pareto distribution, where β is the Pareto index;*18

(51) $\partial^2R(t)/\partial t^2/\partial R(t)/\partial t = -\{(\beta-1)-2\beta t\}/\beta(1-t)t < (1+\alpha)/(1-t) \lesseqgtr 0$

as $t \lesseqgtr (\beta-1)/\beta(1-\alpha)$. Hence, if $\beta \leq 1+(1-\alpha)$, $\partial^2R(t)/\partial t^2/\partial R(t)/\partial t \geq (1+\alpha)/(1-t)$ for t such that $(\beta-1)/\beta(1-\alpha) < t \leq 1/\beta$, $\partial t^{i*}/\partial R^i > 0$.

Except in case otherwise stated, we assume the uniqueness. Under the assumption, we shall examine how those rates are ordered relatively to the incentive losing rate $\underline{t}^i$, in particular to the smallest one $\underline{t}$.

5.1 Where the Individually Optimal Rates, t^{i*}'s, are Situated Relatively to the Critical Rate $\underline{t}$?

We show first that t^{i*} is larger than $\underline{t}$, if $R^i/E(R) < (1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}$. Let $\phi(t) = \alpha^2t^2 - [\{2\alpha R^i-(1+\alpha)E(R)\}/\{R^i-E(R)\}]t + 1$. Then, $\phi(t)=0 \leftrightarrow \partial V^i(t)/\partial t/V^i(t) = 0$, if $t \leq \underline{t}$. Note $\phi(0)=1, \phi(1)<0$. Then, $\phi(\underline{t}^i) = [\alpha(1-\alpha)E(R)/\{E(R)-R^i\}\{\alpha R^i+(1-\alpha)E(R)\}^2]\psi(R^i) \geq 0$ if $\psi(R^i) \geq 0$, where $\psi(R^i) = -\alpha R^{i2} - 2(1-\alpha)E(R)R^i + (1-\alpha)E(R)$, $\psi(E(R)) = -E(R)$, and $\psi(R^i) \geq 0$ if $R^i \leq [(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}]E(R)$. Also note $\partial\phi(t)/\partial t < 0$.

Thus, suppose $\underline{R} < [(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}]E(R)$, then, for every R^i such that $\underline{R} \leq R^i \leq [(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}]E(R)$, $\phi(\underline{t}^i) \geq 0$. Here recall $\underline{t}^i$ is the rate at which individual i of R^i will not work at all. Thus, $0 < \underline{t} < t^{i*}$, for any i such that $R^i < [(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}]E(R)$. This argument is of course supported by the fact that such t^{i*} does exist. See Section 2.2 for the existence.

Secondly, what about t^i* for i such that

$$R^i > [(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}]E(R)$$

? $\phi(\underline{t})=0$ for $\underline{t}=\underline{R}/\{\alpha\underline{R}+(1-\alpha)E(R)\}$ will give $\underline{R}\dagger$;

(52) $\underline{R}\dagger = E(R)-[\{\alpha\underline{R}+(1-\alpha)E(R)\}/(1-\alpha)E(R)]\underline{R}$.

That $R^i = \underline{R}\dagger$ is equivalent to that $t^i* = \underline{t}$.

Note $\underline{R}\dagger$ is larger than $\underline{R}$ if and only if $\underline{R}<[(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}]E(R)$.
$\underline{R}\dagger = \underline{R} \leftrightarrow \underline{R} = [(1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}]E(R)$. We specify t^i* by computing $\phi(t^i*)=0$. That is, if $t^i* \leqq \underline{t}$,

(53) $t^i* = \xi(R^i, E(R));\alpha) = \alpha^{-1}[\{R^i-(1+\alpha)E(R)/2\alpha\}/\{R^i-E(R)\}$
$- \sqrt{\{\{R^i-(1+\alpha)E(R)/2\alpha\}/\{R^i-E(R)\}\}^2 - 1}\ ]$.

Let $\theta = R^i-(1+\alpha)E(R)/2\alpha$, then, $\theta > 1$ if $R^i<E(R)$,

(54) $\xi(R^i, E(R); \alpha) = \theta - \sqrt{\theta^2-1} > 0$,

where $\partial\theta/\partial R^i = (1-\theta)/\{R^i - E(R)\} > 0$, hence, $\partial\xi/\partial t = (\partial\xi/\partial R^i)(1-\theta/\sqrt{\theta^2-1}) < 0$,

and, $\xi(0, E(R);\alpha) = \{(1+\alpha)-\sqrt{(1-\alpha)(1+3\alpha)}\}/2\alpha^2 > 0$, $\xi(\lim R^i, E(R);\alpha)=0$

for $\lim R^i = \lim_{R^i\to E(R)} R^i$.

Thirdly, let $\underline{t}^{i\circ}$ be such that $R(\underline{t}^{i\circ})=R^{i\circ}$ and $\partial V^{i\circ}(\underline{t}^{i\circ})/\partial t=0$.
Such a rate $\underline{t}^{i\circ}$ exists. In view of (31), it must be that $\underline{t}^{i\circ}=t^{\circ\circ}$.

From the unique (inverse) correspondence of t^i* to R^i (which is satisfied for any t, $0<t\leqq\underline{t}$, but not for some t, $\underline{t}<t$, therefore assumed) *19
the above argument can be summarized neatly;

(55) $t^i*=0 \leftrightarrow R^i \geqq E(R)$,

(56) $0<t^i*<\underline{t} \leftrightarrow E(R)>R^i>\underline{R}\dagger$,

and

(57) $0<\underline{t}\leqq t^i* \leftrightarrow \underline{R}\dagger \geqq R^i > \underline{R}$.

Observe that the last equivalence (57) holds if and only if

(58) $\underline{R}/E(R) < (1-\alpha)/\{(1-\alpha)+\sqrt{1-\alpha}\}$.

Otherwise, for every i, (55) or (56) holds. We use this $\underline{R}/E(R)$ as the inequality measure in the present analysis.

5.2 Tax Revenues Take Its Maximum at $t^{\circ\circ}$, at Which There are Some Workers Who Will Reserve Employment

We shall see here that the rate $t^{\circ\circ}$ is higher than the critical rate $\underline{t}$, under the condition (58).

Let $\Phi(t)=\alpha t^2-2t+1$. Then, since, if $t^{\circ\circ}\leqq \underline{t}$, $\Phi(t^{\circ\circ})=0 \leftrightarrow t=t^{\circ\circ}$, and since $\Phi(0)=1$, $\Phi(1)=-(1-\alpha)<0$, it follows that $\Phi(\underline{t})>0 \leftrightarrow 0<\underline{R}/E(R) < (1-\alpha)/(1-\alpha+\sqrt{1-\alpha})$. Thus, if $\underline{R}/E(R)<(1-\alpha)/(1-\alpha+\sqrt{1-\alpha})$, then,

(59) $t^{\circ\circ} > \underline{t}$.

