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Economies with Labor Indivisibilities Part II - Competitive Equilibria under Tax Schedules

by

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Abstract: In Part II, we prove that there is a competitive equilibrium under any net tax function, some amount of subsidy and production plan of public goods. In doing so, we need the assumptions of Part I and certain additional assumptions on utility functions and labor production functions. The additional assumptions are rather innocuous and do not prohibit non-convexities and indivisibilities on labor supplies. Further a necessary and sufficient condition for active competitive equilibria (positive production equilibria) is considered.

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1. Introduction

Part I of this paper formulates an optimal income taxation problem, in which the optimal tax schedule is defined to be a schedule which yields a competitive equilibrium with the maximal social welfare. In Part I, the existence of an optimal tax schedule is proved on condition (Assumption R) that there is at least one tax schedule having a competitive equilibrium. However if the set of such tax schedules was empty or small, then the result of Part I would be vacant. The standard existence proof of a competitive equilibrium can not be applied to the model of Part I even with the trivial tax schedule (no tax and no public goods), because it includes nonconvexities and indivisibilities on labor supplies.[1] Even if the existence is proved under the trivial tax schedule, it might be the case that only restrictive tax schedules yield competitive equilibria because of the nonconvexities of tax functions. Therefore we have to examine the class of tax schedules having competitive equilibria. The purpose of Part II is to give conditions for the existence of a competitive equilibrium.

In Part II, it will be shown under the assumptions of Part I and certain additional assumptions on utility functions and labor production functions that there exists a competitive equilibrium under any net tax function, some amount of subsidy and production plan of the public good. The additional assumptions are natural and innocuous, and still permit indivisibilities on labor supplies entirely. This existence result is in contrast with Mas-Collel (1977) and Yamazaki (1978), where the existence was proved in an exchange economy with indivisibilities on consumption sets under a certain

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1) Kaneko (1981) ensured the existence of a tax schedule having a competitive equilibrium by applying the standard proof to the case of the trivial tax schedule.

measure theoretic assumption.[2] This result means that the tax schedules having competitive equilibria form a wide class, which ensures that the maximization problem of social welfare in Part I is meaningful. Further we will examine a necessary and sufficient condition for active competitive equilibria (positive production equilibria).

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2) Of course, these models permit the nonconvexities of utility functions. It has been well-known since Aumann (1966) that the existence of a competitive equilibrium can be proved without any convexity assumption on utility functions.

Assumption H For any $r \in L$, there exists a subset S of I with $\mu(S) > 0$ having the property that for any $i \in S$ and $k \in C$, there are $t \in H$ and $\lambda > 0$ such that $f_r^i(t) > 0$ and $U^i(t, \lambda e^k, 0) > U^i(0,0,0)$.[4] Here e^k is the k-th unit vector of R^c.

Assumption I For any $i \in I$, $r \in L$ and $(t,x,Q) \in H \times R_+^{c+1}$ with $f_r^i(t) > 0$, there is a $t' \in H$ such that $f_k^i(t') \geqq f_k^i(t)$ for any $k \neq r$ and $U^i(t',x,Q) > U^i(t,x,Q)$.

Assumption H would be self-explanatory, but note that the choice of t and λ is not necessarily uniform on S. If we add condition $f_r^i(t')=0$ to the assertion of Assumption I, then Assumption I can be well-interpreted: When individual i is working in the k-th job and possibly in some others, he can increase his utility and ensure the present labor levels for the other jobs by resigning the k-th job on condition that the consumption level is fixed. But for the purpose of this paper, the additional condition is not necessary.

Remark 1. Assumption H implies that for all $r \in L$, there is a measurable function $h(i)$ such that $h(i) \in H$ for all $i \in I$ and $\int_I f_r^i(h(i)) > 0$.

Before the next assumption, we need further notations; $L(i)=\{r \in L \mid \text{there is a } t \in H \text{ with } f_r^i(t) > 0\}$ for $i \in I$ and $B(\varepsilon)=\{t \in R_-^{\ell} \mid 0 < \|t\| \leqq \varepsilon\}$ for $\varepsilon > 0$.[5] Note that $0 \notin B(\varepsilon)$ for all $\varepsilon > 0$.

Assumption J Either (J-1) or (J-2) holds.

(J-1) There exists an $\varepsilon_0 > 0$ such that $H \cap B(\varepsilon_0) = \phi$.

(J-2) For any $\varepsilon > 0$, $r \in L$ and all $i \in I$ with $L(i) \neq \phi$, there is a $t \in H \cap B(\varepsilon)$ with $f_r^i(t) > 0$.

Remark 2. In fact, it follows from Assumption J-2 that $L(i)=\phi$ or $L(i)=L$ for all $i \in I$. By Remark 1, the set $S^0 \equiv \{i \in I \mid L(i)=L\}$ has a positive measure.

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4) L and C are the index sets of labor and consumption goods respectively, i.e., $L=\{1,2,\ldots,\ell\}$, $C=\{\ell+1,\ell+2,\ldots,\ell+c\}$.

5) $\|\cdot\|$ is the Euclid norm.

Now we have to give comments on Assumptions (S-1) and (S-2). Assumption (S-1) requires that the upper bound of subsidies be larger than that of the government's possible revenues. Assumption (S-2) ensures the existence of at least one production possibility ray of the public good. If competitive equilibrium given in the main theorem is non-active, then the government's revenue is zero, but if it is active, then its revenue is positive unless the tax function is trivial (no tax). In this case, the government must subsidize individuals or produce the public good to equalize its budget. Assumption (S-1) or (S-2) allows the government to equalize its budget by subsidizing individuals or producing the public good. This reasoning yields the following proposition.

Proposition 1. Assume that $T(y) > 0$ for all $y > 0$ and that a competitive equilibrium (p,γ) under $(T-b, z^0)$ is active. Then

(1) if $Z^0=\{0\}$, then $b > 0$;

(2) if $B=0$ and $[\ z \in Z^0$ and $(z_L, z_C) < 0 \Rightarrow z_Q > 0\ ]$, then $Q > 0$.

The main theorem ensures the existence of a competitive equilibrium in both cases (1) and (2) of Proposition 1, because they are covered by Assumptions (S-1) and (S-2), respectively. The next proposition gives a necessary and sufficient condition for active competitive equilibria.

Proposition 2. Let T be fixed. Any competitive equilibrium (p^*, γ^*) under any tax schedule $(T-b^*, z^{0*})$ is active if and only if

(i) $p \in R_+^{\ell+c}$ and $p \cdot z \leqq 0$ for all $z \in \sum_{j=1}^{m} Z^j$,

implies

(ii) there is a subset S of I with a positive measure such that for all $i \in S$, $U^i(t,x,Q) > U^i(0,0,0)$ for some $(t,x) \in H \times R_+^c$ with $p_C \cdot x \leqq (1-T)[p_L \cdot f^i(t)]$.

Proof. Necessity: Let (i) hold but not (ii). Put $b^*=0$ and $z^{0*}=0$. Then, $(t^*(i),x^*(i)) = (0,0)$ maximizes i's utility under the budget constraint $p_C \cdot x \leqq (1-T)[p_L \cdot f^i(t)]$ for almost all $i \in I$. If $p \cdot z^j > 0$ for some $z^j \in Z^j$, then $z=0+0+\ldots+z^j+\ldots+0 \in \sum_{j=1}^m Z^j$ gives a positive profit $p \cdot z = p \cdot z^j > 0$, which is a contradiction to (i). Hence $z^{j*}=0$ gives the j's maximal profit $p \cdot z^{j*}=0$. Therefore $(p,(t^*(i),x^*(i),0),z^{1*},z^{2*},\ldots,z^{m*})$ and (b^*,z^{0*}) form a non-active competitive equilibrium under $(T-b^*,z^{0*})$. This is a contradiction.

Sufficiency: Suppose there is a non-active competitive equilibrium (p^*,γ^*) under $(T-b^*,z^{0*})$, i.e., $(t^*(i),x^*(i))=(0,0)$ a.e. $z^{1*}=z^{2*}=z^{3*}=\ldots=z^{m*}=0$, $b^*=0$ and $z^{0*}=0$. Since $p^* \cdot z^j \leqq p^* \cdot z^{j*}=0$ for all $z^j \in Z^j$ $(j=1,2,\ldots,m)$, $p^* \cdot z = p^* \cdot \sum_{j=1}^m z^j \leqq p^* \cdot \sum_{j=1}^m z^{j*}=0$ for all $z=\sum_{j=1}^m z^j \in \sum_{j=1}^m Z^j$, i.e., (i) holds. Therefore (ii) holds for p^*. This contradicts that (0,0) maximizes i's $(i \in S)$ utility under his budget. Hence every competitive equilibrium (p^*,γ^*) under $(T-b^*,z^{0*})$ is active. Q.E.D.

