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ON A FOUNDATION OF THE ECONOMIC THEORY OF LOCATION — *Transport Distance vs. Technological Substitution*

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Economic Theory of Location

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1. *Introduction*

1.1. *Historical Notes*

How to locate a plant in relation to given fixed market sites with the aim of minimizing total transportation cost is the fundamental problem in location theory of the firm. The first proposal to come to our attention was the famous physical triangle presentation by Weber *2 [1929]. The economics of spatial location was further developed by Isard [1956]. But it was Moses [1958] who integrated location theory with economic theory of production in a way that attracted the attention of economists as well as urban or regional scientists.

Moses examined implication, *spatial* as well as *technological*, of input substitution in the firm's locational decision. The theory has continued to develop in the Moses production model in the strict sense of incorporating transportation costs of inputs and output in the *neo-classical* theory of production. A critical review and survey of recent contributions was made by Miller and Jensen [1978]. Sakashita [1967], Emerson[1973] Woodward [1973], and Mathur [1979], among others, worked in a *linear* (one-dimensional) transportation space. Bradfield [1967], Khalili, Mathur, and Bodenhorn [1973, 1974], Hurter, Jr., Martinich and Venta [1980], among others, worked in a *triangle* (two-dimensional) space.

In the linear space, a somewhat surprising fact was established:

a cost-minimizing firm would locate its plant at one of the fixed market sites, never at an intermediate (interior) location between them. This result is due exclusively to the *one-dimensional* nature of the space. In that space, technological substitution and concavity of input and output transport costs make the total transportation cost function concave.

In case two or more input markets and one output market are situated at the vertices of the triangle space, the following partial results are obtained.

(a) The firm would never choose an intermediate point on the line joining the output to either of the input markets (e.g., Khalili, et al.; Miller and Jensen).

(b) Being constrained to remain a fixed distance from a market site, the firm would, for any level of output, continue to stay where its cost function takes its minimum value, if and only if the firm's production is homothetic (Khalili et al.). See also Hurter,Jr. et al.

(c) Once the firm chooses a point optimal for a given level of output, the firm will stay there for any other level of output, if and only if production is governed by constant returns to scale (Woodward [1973] in the linear space; Khalili et al. [1974]).

(d) Suppose the production function is homogeneous but with variable returns to scale; then, as the level of output increases, the firm would move toward the output market under increasing returns to scale; and away from the output market under decreasing returns (Khalili et al. [1974], and others).

Most recently, Eswaran, Kanemoto and Ryan [1980] proposed a dual approach to both the one and two-dimensional cases. They extended the results by using an arbitrary number of inputs, and derived additional comparative statics results, examining the effects on the firm's decision of exogenous changes in market prices or transportation rates.

1.2 *A Contribution of the Present Analysis*

However, on the central problem of whether the firm locates at an interior or extreme point of the triangle space, little literature is available (see, for instance, Miller and Jensen [1978]). Instead of explaining the reasons for locating at either an interior or extreme point, previous authors assume the optimality of one location or the other. We wish to answer this basic, but still open, question in a simple triangle space; namely, under what condtitions does the firm find an interior or extreme point optimal ? The rough answer is; if at each market site the total cost is decreasing as the firm moves toward other market site, the firm will find an interior location optimal.

The present analysis thus integrates the spatial property of location and the substitutive input demands of a firm producing a fixed output.

We derive a set of the sufficient conditions for interior location involving the first derivatives of the the cost function at the vertices of the space *only*. This is our present result.

More specifically, we shall argue the following:

On a location decision of the firm:

(i) Account must be taken of two conflicting properties, *spatial* and *technological*. One is the (strict) convexity property of transport distance which will make an interior point optimal. This property will in fact make it impossible for an intermediate location to be optimal between any two *isolated* markets. No intermediate location is chosen by the firm. The other is the input substitution property with respect to location (hence, the strict quasi-concavity of production technology) will attract the firm to locate at one site (vertex) of the substituting input markets.

(ii) Regardless of whether or not input and output transport costs (C.I.F. prices) may be increasing at decreasing rates, the spatial property will overcome the substitution property, if the total cost at every vertex decreases in a direction toward intermediate or interior location. Hence, minimizing the total cost will then make the firm locate its production plant at an interior point, but never at a vertex nor at an intermediate point.

(iii) Another typical but trivial case is one in which the total cost is so partially high at a certain vertex, compared to other vertices that the location may chosen at that vertex in order to save the unfavourable transport cost, or one in which one or two markets are perfectly substituted by a certain input market. Hence, for example, the optimal location will be taken at the market site, at which two (or many) markets are spatially agglomerated.

1.3. *Summaries of Sections*

In section 2 through 3, an economic theory of production is reconsidered[*3] in an equilateral[*4] triangle space of transportation, at every vertex of which at least one market is distributed. A (production) location with its inputs is defined as a pair of distance and direction angle from a fixed (at one vertex) market site. Given ad hoc transport cost rates, the firm's total cost function for a fixed level of output is shown to be a continuous function of the location in the triangle.

Section 3 investigates the continuity property of the total cost function, hence, the existence of an optimal location which minimizes total cost, the effects of a change in location distance on cost, etc. Two conflicting effects in terms of the derivatives are distinguished: (i) the (negative) substitution effect and (ii) the transport cost effect. The transport cost effect is further divided into two parts; the negative or positive effect due to the ad hoc transport cost rates assumption; and the positive effect due to the intrinsic property of transport distance.

Section 4 discusses the spatial property i.e. how the transport distance function takes its global minimum in the interior of the triangle space.

With the results so far obtained, Section 5 proves two theorems, one for non-intermediate location and the other for interior location. Consequently, Theorem 1 implies that an optimal location is possible only at an interior point or a vertex, while, Theorem 2 does that a boundary location is possible only at a vertex.

Section 6 is concluding remarks.

2. *The Abstract Economic Location Problem*

We assume that the output market and a first input market are distributed at the same place (site). Hence we assume in principle that four markets, one output market and three input markets form a triangle distribution, which depicts as in Figure 0, the location problem. At this abstract level, we make the triangle regular (equilateral) without losing any generality.[*4] The whole problem now reduces to: *Where does the firm find its production location F in the space included in the regular triangle, from the point of view of cost minimization?* Let us denote by x the distance between the firm's candidate location and a fixed market site (we shall take the output market), and by α the angle which measures a direction from the output market site towards the firm's location. Then, the candidate location is always shown as the variable pair (x,α) in the space $X \times A$, where $X = [x \in R;\ x \geqq 0]$, and $A = [\alpha \in R;\ 0 \leqq \alpha \leqq 60°]$. For each location (x,α), the input and output transport cost functions, which are exogenously given, specify the C.I.F. prices for output and inputs. Take the output as a numeraire $(\bar{P}_0 = 1)$ and market (input) prices, $\bar{P}_i$, i=1,2,3, as given. Then, for each vector of such prices, the cost minimizing firm will find optimal quantity of inputs for a fixed level, Q, of output. We shall call the location (x,α) of the firm with its optimal inputs $m_i(x,\alpha)$, i=1,2,3 depending on location (x,α), for the output level, the (*production*) *location*. Then, the problem is how to find an optimal location (x^*,α^*) which minimizes total (transportation) cost; $P_0(x,\alpha)Q + \sum_{i=1}^{3} P_i(x,\alpha)\, m_i(x,\alpha)$, of the firm with respect to location (x,α).

Figure 0

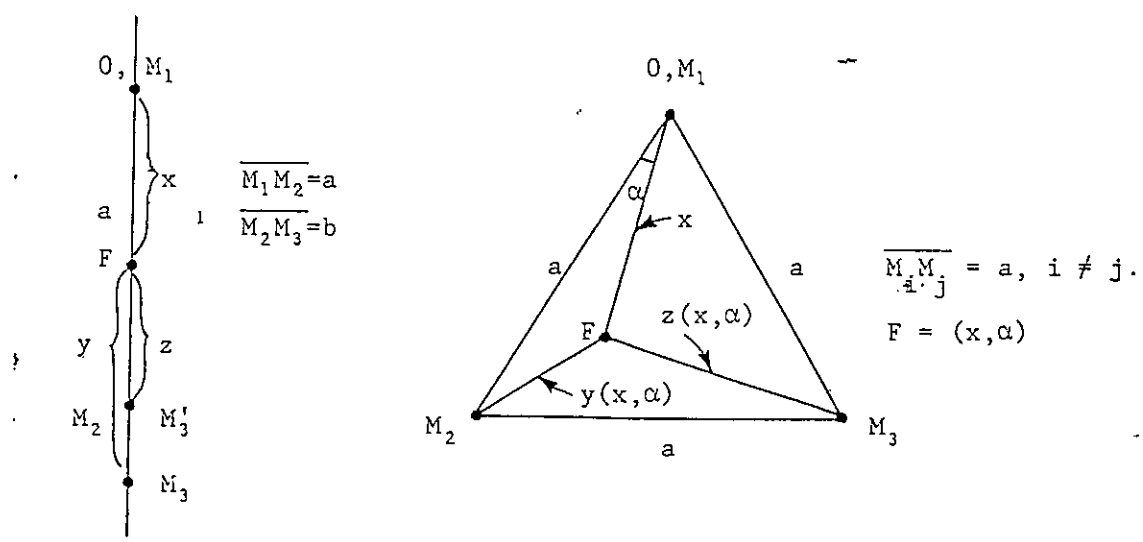


Figure 0 a: (One-Dimensional Space)

Figure 0 b: $\overline{OF} = x$, $\overline{FM_2} = y$, $\overline{FM_3} = z$. (Two-Dimensional Space)

To keep the problem methematically tractable, we specify, in the rest of this section, the assumptions and properies of input and output transport cost functions and production function, mainly those of continuity and differentiability.