From (44) and (44'), we have,

(60) $\partial V^i(t^{\circ\circ})/\partial t/V^i(t^{\circ\circ}) = 0 \quad \text{if } R^i \leqq R(t^{\circ\circ})$,

(60') $\partial V^i(t^{\circ\circ})/\partial t/V^i(t^{\circ\circ}) = -(1-\alpha)\{R^i-R(t^{\circ\circ})\}/(1-t^{\circ\circ})\{\alpha R(t^{\circ\circ})+(1-\alpha)R^i\}$

$$\leqq 0 \quad \text{if } R^i \geqq R(t^{\circ\circ}).$$

We have also from (46),

(61) $\partial V(t^{\circ\circ})/\partial t/V(t^{\circ\circ})=-\alpha\partial R(t^{\circ\circ})/\partial t\}\{\tilde{R}(t^{\circ\circ})-R(t^{\circ\circ})\}/R(t^{\circ\circ})\{(1-\alpha)\tilde{R}(t^{\circ\circ})$

$$+\alpha R(t^{\circ\circ})\}+(1-\alpha)\partial\tilde{R}(t^{\circ\circ})/\partial t/\{(1-\alpha)\tilde{R}(t^{\circ\circ})+\alpha R(t^{\circ\circ})\},$$

which is negative if $\tilde{R}(t^{\circ\circ})\geqq R(t^{\circ\circ})$ from (47).

The intended interpretation for (60'), for example, is that individual i, who is at any rate working at tax rate $t^{\circ\circ}$, will be worse off, whereas, the welfare of the voluntarily unemployed will be maximized at that rate. From (61), $\partial V(t^{\circ\circ})/\partial t< 0$ whenever $\tilde{R}(t^{\circ\circ})\geqq R(t^{\circ\circ})$, hence, the rate $t^{\circ\circ}$ can not be the rate t^*.

5.3 How the Aggregate Welfare Maximizing Tax Rate t* Is Determined ?

An intuition would tell us that, through (55) to (61), the tax rate t*, to be determined by taking (46) to be 0, would be somewhere between 0 and $t^{\circ\circ}$; that is, $0 \leqq t^* \leqq t^{\circ\circ}$, under the assumption (50). This will be easily confirmed.

Suppose $t^* < \underline{t}$, then, $\tilde{R}(t)=\tilde{R}(t^*)=\tilde{R}$ from (43'). Let, in Sub-section 5.1, $\phi(t)=\phi(R^i,t)$. Then, $\phi(R,t^*)=0 \leftrightarrow \partial V(t^*)/\partial t/V(t^*)=0$. Hence, we are able to apply the results (55)&(56) to obtain,

(63) $0 < t^* \leqq \underline{t} \leftrightarrow E(R) > \tilde{R} \geqq \underline{R}\dagger$.

By the unique (inverse) correspondence of t^{i*} to R^i, this t* is uniquely determined. But what about the general case ? By (47) & (47'), R(t) will decrease as t increases when $t > \underline{t}$. To handle the case, define R*(t) to be such that $(1-t)^{-1}=\partial R(t)/\partial t/\{(1-\alpha)R^*(t)+\alpha R(t)\}$ for t such that $0 \leqq t \leqq t^{\circ\circ}$, where $R(t^{\circ\circ}) \geqq R^* \rightarrow R^*(t^{\circ\circ})=R^*$.

Then, the rate t* must satisfy one of the following;

(64) $-\partial\tilde{R}(t^*)/\partial t/\partial R(t^*)/\partial t = \alpha\{R^*(t^*)-\tilde{R}(t^*)\}/\{(1-\alpha)R^*(t^*)+\alpha R(t^*)\} > 0$

(64') $-\partial\tilde{R}(t^{\circ\circ})/\partial t/\partial R(t^{\circ\circ})/\partial t = \alpha\{R^*-\tilde{R}(t^{\circ\circ})\}/\{(1-\alpha)R^*+\alpha R(t^{\circ\circ})\} > 0.$

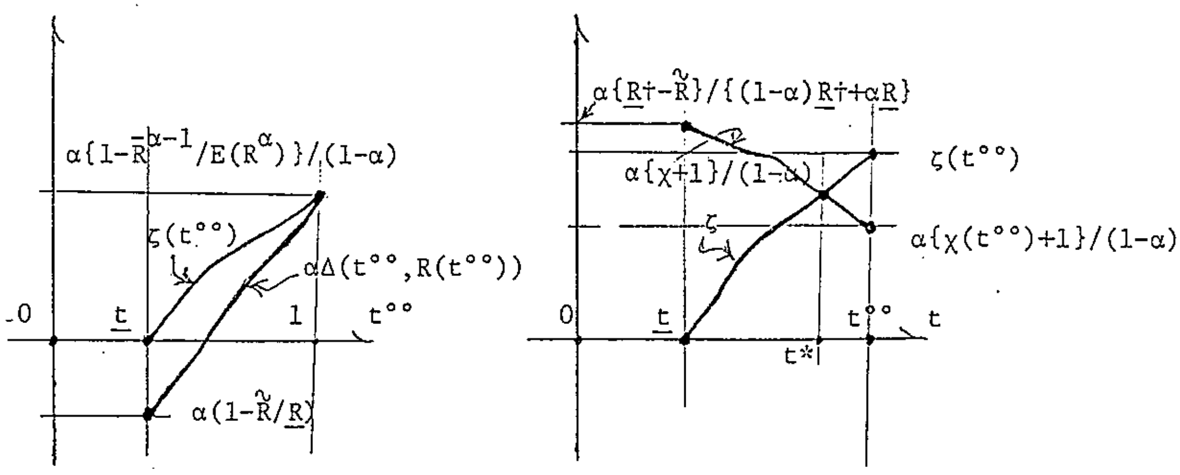
The sign follows from (24)and(47).

Let $\zeta(t)=\{\alpha\tilde{R}/(1-\alpha)E(R^\alpha)\}\int_{\underline{R}}^{R(t)}\{R^{\alpha-1}-R(t)^{\alpha-1}\}g(R)dR$, and, let $\chi(t)= -\{(1-\alpha)\tilde{R}(t)+\alpha R(t)\} / \{(1-\alpha)R^*(t)+\alpha R(t)\}$. Then, for the uniqueness, we shall see it suffices to show that $\partial\zeta(t)/\partial t>0$, and, $\partial\chi(t)/\partial t< 0$.

The rate t* will be determined as a solution to $\{(1-\alpha)/\alpha\}\zeta(t)=1+\chi(t)$. We know already such t* exists. We also know easily that $\zeta(\underline{t})=0$,

$\zeta(t^{\circ\circ})>0$, $\chi(\underline{t})>0$ (otherwise see (63)). To see the value of $\chi(t^{\circ\circ})$, we must extend a tedious calculation. But, if $\tilde{R}(t^{\circ\circ})\geqq \underline{R}$, there exists a R^* such that $\{\alpha/(1-\alpha)\}\ \{\chi(t^{\circ\circ})+1\}\geqq\alpha\{R^*-\tilde{R}(t^{\circ\circ})\}/\{(1-\alpha)R^*+\alpha R(t^{\circ\circ})\}\geqq 0$, $\underline{R}\leqq R^*\leqq R(t^{\circ\circ})$, and (64) or (64') holds. Thus, as in Figure 6, the rate t^* will be uniquely determined. Formally, from (63), $\underline{t}<t^* \leftrightarrow \underline{R}\dagger>\tilde{R}>\underline{R}$. Hence, if $\tilde{R}(t^{\circ\circ})\geqq\underline{R}$, then, $\underline{R}\dagger>\tilde{R}(t^*)\geqq\underline{R}$. Further, in view of (61), $\tilde{R}(t^{\circ\circ})\geqq R(t^{\circ\circ})$ implies $\underline{t}<t^*<t^{\circ\circ}$.