The necessary and sufficient condition can be varbally stated: When $p \cdot z \leqq 0$ for all $z \in \sum_{j=1}^m Z^j$, i.e., the labor prices are relatively higher than the prices of consumption goods, individuals with a positive measure supply positive amounts of labor and make positive consumptions.

Finally we give a schetch of the proof of the main theorem, for the proof is quite long and divided into many steps.

Sketch of the proof of the main theorem. The proof is divided into three main steps.

The first step (section 3): We approximate the consumption sets and production possibility sets by some large bounded sets. Under Assumption (J-2), it is proved that there exists a competitive equilibrium in the bounded economy.

The second step (section 4): It is proved that a limit of competitive equilibria in approximate bounded economies forms a true competitive equilibrium in the economy with (J-2).

The third step (section 5): An economy with (J-1) is approximated by economies with (J-2). It is shown that a limit of competitive equilibria in economies with (J-2), whose existence are proved in the second step, forms a competitive equilibrium in the economy with (J-1).

3. Auxiliary Theorem.

3.1 First we state the following lemma due to Aumann (1965, Theorems 1, 2,4) or Hildenbrand (1974,p.62,Theorems 2,3 and p.73,Proposition 7).

Lemma 3.1. Let $F(i)=f^i(H)$ for all $i \in I$. Then

$$\int_I F(i) = \{\int_I f^i(t(i)) \mid t(i) \text{ is a measurable function with } t(i) \in H \text{ a.e.}\}$$

is a non-empty compact convex set.

Let

$$\bar{Z}^j=\{z^j \in Z^j \mid \sum_{h=0}^{m} z_C^h \geqq 0,\ \sum_{h=0}^{m} z_L^h \in \int_I F(i),\ z_Q^0 \geqq 0 \text{ for some } z^h \in Z^h\ h=0,1,\ldots,m, h\neq j\}$$

for $j=0,1,2,\ldots,m$. It is easily verified that

$$\bar{Z}^j \text{ is a non-empty compact convex set for } j=0,1,\ldots,m. \tag{1}$$

Let $\alpha = \min(\mu(S^0),1) > 0$, where S^0 is the set given in Remark 2. From (1) we can take a positive number M such that

$$H \subset [-M,0]^{\ell},\ \sum_{j=1}^{m} \bar{Z}^j \subset (-\alpha M,\alpha M)^{\ell+c},\ \bar{Z}^0 \subset (-M,M)^{\ell+c+1}. \tag{2}$$

Note that $(-\alpha M,\alpha M)^{\ell+c}$ and $(-M,M)^{\ell+c+1}$ are open sets. For any positive integer h, we define

$$E_h = [-hM,hM]^{\ell+c},\quad \bar{E}_h = [-hM,hM]^{\ell+c+1}. \tag{3}$$

Definition 2. Let h be a positive integer. A pair $(p,\gamma)=(p,(t(i),x(i),Q), z^1,z^2,\ldots,z^m)$ is called an h-bounded competitive equilibrium under a tax schedule $(T-b,z^0)$ if

(i) $p=(p_L,p_C) \in P=\{p \in R_+^{\ell} \times R_+^{c} \mid \Sigma p_k=1\}$;

(ii) $t(i)$ and $x(i)$ are measurable functions of i with $(t(i),x(i),Q) \in \Omega \cap \bar{E}_h$ for all $i \in I$, $z^j \in Z^j \cap E_h$ for $j=1,2,\ldots,m$, $z^0 \in Z^0 \cap \bar{E}_h$, $z_Q^0 = Q$, $\sum_{j=0}^{m} z_C^j \geqq \int_I x(i)$ and $-\sum_{j=0}^{m} z_L^j \leqq \int_I f^i(t(i))$;

(iii) $p_C \cdot x(i) \leqq (1-T)[p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j^h(p)] + b$ for almost all $i \in I$;

(iv) for almost all $i \in I$, $U^i(t(i), x(i), Q) \geqq U^i(t, x, Q)$ for all $(t, x) \in (H \times R_+^C) \cap E_h$ with $p_C \cdot x \leqq (1-T)[p_L \cdot f^i(t) + \Sigma\theta_j(i)\pi_j^h(p)] + b$;

(v) $\pi_j^h(p) \equiv p \cdot z^j \geqq p \cdot z$ for all $z \in Z^j \cap E_h$ for $j=1,2,\ldots,m$.

This section proves the following theorem.

Auxiliary Theorem. Make Assumptions A ∿ I, J-2, K ∿ R and either S-1 or S-2. Then for any positive integer h, there exists an h-bounded competitive equilibrium (p, γ) under a tax schedule $(T-b, z^0)$ for some $b \in [0,B]$ and $z^0 \in Z^0 \cap \bar{E}_h$. Moreover it holds that

(i) $p_L \geqq 0$ and $p_C > 0$;

(ii) $\pi_j^h(p) = 0$ for $j=1,2,\ldots,m$;

(iii) if Assumption S-1 holds, then

$$-p_C \cdot z_C^0 - p_L \cdot z_L^0 + b\mu(I) = [p_L \cdot \int_I f^i(t(i)) - p_C \cdot \int_I x(i) + b\mu(I)]_0^B ;$$

(iv) if Assumption S-2 holds, then

$$-p_C \cdot z_C^0 - p_L \cdot z_L^0 = p_L \cdot \int_I f^i(t(i)) - p_C \cdot \int_I x(i) .$$

Here $[\cdot]_0^B$ denotes

$$[y]_0^B = B\mu(I) \quad \text{if } y > B\mu(I),$$
$$= y \quad \text{if } 0 \leqq y \leqq B\mu(I),$$
$$= 0 \quad \text{if } y < 0,$$

for $y \in R_+$.

The following two subsections prove this theorem in the case of Assumption S-1, and subsection 3.3 does in the case of Assumption S-2.

3.2. First for any positive integer h, we prove that there exist $(p^*,(t^*(i), x^*(i),Q^*),z^{1*},z^{2*},\dots,z^{m*})$, $b^*\in[0,B]$ and $z^{0*}\in Z^0\cap\bar{E}_h$ which satisfy (i),(ii),(iii),(v) of Definition 2 and

(iv)' for almost all $i\in I$, if $p_r^*>0$ for some $r\in L(i)$ or $b^*>0$, then $U^i(t^*(i),x^*(i),Q^*)\geqq U^i(t,x,Q^*)$ for all $(t,x)\in B_{p^*}^h(i)$, and if $L(i)=\phi$ and $b^*=0$, then $t^*(i)=0$.

Here $B_p^h(i)=\{(t,x)\in(H\times R_+^C)\cap E_h \mid p_C\cdot x\leqq(1-T)[p_L\cdot f^i(t)+\Sigma\theta_j(i)\pi_j^h(p)]+b\}$.

We define demand and supply functions $D^i(p,Q,b)$ $(i\in I)$, $D(p,Q,b)$, $S^j(p)$ $(j=1,2,\dots,m)$ and $S(p)$ by

$$
\begin{aligned}
D^i(p,Q,b) &= \{(t',x')\in B_p^h(i)\mid U^i(t',x',Q)\geqq U^i(t,x,Q) \text{ for all } (t,x)\in B_p^h(i)\} \\
&\qquad\qquad \text{if } p_r>0 \text{ for some } r\in L(i) \text{ or } b>0, \\
&= \{(t',x')\in B_p^h(i)\mid t'=0\} \quad \text{if } L(i)=\phi \text{ and } b=0, \\
&= B_p^h(i) \quad \text{otherwise;}
\end{aligned}
\tag{4}
$$

$$D(p,Q,b) = \{(-\int_I f^i(t(i)),\ \int_I x(i)) \mid \text{there are measurable functions } t(i) \text{ and } x(i) \text{ with } (t(i),x(i))\in D^i(p,Q,b) \text{ a.e.}\};$$

$$S^j(p) = \{z^j\in Z^j\cap E_h \mid p\cdot z^j\geqq p\cdot z \text{ for all } z\in Z^j\cap E_h\};$$

$$S(p) = \sum_{j=1}^m S^j(p).$$

The following lemma is verified in the standard way. See Nikaido (1970, Section 46) for example.