2.1. *The Cost Minimizing Inputs at a Location* (x,α)

The cost minimization problem is set out; for a given location (x,α) and for a given output level Q,

$$\text{(P.I.)} \qquad \min_{(m_1,m_2,m_3;\ x,\alpha,Q)} \quad C = c(m_1,m_2,m_3;\ x,\alpha,Q)$$

subject to

$$Q \leqq f(m_1,m_2,m_3),\ Q > 0,\ m_1 \geqq 0,\ m_2 \geqq 0,\ m_3 \geqq 0,$$

where $c(.) = R_0(x,\alpha) + \sum_{i=1}^{3} P_i(x,\alpha)m_i$ and $f(.)$ is the production function. Here, m_i, i=1,2,3, x, α, and Q, denote, respectively, the quantity used of i th input, distance (range) between output market and the firm's production location, its direction angle, quantity of output. Transport costs of output R_0, and of input P_i, i=1,2,3, may depend on range x; that is, they are real valued functions of range x, while P_i, i=2,3 depends also on angle α while the transport costs of output and the first input do not. We shall see this dependence in full detail later.

2.2. *Input and Output Transport Cost Functions of Location* (x,α)

We assume that the output transport cost function $R_0(.)$, the input transport cost functions, $R_i(.)$, i=1,2,3, take the forms,

(A. 1) $R_0(x) = P_0(x) - \bar{P}_0 = r_0(x)x,$

$$R_i(u_i) = \rho_i(u_i) - \bar{P}_i = \tau_i(u_i)u_i \quad i = 1, 2, 3,$$

where range variable u_i represents the range between the firm's location which is *variable* and the i th input market which is *fixed*. For $x>0$, the form $r_0(x)x$ is not a restriction. But a definition of $r_0(.)$ gives $R_0(.)$, namely; $r_0(x) = R_0(x)/x$; and similarly for $\rho_i(.)$, i=1,2,3.

$\bar{P}_i$, i=1,2,3, is the *(F.O.B.)* price established at the i th input market, so that $\rho_i(u_i)$ may be interpreted as a *C.I.F.* price. We shall *assume that input and output transport cost functions are twice continuously differentiable in (non-negative) ranges (of class C^2).*

Take any transport cost as positive and also as a non-decreasing function of the range, so that, for each $x > 0$,

(A. 2)-1 $R_0(x) > 0,\ R_0'(x) = r_0'(x)x + r_0(x) \geqq 0,$

and for each $u_i > 0$, i=1,2,3,

(A. 2)-2 $R_i(u_i) > 0,\ \rho_i'(u) = \tau_i'(u_i)u_i + \tau_i(u_i) \geqq 0.$

Also, unless stated to the contrary, we assume that marginal transport cost is non-increasing in range, that is, unless stated to the contrary, we always assume that

(A. 3) $\begin{cases} R_0''(x) \leqq 0. \\ R_i''(u_i) \leqq 0, \quad i = i, 2, 3. \end{cases}$

We now easily see that, for each location variable pair (x,α), the range variable u_i continuously depends on the location variable (x,α); that is, $u_i = u(x,\alpha_i)$,
hence, we shall define the C.I.F. prices, P_i to be as a function of (x,α).

$$\rho_i(u_i) = \rho_i(u(x,\alpha_i)) \stackrel{\text{def}}{=} P_i(x,\alpha) \qquad (1)\text{-}1$$

Similarly for transportation cost rates, τ_i to be

$$\tau_i(u_i) = \tau_i(u(x,\alpha_i)) \stackrel{\text{def}}{=} r_i(x,\alpha). \qquad (1)\text{-}2$$

Specifically, $u_1 = x$, and for i=2,3,

$$u(x,\alpha_i) = \sqrt{(x - a\cos\alpha_i)^2 + a^2\sin^2\alpha_i} \qquad (1)\text{-}3$$

$\alpha_2 = \alpha$, and $\alpha_3 = 60° - \alpha$.

In Figure 0, we see $u(x,\alpha_2) = y(x,\alpha)$, and $u(x,\alpha_3) = z(x,\alpha)$.

2.3. *Neo-Classical Production Function*

We assume here, except otherwise stated, that production function f well defined on an input factor space (a non-negative orthant in a Euclidean space), possesses a Hessian matrix, bordered with its first derivatives, which is everywhere invertible, and that its inverse is negative semidefinite. Then, *input* demand functions (optimal input factors), generated from minimization of the cost (P.I.), are continuously differentiable with respect to pair (x,α), provided that *the C.I.F. price(functions)* (that is, input and output transport cost functions) are continuously differentiable in (x,α). We give a sketch of proof for the argument in Appendix A. See Debreu [1972], and Kusumoto [1973] for details. See also Rader [1973] for conditions insuring differentiability *almost everywhere*.*5 We shall hereinafter call this production function f a *neoclassical* one.

3. *Global Properties of Total Cost Function of the Firm with Respect to Distance* x

We shall investigate here the properties of the total cost function, hence of the optimal location which minimizes the total cost C^* in the space. Let there be given Q and $\bar{P}_i$, $i=1,2,3$. Then, the solutions for factor inputs to the problem (P.I.) for each fixed location (x,α), are written; $\phi_i(P_1(x),\ P_2(x,\alpha),\ P_3(x,\alpha),\ Q) = m_i(x,\alpha)$, $i=1,2,3$, and the optimal cost C^* for each location (x,α) may be expressed by

$$C^* = c(m_1(x,\alpha),\ m_2(x,\alpha),\ m_3(x,\alpha),\ x,\ Q)$$

$$= R_0(x)Q + \sum_{i=1}^{3} P_i(x,\alpha)m_i(x,\alpha). \qquad (2)$$

Making explicit the cost function; $C^* : (x,\alpha) \to C^*(x,\alpha) \in R_+$, for each (x,α) in the space $X \times A \supset T$, we may view $m_i(x,\alpha)$ as a continuously differentiable function in (x,α), hence, the optimal cost function $C^*(.)$ continuously depends on location (x,α). We wish to see how the firm's cost $C^*(.)$ varies when the pair (x,α) varies in the space of candidate locations. (Choice of location is not restricted in the triangle.)

Lemma 1. (Existence of the Solution): ((Weierstrass); Debreu [1959, p. 16], Nikaido [1970, p. 103].) $C^*(.)$ varies continuously with location (x,α). It takes a maximum and a minimum on the triangle space.

Now, suppose $C^*(.,.)$ is a concave function in $X \times A$, where $X = [x \in R;\ x \geqq 0]$, $X(\alpha) = [x \in X,\ 0 \leqq x \leqq a(\sqrt{3}/2/\sin(60^\circ + \alpha)]$ and $A = [\alpha;\ 0^\circ \leqq \alpha \leqq 60^\circ]$. Let $T = [(x,\alpha) \in XxA;\ x \in X(\alpha);\ \alpha \in A]$.

The triangle space is now given by T. Then, we have;

Lemma 2. If the cost function $C^*(.)$ is concave and not constant in T, then, for it to attain its minimum on T, the point (x^*,α^*) must not be an interior point of T; i.e., $(x^*,\alpha^*) \notin T^\circ$ *where* $T^\circ = [(x,\alpha) \in R^2;\ 0 < x < a\sqrt{3}/2/\sin(60^\circ + \alpha),\ 0^\circ < \alpha < 60^\circ]$. We call T° the *interior* of the triangle. For a proof, make use of Berge [1963, p. 194] for instance.

However, with respect to T, the concavity condition in Lemma 2 will turn out to be never satisfied. It is, in fact, overly strong for excluding all interior locations in the triangle space. This lemma may be applied to the cost function $C^*(.,\alpha)$ for a fixed $\alpha \in A$, or $C^*(x,.)$ for a fixed $x \in X$, and will in fact be so used. Since $C^*(.)$ is twice continuously[*6] differentiable in (x,α), the sufficient conditions for a boundary point to be an optimal location can be given in terms of the first and second partial derivatives of C^* at each (x,α). Hence, it holds with respect only to distance variable $x \in X$, for each fixed angle $\alpha \in A$, that;

$$\frac{\partial C^*(x,\alpha)}{\partial x} = R_0'(x)Q + \{P_1'(x)m_1(x,\alpha) + \sum_{i=2}^{3} \frac{\partial P_i(x,\alpha)}{\partial x} m_i(x,\alpha)\}$$

$$\leqq 0, \quad \text{or,} \quad \geqq 0; \tag{4}$$

(*monotonicity* with respect to $x \in X(\alpha)$)

or,

$$\frac{\partial^2 C^*(x,\alpha)}{\partial x^2} = [R_0''(x)Q + \{P_1''(x)m_1 + \sum_{i=2}^{3} \frac{\partial^2 P_i(x,\alpha)}{\partial x^2} m_i(x,\alpha)\}]$$

$$+ \{P'(x) \frac{\partial m_1(x,\alpha)}{\partial x} + \sum_{i=2}^{3} \frac{\partial P_i(x,\alpha)}{\partial x} \frac{\partial m_i(x,\alpha)}{\partial x}\} \leqq 0 \tag{5}$$

(*concavity* with respect to $X(\alpha)$).