However, the assumption that $\tilde{R}(t)\geqq\underline{R}$ is rather strong. In general, the case $\tilde{R}(t)<\underline{R}$ may be possible. In fact, we shall see that this assumption is redundant. To this end, suppose $t^{\circ\circ}$ is able to take any number between $\underline{t}$ and 1.[*20] Then, $\zeta(1)=\alpha\{1-R^{\alpha-1}/E(R^{\alpha-1})\}/(1-\alpha)$, where $R(1)=\bar{R}$ from (23). Also note that $\Delta(1,\bar{R})=\zeta(1)/\alpha$, where $\Delta(t^{\circ\circ},R(t^{\circ\circ}))=\{\chi(t^{\circ\circ})+1\}/(1-\alpha)$. Because $\alpha\Delta(\underline{t},\underline{R})=\alpha(1-\tilde{R}/\underline{R})$ and[*21] $\alpha\Delta(t^{\circ\circ},R(t^{\circ\circ}))-\zeta(t^{\circ\circ})$ is strictly increasing in $t^{\circ\circ}$, the assumption that $\tilde{R}>\underline{R}$ will imply the uniqueness of t^* such that $t^*<t^{\circ\circ}$. See Figure 6 a & b.



a Figure 6 b

Formally, we have

(65) $\underline{t} < t^* < t^{\circ\circ} \leftrightarrow \underline{R} < \tilde{R} < \underline{R}\dagger \leftrightarrow \tilde{R}(t^*) < \underline{R}\dagger$,

where the last one implies the first one. Observe that t^* will approach $t^{\circ\circ}$ as $\tilde{R}$ approaches $\underline{R}$. Cf–the case of Pareto distribution. Lastly, we must show

(66) $\partial\zeta(t)/\partial t = \{\alpha\tilde{R}/E(R^{\alpha})\}R(t)^{\alpha-2}\int_{\underline{R}}^{R(t)} g(R)\,dR > 0$,

and,

(67) $\partial\chi(t)/\partial t = (1-\alpha)/\{(1-\alpha)R^*(t)+\alpha R(t)\}\partial R^*(t)/\partial t < 0$,

where the sign follows from the fact that $\partial R^*(t)/\partial t < 0$.*22

5.4 The Degree of Inequality: Alternative Measures

In view of the analysis and results in Subsection 5.1 through 5.3, the degree could be representable by the ratio, $E(R)/\underline{R}$. In fact, the ratio was made use of, explcitly or implicitly, as an inequality measure. (58) and (59) are also equivalent to $\underline{t} \leq 1/\{1+\sqrt{1-\alpha}\}$. From (32),

(68) $\underline{t} < 1/\{1+\sqrt{1-\alpha}\} < t^{\circ\circ} < 1$;

hence, from (32'),

(69) $\underline{t} \geq 1/\{1+\sqrt{1-\alpha}\} = t^{\circ\circ}$.

Thus, the inverse of the smallest incentive-losing rate, $1/\underline{t}$, is able to be another measure of inequality, completely substitutable for $E(R)/\underline{R}$. See the case of Pareto distribution in Section 6. The higher the degree of inequality, the smaller the incentive losing rate.

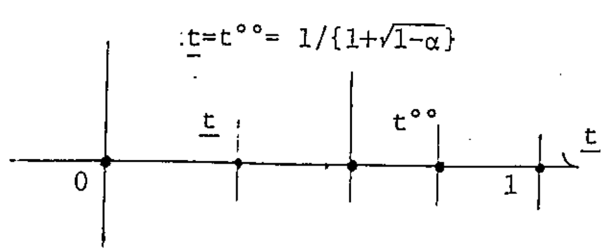


Figure 7

Observe that the smaller the rate $\underline{t}$, the larger the number of the unemployed $(\int_{\underline{R}}^{R(t^{\circ\circ})} g(R)dR)$, hence the smaller the number of working individuals $(\int_{R(t^{\circ\circ})}^{\bar{R}} g(R)dR)$, hence the higher the rate $t^{\circ\circ}$. Thus, the higher the degree of inequality, the larger the tax rate $t^{\circ\circ}$ which maximizes the tax revenues. This rate $t^{\circ\circ}$ also may serve as a useful index of inequality. In fact, $t^{\circ\circ}=1/\beta$ for the Pareto index β.

5.5 How the Income Maximizing Tax Rates, $t^{i\circ}$'s, Are Ordered ?

From (2), the income $I^i(t)$, consumption $y^i(t)$, and labor income plus transfer $(1-t)L^i(t)+T(t)$, of a working individual i change exactly in the same way when t changes. Also recall the income $I^i(t)$ of each working individual i attains its maximum at the rate $t^{i\circ}$ lower than his welfare maximizing rate t^{i*}; whereas, that of the unemployed at the revenues maximizing rate $t^{\circ\circ}$. See (44)&(44') for this.

Take $\phi^\circ(t)=\alpha^2t^2-2\alpha t+\eta$, where $\eta=\{R^i-\alpha E(R)\}/\{R^i-E(R)\}$, and let $R\dagger=\alpha\underline{R}\dagger$. Then, the argument extended in Subsection 5.1 may apply here if we take $\phi^\circ(t)$ and $R\dagger$, instead of $\phi(t)$ and $\underline{R}\dagger$, respectively. The results may be summarized as follows:

(70) $R\dagger > \underline{R} \leftrightarrow \underline{R}/E(R) < 1/\{(1+1/\alpha)/2+\sqrt{\{(1+1/\alpha)/2\}^2+\alpha/(1-\alpha)}\}$.

Note the right hand side of the inequality is less than $1/\{1+(\sqrt{1-\alpha})^{-1}\}$.

(71) $t^{i\circ} = 0 \leftrightarrow R^i > \max\{\alpha E(R),R\}$,

(72) $0<t^{i\circ}< \underline{t} \leftrightarrow \alpha E(R) > R^i > R\dagger$,

and

(73) $0< \underline{t}< t^{i\circ} \leftrightarrow R^i \leqq R\dagger$.

The relations (55)-(57) may be combined with (71)-(73) to obtain; for example, $0<\underline{t}\leq t^{i\circ}<t^{i}*\leq t^{\circ\circ} \leftrightarrow \underline{R} \leq R^{i} \leq R\dagger$, implying that both the income and welfare of individual i whose productivity is lower than $R\dagger$ will take their maximum values at tax rates $t^{i\circ}$ and $t^{i}*$, respectively, at which some individuals whose productivities are less than $R(t^{i}*)$ won't work at all.

6. Pareto Distribution, the Pareto Index and Taxation Policy: Application

We shall apply first the results thus far obtained to the case of a Pareto distribution and then extend them in a direction towards incorporating the non-uniqueness case. See Appendix for the Pareto case.