Lemma 3.2. $S^j(p)$ $(j=1,2,\dots m)$ and $S(p)$ are non-empty convex valued and upper-semicontinuous correspondences on P. $\pi_j^h(p)$ $(j=1,2,\dots,m)$ are continuous functions on P.

Lemma 3.3. $D^i(p,Q,b)$ is a non-empty valued and upper-semicontinuous correspondence on $P \times [-hM,hM] \times [0,B]$ for all $i \in I$.

Proof. Clearly $D^i(p,Q,b)$ is non-empty for any $(p,Q,b) \in P \times [-hM,hM] \times [0,B]$. We show the upper-semicontinuity of $D^i(p,Q,b)$. Let $t^s \to t^0$, $x^s \to x^0$, $p^s \to p^0$, $Q^s \to Q^0$, $b^s \to b^0$ as $s \to \infty$ and $(t^s,x^s) \in D^i(p^s,Q^s,b^s)$ for all s. We will show $(t^0,x^0) \in D^i(p^0,Q^0,b^0)$. First note that

$$p_C^0 \cdot x^0 \leqq (1-T)[p_L^0 \cdot f^i(t^0) + \Sigma\theta_j(i)\pi_j^h(p^0)] + b^0, \text{ i.e., } (t^0,x^0) \in B_{p^0}^h(i), \tag{5}$$

because $p_C^s \cdot x^s \leqq (1-T)[p_L^s \cdot f^i(t^s) + \Sigma\theta_j(i)\pi_j^h(p^s)] + b^s$ for all s.

If $p_r^0=0$ for any $r \in L(i) \neq \phi$ and $b^0=0$, then $(t^0,x^0) \in B_{p^0}^h(i) = D^i(p^0,Q^0,b^0)$ by (4). Consider the case $p_r^0 > 0$ for some $r \in L(i)$ or $b^0 > 0$. Then for large s, $p_r^s > 0$ or $b^s > 0$. Hence

$$U^i(t^s,x^s,Q^s) \geqq U^i(t,x,Q^s) \quad \text{for all } (t,x) \in B_{p^s}^h(i). \tag{6}$$

We prove $U^i(t^0,x^0,Q^0) \geqq U^i(t,x,Q^0)$ for all $(t,x) \in B_{p^0}^h(i)$. Suppose there is a $(\bar{t},\bar{x}) \in B_{p^0}^h(i)$ such that $U^i(t^0,x^0,Q^0) < U^i(\bar{t},\bar{x},Q^0)$. If $0 < (1-T)[p_L^0 \cdot f^i(\bar{t}) + \Sigma\theta_j(i) \times \pi_j^h(p^0)] + b^0$, then there exists an x' near $\bar{x}$ such that $U^i(t^0,x^0,Q^0) < U^i(\bar{t},x',Q^0)$ and $p_C^0 \cdot x' < (1-T)[p_L^0 \cdot f^i(\bar{t}) + \Sigma\theta_j(i)\pi_j^h(p^0)] + b^0$, which implies for large s

$$U^i(t^s,x^s,Q^s) < U^i(\bar{t},x',Q^s) \text{ and } p_C^s \cdot x' < (1-T)[p_L^s \cdot f^i(\bar{t}) + \Sigma\theta_j(i)\pi_j^h(p^s)] + b^s.$$

This contradicts (6). Let $(1-T)[p_L^0 \cdot f^i(\bar{t}) + \Sigma\theta_j(i)\pi_j^h(p^0)] + b^0 = 0$. Since $b^0=0$ by this equation, $p_r^0 > 0$ for some $r \in L(i)$ because we are considering the case $p_r^0 > 0$ for some $r \in L(i)$ or $b^0 > 0$. By Assumption B(ii), we have

$$U^i(t^0,x^0,Q^0) < U^i(\bar{t},\bar{x},Q^0) \leqq U^i(0,\bar{x},Q^0).$$

Hence there is an $\varepsilon > 0$ such that $U^i(t^0,x^0,Q^0) < U^i(t,\bar{x},Q^0)$ for all t with $\|t\| < \varepsilon$. Then by Assumption J-2, there is a $t' \in H$ with $\|t'\| < \varepsilon$ and $f_r^i(t') > 0$. Hence we have

$$U^i(t^0,x^0,Q^0) < U^i(t',\bar{x},Q^0) \quad \text{and} \quad f_r^i(t') > 0. \tag{7}$$

Since $p_L^0 \cdot f^i(t') + \Sigma\theta_j(i)\pi_j^h(p^0) \geqq p_r^0 f_r^i(t') > 0$, $(1-T)[p_L^0 \cdot f^i(t') + \Sigma\theta_j(i)\pi_j^h(p^0)] + b^0 > 0$ by Assumption R. This implies

$$p_C^0 \cdot \overline{x} = 0 < (1-T)[p_L^0 \cdot f^i(t') + \Sigma\theta_j(i)\pi_j^h(p^0)] + b^0.$$

Hence, $p_C^s \cdot \overline{x} < (1-T)[p_L^s \cdot f^i(t') + \Sigma\theta_j(i)\pi_j^h(p^s)] + b^s$ for large s. Noting (6), we have $U^i(t^s, x^s, Q^s) \geqq U^i(t', \overline{x}, Q^s)$. As $s \to \infty$, we have $U^i(t^0, x^0, Q^0) \geqq U^i(t', \overline{x}, Q^0)$, which is a contradiction to (7).

Finally we consider the case $L(i) = \phi$ and $b^0 = 0$. If $b^s = 0$, then $(t^s, x^s) \in D^i(p^s, Q^s, b^s) = \{(t,x) \in B_{p^s}^h(i) \mid t = 0\}$. This implies $t^s = 0$ for all s with $b^s = 0$. Let $b^s > 0$. Then (6) holds by (4). We show $t^s = 0$. Suppose $t^s \neq 0$. By Assumption B(ii), $U^i(0, x^s, Q^s) > U^i(t^s, x^s, Q^s)$. From (6), $p_C^s \cdot x^s > (1-T)[p_L^s \cdot f^i(0) + \Sigma\theta_j(i)\pi_j^h(p^s)] + b^s = (1-T)[\Sigma\theta_j(i)\pi_j^h(p^s)] + b^s$. But it follows from $L(i) = \phi$ and $(t^s, x^s) \in B_{p^s}^h(i)$ that

$$p_C^s \cdot x^s \leqq (1-T)[p_L^s \cdot f^i(t^s) + \Sigma\theta_j(i)\pi_j^h(p^s)] + b^s = (1-T)[p_L^s \cdot 0 + \Sigma\theta_j(i)\pi_j^h(p^s)] + b^s$$
$$= (1-T)[\Sigma\theta_j(i)\pi_j^h(p^s)] + b^s.$$

This is a contradiction. Hence $t^s = 0$ for all s with $b^s > 0$. Consequently we have $t^s = 0$ for all s, which implies $t^0 = 0$. Noting (4), we have $(t^0, x^0) \in \{(t,x) \in B_{p^0}^h(i) \mid t = 0\} = D^i(p^0, Q^0, b^0)$.

Q.E.D.

It can be proved in the standard way that $D^i(p,Q,b)$ is a measurable correspondence of i. Then the next lemma can be proved by using Theorems 1,2,4 in Aumann (1965) or Theorems 2,3 (p.62), Proposition 7 (p.73) in Hildenbrand (1974).

Lemma 3.4. $D(p,Q,b)$ is a non-empty convex valued and upper-semicontinuous correspondence on $P \times [-hM, hM] \times [0,B]$.

For $p \in P$, $y \in E_h$, $b \in [0,B]$, we define the government's excess supply function $G(p,y,b)$ by

$$G(p,y,b) = \{(\tilde{z}^0, \tilde{b}) \in (\overline{E}_h \cap Z^0) \times [0,B] \mid -p_C \cdot \tilde{z}_C^0 - p_L \cdot \tilde{z}_L^0 + \tilde{b}\mu(I) = [p \cdot y + b\mu(I)]_0^B\}. \quad (8)$$

Lemma 3.5. $G(p,y,b)$ is a non-empty convex valued and upper-semicontinuous correspondence on $P \times E_h \times [0,B]$.