3.1. *The Substitution and Transport Cost Effects*

Observe that, in the right hand side of (5), the last blacket may be called the *substitution effect* (term), whereas, the first square blacket *transport cost effect*. Further, the transport cost effect term of (5) can be reduced to, by making use of (9), (11), (14) and (15),

$$\{R_0''(x)Q + P_1''(x)m_1(x,\alpha) + \Sigma_{i=2}^{3} \rho_i''(u)(\frac{\partial u}{\partial x})^2 m_i(x,\alpha)\} + \Sigma_{i=2}^{3} \rho_i'(u)\frac{\partial^2 u}{\partial x^2} m_i(x,\alpha). \tag{5'}$$

Here, the first blacket is the effect due to the ad hoc transport cost rates, whereas, the second the one due to the spatial property of transport distance. The substitution effect takes, for each (x,α), non-positive value, if production function f is neoclassical. Denote this by $S_x(x,\alpha)$, then, we can write, for a real valued function ϕ_i in prices P, such that $\phi_i(P(x,\alpha)) = m_i(x,\alpha)$, $P(x,\alpha) = (P_1(x,\alpha),$ $P_2(x,\alpha),$ $P_3(x,\alpha))$, and $P_1(x,\alpha) = P_1(x)$;

$$S_x(x,\alpha) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial P_i(x,\alpha)}{\partial x} \frac{\partial \phi_i(P(x,\alpha))}{\partial P_j} \frac{\partial P_j(x,\alpha)}{\partial x}, \tag{6}$$

where we apply the chain rule and obtain,

$$\sum_{j=1}^{3} \{\partial\phi_i(P(x,\alpha))/\partial P_j\}\{\partial P_j(x,\alpha)/\partial x\} = \partial m_i(x,\alpha)/\partial x.$$

A 3 by 3 matrix, whose typical element $\partial\phi_i/\partial P_j$, is negative semidefinite and equal to a submatrix of the inverse of the bordered Hessian of f. The intended interpretation for the negative $S_x(x,\alpha)$ is that the most generalized change in transport prices, due to the change in location variable, must set up a change in demands for inputs in the opposite direction. See for example, Hicks [1965, p. 52] for detail. If we assume that the transport cost functions are all concave w.r.t. X, at any $x \in X$, then, it is sufficient for (5) to hold for each (x,α) that,

$$\partial P_i^2(x,\alpha) / \partial x^2 \leqq 0, \quad i = 1, 2, 3. \tag{7}$$

This, however, hardly holds, in general. The case $\alpha_i \equiv 0°$ i=2,3, is the one-dimensional case, including perfect substitution.*7 This one-dimensional case in fact satisfies the inequality (5) under (A. 3).

Remark 0: The signs of $\partial^2 P_i(x,\alpha)/\partial x^2$, i=2,3, *are in general indeterminate, except in special cases, such as* $\rho_i''(u) \geqq 0$, *in which case it is positive (or zero if* $\alpha_i \equiv 0°$*).*

This exception, which is, however, a fundamental case, anticipates the existence of *a difficulty* feature, in case finding optimal point at an input market.

3.2. *Properties of C.I.F. Prices With Respect To Distance* x

Variations of the transport costs with respect to x will be listed below, with which we shall see the proof of Remark 0. They will be repeatedly utilized in the following sections.

Let $u = u_i$, then,

$$\frac{\partial P_i}{\partial x} = \rho_i'(u)\frac{\partial u}{\partial x}, \tag{8}$$

$$\frac{\partial^2 P_i}{\partial x^2} = \rho_i''(u)\ \{\frac{\partial u}{\partial x}\}^2 + \frac{\rho_i'(u)}{u}\ \{1 - \{\frac{\partial u}{\partial x}\}^2\} \tag{9}$$

$$= \{\rho''(u) - \frac{\rho_i'(u)}{u}\}\{\frac{\partial u}{\partial x}\}^2 + \frac{\rho_i'(u)}{u}, \tag{9'}$$

where $P_i = P_i(x,\alpha) = \rho_i(u(x,\alpha_i))$, $i=2,3$, and,

$$\frac{\partial u}{\partial x} = \frac{(x - a\cos\alpha_i)}{u}, \quad \alpha_2 = \alpha, \quad \alpha_3 = 60° - \alpha, \tag{10}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{u}\{1 - (\frac{\partial u(x,\alpha_i)}{\partial x})^2\} \geqq 0, \quad \text{(convexity w.r.t. } X(\alpha_i)) \tag{11}$$

since

$$|\frac{\partial u}{\partial x}| \leqq 1.$$

Under the assumption (A. 1) - (A. 3), in view of (2), if $\alpha \leqq 30°$, then,

$$\frac{\partial P_2(x,\alpha)}{\partial x} \leqq 0, \quad \text{for any } x \text{ such that } 0 \leqq x \leqq a\cos\alpha, \tag{12}$$

$$\frac{\partial P_3(x,\alpha)}{\partial x} \leqq 0, \quad \text{for any } x \text{ such that } 0 \leqq x \leqq a\cos(60° - \alpha),$$

or, $\qquad > 0$ otherwise. (13)

Let $u(x,\alpha_2) = y(x,\alpha)$ and $u(x,\alpha_3) = z(x,\alpha)$, then,

$$\frac{\partial u(x,\alpha_2)}{\partial x} = \frac{\partial y(x,\alpha)}{\partial x} \leqq 0, \quad \text{if} \quad 0 \leqq x \leqq a\cos\alpha, \tag{14}$$

$$\frac{\partial u(x,\alpha_3)}{\partial x} = \frac{\partial z(x,\alpha)}{\partial x} \leqq 0, \quad \text{if} \quad 0 \leqq x \leqq a\cos(60° - \alpha),$$

or, $\qquad > 0$ otherwise. (15)

Remark of Discontinuity: Note that discontinuity of $\partial u(x,\alpha_i)/\partial x$ appears at the vertex $(a,\underline{\alpha}_i)$, where $\underline{\alpha}_2 = 0°$, $\underline{\alpha}_3 = 60°$. $\partial u^+(x,\alpha_i)/\partial x = \lim_{\varepsilon\to 0+}\{u(x+\varepsilon,\alpha_i) - u(x,\alpha_i)\}/\varepsilon = 1$, $\partial u^-(x,\alpha_i)/\partial x = \lim_{\varepsilon\to 0-}\{u(x+\varepsilon,\alpha_i) - u(x,\alpha_i)\}/\varepsilon = -1$, at $(a,\underline{\alpha}_i)$. Hence, the discontinuity of $\partial P_i(x,\alpha)/\partial x$, $\partial C^*(x,\alpha)/\partial x$, $\partial^2 C^*(x,\alpha)/\partial x^2$ etc. appears there. It depends on where the origin is taken. Discontinuity does not happen to the derivatives with respect to $\alpha\epsilon A$; see (22)-(26).

Since $y(x,\alpha)$ and $z(x,\alpha)$ can be symmetrically treated, the same properties as in (12) & (13) of P_i, i=2,3, with respect to x are obtained for the case in which $\alpha \geqq 30°$.

Suppose here, as a very special case which is compatible with (A. 2)2, $\rho_i'(u) = 0$, i=1,2,3. Then, every C.I.F. price is constant, and, the location problem will disappear. This assumption thus will be made only for some i but not all i. In view of (9), (10) and (11), (A. 2)2 and (A. 3) together will imply the first term in the right hand side of (9) is negative while the second positive, except in case $\rho_i'(u) = 0$, or $\rho_i''(u) = 0$, which completes a proof of Remark 0.

4. *Transport Distance Function*

In an essentially one-dimensional case, where the sites M_2, and M_3 are on the same line, the (total) transport distance $x + y(x,\alpha) + z(x,\alpha)$, can be viewed as linear, as the location changes in the space. To see this, take $\alpha_i = 0°$, in (1)-3. Then, $x + y(x,0°) + z(x,60°) = 2a - x$. In case $M_2 \neq M_3$ as in Figure 0a, $x + y(x,\theta°) + z(x,60°) = a + b + |x - a|$. In the two-dimensional case, however, the distance, denoted by $d(x,\alpha)$, is not linear and continuously varies with the location variables (x,α) in the space, so that it attains a maximum and a minimum on a (closed and) bounded set T by Lemma 1.

Lemma 3. (1) For each direction angle $\alpha \in A$, the total transport distance $x + d(x,\alpha)$ for our market distribution attains its minimum at the output market, at which markets are agglomerated.

(2) For each direction angle $\alpha \in A$, the transport distance $d(x,\alpha)$ attains its maximum at a boundary of the space T. To see the first remark, in Figure 1, observe that $a \leqq x + y(x,\alpha)$ and $a \leqq x + z(x,\alpha)$. Hence, $2a \leqq 2x + y(x,\alpha) + z(x,\alpha)$, with equality when $x = 0$. The second follows from Lemma 2, with the convexity (21) below.