6.1 Pareto Index, Social Productivity and the Unique Tax Rate t*

For the Pareto index $\beta>1$, we have

$E(R)/\underline{R}=\beta/(\beta-1)$, $\tilde{R}=E(R^{\alpha})/E(R^{\alpha-1})=(1+\beta-\alpha)/(\beta-\alpha)$, where $E(R^{\alpha})/\underline{R}^{\alpha}=\beta/(\beta-\alpha)$ and $E(R^{\alpha-1})/\underline{R}^{\alpha-1}=\beta/(1+\beta-\alpha)$. (52) can be expressed in terms of (β,α) and hence,

(74) $(\tilde{R}-\underline{R}\dagger)/\underline{R} = \{(\beta^2-2\beta+\alpha)(\beta-\alpha)+\beta(\beta-1)(1-\alpha)\}/(\beta-\alpha)(1-\alpha)(\beta-1)^2.$

Let us denote by $\gamma(\beta)$ the numerator of the RHS of (74), then, $\gamma(1)=-(1-\gamma)<0$, $\gamma(1+\sqrt{1-\alpha})>0$, by continuity, there exists $\underline{\beta}\dagger$ for which $\gamma(\underline{\beta})=0$ and $1<\underline{\beta}\dagger<1+\sqrt{1-\alpha}$. Hence,

(75) $\tilde{R} \gtreqless \underline{R}\dagger$ as $\beta \gtreqless \underline{\beta}\dagger$.

Similarly,

(7.6) $\tilde{R}-R\dagger \gtreqless 0$ as $\beta \gtreqless \beta\dagger$,

where $\delta(\beta\dagger)=0$, $\delta(\beta)=1+1/(\beta-\alpha)+\alpha^2(\beta-1)/(1-\alpha)\beta+\alpha(\beta-2)/(\beta-1)$, $\delta(1)<0$, $\delta(1+\sqrt{1-\alpha})>0$, $1<\beta\dagger<1+\sqrt{1-\alpha}$.

Let $\tilde{t}^{\circ}$ be such that $\partial I^{i}(\tilde{t}^{\circ})/\partial t/I^{i}(\tilde{t}^{\circ}) = 0$ for $R^{i}=\tilde{R}$.

Then, since $\underline{\beta}\dagger>\beta\dagger$, it follows that

(77) $\quad \beta > \underline{\beta}\dagger \quad \leftrightarrow \quad \underline{R}\dagger<\tilde{R}< E(R) \leftrightarrow \quad 0 = \tilde{t}^{\circ}< t^{*}< \underline{t},$

(78) $\quad \underline{\beta}\dagger\geqq\beta>\beta\dagger \quad \leftrightarrow \quad R\dagger<\tilde{R}\leqq \underline{R}\dagger \quad \leftrightarrow \quad 0 <\tilde{t}^{\circ}< \underline{t} \leqq t^{*}$

(79) $\quad \beta\dagger\geqq\beta \quad \leftrightarrow \quad \underline{R} <\tilde{R}\leqq R\dagger \quad \leftrightarrow \quad 0 <\underline{t} \leqq \tilde{t}^{\circ}< t^{*}.$

We shall confirm the argument extended in Subsection 5.3 for the case of Pareto distribution. To see further

(80) $\quad \underline{R}\dagger>\tilde{R}(t^{*}) \leftrightarrow \underline{R}<\tilde{R}<\underline{R}\dagger \quad \leftrightarrow \underline{t}<t^{*}\leqq t^{\circ\circ},$

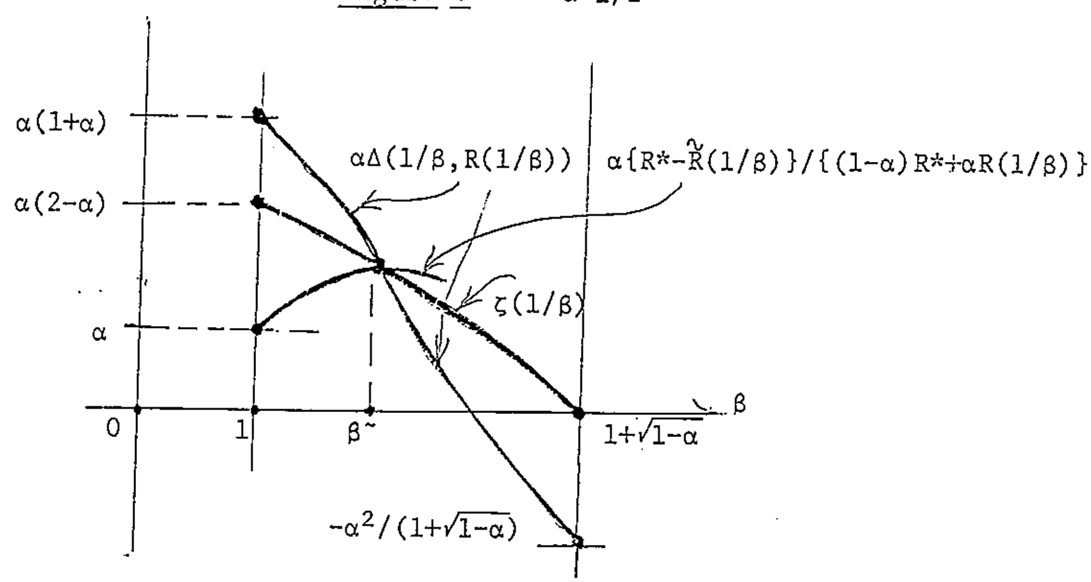
We must extend a tedious calculation;

Let $\Delta(t,R(t))=\{R(t)-\tilde{R}(t)\}/R(t)$. For a Pareto distribution, $\Delta(t^{\circ\circ}, R(t^{\circ\circ}))=1-\tilde{R}(t^{\circ\circ})/R(t^{\circ\circ})= 1+\alpha/\beta+\alpha(R(t)/\underline{R})^{\alpha-\beta-1}/\beta(\beta-\alpha)-\tilde{R}(1+\alpha/\beta)/\underline{R}(R(t)/\underline{R})$, where $\tilde{R}=1+1/(\beta-\alpha)$. On the other hand, we may have; $\zeta(t^{\circ\circ})=\{\alpha\tilde{R}/\underline{R}/E(R^{\alpha})/\underline{R}^{\alpha}\}\{1-(R(t)/\underline{R})^{-(\beta-\alpha+1)/\beta}\}$ at $t=t^{\circ\circ}=1/\beta$. Note that $R(t^{\circ\circ})/\underline{R}=\{(1-\alpha)/(\beta-1)^{2}\}^{1/\beta}>1$ if $\beta<1+\sqrt{1-\alpha}$. We like to see how the values, $\Delta(1/\beta,R(1/\beta))$ and $\zeta(1/\beta)$, will change with β, where $1<\beta<1+\sqrt{1-\alpha}$. $\lim_{\beta\to 1}\alpha\Delta(1/\beta,R(1/\beta))=\alpha(1+\alpha)$, $\lim_{\beta\to 1+\sqrt{1-\alpha}}\Delta(1/\beta,R(1/\beta))=-\alpha^{2}/(1+\sqrt{1-\alpha})$ Also we have, $\lim_{\beta\to 1}\zeta(1/\beta)=\alpha(2-\alpha)$, $\lim_{\beta\to 1+\sqrt{1-\alpha}}\zeta(1/\beta)=0$. Since both Δ and ζ are decreasing in β, they will have an intersection if $\alpha\Delta(.)>\zeta(.)$ at $\beta=1$. Otherwise $\alpha\Delta(.)<\zeta(.)$ for any β such that $1<\beta<1+\sqrt{1-\alpha}$. In fact $\alpha\Delta(.)-\zeta(.)$ is decreasing in β since $\partial\{\alpha\Delta(.)-\zeta(.)\}/\partial t= (\alpha\tilde{R}/\underline{R}\beta)(R(t)/\underline{R})^{-2}\{1-(\beta-1)(R(t)/\underline{R})^{-(\beta-\alpha)}+\alpha(R(t)/\underline{R})^{-(\beta-\alpha)}\}>0$ if $(\beta-1)<\sqrt{1-\alpha}$. Thus, two cases are classified, depending upon whether or not $\alpha<1/2$. For the case $\alpha<1/2$, there is a t^{*} such that $t^{*}<t^{\circ\circ}$, $\{\alpha/(1-\alpha)\}\{$