Proof. The convexity of $G(p,y,b)$ is clear. The upper-semicontinuity of $G(p,y,b)$ is easily verified from the continuity of $[\cdot]_0^B$. Since $0 \leqq [p\cdot y+b\mu(I)]_0^B \leqq B\mu(I)$, $0 \leqq \frac{[p\cdot y+b\mu(I)]_0^B}{\mu(I)} \leqq B$. Hence $\tilde{z}^0=0$ and $\tilde{b}=\frac{[p\cdot y+b\mu(I)]_0^B}{\mu(I)}$ satisfy $(\tilde{z}^0,\tilde{b}) \in (\overline{E}_h \cap Z^0) \times [0,B]$ and $-p_C\cdot\tilde{z}_C^0 - p_L\cdot\tilde{z}_L^0 + \tilde{b}\mu(I) = \tilde{b}\mu(I) = [p\cdot y+b\mu(I)]_0^B$, which shows the non-emptiness of $G(p,y,b)$. Q.E.D.

Put

$$\xi(a) = \{p \in P \mid p\cdot a \leqq q\cdot a \text{ for all } q \in P\} \tag{9}$$

for any $a \in E_h$. Then $\xi(a)$ is a non-empty convex valued and upper-semicontinuous correspondence on E_h. We define a correspondence ψ from $P\times E_h \times \overline{E}_h \times [0,B]$ to itself by

$$\psi(p,y,(z,Q),b) = \xi(y+z) \times [S(p)-D(p,Q,b)] \times G(p,y,p)$$

for $(p,y,(z,Q),b) \in P\times E_h \times \overline{E}_h \times [0,B]$. It follows from Lemmas 3.2, 3.4 and 3.5 that $\psi(p,y,(z,Q),b)$ is a non-empty convex valued and upper-semicontinuous correspondence on the compact set $P\times[0,B]\times E_h \times \overline{E}_h$ to itself. By Kakutani's Fixed Point Theorem [Nikaido (1970, Theorem 44.1)], there exists a fixed point $(p^*,y^*,(z^*,Q^*),b^*) \in \psi(p^*,y^*,(z^*,Q^*),b^*)$, that is,

$$p^* \in \xi(y^*+z^*) \tag{10}$$

$$y^* \in S(p^*) - D(p^*,Q^*,b^*) \tag{11}$$

$$((z^*,Q^*),b^*) \in G(p^*,b^*,y^*). \tag{12}$$

Lemma 3.6. (Walrass law) For all $p \in P$, $Q \in [-hM,hM]$, $b \in [0,B]$,

$$p\cdot\tilde{y} + b\mu(I) \geqq 0 \quad \text{for any } \tilde{y} \in S(p)-D(p,Q,b).$$

Proof. Since $\tilde{y} \in S(p)-D(p,Q,b)$, there are $z^j \in Z^j$ $(j=1,2,\ldots,m)$ and measurable functions $t(i)$, $x(i)$ such that $z^j \in S^j(p)$ $(j=1,2,\ldots,m)$ and $(t(i),x(i)) \in D^i(p,Q,b)$ a.e. and $\tilde{y}=\sum_{j=1}^m z^j-(-\int_I f^i(t(i)), \int_I x(i))$. Hence

$$\begin{aligned} p_C\cdot x(i) &\leqq (1-T)[p_L\cdot f^i(t(i))+\sum_{j=1}^m \theta_j(i)\pi_j^h(p)]+b \\ &\leqq p_L\cdot f^i(t(i))+\sum_{j=1}^m \theta_j(i)\pi_j^h(p)+b, \end{aligned}$$

for almost all $i \in I$. This implies

$$p_C\cdot\int_I x(i) \leqq p_L\cdot\int_I f^i(t(i)) + \sum_{j=1}^m\int_I\theta_j(i)\pi_j^h(p)+b\mu(I)=p_L\cdot\int_I f^i(t(i))+\sum_{j=1}^m p\cdot z^j+b\mu(I).$$

Hence we have

$$p\cdot\tilde{y}+b\mu(I)=-p_C\cdot\int_I x(i)+p_L\cdot\int_I f^i(t(i))+\sum_{j=1}^m p\cdot z^j+b\mu(I) \geqq 0. \qquad \text{Q.E.D.}[6]$$

By this lemma and (11), we have $p^*\cdot y^*+b^*\mu(I) \geqq 0$. Hence it follows from (8) and (12) that

$$-p_C^*\cdot z_C^*-p_L^*\cdot z_L^*+b^*\mu(I)=[p^*\cdot y^*+b^*\mu(I)]_0^B \leqq p^*\cdot y^*+b^*\mu(I),$$

which implies $-p^*\cdot z^* \leqq p^*\cdot y^*$, i.e., $p^*\cdot(y^*+z^*) \geqq 0$. But by (9) and (10),

$$q\cdot(y^*+z^*) \geqq p^*\cdot(y^*+z^*) \geqq 0 \quad \text{for all } q \in P.$$

If $q=(0,0,\ldots,\overset{k}{1},\ldots,0)$ for $k \in C \cup L$, then $y_k^*+z_k^* \geqq 0$. This implies

$$y^* + z^* \geqq 0. \tag{13}$$

Let $z^{0*}=(z^*,Q^*)$. Then $z^{0*} \in \bar{E}_h \cap Z^0$ by (12). It follows from (11) and

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6) The exact form of Walrass law should be;

$$p\cdot\tilde{y}+b\mu(I)-\int_I T[p_L\cdot f^i(t(i))+\sum_{j=1}^m \theta_j(i)\pi_j^h(p)] \geqq 0 \quad \text{for any } \tilde{y} \in S(p)-D(p,Q,b),$$

but the above form is enough for our purpose.

the definitions of $S(p^*)$ and $D(p^*,Q^*,b^*)$ that there exist $z^{j*} \in S^j(p^*)$ ($j=1,2,\ldots m$) and measurable functions $t^*(i)$, $x^*(i)$ such that $(t^*(i),x^*(i)) \in D^i(p^*,Q^*,b^*)$ a.e. and $y^* = \sum_{j=1}^m z^{j*} - (-\int_I f^i(t^*(i)), \int_I x^*(i))$. These $(p^*,(t^*(i), x^*(i),Q^*), z^{1*}, z^{2*},\ldots,z^{m*})$, b^* and z^{0*} satisfy (i)-(iii), (v) of Definition 2 and (iv)'. Indeed, (v) is equivalent to $z^{j*} \in S^j(p^*)$ ($j=1,2,\ldots,m$) and (iii), (iv)' follow from $(t^*(i),x^*(i)) \in D^i(p^*,Q^*,b^*)$ a.e. Moreover (13) and $y^* = \sum_{j=1}^m z^{j*} - (-\int_I f^i(t^*(i)), \int_I x^*(i))$ imply

$$\left(\sum_{j=0}^m z_L^{j*} + \int_I f^i(t^*(i)),\ \sum_{j=0}^m z_C^{j*} - \int_I x^*(i)\right)$$

$$= (z_L^{0*}, z_C^{0*}) + \sum_{j=1}^m (z_L^{j*}, z_C^{j*}) - \left(-\int_I f^i(t^*(i)), \int_I x^*(i)\right) = y^* + z^* \geqq 0,$$

which is (ii) of Definition 2.

<u>3.3</u>. This subsection completes the proof of the case (S-1). Lemma 3.7 proves (ii) of the Auxiliary Theorem.

<u>Lemma 3.7</u>. $\pi_j^h(p^*)=0$ for $j=1,2,\ldots,m$.

<u>Proof</u>. Since $z^{j*} \in \bar{Z}^j$, we have $z^{j*} \in \text{Int } E_h$ ($j=1,2,\ldots,m$) by (2) and (3).[7] Suppose $\pi_j^h(p^*) = p^* \cdot z^{j*} > 0$ for some j. Then for a sufficiently small $\alpha > 0$, $(1+\alpha) z^{j*} \in E_h$ and also $(1+\alpha) z^{j*} \in Z^j$ by Assumption L. Hence $p^* \cdot (1+\alpha) z^{j*} = (1+\alpha) p^* \cdot z^{j*} > p^* \cdot z^{j*}$, which contradicts (v) of Definition 2. Q.E.D.

The following two lemmas prove (i) of the Auxiliary Theorem.

<u>Lemma 3.8</u>. $p_L^* \geq 0$.

<u>Proof</u>. Suppose $p_L^*=0$. Then $p_C^* \geq 0$ by $p^* \in P$. By Assumption M, there is a $\bar{z} = \sum_{j=1}^m \bar{z}^j \in \sum_{j=1}^m Z^j$ such that $\bar{z}_C > 0$. Assumption L implies $\alpha\bar{z} \in \sum_{j=1}^m Z^j$ for all $\alpha>0$.

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7) Int X denotes the interior of X.