4.1. *Global Properties of Transport Distance Function in Location Space*

We shall take a look at how the distance function $d(.)$ achieves its minimum distance at an interior point of the triangle space $T°$. The necessary condition for a minimum distance in the interior (open interval) $X°(\alpha)$ for each α is that, for some $\bar{x} \in X°(\alpha)$

$$\partial d(\bar{x},\alpha)/\partial x = 0 \,. \tag{18}$$

To inquire into whether or not (18) holds, we shall list up derivative relations of the distance function $d(.,\alpha)$ on X for each α.

$$d(x,\alpha) = x + y(x,\alpha) + z(x,\alpha) > 0 \tag{19}$$

where the total transport distance is $x + d(x,\alpha)$ for our market distribution

$$\frac{\partial d(x,\alpha)}{\partial x} = 1 + \frac{\partial y(x,\alpha)}{\partial x} + \frac{\partial z(x,\alpha)}{\partial x}, \quad \frac{\partial d(x(\alpha),\alpha)}{\partial x} \geqq \frac{1}{2}, \tag{20}$$

$$\frac{\partial^2 d(x,\alpha)}{\partial x^2} = \frac{\partial^2 y(x,\alpha)}{\partial x^2} + \frac{\partial^2 z(x,\alpha)}{\partial x^2} > 0 \quad \text{(Convexity with respect to X)}. \tag{21}$$

Now, we can investigate what are the values of $d(.,\alpha)$, the sign of the first derivatives (20), at each $x \in X$ such that $x = 0$, $a/2$, a, $\sqrt{3}/2\,a$ and $x(\alpha)$, where $x(\alpha) = \{(\sqrt{3}/2)/\sin(60° + \alpha)\}a$. Use (1) and (10), then, we can directly calculate and get those values and signs. Consult with Appendix B and C.

Make use of the symmetry property of an equilateral triangle, i.e., that of $y(x,\alpha)$ and $z(x,60°-\alpha)$ with respect to the plane $[(x,\alpha);\, x \in X,\, \alpha = 30°]$, we have a result on the properties of function $d(.)$, which can be summarized in a lemma below;

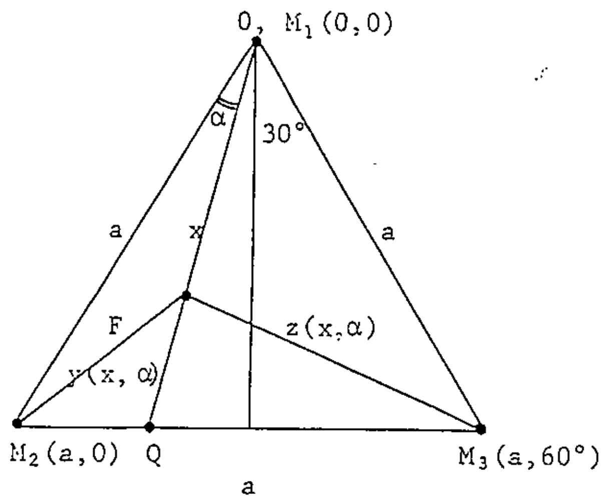
Lemma 4. For each direction angle $\alpha \in A$, $\alpha \leqq 30°$, *the transport distance* $d(.,\alpha)$ *attains its minimum at an interior point* $(\bar{x}(\alpha),\alpha)$.

The interior point, at which the distance $d(.,\alpha)$ attains its minimum for each $\alpha \in A$, is located along the linear segment whose boundaries are $(1/2\,a,\ 0)$ and $(\sqrt{3}/3\,a,\ 30°)$. Thus, the locus of such minimizing points $[(\bar{x}(\alpha),\alpha);\ \alpha \in A]$ is drawn in Figure 1b as the solid line segment.

We can also prove the following lemma, which seems intuitively true, and is illustrated below in Figure 1b.

Lemma 5. The distance function d(.) *attains its minimum at the gravity point* $G((\sqrt{3}/3)a, 30°)$ *in the triangle* T.

We call the properties rivealed in Lemma 3, 4, 5 and (21) of the distance function d the *spatial* property. This property disappears in the one dimensional case.



Q = (x(α),α)   0Q = X(α)

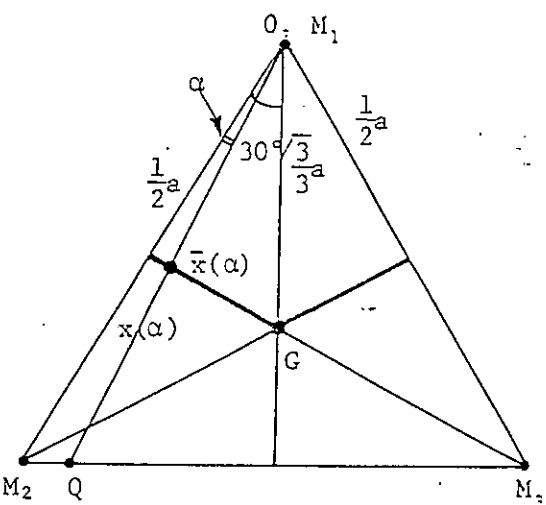


Figure 1a
Transport Distance for Given Location (x, α)

Figure 1b
The Locus of Minimum Transport Distance

5. *Theorems for Optimal Location*

We investigate here the conditions for an interior or intermediate point to be optimal, and establish two theorems; one for non-intermediate location, and the other for interior location.

5.1 *Conditions for an Interior or Intermediate Point*

The following lemma is for an interior or intermediate point, a proof of which may be easily completed by applying the theorem of intermediate value; see Nikaido [1970, p. 103] or Apostol [1957, pp. 88-89], and Lemma 1.

Lemma 6; a) *In order that an interior or intermediate location is chosen by the firm as its production location, it suffices to show that the cost function* $C^*(.,\alpha)$ *is strictly decreasing in* x *at* $x = 0$, *and, it is non-decreasing in* x *at* $x = x(\alpha)$, *i.e., for each* $\alpha \in A$, $x(\alpha) = a(\sqrt{3}/2)/\sin(60°+\alpha)$,

$$\partial C^{*+}(0,\alpha)/\partial x < 0, \textit{ and } \partial C^{*-}(x(\alpha),\alpha)/\partial x \geqq 0 .$$

b) *The chosen location is strictly interior of the interval* $X(\alpha)$, *i.e.,* $0 < \bar{x}(\alpha) < x(\alpha)$, *and* $\partial C^*(\bar{x}(\alpha),\alpha)/\partial x = 0$ *if* $\partial C^{*-}(x(\alpha),\alpha)/\partial x > 0$ *for each* $\alpha \in A$ *further.*

where $\partial C^{*+}(x,\alpha)/\partial x = \lim_{\varepsilon\to 0+}\{C^*(x+\varepsilon,\alpha)-C^*(x,\alpha)\}/\varepsilon$, and $\partial C^{*-}(x,\alpha)/\partial x = \lim_{\varepsilon\to 0-}\{C^*(x+\varepsilon,\alpha)-C^*(x,\alpha)\}/\varepsilon$. Note $\partial C^{*+}(x(0),0)/\partial x > \partial C^{*-}(x(0),0)/\partial x$, and similarly for $\alpha=60°$; this discontinuity arises due to : $\partial y^{-}(x(0),0)/\partial x = -1$, and $\partial y^{+}(x(0),0)/\partial x = 1$, for example, at the vertex M_2. See Section 3.2.

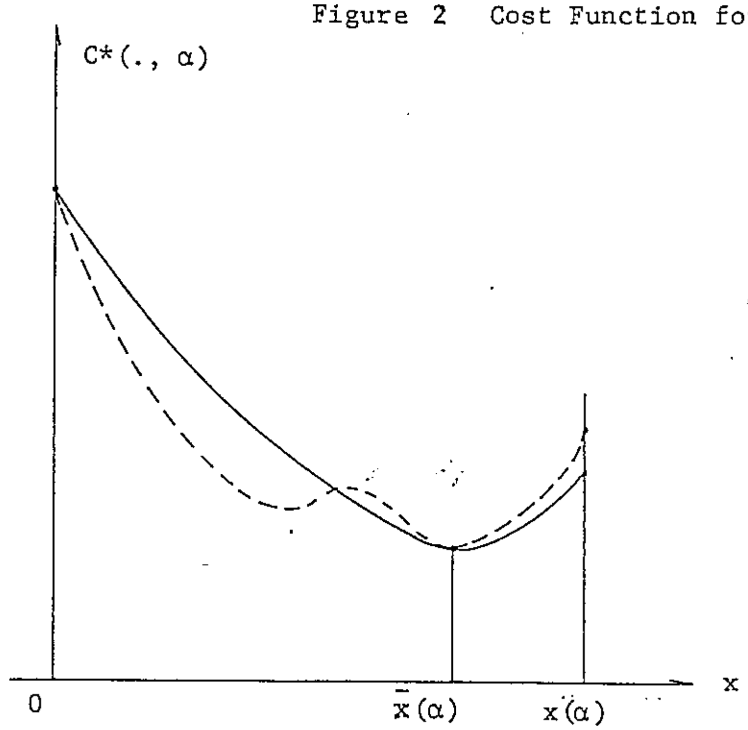
The next lemma is a result of Lemma 6b applied to $C^*(x,.)$ with respect to A. It is very important fact. Let $\hat{X}(0)° = \{x \in X;\ 0<x<a\}$.

Lemma 7: For every positive distance $x \in \hat{X}(0)°$ *the cost function* $C^*(x,.)$ *takes its minimum in the interior of A, if distance x is fixed.*

Proof will follow the remark;

This result holds true, whether or not $\angle M_2OM_3=60°$.*8 We can take the location origin O(0,0) at any other vertex, without a change in the result.