*23

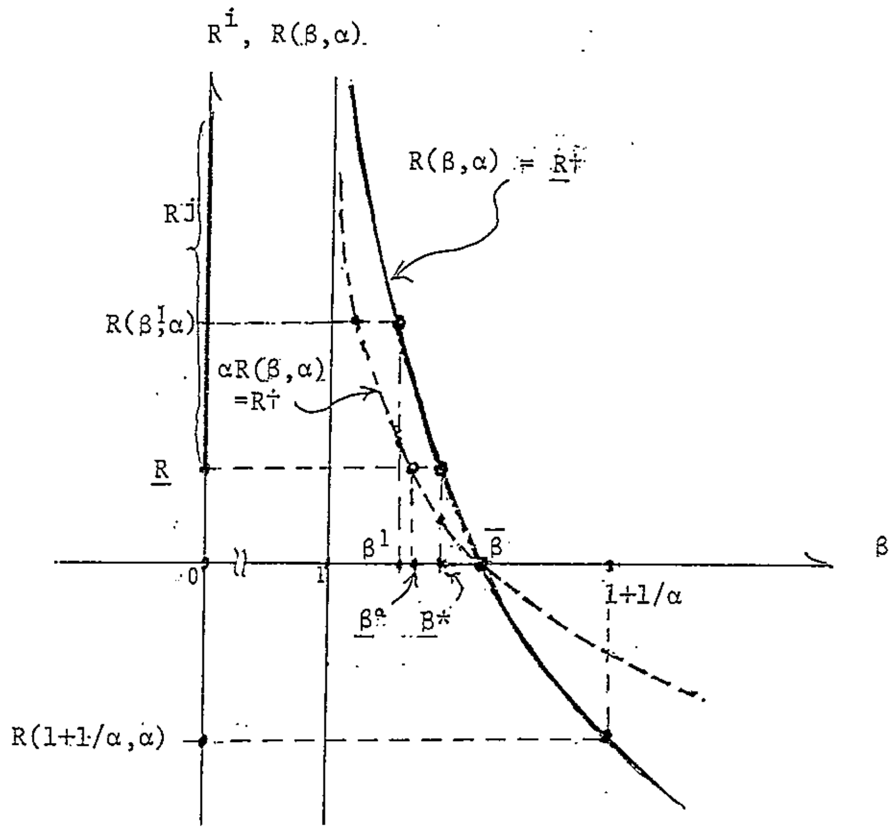
$\chi(t^*)+1\}=\zeta(t^*)$. To see the other case, consider, in view of (64'), the value of $\alpha\{R^*(t^{\circ\circ})-\tilde{R}(t^{\circ\circ})\}/\{(1-\alpha)R^*(t^{\circ\circ})+\alpha R(t^{\circ\circ})\}$ at $t=1$. Since $\underline{R}\leq R^*\leq R(t^{\circ\circ})$, this is larger than or equal to $\alpha\{\underline{R}-\tilde{R}(t^{\circ\circ})\}/\{(1-\alpha)\underline{R}+\alpha R(t^{\circ\circ})\}$ which converges $-\alpha$ as $t^{\circ\circ}\to 1$, because it approaches to $-\tilde{R}(1)/R(1)=-1$. Thus, the value of $\zeta(.)$ coincides with that of $\alpha\{R^*(.)-R(.)\}/\{(1-\alpha)R^*(.)+\alpha R(.)\}$ at some R^*. See the figure below. This means; as $\beta\to 1$, the rate $t^*\to t^{\circ\circ}=1$. Note also that $\partial\{(R^*-\tilde{R}(t)/\{(1-\alpha)R^*+\alpha R(t)\}/\partial t = \{\zeta(t)-\alpha\Delta(t,R(t))\}\partial R(t)/\partial t/\{(1-\alpha)R^*+\alpha R(t)\}\lesseqgtr 0$ as $\zeta(t)\gtreqless\alpha\Delta(t,R(t))$. Hence, it is increasing in β, attaining its largest value at a certain $\beta^{\sim}$, where $\beta^{\sim}$ is such that $\zeta(1/\beta^{\sim})=\alpha\Delta(1/\beta^{\sim},R(1/\beta^{\sim}))$. This means; for any $\beta<\beta^{\sim}$, $\alpha\Delta(1/\beta,R(1/\beta))>\zeta(1/\beta)>\alpha\{R^*-\tilde{R}(1/\beta)\}/\{(1-\alpha)R^*+\alpha R(1/\beta)\}$ and $t^*=t^{\circ\circ}$. If $\beta>\beta^{\sim}$, $\alpha\Delta(1/\beta,R(1/\beta))<\zeta(1/\beta)$ and the unique t^* is less than $t^{\circ\circ}$.

Figure 8 $\quad\alpha<1/2$



From these, $t^{\circ\circ}=1/\beta$, $\underline{t}=(\beta-1)/(\beta-\alpha)$, $R^{i\circ}/\underline{R}=\{(1-\alpha)/(\beta-1)^2\}^{1/\beta}$, etc.

Figure 9



$\underline{\beta}^\circ=\{1+\alpha+\sqrt{(1-\alpha)(1+3\alpha)}\}/2 < \underline{\beta}^*$

Similarly for the income $I^i(t)$, for which $R\dagger=\underline{R}\dagger$, $\alpha R(\beta,\alpha)$, $\alpha\nu(t,\beta,\alpha)$ $\alpha\mu(t,\beta,\alpha)$ etc. are taken instead. The broken curves in Figure 9 & 10 are for the income $I^i(t^{i\circ})$.

It is somewhat surprising to have the case in which $t^* = t^{\circ\circ}$ whenever $1<\beta\leq\beta^{\sim}$. This result is due to relaxing the uniqueness.

(81) $1<\beta\leq\beta^{\sim} \leftrightarrow t^*=t^{\circ\circ}$, and, $\beta^{\sim}<\beta<\underline{\beta}^{\dagger} \leftrightarrow \underline{t} < t^* < t^{\circ\circ}$.

6.2 <u>The Pareto Index and Individually Optimal Tax Rates</u>, t^{i*}.

From (52), we have

(82) $\underline{R}^{\dagger}=[-\{\alpha\beta^2-(1+\alpha)\beta+\alpha\}/(1-\alpha)\beta(\beta-1)]\underline{R}$.