Clearly, $\alpha\bar{z} \in E_h \cap \sum_{j=1}^{m} Z^j$ for sufficiently small $\alpha>0$. Then $p^* \cdot (\alpha\bar{z}) = \alpha p^* \cdot \bar{z} > 0$ because $p_L^* = 0$, $p_C^* \geq 0$ and $\bar{z}_C > 0$. This implies that $\pi_j^h(p^*) \geqq p^* \cdot \alpha\bar{z}^j > 0$ for some j, which is a contradiction to Lemma 3.7. Q.E.D.

<u>Lemma 3.9</u>. $p_C^* > 0$.

<u>Proof</u>. Suppose $p_k^*=0$ for some $k \in C$. For any individual i in the set S^0 given in Remark 2, we have, by (iv)' of subsection 3.2 and $p_L^* \geq 0$,

$$U^i(t^*(i), x^*(i), Q^*) \geqq U^i(t, x, Q^*) \quad \text{for all } (t,x) \in B_{p^*}^h(i).$$

Since hM is the maximum of the k-th consumption goods that each individual i can get, it follows from Assumption B(i) and $p_k^* = 0$ that $x_k^*(i) = hM$. Hence by (ii) of Definition 2,

$$\sum_{j=0}^{m} z_k^{j*} \geqq \int_I x_k^*(i) \geqq \int_{S^0} x_k^*(i) = hM\mu(S^0).$$

On the other hand, we have $\sum_{j=0}^{m} z_k^{j*} < \alpha M \leqq M\mu(S^0) \leqq hM\mu(S^0)$ by $z^{j*} \in \bar{Z}^j$ and (2). This is a contradiction. Q.E.D.

<u>Lemma 3.10</u>. For almost all $i \in I$, $U^i(t^*(i), x^*(i), Q^*) \geqq U^i(t, x, Q^*)$ for all $(t,x) \in B_{p^*}^h(i)$.

<u>Proof</u>. Since the above assertion is always true in the case $b^* > 0$ by (iv)' of subsection 3.2, we may assume $b^*=0$.

Let $L(i) \neq \phi$. Then $L(i)=L$ by Remark 2. Since $p_r^* > 0$ for some $r \in L$ by Lemma 3.8, (iv)' of subsection 3.2 together with $L(i)=L$ implies the above assertion.

Let $L(i) = \phi$. Then $t^*(i)=0$ from $b^*=0$ and (iv)' of subsection 3.2. Since $L(i)=\phi$ and $b^*=0$, we have, by Lemma 3.7,

$$0 \leqq p_C^* \cdot x \leqq (1-T)[p_L^* \cdot f^i(t) + \Sigma\theta_j(i)\pi_j^h(p^*)] + b^* = 0 \quad \text{for any } (t,x) \in B_{p^*}^h(i).$$

By this and Lemma 3.9, x=0 for any $(t,x) \in B^h_{p*}(i)$. Hence from Assumption B(ii), $U^i(0,x^*(i),Q^*) \geqq U^i(0,0,Q^*) \geqq U^i(t,0,Q^*)$, which shows $U^i(t^*(i),x^*(i),Q^*) \geqq U^i(t,x,Q^*)$ for any $(t,x) \in B^h_{p*}(i)$. Q.E.D.

Thus we have shown that (iv) of Definition 2 besides (iv)' is true. This togeter with the assertion of subsection 3.2 says that $(p^*,(t^*(i), x^*(i),Q^*),z^{1*},z^{2*},\ldots z^{m*})$ is an h-bounded competitive equilibrium under $(T-b^*,z^{0*})$.

Finally we show (iii) of the Auxiliary Theorem. By (8), we have $-p^*_L\cdot z^{0*}_L -p^*_C\cdot z^{0*}_C+b^*\mu(I) = [p^*\cdot y^* + b^*\mu(I)]^B_0$. Here $y^*=(\sum_{j=1}^m z^{j*}_L+\int_I f^i(t^*(i)),\ \sum_{j=1}^m z^{j*}_C-\int_I x^*(i))$.

Hence

$$-p^*_L\cdot z^{0*}_L-p^*_C\cdot z^{0*}_C+b^*\mu(I)=[\sum_{j=1}^m (p^*_L,p^*_C)\cdot(z^{j*}_L,z^{j*}_C)+p^*_L\cdot\int_I f^i(t^*(i))-p^*_C\cdot\int_I x^*(i)+b^*\mu(I)]^B_0$$

$$=[p^*_L\cdot\int_I f^i(t^*(i))-p^*_C\cdot\int_I x^*(i)+b^*\mu(I)]^B_0.$$

This completes the proof of the Auxiliary Theorem in the case (S-1).

<u>3.4</u>. In the case of Assumption S-2, we have to modify the definitions of $\overline{E}_h$ as follows;

$$\overline{E}'_h = [-hM,\ hM]^{\ell+c} \times [0,\eta],$$

where $\eta = \max(\ hM,\ \sup\{z^0_Q \mid (-\lambda,-\lambda,-\lambda,\ldots,-\lambda,z^0_Q)\in Z^0$ for some $\lambda\ (0\leqq\lambda\leqq M)\}\ )$.
We also modify G(p,y,b) into G*(p,y,b);

$$G^*(p,y,b)=\{(\tilde{z}^0,\tilde{b})\in(\overline{E}'_h\cap Z^0)\times[0,B] \mid -p_C\cdot\tilde{z}^0_C-p_L\cdot\tilde{z}^0_L+b\mu(I)=[p\cdot y+b\mu(I)]^*\},$$

where
$$\begin{aligned}[p\cdot y+b\mu(I)]^* &= M + B\mu(I) && \text{if } p\cdot y + b\mu(I) > M+B\mu(I),\\ &= p\cdot y+ b\mu(I) && \text{if } 0\leqq p\cdot y+b\mu(I) \leqq M+B\mu(I),\\ &= 0 && \text{if } p\cdot y+b\mu(I) < 0.\end{aligned}$$

Then $G^*(p,y,b)$ is non-empty convex valued and upper-semicontinuous correspondence. Indeed, if $[p\cdot y+b\mu(I)]^* \leqq B\mu(I)$, then $((0,0,\ldots,0),\frac{[p\cdot y+b\mu(I)]^*}{\mu(I)}) \in G^*(p,y,b)$. Let $[p\cdot y+b\mu(I)]^* > B\mu(I)$. Then put $\lambda=[p\cdot y+b\mu(I)]^*-B\mu(I)$. Assumption S-2 ensures that $(-\lambda,-\lambda,\ldots,-\lambda, z_Q^0) \in Z^0$ for some z_Q^0. Since $0<\lambda \leqq M$, we have $(-\lambda,-\lambda,\ldots,-\lambda,z_Q^0) \in Z^0 \cap \bar{E}_h'$ by the definition of $\bar{E}_h'$. Then $((-\lambda,-\lambda,\ldots,-\lambda, z_Q^0),B)$ satisfies $-p_C\cdot(-\lambda,-\lambda,\ldots,-\lambda)-p_L\cdot(-\lambda,-\lambda,\ldots,-\lambda)+B\mu(I)= \lambda+B\mu(I)=[p\cdot y+b\mu(I)]^*$, which implies $((-\lambda,-\lambda,\ldots,-\lambda,z_Q^0),B) \in G^*(p,y,b)$. Thus $G^*(p,y,b)$ is non-empty. The convexity and upper-semicontinuity of G^* are verified easily. Replacing G by G^*, the above proof of the Auxiliary Theorem holds without any change.

The statement (iv) is obtained as follows. It follows from the individuals' budget constraint that

$$p^*\cdot y^* + b^*\mu(I) = \sum_{j=1}^{m} p^*\cdot z^{j*} + p_L^*\cdot\int_I f^i(t^*(i)) - p_C^*\cdot\int_I x^*(i)+b^*\mu(I)$$

$$\geqq \int_I T[p_L^*\cdot f^i(t^*(i))] \geqq 0.$$

By the definition of M and B,

$$p^*\cdot y^*+b^*\mu(I) = \sum_{j=1}^{m} p^*\cdot z^{j*}+p_L^*\cdot\int_I f^i(t^*(i)) - p_C^*\cdot\int_I x^*(i) + b^*\mu(I) \leqq M+B\mu(I).$$

Hence

$$-p_C^*\cdot z_C^{0*}-p_L^*\cdot z_L^{0*}+b^*\mu(I) = p^*\cdot y^*+b^*\mu(I)=\sum_{j=1}^{m} p^*\cdot z^{j*}+p_L^*\cdot\int_I f^i(t^*(i))-p_C^*\cdot\int_I x^*(i)+b^*\mu(I).$$

This implies (iv) by Lemma 3.7.