Figure 2 Cost Function for Fixed α



Proof: For each $x \in X$, $y(x,.)$ and $z(x,.)$ vary continuously with direction angle $\alpha \in A$, hence, the cost function $C^*(x,.)$ takes its minimum and maximum somewhere on A. Since

$$\frac{\partial C^*(x,\alpha)}{\partial \alpha} = \sum_{i=2}^{3} \frac{\partial P_i(x,\alpha)}{\partial \alpha} m_i(x,\alpha), \text{ and, } x > 0 \tag{22}$$

it follows that

$$\frac{\partial C^*(x,0)}{\partial \alpha} = -\rho_3'(z) \frac{ax \sin 60°}{z(x,0)} m_3(x,0) < 0 , \tag{23}$$

and

$$\frac{\partial C^*(x,60°)}{\partial \alpha} = \rho_2'(y) \frac{ax \sin 60°}{y(x,60°)} m_2(x,60°) > 0, \tag{24}$$

where observe $ax\sin 60°/u_i$, $i = 2, 3$ are symmetric, but not $\rho_i'(u)m_i(x,\alpha_i)$, and, differentiate (1)-3 with respect to α, then we have,

$$\frac{\partial u(x,\alpha_i)}{\partial\alpha} = \frac{ax\sin\alpha_i}{u(x,\alpha_i)}\frac{\partial\alpha_i}{\partial\alpha}, \quad \alpha_2 = \alpha,\ \alpha_3 = 60°-\alpha, \tag{25}$$

and

$$\frac{\partial P_i(x,\alpha)}{\partial\alpha} = \rho_i'(u)\,\frac{\partial u(x,\alpha_i)}{\partial\alpha}, \quad i = 2, 3 . \tag{26}$$

5.2 *Theorem for Non-Intermediate Location*

We call an optimal location *non-intermediate*, if it is not on either of the *open* segments, $\overline{OM_2}$, $\overline{M_2M_3}$, and $\overline{OM_3}$.

Theorem 7. (Non-Intermediate Location): The cost function $C^*(.,.)$ takes its minimum in the interior of T, or at a vertex, but never at an intermediate point. An intermediate location is never chosen by the firm.

Proof: By Lemma 7, neither of $(x,0°)$ nor $(x,60°)$ is optimal for all x such that $0<x<a$. Nor is $(x(\alpha),\alpha)$, where $\alpha\in A°$. To see the latter, take the origin $(0,0)$ at the vertex M_2. Location F is measured from this new origin as (y,β), where $\beta=\angle M_3M_2F$. The cost function may be redefined for each such location (y,β) without any change in its properties; $D^*:(y,\beta)\rightarrow D^*(y,\beta)$ must satisfy $D^*(y,\beta) = C^*(x,\alpha)$, where $y=y(x,\alpha)$, $\beta=\beta(x,\alpha)$. The cost $D^*(.,.)$ varies over $S = [(y,\beta);\in Y\times B,\ y\in Y(\beta),\beta\in B]$,[*9] just as $C^*(.,.)$ does over T. Apply Lemma 7 to $D^*(y^*,.)$ with respect to B. Then, for some $\beta°\in B$, $D^*(y^*,\beta°)<D^*(y^*,\beta^*)=C^*(x^*,\alpha^*)$ where $x^*=x(\alpha^*)$, (that is, the optimal location (x^*,α^*) was supposed to be on the base,) $y^*=y(x^*,\alpha^*)$, and $\beta^*=\beta(x^*,\alpha^*)$. Hence, $(x°,\alpha°)$ exists in $T°$, where $x°=x(y^*,\beta°)$, $\alpha°=\alpha(y^*,\beta°)$. Hence, $D^*(y^*,\beta°)=C^*(x°,\alpha°)<C^*(x^*,\alpha^*)$. *Q.E.D.*

Note that $x \in X(\alpha)^\circ, \alpha \in A^\circ \leftrightarrow (x,\alpha) \in T^\circ$, since T is a convex set.

Thus, in order that, at an interior point $(x,\alpha) \in T^\circ$, $\partial C^*(x,\alpha)/\partial x = 0$, it is sufficient to show that $\partial C^{*+}(0,\alpha)/\partial x < 0$ and $\partial C^{*-}(x(\alpha),\alpha)/\partial x > 0$ for each $\alpha \in A^\circ$.

5.3 *Theorem for " Interior " Location*

We shall establish here a theorem that, under specified conditions, a location, interior in the triangle T, is optimal for the firm. We shall call this location " interior ".

Theorem 2 (Interior Location): The total transportation cost function $C^*(.,.)$ takes its minimum at an interior point (x^*, α^*) of the triangle space T (i.e. an interior location is chosen by the firm), provided that the following three conditions (i-1, 2, 3) hold;

for some $\alpha \in A$ (for example, $\alpha = 0^\circ$ or 60°),

$$\partial C^{*+}(0,\alpha)/\partial x < 0, \qquad (i\text{-}1)$$

(which is equivalent to;

$$R_l'(0)Q + P_l'(0)m_l(0,\alpha) < \rho_2'(a) \cos\alpha \; m_2(0,\alpha) + \rho_3'(a) \cos(60^\circ - \alpha) \; m_3(0,\alpha), \qquad (27\text{-}1)),$$

$$\partial C^{*-}(a,0)/\partial x > 0, \qquad (i\text{-}2)$$

(which is equivalent to;

$$R_l'(a)Q + P_l'(a)m_l(a,0) > \rho_2'(0)m_2(a,0) - \rho_3'(a)/2 \; m_3(a,0) \qquad (27\text{-}2)),$$

and,

$$\partial C^{*-}(a,60^\circ)/\partial x > 0, \qquad (i\text{-}3)$$

(which is equivalent to;

$$R_l'(a)Q + P_l'(a)m_l(a,60^\circ) > -\rho_2'(a)/2 \; m_2(a,60^\circ) + \rho_3'(0)m_3(a,60^\circ) \qquad (27\text{-}3)),$$

where the one-sided partial derivatives are defined as;

$$\partial C^{*+}(x,\alpha)/\partial x = \lim_{\varepsilon\to 0+}\{C^*(x+\varepsilon,\alpha)-C^*(x,\alpha)\}/\varepsilon,$$
$$\partial C^{*-}(x,\alpha)/\partial x = \lim_{\varepsilon\to 0-}\{C^*(x+\varepsilon,\alpha)-C^*(x,\alpha)\}/\varepsilon,$$

and, from (8) and (10), $\rho_i'(a)\cos\alpha_i=\partial P_i(0,\alpha)/\partial x,\ i=2,3,\alpha_2=\alpha,\alpha_3=60^\circ-\alpha,$ $\rho_i'(0)=\partial P_i(a,\underline{\alpha}_i)/\partial x, i=2,3, \rho_i'(a)/2=\partial P_i(a,\underline{\alpha}_i)/\partial x,\ i=2,3,\underline{\alpha}_2=0^\circ,$ *and* $\underline{\alpha}_3=60^\circ$.

Remark 1: The set of conditions (i-1,2,3) implies that the three vertices are not even locally optimal, hence *a fortiori* not globally optimal. By theorem 1, the optimal point must be interior.

We shall rigorously prove this here, by showing, in terms of the first derivatives, an interior point globally optimal.

In view of (10), (12)-(15) with $3/2 \leqq \eta(\alpha) \leqq \sqrt{3}$, where $\eta(\alpha) = \cos\alpha + \cos(60°-\alpha)$, it is natural that many interesting cases should satisfy the conditions, at the same time. The conditions are satisfied in Corollary 1.

Remark 2: Any one of all these "derivative" conditions at the vertices; Conditions (27-i), i=1,2,3, will become, however, not likely to be satisfied, as (relatively) more input markets for indispensable factors, M_4, M_5, ..., are spatially agglomerated at the corresponding vertex (the output market site 0, for example). In fact, for $i = 1$ in (27-i), $R_0'(0)\,Q + P_1'(0)m_1(0,\alpha) + \sum_{j=4}^{s} P_j'(0)m_j(0,\alpha)$ will become larger definitely than $P_2'(0,\alpha)(\cos\alpha)m_2(0,\alpha) + P_3'(0,\alpha)(\cos(60°-\alpha))m_3(0,\alpha)$, if the number s becomes larger enough. Then, the location will be easily found at the output site, regardless of whether technology is of a Leontief type or of a neoclassical type.

Proof: Observe first $\partial C^*(\tilde{x}(\alpha),\alpha)/\partial x > 0$ (=0, if $R_0'(x)=P_1'(x)=0$ and $\alpha=30°$), $0° \leqq \alpha \leqq 30°$, where $\tilde{x}(\alpha) = a \cos \alpha$. By (i-1) $\partial C^*(0,\alpha)/\partial x < 0$ and Lemma 6b, there exists a point $(\underline{x},\underline{\alpha})$ such that $\underline{x} \in \tilde{X}(\underline{\alpha})°$, where $\tilde{X}(\underline{\alpha})°$ is the interior of $\tilde{X}(\underline{\alpha}) = [0,\tilde{x}(\underline{\alpha})], \alpha \in A°$, and such that $\partial C^*(\underline{x},\underline{\alpha})/\partial x = 0$. This is true for every α in $N_\varepsilon(\underline{\alpha}) \cap A$, where $N_\varepsilon(\underline{\alpha})$ is the spherical ε-neighborhood, by continuity of $\partial C^*(.,.)/\partial x$ on $[(x,\alpha) \in X x A \; ; \; x \in \tilde{X}(\alpha)], \alpha \in A$. On the other hand, even without (i-1), $C^*(.,\alpha)$ takes its minimum in the closed interval $X(\alpha)$ by Lemma 1. Let point$(x^*(\alpha),\alpha)$ be a point at which $C^*(x^*(\alpha),\alpha) \leqq C^*(x,\alpha)$, $x \in \tilde{X}(\alpha), \alpha \in A$. Then, $\partial C^{*+}(x^*(\alpha),\alpha)/\partial x = \lim_{\varepsilon \to 0+} \{C^*(\alpha)+\varepsilon,\alpha) - C^*(x^*(\alpha),\alpha)\}/\varepsilon \geqq 0$ *i), and, $\partial C^{*-}(x^*(\alpha),\alpha)/\partial x = \lim_{\varepsilon \to 0-} \{C^*(x^*(\alpha)+\varepsilon,\alpha) - C^*(x^*(\alpha),\alpha)\}/\varepsilon \leqq 0$ *ii).