Let $\underline{\beta}^*$ be such that $R(\underline{\beta}^*,\alpha) = \underline{R}$, where $R(\beta,\alpha)=\underline{R}^{\dagger}$. Then, $\underline{\beta}^* = 1+\sqrt{1-\alpha}$. If $\underline{\beta}^* > \beta$, then, $\partial V^i(\underline{t})/\partial t/V^i(t) \lesseqgtr 0$, as $R^i \gtreqless R(\beta,\alpha)$. Otherwise $(\underline{\beta}^* \leq \beta)$ $\partial V^i(\underline{t})/\partial t/V^i(\underline{t}) \leq 0$

Let $\overline{\beta} = \{1+\alpha+\sqrt{(1-\alpha)(1+3\alpha)}\}/2\alpha$, then, $\underline{R}^{\dagger}=R(\overline{\beta},\alpha)=0$, $R(\beta,\alpha)>0$ for $\beta<\overline{\beta}$, $R(\beta,\alpha)\leq 0$ for $\beta\geq\overline{\beta}$. Figure 9 below shows the productivity R^i is related with the index β. For example, the welfare of i of $R^i=R(\beta_i,\alpha)$ will take its maximum at $\underline{t}$, and that of i of $R^i<R(\beta_1,\alpha)$ at $t^*>\underline{t}$.

For this distribution, $R(t)$ can be specified;

(83) $R(t) = (1-\alpha)t\beta\underline{R}/(1-\alpha t)(\beta-1) \qquad t\leq \underline{t}$,

(84) $R(t) =\{(1-\alpha)t/(1-t)(\beta-1)\}^{1/\beta}\underline{R}, \quad t\geq \underline{t}$,

where both are equal at $\underline{t}$. Consequently, for $t\geq\underline{t}$,

(24") $\partial R(t)/\partial t/R(t)=\{\beta t(1-t)\}^{-1}> 0$

and for $t\leq\underline{t}$ we have (15), and note that both are equal at $\underline{t}$, showing the differentiability of $R(t)$ at $\underline{t}$. From (44)&(44'),

(85) $\partial V^i(t^{i*})/\partial t=0 \leftrightarrow R^i/\underline{R}=\nu(t^{i*},\beta,\alpha), \quad t^{i*}\leq\underline{t}$,

where $\nu(t,\beta,\alpha)=\beta\{\alpha t^2-(1+\alpha)t+1\}/(1-\alpha t)^2(\beta-1)$,

(86) $\partial V^i(t^{i*})/\partial t=0 \leftrightarrow R^i/\underline{R}=\mu(t^{i*},\beta,\alpha), \quad t^{\circ\circ}\geq t^{i*}\geq\underline{t}$,

where $\mu(t,\beta,\alpha)=\{(1-\alpha\beta t)/\beta t\}\{(1-\alpha)t/(\beta-1)(1-t)\}^{1/\beta}(1-\alpha)^{-1}$, or,

(87) $\partial V^i(t^{i*})/\partial t=\partial T(t^{\circ\circ})/\partial t=0 \leftrightarrow t^{i*}=t^{\circ\circ}=1/\beta$.

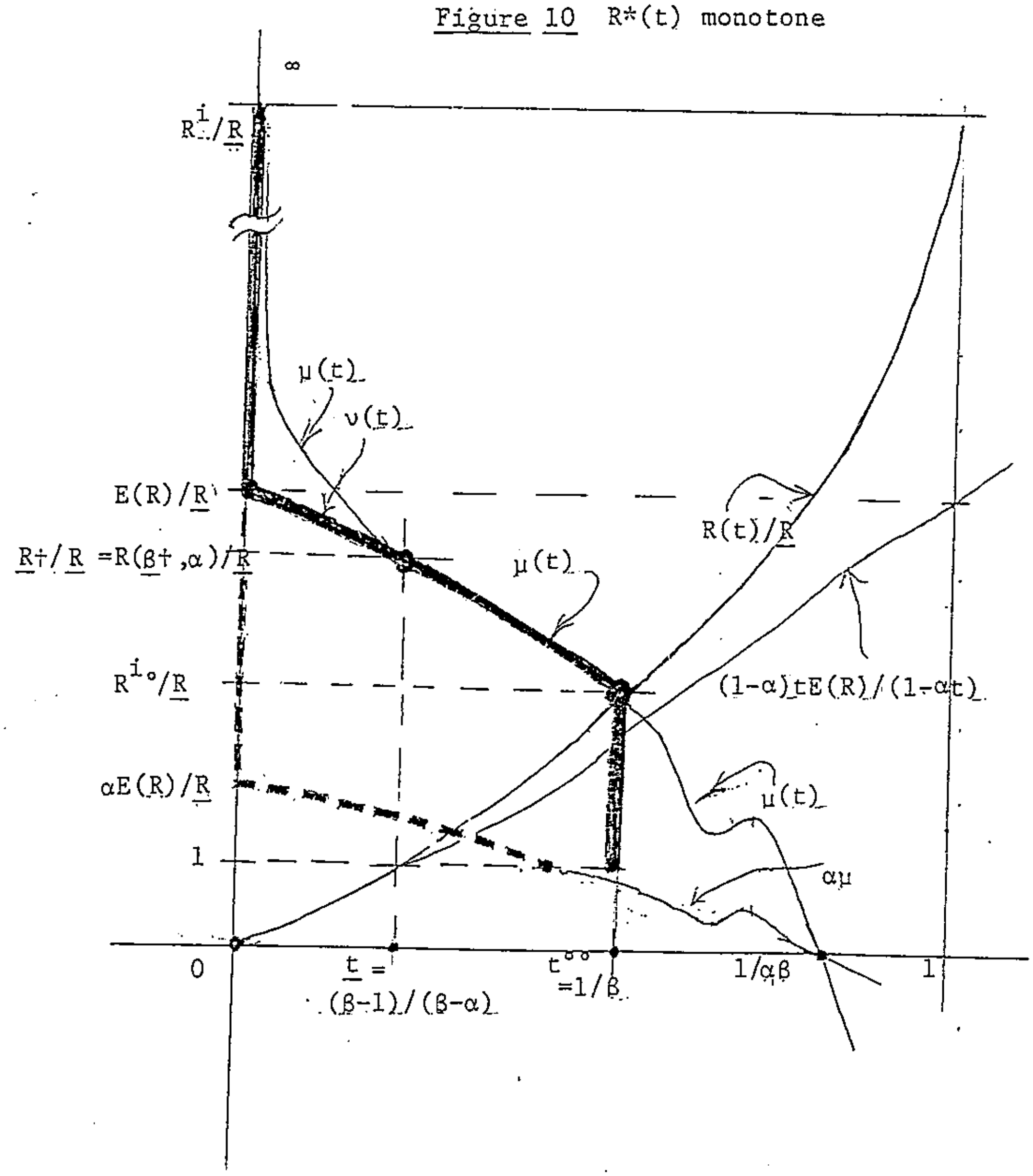
We like to see how ν and μ will change with t. We have, for ν,

$\partial\nu/\partial t < 0$, $\nu(\underline{t})=\mu(\underline{t}) = -\{\alpha\beta^2-(1+\alpha)\beta+\alpha\}/\beta(\beta-1)(1-\alpha) = R(\beta,\alpha)/\underline{R}$, $\nu(0,\beta,\alpha)=\beta/(\beta-1)$, $\nu((1-\sqrt{1-\alpha})/\alpha)=0$ and $\nu(1,\beta,\alpha)=-\beta/(\beta-1)(1-\alpha)<0$.