4. The case of Assumption J-2.

This section will prove the following theorem.

Theorem 1. Make Assumptions A∿I, J-2, K∿R and either S-1 or S-2. Then there exists a competitive equilibrium under a tax schedule $(T-b,z^0)$ for some $b \in [0,B]$ and $z^0 \in Z^0$.

4.1. The Auxiliary Theorem ensures the existence of an h-bounded competitive equilibrium $(p,(t(i),x(i),Q),z^1,z^2,\dots,z^m)$ under $(T-b,z^0)$ for some b and z^0. Hence there are subsequences $(p^s,\gamma^s) = (p^s,(t^s(i),x^s(i),Q^s),z^{1s},z^{2s},\dots,z^{ms})$ and (b^s,z^{0s}) such that for $s=1,2,\dots$, (p^s,γ^s) is an s-bounded competitive equilibrium under $(T-b^s,z^{0s})$.

Remember the utility maximization in the budget set $B^s_{p^s}(i)$ is required for almost all individuals in the Auxiliary Theorem. Let $I^s=\{i \in I \mid (t^s(i), x^s(i))$ maximizes i's utility in his budget set $B^s_{p^s}(i)\}$. Then since $\mu(I^s)=\mu(I)$ for all s, $\mu(\bigcap_{s=1}^{\infty} I^s)=\mu(I)$. Therefore we can confine ourselves to the set $\bigcap_{s=1}^{\infty} I^s$ instead of I. In the following we assume;

$$\text{for all } i \in I, \text{ all } s,\ U^i(t^s(i),x^s(i),Q^s) \geqq U^i(t,x,Q^s) \text{ for all } (t,x) \in B^s_{p^s}(i) \text{ and } (t^s(i),x^s(i)) \in B^s_{p^s}(i). \tag{14}$$

Since $p^s \in P$, $z^{js} \in \bar{Z}^j (j=0,1,\dots,m)$, $b^s \in [0,B]$ and $Q^s=z^{0s}_Q$ for $s=1,2,\dots$, the sequences $\{p^s\}$, $\{z^{js}\}$, $\{b^s\}$, $\{Q^s\}$ have cluster points $p^* \in P$, $z^{j*} \in \bar{Z}^j$, $b^* \in [0,B]$ and $Q^*=z^{0*}_Q$. Hence these have convergent subsequences. We assume $\{p^s\}$, $\{z^{js}\}$, $\{b^s\}$, $\{Q^s\}$ themselves converge to p^*, z^{j*}, b^*, Q^*, $(j=0,1,\dots,m)$. Since $(1-T)[p^s_L \cdot f^i(t^s(i))] \leqq p^s_L \cdot f^i(t^s(i)) \leqq \ell g(i)$ for all s and $i \in I$, $\{\int_I (1-T)[p^s_L \cdot f^i(t^s(i))]\}$ has a convergent subsequence. We assume $\{\int_I (1-T)[p^s_L \cdot f^i(t^s(i))]\}$ itself converges.

It is easy to verify the following lemma.

Lemma 4.1. $\pi_j(p^*)=p^*\cdot z^{j*} \geqq p^*\cdot z$ for all $z \in Z^j$ and $\pi_j(p^*)=0$ for $j=1,2,\ldots,m$.

The next lemma is proved in the same way as the proof of Lemma 3.8.

Lemma 4.2. $p_L^* \geq 0$.

4.2. This subsection will show $p_C^* > 0$. First we show the following lemma.

Lemma 4.3. (i) For any subsequence $\{(x^{s^\nu}(i), t^{s^\nu}(i))\}$, $x^{s^\nu}(i) \to 0$ implies $t^{s^\nu}(i) \to 0$.

(ii) $\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))]=0$ if and only if $\lim_s \int_I x^s(i)=0$.

Proof. (i) Suppose $t^{s^\nu}(i) \to \bar{t} \leq 0$. Since $U^i(\bar{t},0,Q^*) < U^i(0,0,Q^*)$ and $t^{s^\nu}(i) \to \bar{t}$, $x^{s^\nu}(i) \to 0$, there is a ν^0 such that $U^i(t^{s^\nu}(i), x^{s^\nu}(i), Q^{s^\nu}) < U^i(0,0, Q^s)$ for all $\nu \geqq \nu^0$. This is a contradiction to (14) because (0,0) always satisfies the budget constraint.

(ii) Sufficiency: Since $\lim_s \int_I x^s(i)=0$, we hava a subsequence $\{x^{s^\nu}(\cdot)\}$ of $\{x^s(\cdot)\}$ such that $\lim_\nu x^{s^\nu}(i)=0$ a.e. by Dunford and Schwartz (1957, Theorem III 3.6 and Corollary III 6.13 (a)). Then by (i), $\lim t^{s^\nu}(i)=0$ a.e., which implies $\lim_\nu f^i(t^{s^\nu}(i))=0$ a.e. Hence $\lim_\nu p_L^{s^\nu} \cdot f^i(t^{s^{\nu^\nu}}(i))=0$, that is, $\lim_\nu (1-T)[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]=0$ a.e. Then since it has been assumed that $\int_I (1-T)[p_L \cdot f^i(t^s(i))]$ converges, we have

$$\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))] = \lim_\nu \int_I (1-T)[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]=0.$$

Necessity: Since $\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))]=0$, we have a subsequence $\{(1-T)[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(\cdot))]\}$ such that $\lim_\nu (1-T)[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]=0$ a.e. by Dunford and Schwartz (1957, Theorem III 3.6 and Corollary III 6.13 (a)).

By Assumption R, $\lim_{\nu} p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))=0$ a.e. Since $p_r^* > 0$ for some $r \in L$ by Lemma 4.2, $\lim_{\nu} f_r^i(t^{s^\nu}(i))=0$ a.e., which shows $\lim_{\nu} \int_I f_r^i(t^{s^\nu}(i))=0$. By (ii) of Definition 2, $-\sum_{j=0}^{m} z_L^{js^\nu} \leq \int_I f^i(t^{s^\nu}(i))$. This togeter with Assumptions K and O implies

$$0 \leqq -\sum_{j=1}^{m} z_r^{j*} \leqq -\sum_{j=0}^{m} z_r^{j*} = \lim_{\nu} \left(-\sum_{j=0}^{m} z_r^{js^\nu}\right) \leqq \lim_{\nu} \int_I f_r^i(t^{s^\nu}(i)).$$

These show $0 \leqq -\sum_{j=1}^{m} z_r^{j*} \leqq \lim_{\nu} \int_I f_r^i(t^{s^\nu}(i)) = 0$. Hence we have $\sum_{j=1}^{m} z_C^{j*}=0$ by Assumption N. Then

$$0 \leqq \int_I x^s(i) \leqq \sum_{j=0}^{m} z_C^{js} \leqq \sum_{j=1}^{m} z_C^{js} \to \sum_{j=1}^{m} z_C^{j*} = 0 \quad \text{as } s \to \infty.$$

This implies $\lim_{s} \int_I x^s(i)=0$. Q.E.D.

<u>Lemma 4.4</u>. If $p_k^*=0$ for some $k \in C$, then $\lim_{s} \int_I (1-T)[p_L^s \cdot f^i(t^s(i))] > 0$.

<u>Proof</u>. Suppose $p_k^*=0$ and $\lim_{s} \int_I (1-T)[p_L^s \cdot f^i(t^s(i))]=0$. This is equivalent to $\lim_{s} \int_I x^s(i)=0$ by Lemma 4.3 (ii). Then by Dunford and Schwartz (1957, Theorem III 3.6 and Corollary III 6.13 (a)), we have a subsequence $\{x^{s^\nu}(\cdot)\}$ such that $x^{s^\nu}(i) \to 0$ a.e. We assume $\{x^s(\cdot)\}$ itself converge to 0 a.e. By Lemma 4.3 (i), we have $t^s(i) \to 0$ a.e., which implies $f^i(t^s(i)) \to 0$ a.e. Hence, by the uniform integrability of $f^i(t^s(i))$ and Lebesgue's Convergence Theorem, $\int_I f^i(t^s(i)) \to 0$. By Assumptions K,O,M and Q, we have $z^{0*}=0$.