Now, suppose the point $(x^*(\alpha^*),\alpha^*)$ be an optimal point, which minimizes $C^*(.,.)$ over$[(x,\alpha) \in X x A; \; x \in \tilde{X}(\alpha), \alpha \in A]$. Suppose also it be in the closed interval $[x(\alpha^*),\tilde{x}(\alpha^*)]$. Note this point also minimizes $C^*(.,\alpha^*)$ over $\tilde{X}(\alpha^*)$. Then, suppose further it be in the interval $(x(\alpha^*),\tilde{x}(\alpha^*)]$, where $x(\alpha) = a\sqrt{3}/2/\sin(60°+\alpha)$ as defined before, then, there is a point $(x,\gamma) \in T°$, such that $x^* \geqq x$, $y(x^*,\alpha^*) \geqq y(x,\gamma)$, and, $z(x^*,\alpha^*) \geqq z(x,\gamma)$, where $x^* = x^*(\alpha^*)$. See Figure 3, where K is the point (x,γ).

Hence, for any (x,γ) on the closed line segment $\overline{M_2M_3}$,

$$R_0(x^*) \geqq r_0(x^*)x \geqq r_0(x)x \; ; \quad P_1(x^*) \geqq \bar{P}_1 + r_1(x^*)x \geqq \bar{P}_1 + r_1(x)x \; ;$$

$$P_i(x^*,\alpha^*) = \bar{P}_i + r_i(x^*,\alpha_i^*)u(x^*,\alpha_i^*) \geqq \bar{P}_i + r_i(x,\gamma)u(x,\gamma_i), \gamma_2=\gamma, \gamma_3=$$

$60°-\gamma$. Every 2nd inequality follows from the monotonicity (A-2).

Define a cost function $C(x,\gamma)$ of such location (x,γ) to be;

$$C(x,\gamma) = R_0(x)Q + P_1(x)m_1(x^*,\alpha^*) + \sum_{i=2}^{3} P_i(x,\gamma)m_i(x^*,\alpha^*).$$

Then, $C^*(x^*,\alpha^*) \geqq C(x,\gamma)$ with equality only when $x^* = x(\alpha) = x(\beta)$, where β is the location angle of the point $H(x(\beta),\beta)$. Thus, (x^*,α^*) can not be outside of the space T. That is, $0 \leqq x^* \leqq x(\alpha^*)$.

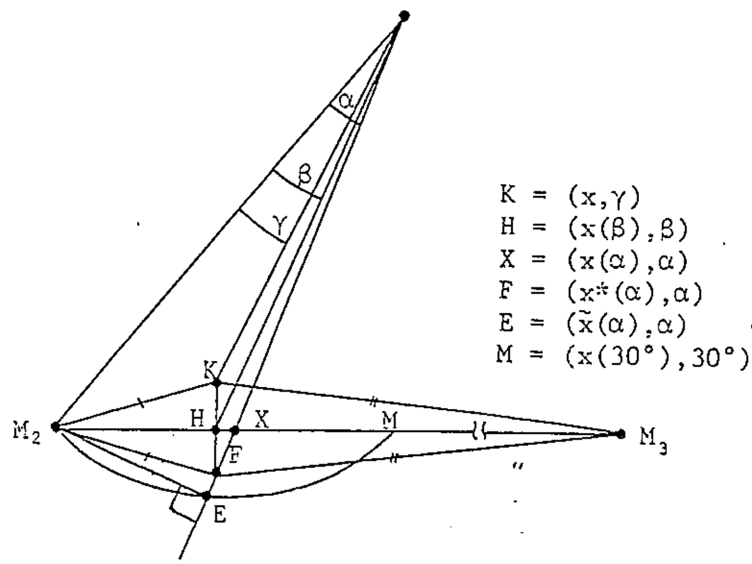


Figure 3
Location Outside of the Triangle Space

Suppose it be a vertex, then, immediately a contradiction follows by (i-1,2, or 3). At the output market site, O, there is $\alpha\epsilon A$, such that $\partial C^{*}_{+}(0,\alpha)/\partial x< 0$, contradicting to *i) above. Similarly for the input market sites, for example, at the site, M_2, Condition (i-2) contradicts to *ii) above derived. Hence, by Theorem 1, the optimal location is possible only at interior points. For each intermediate point $(x(\alpha),\alpha)$ on the base $\overline{M_2 M_3}$, then, by (the proof of) Theorem 1, there is a point $(x°, \alpha°)\epsilon T°$, $C^*(x°,\alpha°)<C^*(x(\alpha),\alpha)$, implying;

$$\varepsilon_1(\partial C^*(x(\alpha),\alpha)/\partial x) +\varepsilon_2(\partial C^*(x(\alpha),\alpha)/\partial \alpha) + o\rho< 0, \sqrt{\varepsilon_1^2+\varepsilon_2^2} = \rho,$$

by the existence and continuity of $\partial C^*(x,\alpha)/\partial x$, or $\partial C^*(x,\alpha)/\partial \alpha$ (except at the vertices).(Subsection 3.2 for the discontinuity.) Hence, $\partial C^{*-}(x(\alpha),\alpha)/\partial x> 0$ for every $\varepsilon_1<0$, and $\partial C^{*-}(x(\alpha),\alpha)/\partial\alpha>0$ for every $\varepsilon_2<0$.

By (i-1), for an $\alpha\epsilon A$, $\partial C^{\ddagger}_{*}(0,\alpha)/\partial x < 0$. Suppose $\alpha \leq 30°$. Also note $\partial C^{\ddagger}_{*}(x,0)/\partial\alpha < 0$ for every $x > 0$. Then, Lemma 6b, applied to the one-sided partial derivatives which are positive at $(x(\alpha),\alpha)$, implies, by continuity, the existence of a point $(x^*,\alpha^*)\epsilon T°$, such that $\partial C^*(x^*,\alpha^*)/\partial x = 0$ and $\partial C^*(x^*,\alpha^*)/\partial\alpha = 0$. Similarly for the case where $\alpha \leq 30°$ and $\partial C^{\ddagger}_{*}(x(\alpha),\alpha)/\partial\alpha < 0$ for $\varepsilon_2 > 0$, $\partial \bar{C}^{*}(x,60°)/\partial\alpha > 0$. *Q.E.D.*

Consulting with Lemma 1, we may have,

Remark 3: A boundary location is possible only at one of the vertices. *This is closely related to the well-known presentation of the physical Weber triangle.* See Weber [1929] 's Mathematical Appendix.

Let in (i-1) in Theorem 2

$$\omega(\alpha) = \rho_2^{\prime}(a)\cos\alpha\, m_2(0,\alpha) + \rho_3^{\prime}(a)\cos(60°-\alpha)m_3(0,\alpha).$$

Then, since $\frac{\partial m_i(0,\alpha)}{\partial\alpha} = \sum_{j=1}^{3}\partial\phi_i(P(0,\alpha))/\partial P_j\, \partial P_j(0,\alpha)/\partial\alpha = 0$, from (25)-(26),

$$\omega'(\alpha) = -\rho_2^{\prime}(a)\sin\alpha\, m_2(0,\alpha) + \rho_3^{\prime}(a)\sin(60°-\alpha)m_3(0,\alpha)$$

and,

$$\omega''(\alpha) = -\rho_2^{\prime\prime}(a)\cos\alpha\, m_2(0,\alpha) - \rho_3^{\prime\prime}(a)\cos(60°-\alpha)m_3(0,\alpha) < 0.$$

Thus,

$$\partial C^*(0,\alpha)/\partial x = R_0^{\prime}(0)Q + P_1^{\prime}(0)m_1(0,\alpha) - \omega(\alpha),$$

$$\partial^2 C^*(0,\alpha)/\partial\alpha\partial x = -\omega'(\alpha),\ -\omega'(0) < 0,\ -\omega'(60°) > 0, \text{ and,}$$

$$\partial^2(\partial C^*(0,\alpha)/\partial x)/\partial\alpha^2 = -\omega''(\alpha) > 0;$$ Convexity of $\omega(.)$ w.r.t. A.

Hence, (i-1) can be strengthened as follows;

(i-1') $\max\{\partial C^*(0,0)/\partial x,\ \partial C^*(0,60°)/\partial x\} < 0.$

This implies (i-1), but not vice versa. It also implies $\partial C^*(0,\alpha)/\partial x < 0$ for every $\alpha\epsilon A$.