On the other hand, we have, for μ, $\mu(0,\beta,\alpha)=+\infty$, $\mu(1/\beta,\beta,\alpha)=0$, $\mu(1,\beta,\alpha)=-\infty$, and $\partial\mu/\partial t<0$ if $t<1/\beta$ provided that $2-\alpha<\beta<1+\sqrt{1-\alpha}$.

Thus, we can draw how $R^*(t)$ changes with tax rate t, as in Figure 10 below, where $R^*(t)/\underline{R}$ is given by (85)-(87).

Figure 10 $R^*(t)$ monotone



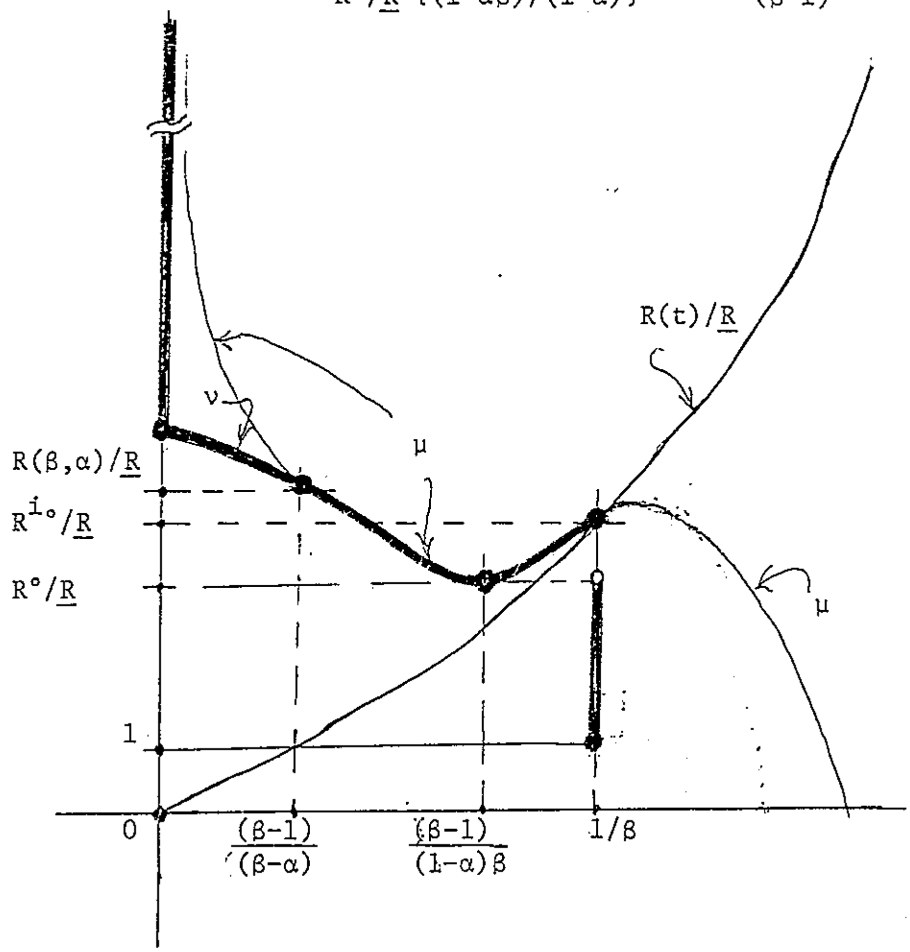
6.3 The Case In Which Individually Optimal Tax Rate Is Not Unique

However, the uniqueness need not be true in case the Pareto index β is smaller. Suppose that $1<\beta\leq 2-\alpha$. Then, $\partial\mu/\partial t \leq 0$ for each t such that $0<t\leq(\beta-1)/\beta(1-\alpha)$, whereas, $\partial\mu/\partial t>0$ for each larger t. Figure 10b below illustrates how $\mu(t,\beta,\alpha)$ will change in t. Here note $\mu(t,\beta,\alpha)>R(t)/\underline{R}$ at every $t<1/\beta$ and at $t=1/\beta$ they are equal.

Figure 10b R*(t) not monotone

$1<\beta<2-\alpha$

$R°/\underline{R}=\{(1-\alpha\beta)/(1-\alpha)\}^{(\beta-1)/\beta}(\beta-1)^{-1}$



We shall see also that there exists a $\beta°$ such that $1<\beta°<2-\alpha$, if and only if $1<\beta\leq\beta° \leftrightarrow E(R)/\underline{R} = \beta/(\beta-1) \leq R^{i°}/\underline{R} = \{(1-\alpha)/(\beta-1)^2\}^{1/\beta}$, and $\beta°<\beta <2-\alpha \leftrightarrow \beta/(\beta-1)>\{(1-\alpha)/(\beta-1)^2\}^{1/\beta}$. Since $\beta°$ satisfies $\{\beta/(\beta-1)\}^{\beta}(\beta-1)^2 =(1-\alpha)$, we like to see how the left hand side will change in β. Let $\nabla(\beta)=\{\beta/(\beta-1)\}^{\beta}(\beta-1)^2$. Then, $\partial\nabla(\beta)/\partial\beta = \beta^{\beta}(\beta-1)^{1-\beta}[\log(\beta/(\beta-1))+(\beta-1)^{-1}] >0$, where $\partial\nabla(1)/\partial\beta=\infty$, $\partial\nabla(2-\alpha)/\partial\beta> 0$, $\nabla(1)=0$, $\nabla(2-\alpha)>1$, and hence such $\beta°$ is uniquely determined, as an intersection of $\nabla(\beta)=(1-\alpha)$.

Collecting the above partial results together, we are able to say what follows: See Figure 10b;

Individual of such a high productivity that $R^i>R^{i°}$, would find a unique rate $t^{i*}\geq 0$ optimal, whereas, individual of productivity R^i such that $R°\leq R^i\leq R^{i°}$ would find two distinct rates optimal. Note individual of the high productivity $R^{i°}$ would find the rate $t°°$ as optimal as well as his incentive losing rate $\underline{t}^{i°}$. Individuals of lower productivities, such that $\underline{R}\leq R^i<R°$, would find the rate $t°°$ optimal, at which they are voluntarily unemployed. Hence, in a very high inequality prevailing, a major part of the population, to each of which income productivity is endowned lower than the average, will find high tax rates (the higher the productivity, the higher the optimal tax rates).[*24] This will be true for the population that contains individuals, to whom even higher than the average productivities are endowned, if the distribution is given in a more inequal way. This should be a good reason why, in a very inequal society, the authority will take a very high tax rate t* under the objective of maximizing the aggregate welfare, which is very close or equal to the rate $t°°$.

7. Concluding Remarks

We have studied, both from welfare and from work-incentive view points, an inequal society, in which the structure of the income distribution is so dissimilar that the redistributive taxation policy may well be justified from the social welfare judgement. A relevant conclusion is that the moral hazard problem, pointed out in Subsection 3.2, will be aggravated in such an inequal structure of the income productivities. See also the last paragraph in Subsection 6.3.