By Lemma 4.2, $p_r^* > 0$ for some $r \in L$. Let i be an individual in the set S of Assumption H such that $\lim_{s} x^s(i)=0$ and $\lim_{s} t^s(i)=0$. By Assumption H, there are $\bar{t} \in H$ and $\lambda > 0$ such that $f_r^i(\bar{t}) > 0$ and $U^i(\bar{t},\lambda e^k,0) > U^i(0,0,0)$. Since $t^s(i) \to 0$, $x^s(i) \to 0$ and $Q^s \to 0$, $U^i(\bar{t},\lambda e^k,0) > U^i(t^s(i),x^s(i),Q^s)$ for sufficiently large s. Since $p_r^s \to p_r^* > 0$ and $p_k^s \to p_k^*=0$, we have

$$p_C^s \cdot \lambda e^k < (1-T)[p_L^s \cdot f^i(\bar{t})] \leqq (1-T)[p_L^s \cdot f^i(\bar{t})]+b^s \quad \text{for sufficiently large } s.$$

This is a contradiction to (14). Q.E.D.

Lemma 4.5. If $\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))] > 0$, then $p_C^* \geq 0$.

Proof. Suppose $p_C^* = 0$. Then $\sum_{j=1}^m z_C^{j*} = 0$ because if $\sum_{j=1}^m z_C^{j*} \geq 0$, then $\sum_{j=1}^m z_L^{j*} < 0$ by Assumption N, and so, $p^* \cdot \sum_{j=1}^m z^{j*} = p_C^* \cdot \sum_{j=1}^m z_C^{j*} + p_L^* \cdot \sum_{j=1}^m z_L^{j*} = 0 + p_L^* \cdot \sum_{j=1}^m z_L^{j*} < 0$ by $p_L^* \geq 0$, which is a contradiction to Lemma 4.1. Hence we have, by (ii) of Definition 2, $0 \leqq \int_I x^s(i) \leqq \sum_{j=0}^m z_C^{js} \leqq \sum_{j=1}^m z_C^{js} \to \sum_{j=1}^m z_C^{j*} = 0$. This shows $\int_I x^s(i) \to 0$, which contradicts Lemma 4.3 (ii). Q.E.D.

Lemma 4.6. $p_C^* > 0$.

Proof. Suppose $p_k^* = 0$ for some $k \in C$. Then by Lemma 4.4, we have $\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))] > 0$. This implies , by Lemma 4.5, that $p_C^* \geq 0$. The following lemma claims that $p_k^*=0$ and $p_C^* \geq 0$ imply $\lim_\nu \int_I (1-T)[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]=0$ for some subsequence $\{t^{s^\nu}(\cdot)\}$, which contradicts $\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))] > 0$.

Q.E.D.

Lemma 4.7. If neither $p_C^* > 0$ nor $p_C^* = 0$, then there is a subsequence $\{t^{s^\nu}(\cdot)\}$ such that $\lim_\nu \int_I (1-T)[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]=0$.

Sketch of the proof. This lemma can be proved by repeating the proof of Lemma 11 of Part I with suitable changes. In the proof, for gross tax function T^s, a contradiction is derived from the supposition that any subsequence of $\{\int_I (1-T^s)[p_L^s \cdot f^i(t^s(i))]\}$ does not converge to zero. Although T is a net tax function in this lemma, it is seen that if any subsequence of $\{\int_I (1-T)[p_L^s \cdot f^i(t^s(i))]\}$ does not converges to zero, then $\{\int_I (1-(T-b^s))[p_L^s \cdot f^i(t^s(i))]\} = \{\int_I (1-T)[p_L^s \cdot f^i(t^s(i))]+b^s\mu(I)\}$ has the same property for gross tax functions $T-b^s$. Besides this difference, (p^s,γ^s) are competitive equilibria in Lemma 11 but bounded competitive equilibria in this lemma. However all the assertions of the proof of Lemma 11 except Assertion 3 hold without any essential change. The proof of Assertion 3 is given in the Appendix.

4.3. Since $p_k^* > 0$ for any $k \in C$, we can take a $\delta > 0$ such that $\delta < \min_{k \in C} p_k^*$. Then for some s^0, $p_k^s > \delta$ for any $k \in C$ and all $s \geqq s^0$. It follows from the budget constraint and Assumption G that for all $k \in C$, $s \geqq s^0$,

$$0 \leqq x_k^s(i) \leqq \frac{(1-T)[p_L^s \cdot f^i(t^s(i))]}{\delta} \leqq \frac{p_L^s \cdot f^i(t^s(i))-K}{\delta} \leqq \frac{g(i)-K}{\delta} .$$

Hence $x^s(i)$ is uniformly integrable for all $s \geqq s^0$ and $\{(t^s(i),x^s(i))\}$ has a cluster point.

The next lemma can be proved in the same way as the proof of Lemma 7 of Part I.

Lemma 4.8. Let i be any individual. For any cluster point $(\tilde{t},\tilde{x})$ of $\{(t^s(i),x^s(i))\}$, $U^i(\tilde{t},\tilde{x},Q^*) \geqq U^i(t,x,Q^*)$ for all $(t,x) \in H \times R_+^C$ with $p_C^* \cdot x \leqq (1-T)[p_L^* \cdot f^i(t)] + b^*$, and $p_C^* \cdot \tilde{x} \leqq (1-T)[p_L^* \cdot f^i(\tilde{t})]+b^*$.

Since $\{\int_I f^i(t^s(i))\}$, $\{\int_I x^s(i)\}$ are bounded, they have convergent subsequences.

We assume $\{\int_I f^i(t^s(i))\}$, $\{\int_I x^s(i)\}$ themselves converge. It can be shown in exactly the same way as subsection 3.4 of Part I that there are measurable functions $t^*(\cdot)$ and $x^*(\cdot)$ such that

$$\text{for almost all } i \in I,\ (t^*(i),x^*(i)) \text{ is a cluster point of } \{(t^s(i),x^s(i))\},$$

$$\lim_s \int_I f^i(t^s(i)) = \int_I f^i(t^*(i)) \quad \text{and} \quad \lim_s \int_I x^s(i) = \int_I x^*(i). \tag{15}$$

Then it follows from Lemma 4.8 that for almost all $i \in I$,

$$U^i(t^*(i),x^*(i),Q^*) \geqq U^i(t,x,Q^*) \text{ for all } (t,x) \text{ with } p^*_C\cdot x \leqq (1-T)[p^*_L\cdot f^i(t)]+b^*,$$
$$\text{and} \quad p^*_C\cdot x^*(i) \leqq (1-T)[p^*_L\cdot f^i(t^*(i))]+b^*. \tag{16}$$

<u>4.4</u> We have already constructed $(p^*,(t^*(i),x^*(i),Q^*),z^{1*},z^{2*},\ldots,z^{m*})$ and (z^{0*},b^*) with properties (i), (iii), (iv) and (v) of Definition 1. This subsection shows that $((t^*(i),x^*(i),Q^*),z^{0*},z^{1*},\ldots,z^{m*})$ is an allocation, which completes the proof that $(p^*,(t^*(i),x^*(i),Q^*),z^{1*},z^{2*},\ldots,z^{m*})$ is a competitive equilibrium under $(T\text{-}b^*,z^{0*})$. For this purpose it suffices to show that

$$\sum_{j=0}^m z_C^{j*} = \int_I x^*(i) \quad \text{and} \quad -\sum_{j=0}^m z_L^{j*} = \int_I f^i(t^*(i)). \tag{17}$$

First we show

$$-p^*_C\cdot z_C^{0*} - p^*_L\cdot z_L^{0*} = p^*_L\cdot\int_I f^i(t^*(i)) - p^*_C\cdot\int_I x^*(i). \tag{18}$$

If Assumption S-2 holds, we have, by (iv) of the Auxiliary Theorem,

$$p_C^s\cdot z_C^{0s} - p_L^s\cdot z_L^{0s} = p_L^s\cdot\int_I f^i(t^s(i)) - p_C^s\cdot\int_I x^s(i) \qquad \text{for all } s.$$

As $s\to\infty$, we have (18) by (15). Next consider the case of Assumption S-1.

The following lemma states that for large s

$$0 \leqq p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I) < B\mu(I),$$

which implies $[p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I)]_0^B = p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I)$.

Then we have, by (iii) of the Auxiliary Theorem,

$$-p_C^s \cdot z_C^{0s} - p_L^s \cdot z_L^{0s} + b^s \mu(I) = p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I),$$

i.e.,

$$-p_C^s \cdot z_C^{0s} - p_L^s \cdot z_L^{0s} = p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i)$$

for large s. Taking the limit, we have (18) by (15).