We have a corollary to Theorem 2;

Corollary 1 (Interior Location): Theorem 2 holds, if (i-1') is satisfied instead of (i-1). Here, there is a point $(x^*(\alpha),\alpha)\epsilon T$ such that $0 < x^*(\alpha) < x(\alpha)$ and $C^*(x^*(\alpha)\alpha) \leq C^*(x,\alpha)$, $0 \leq x \leq x(a)$, for every $\alpha\epsilon A$.

6. *Concluding Remarks*

The two problems of optimal choice by an economic agent of locating and designing a facility have been treated simultaneously and a solution provided for the case where the agent is a *competitive* firm. The firm finds the optimal location as well as the optimal quantity of each input for production.

In the analysis throughout the preceding sections, concerning whether an extreme or interior point is optimal in the triangle space, we find that the propositions listed in Subsection 1.2 hold true. By Theorem 1, being not an interior location is equivalent to being an *extreme point* location. Hence the result may trivially apply to one-dimensional case. Sufficient conditions for extreme points have not fully been discussed, however, in the triangle space.*10

Alternatively, we may take as the agent a cost-minimizing consumer or worker who for example is looking for a house to live in. In terms of the formulation extended above, one has only to interpret the inputs as consumption goods or commodities, and the production function as utility function or labor reproduction function.

Now, the global investigation of the basic properties of optimal location with inputs reinforces the foundations of the economic theory of location, by combining the latent qualities in the *two-dimensional* space with the contributions of the various authors referred above. We thereby unify traditional location and production (utility) analyses.

Appendix A

Lemma for Continuous Differentiability of Input Demands :

Input demand functions are continuously differentiable with respect to location variables (x,α), if production function is neoclassical.

Let f be a real valued function which may take an infinite number, and whose domain is a subset of an eucleadean space R^n. We take as the subset the positive orthan R^n_{++}. We assume that;

(i) f is finite and twice continuously differentiable throughout R^n_{++} (of class C^2). Also f is continuous where it is finite on R^n_+, where R^n_+ is the closure of R^n_{++}.

(ii) The first-order partial derivatives, $f_i = \partial f/\partial m_i > 0$ for $i=1,2,3,\ldots,n$ and all $m \in R^n_{++}$.

(iii) f is strictly quasi-concave; that is, for any distinct m, and m' in R^n_{++} such that $f(m) = f(m')$, and for any positive real number ν, $0<\nu<1$, $f(\nu m + (1-\nu)m') > f(m)$.

(iv) For any pair i, j, $(\partial f/\partial m_i)/(\partial f/\partial m_j) \to +\infty$ as $m_i \to 0$ and m_j is adjusted to hold f constant.

The last condition ensures that the isoquant surfaces do not intersect the boundary of R^n_{++}. It is made for convenience only. In fact, we shall relax this sometimes in the context.

Let S be the set of all positive, normalized input prices and outputs; $S = [(p,Q) \in R^{n+1}_{++}]$. Then, input demand function, ϕ, is a vector-valued function from S into R^n_+, which minimizes the cost $C(p,Q;\, x,\alpha) = R_0(x)\, Q + \sum_{i=1}^{n} P_i(x,\alpha) m_i$, where $m_i = \phi_i(p,Q)$, $p = (P_i(x,\alpha);\ i=1,2,3,\ldots,n)$. For once a $(p,Q) \in S$ is specified, the corresponding input, $m \in R^n_{++}$, is defined by the tangency between a hyperplane $\tilde{\tilde{C}}^* = p\ m$ and the isoquant surface $Q = f(m)$. Assumption (i)-(iii) will, together with (iv), imply that ϕ is one-to-one, continuous, and possesses the symmetry and negative semi-definiteness properties, where it is differentiable.

The question of the differentiability of ϕ, however, hinge on whether the bordered Hessian determinant $|F(m)|$,

$$|F(m)| = \begin{vmatrix} 0 & f_j \\ f_i & f_{ij} \end{vmatrix}$$

vanishes at a point in R^n_{++}. To see this, apply the Lagrangean method to minimize the cost C, subject to $Q = f(m)$. Then, the inverse demand function is given by

$$P_j = \lambda f_j(m), \quad Q = f(m), \quad j=1,2,3,\ldots,n$$

where λ is a Lagrangean multiplier, and $\lambda = \phi_0(p,Q)$. See Apostol [1957, pp.153-154].

The Jacobian of this is directly related to the bordered Hessian determinant $|F(m)|$ for each $m \in R^n_{++}$. Thus, $m = \phi(p,Q)$ for some $(p,Q) \epsilon S$, and if $|F(m)| \neq 0$ at m, then, ϕ is differentiable at (p,Q) by the

see Apostol [1957, pp.144-14] for instance.) Conversely, if $|F(m)| = 0$, then, ϕ can not be differentiable at (p,Q). Note that this condition holds at m, if the usual second order condition for a constrained minima holds there, hence the determinant does not vanish.

By the given assumptions on transport cost rates, and the results obtained on the ranges u_i $i=1,2,3$, in Subsection 2.2, we may apply the chain rule; see Apostol [1957, pp.88-89], to obtain the differentiability of input demand, m, and the cost function, C*, with respect to location variable (x,α).

Appendix B

Tranport Distance Function

Table 1 ; $d(x, \alpha)$, $\alpha \leqq 30°$.

α \ x	0	$a/2$	$a/\sqrt{3}$	$\bar{x}(\alpha)$	$x(\alpha)$	$(\sqrt{3}/2)a$	a
0	$2a$	$(7/4)a$	$+$		$+$	$+$	$2a$
30°	$2a$	$(\frac{1}{2}+\sqrt{5-a\sqrt{3}})a$	$\sqrt{3}\ a$		$(1+\sqrt{3}/2)a$		
α	$2a$	$\frac{1}{2}a + \mu(\alpha)a$	$+$	$+$	$(1+x(\alpha))a$		

where $\mu(\alpha) = 1/2 + \sqrt{5/4 - \cos\alpha} + \sqrt{5/4 - \cos(60° - \alpha)}$, and

$x(\alpha) = \{(\sqrt{3}/2)/\sin(60° + \alpha)\}a.$

Table 2 ; $\dfrac{\partial d(x, \alpha)}{\partial x}$, $\alpha \leqq 30°$

α \ x	0	$a/2$	$a/\sqrt{3}$	$\bar{x}(\alpha)$	$x(\alpha)$	$(\sqrt{3}/2)a$	a
0	$-1/2$	0	$+$		$+$	$+$	$1/2$
30°	$-(\sqrt{3} - 1)$	$-$	0		$+$	1	
α	$-\eta(\alpha) + 1$	$-$	$-$	0	$1 + \sigma(\alpha)$		

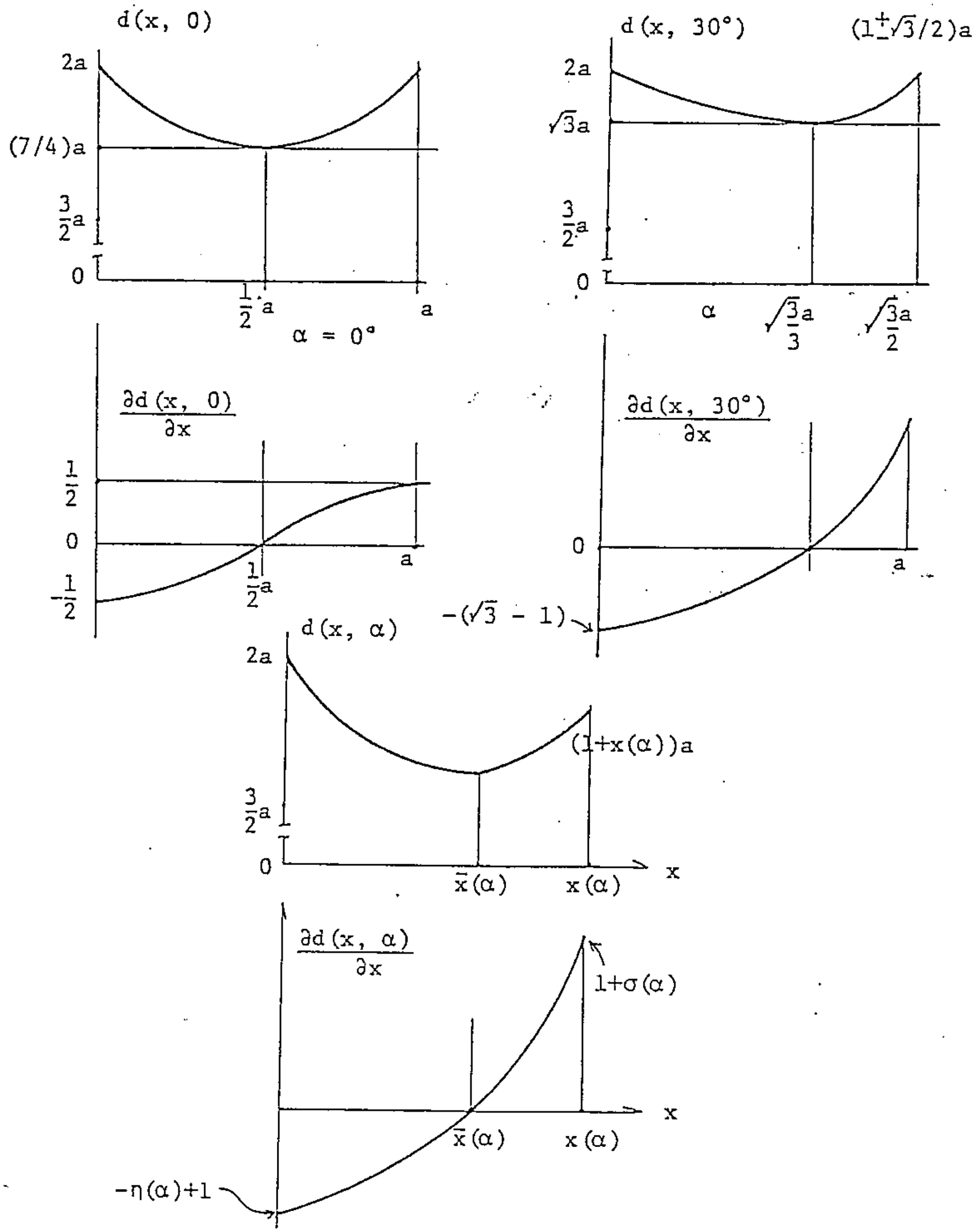
where $\eta(\alpha) = \cos\alpha + \cos(60° - \alpha)$, $3/2 \leqq \cos\alpha + \cos(60° - \alpha) \leqq \sqrt{3}$,