This conclusion would, however, be supported by the fact that the redistributive taxation will not serve as improving the inequality of the income productivities, because the tax revenues, equally transferred to members from the government, are spended only on consumption. Thus, this taxation model would, if well extended, incorporate a general case in which the tax revenues may be spended also on investment to contributing somehow to the improvement and development of the labour productivities, hence changing the income (productivity) distribution which, in the present formulation, is regarded as given externally.

In spite of this reservation pointed out and extension suggested, the analysis undertaken throughout the preceding sections, manifests the latent properties which will always be true in the similar framework of taxation, in so far as the relative difference exists in the productivity of the labor services that member citizens will supply.

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Appendix 1

Pareto Distribution

Pareto's law says, for the number of individuals who possesse at least R, designated N(R),

(1) $N(R) = a R^{-\beta}$; $a>0$, $\beta>1$.

Pareto distribution is said to fit actual distribution well at rather upper levels of incomes, while less well at lower levels. This often has infinite variance and may be suitable for the case where productivities are distributed in a very dissimilar way.

From our assumption,

(2) $N(\underline{R}) = 1$,

where $\underline{R}$ is a lower limit of R's so that $N(\underline{R})=N(R')$ for any $R' \leq \underline{R}$.

(3) $a = \underline{R}^{\beta}$,

and,

(4) $E(R) = \int R|N'(R)|\, dR$,

where $\int$ is $\int_{\underline{R}}^{\infty}$. Also from (1)&(3),

(5) $N'(R) = -\beta\underline{R}^{\beta}R^{-\beta-1}$.

We can now know the value of $E(R^{\alpha-1})E(R)/E(R^{\alpha})$ from this specification;

(6) $E(R) = \{\beta/(\beta-1)\}\underline{R} > \underline{R} > 0$,

(7) $E(R^{\alpha}) = \int R\, (\beta\underline{R}^{\beta}R^{-\beta-1})\, dR$

$$= \{\beta/(\beta-\alpha)\}\underline{R}^{\alpha}$$

Likewise,

(8) $E(R^{\alpha-1}) = \{\beta/(1+\beta-\alpha)(\beta-1)\}\, \underline{R}^{\alpha-1}$.

Appendix 2

Gibrat (Lognormal) Distribution

The density function f is given in the form;

(1) $f(R,\mu,\sigma) = \exp[-\{(\log R - \mu)/\sigma\}^2/2]$

where μ is the average of log R, σ is the variance.

Then, by assumption,

(2) $\int f(R,\mu,\sigma)dR = 1$ where $\int = \int_0^\infty$.

(3) $E(R) = \int Rf(R,\mu,\sigma)dR = \exp(\mu+\sigma^2/2)$,

(4) $E(R^\alpha) = \exp(\alpha\mu+\alpha^2\sigma^2/2)$

(5) $E(R^{\alpha-1}) = \exp\{(\alpha-1)\mu+(\alpha-1)^2\sigma^2/2\}$

hence,

(6) $E(R^{\alpha-1})E(R)/E(R^\alpha) = \exp\{(1-\alpha)\sigma^2\}$,

which is larger than 1 if $\sigma > 0$.

Footnotes

*2 His 1st article may be irrelevant here, though the content is closely related with the optimal taxation problem, from the social welfare viewpoint.

*3 In the following technical analysis extended in later sections, we shall go from general to specific about the productivity distribution. In fact, we shall apply the results to be obtained to the case of Pareto distribution in Section 6.

*4 The term leasure here used may include productive activities, so long as they are outside of the market transaction. Homemaking is an example of leasure often referred to, an activity involving productive services which would be costly at least as much as labor services on the market.

*5 For example, Deaton[1984], in a generalized contex of Atkinson [1973], Atkinson and Stiglitz[1980], calculated the optimal tax rate, which maximizes a social welfare function. The derived formula was given in terms of a measure of the pre-tax inequality the fraction of potential total income required for non-redistributed tax revenues, that for consumer subsistence expenditure and lastly the marginal propensity to spend on leasure.

*6 Under the objective of maximizing the aggregate (average) welfare.

*7 To some of the unemployed, the rate $t^{\circ\circ}$ is not always optimal, if the optimal rate t^{i*} is not unique for R^i.

*8 In a very inequal society, it is possible to have individuals of the upper middle productivity, who think of no taxation as optimal, find also a very high tax rate, close or equal to

$t^{\circ\circ}$ optimal. See the detail in Subsection 6.3.

*9 Hereinafter, we shall some times call $\tilde{R}(t)$ the social productivity for a clear reason.

*10 $1-\alpha$ is the marginal propensity to spend on leasure, which Deaton[1984] called a disincentive factor.

*11 We shall fully discuss this in Subsection 5.3.

*12 The definition of $\underline{R}\dagger$ will appear in the sequel.

*13 Two other measures, substitutable for $E(R)/\underline{R}$, are the rate $t^{\circ\circ}$ and the inverse of rate $\underline{t}$, which will be discussed in Subsection 5.4.

*14 We shall concern ourselves with the welfare (utility) relative to the pre-tax welfare. The levels of individual welfares are, of course, dependent upon tax structure, but, their order follows precisely with that of their productivities.

*15 The central authority is assumed to be able to know the correct value, in advance, of the revenues, for each t, on the basis of information drawn from individuals.

*16 The second equality of (37) gives an interpretation of the net rate in terms of transfer, consumption and tax rate.

*17 See Appendix 2.

*18 See Appendix 1 for β. Observe also that for the Pareto case $\partial^2R(t)/\partial t^2/\partial R(t)/\partial t^2<2/(1-t)$, hence the unique correspondence between $t^{i\circ}$ and R^i is assured to hold for any β such that $1<\beta$.

*19 As referred to before, the assumption need not be true in general, though we derive the results here and in Subsection 5.3 in a more general framework by taking an advantage of the assumption. We shall relax this and extend the case to one incorporating the non-uniqueness case in which a new result will be obtained, in Section 6.

*20 From (32), it follows that $1/\{1+\sqrt{1-\alpha}\} \leqq t^{\circ\circ} < 1$.

*21 $\partial\{\alpha\Delta(t^{\circ\circ},R(t^{\circ\circ}))-\zeta(t^{\circ\circ})\}=\alpha\{-\partial\hat{R}(t^{\circ\circ})/R(t^{\circ\circ})+\tilde{R}\partial R(t^{\circ\circ})/R(t^{\circ\circ})^2\}$
$-\partial\zeta(t^{\circ\circ}) = \tilde{R}\partial R(t^{\circ\circ})\{1-\int_{\underline{R}}^{R(t^{\circ\circ})} R^{\alpha}g(R)\,dR/E(R^{\alpha})\}>0$ from (43) and (47), where the operator ∂ is $\partial/\partial t^{\circ\circ}$.

*22 I conjecture this property that $\partial R^*(t)/\partial t<0$ is not required to obtain an unique tax rate t^*. See Subsection 6.3.

*23 The case in which $\alpha \geqq 1/2$.

*24 That is, the higher the second tax rates, of which they also think as their optimal ones, although their first optima are inversely related with the order of their productivities.