$x(\alpha) - y\,\eta(\alpha) \geqq 0$ if $\alpha \leqq 30°$ with equality when $\alpha = 30°$, and

$\sigma(\alpha) = (1/y(a-y)\{x(\alpha) - y\eta(\alpha)\}$, $\frac{\sqrt{3}}{2}a \leqq x(a) \leqq a.$

Figure B 1

Graphs of Transport Distance Function for Given α



Appendix C

Distance for Given α.

Next, we like to know where $d(\bar{x}(.), .)$ achieves its minimum distance on A. To this end, what we must know is the values of $d(\bar{x}(.), .)$, and the signs of $\frac{\partial d(\bar{x}(.), .)}{\partial \alpha}$, at $\alpha = 0°$, and $\alpha = 30°$ etc.,

We have $y(\bar{x}(\alpha), \alpha) = \bar{x}(\alpha)$, and

$$\frac{\partial^2 d(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} = \frac{\partial^2 y(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} + \frac{\partial^2 z(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} > 0, \qquad (1)$$

since, $0° \leqq \alpha \leqq 30°$,

$$0 \leqq \frac{1}{y} \frac{\partial y(\bar{x}(\alpha), \alpha)}{\partial \alpha} = \frac{a}{\bar{x}(\alpha)} \sin \alpha \leqq 1 \qquad (2)$$

$\sqrt{3} \leq \tan^{-1} \alpha$ and,

$$\sqrt{3}^{-1} \leqq -\frac{1}{z} \frac{\partial z(\bar{x}(\alpha), \alpha)}{\partial \alpha} = \frac{a\bar{x}(\alpha)}{z(x(\alpha), \alpha)^2} \sin(60° - \alpha) \leqq \frac{3}{2} \qquad (3)$$

where let $y = y(\bar{x}(\alpha), \alpha)$ and $x = z(\bar{x}(\alpha), \alpha))$,

$$\frac{\partial^2 y(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} = \frac{\partial y}{\partial \alpha} (\tan^{-1} \alpha - \frac{1}{y} \frac{\partial y}{\partial \alpha}) \qquad (4)$$

$$\frac{\partial^2 z(\bar{x}(\alpha), \alpha)}{\partial \alpha^2} = \frac{\partial z}{\partial \alpha} (\tan^{-1}(60° - \alpha) - \frac{1}{z} \frac{\partial z}{\partial \alpha}). \qquad (5)$$

We also have,

$$\frac{\partial d(\bar{x}(0), 0)}{\partial \alpha} = -\frac{1}{2} a, \quad \frac{\partial d(\bar{x}(30°), 30°)}{\partial \alpha} = 0. \qquad (6)$$

Now, we can illustrate how the distance $d(\bar{x}(.),.)$ varies with direction angle α on A, in what follows:

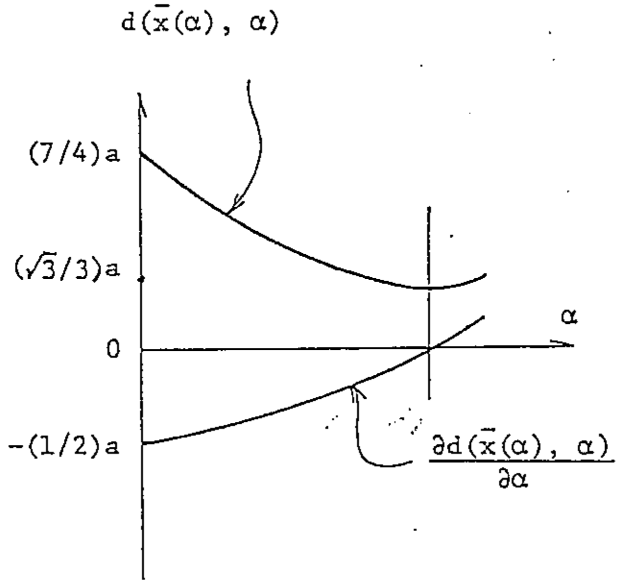


Figure C 1 *Minimum Transport Distance*

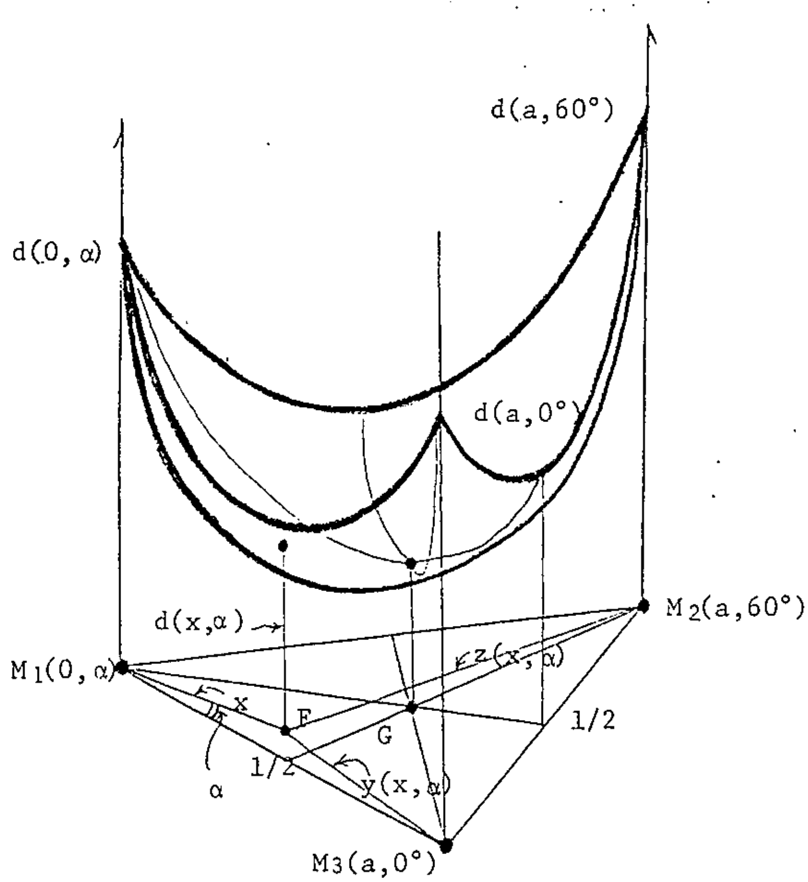


Figure C 2 *Transport Distance*

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FOOTNOTES

*1 This paper partly revised and considerably shortened the previous versions (Kusumoto[1980]). The research is financilly supported by the Tokyo Center for Economic Research. My thanks goes to many colleagues, including Mr.Yoshiro Higano[1980] who let me know this problem, and Dr. Yoshitsugu Kanemoto[1981] especially for suggesting a change in title. I owe also thanks to Professor Robert Miller, the editor, and two reviewers of this Journal for valuable advices on this earlier drafts. Fortunately, I had Professor and Mrs.Leonid Hurwicz of Minnesota to read the draft and work with me on it. They contributed in the exposition of this research. Remaining shortcomings and errors, however, are my responsibility.

*2 Dr.Atsuyuki Okabe of University of Tokyo kindly sent to me a Japanese translation of Weber's book[1929] *Kogyo-Ritti-Ron* translated by Nippon-Sangyo-Kōzo-Kenkyu-sho, Tai-Mei-Do, Tokyo, Japan 1941.

*3 See Debreu[1959 pp.28-35] for an example of treating " location " in economic theory, which abstracts *variable* transportation cost.

*4 The framework of analysis hence derived conclusions could be reformulate in terms of *any* triangle space. See Lemma 7 and its remark for this.

*5 This requires that f be concave.

*6 Except at vertex $M_2(a,0°)$ and $M_3(a,60°)$. See Remark of Discontinuity of Subsection 3.2.

*7 Dr.Kazumi Asako suggested this possibility, in which one or two markets may be perfectly substituted by a certain market for some C.I.F. prices.

*8 By this fact, the present analysis in the regular triangle does not lose any generality.

*9 Take Y = X, and B = A.

*10 We shall discuss this elsewhere. See our paper entitled " Leontief Technology and the Location Theory of the Firm - Specific Localization Theorems".