<u>Lemma 4.9</u>. If Assumption S-1 holds, then

$$0 \leqq p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I) < B\mu(I) \quad \text{for large s.}$$

<u>Proof</u>. Since $p_C^s \cdot x^s(i) \leqq (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s$ a.e., $0 \leqq T[p_L^s \cdot f^i(t^s(i))] \leqq p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s$ a.e. This implies

$$0 \leqq \int_I T[p_L^s \cdot f^i(t^s(i))] \leqq p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I) \quad \text{for all s.}$$

Next we show the other inequality. By Assumption S-1, we have

$$\int_I T[p_L^* \cdot f^i(t^*(i))] \leqq \sup_{p_L \in P_L} \int_I \sup_{t^i \in H} T[p_L \cdot f^i(t^i)] < B\mu(I). \tag{19}$$

Put $C^s = \{i \in I \mid p_C^s \cdot x^s(i) < (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s\}$. Then we show $\mu(C^s) \to 0$ as $s \to \infty$. Since $p_C^s \to p_C^* > 0$, there is a $\delta > 0$ such that for some s_0 and any $s \geqq s_0$, any $k \in C$, $p_k^s > \delta$. On the other hand, for any $i \in I$, there is an $s(i)$ such that $\ell\delta Ms > \ell g(i) + B$ for all $s \geqq s(i)$, where M is definded by (2) and (3). Without loss of generality, we can assume that s(i) is measurable and

$s(i) \geqq s_0$ for all $i \in I$. Then it holds that $i \notin C^s$ for any $s \geqq s(i)$. On the contrary, suppose $i \in C^s$. Remember $(p^s, (t^s(i), x^s(i), Q^s), z^{1s}, z^{2s}, \ldots, z^{ms})$ is an s-bounded competitive equilibrium. Then $x^s(i) = (sM, sM, \ldots, sM)$ holds. Because if $x^s(i) \leq (sM, sM, \ldots, sM)$, there is an $x' \in [-sM, sM]^C$ such that $x^s(i) \leq x'$ and $p_C^s \cdot x' < (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s$, by $i \in C^s$. This x' gives a higher utility than $x^s(i)$ by Assumption B(i), which is a contradiction. Hence since $x_k^s(i) > \delta$ for any $k \in C$,

$$p_C^s \cdot x^s(i) \geqq \ell\delta sM \geqq \ell\delta s(i)M > \ell g(i) + B \geqq (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s$$

for all $s \geqq s(i)$. This is a contradiction to $i \in C^s$. Thus $i \notin C^s$ for any $s \geqq s(i)$. Since $C^s \subset I - \{i \in I \mid s(i) \leqq s\}$, $\mu(C^s) \leqq \mu(I) - \mu(\{i \in I \mid s(i) \leqq s\}) \to 0$ as $s \to \infty$. The uniform integrability of $p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s$ and $\mu(C^s) \to 0$ imply

$$\int_{C^s} (p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s) \to 0 \quad \text{as} \quad s \to \infty. \tag{20}$$

On the other hand, by (19) and $\mu(C^s) \to 0$, we have

$$\int_{I-C^s} T[p_L^s \cdot f^i(t^s(i))] \to \int_I T[p_L^* \cdot f^i(t^*(i))] < B\mu(I). \tag{21}$$

Noting $I - C^s = \{i \in I \mid p_C^s \cdot x^s(i) = (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s\} = \{i \in I \mid T[p_L^s \cdot f^i(t^s(i))] = p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s\}$, we have, by (20) and (21),

$$\begin{aligned}
& p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I) \\
&= \int_{I-C^s} (p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s) + \int_{C^s} (p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s) \\
&= \int_{I-C^s} T[p_L^s \cdot f^i(t^s(i))] + \int_{C^s} (p_L^s \cdot f^i(t^s(i)) - p_C^s \cdot x^s(i) + b^s) \to \int_I T[p_L^* \cdot f^i(t^*(i))] < B\mu(I).
\end{aligned}$$

Hence for large s, $p_L^s \cdot \int_I f^i(t^s(i)) - p_C^s \cdot \int_I x^s(i) + b^s \mu(I) < B\mu(I)$. Q.E.D.

Now it is a position to prove (17). Since $(p^s,(t^s(i),x^s(i),Q^s),z^{1s},\ldots,z^{ms})$ is an s-bounded competitive equilibrium, $\int_I x^s(i) \leqq \sum_{j=0}^m z_C^{js}$ and $-\sum_{j=0}^m z_L^{js} \leqq \int_I f^i(t^s(i))$ for all s. This implies, using (15),

$$\int_I x^*(i) \leqq \sum_{j=0}^m z_C^{j*} \quad \text{and} \quad -\sum_{j=0}^m z_L^{j*} \leqq \int_I f^i(t^*(i)). \tag{22}$$

Suppose $p_C^*\cdot\int_I x^*(i) < p_C^*\cdot\sum_{j=0}^m z_C^{j*}$ or $-p_L^*\cdot\sum_{j=0}^m z_L^{j*} < p_L^*\cdot\int_I f^i(t^*(i))$. Since $p_C^*\cdot\int_I x^*(i) \leqq p_C^*\cdot\sum_{j=0}^m z_C^{j*}$ and $-p_L^*\cdot\sum_{j=0}^m z_L^{j*} \leqq p_L^*\cdot\int_I f^i(t^*(i))$ by (22), it holds that

$$p_C^*\cdot\int_I x^*(i) - p_L^*\cdot\sum_{j=0}^m z_L^{j*} < p_C^*\cdot\sum_{j=0}^m z_C^{j*} + p_L^*\cdot\int_I f^i(t^*(i)),$$

i.e.,

$$-p_L^*\cdot\sum_{j=0}^m z_L^{j*} - p_C^*\cdot\sum_{j=0}^m z_C^{j*} < p_L^*\cdot\int_I f^i(t^*(i)) - p_C^*\cdot\int_I x^*(i).$$

Noting $\sum_{j=1}^m p^*\cdot z^{j*}=0$ by Lemma 4.1,

$$-p_L^*\cdot z_L^{0*} - p_C^*\cdot z_C^{0*} = -\sum_{j=0}^m p_L^*\cdot z_L^{j*} - \sum_{j=0}^m p_C^*\cdot z_C^{j*} < p_L^*\cdot\int_I f^i(t^*(i)) - p_C^*\cdot\int_I x^*(i),$$

which is a contradiction to (18). Thus it holds that

$$p_C^*\cdot\int_I x^*(i) = p_C^*\cdot\sum_{j=0}^m z_C^{j*} \quad \text{and} \quad -p_L^*\cdot\sum_{j=0}^m z_L^{j*} = p_L^*\cdot\int_I f^i(t^*(i)). \tag{23}$$

This implies $\int_I x^*(i)=\sum_{j=0}^m z_C^{j*}$ by $p_C^* > 0$. Finally we prove $-\sum_{j=0}^m z_L^{j*}=\int_I f^i(t^*(i))$. Suppose $\sum_{j=0}^m z_L^{j*} \leq \int_I f^i(t^*(i))$. Then $\sum_{j=1}^m z_L^{j*} < 0$. On the contrary, let $\sum_{j=0}^m z_r^{j*}=0$ for some $r \in L$. By Assumption N, $\sum_{j=1}^m z_C^{j*} = 0$. Hence we have $\int_I x^*(i)=0$ by (22) and Assumption O, i.e., $x^*(i)=0$ a.e. Since $\int_I f^i(t^*(i)) \geq -\sum_{j=0}^m z_L^{j*} \geqq 0$, there is a subset S of I having a positive measure such that $f^i(t^*(i)) \geq 0$ for all $i \in S$, i.e., $t^*(i) \leq 0$ for all $i \in S$. This implies , by Assumption B(ii), that $U^i(t^*(i),x^*(i),Q^*) < U^i(0,0,0)$, which is a contradiction to (16). Hence

$\sum_{j=1}^{m} z_L^{j*} < 0$. Next we show $p_L^* > 0$. Suppose $p_r^* = 0$ for some $r \in L$. Let i be an individual who satisfies the utility maximization (16) and $f_r^i(t^*(i)) > 0$. Then, by Assumption I, there is a $t' \in H$ such that $f_k^i(t') = f_k^i(t^*(i))$ for any $k \in L$ $(k \neq r)$ and $U^i(t', x^*(i), Q^*) > U^i(t^*(i), x^*(i), Q^*)$. Clearly, $(t', x^*(i))$ satisfies the budget constraint, which is a contradiction to (16). Hence $f_r^i(t^*(i)) = 0$ a.e. Since $\sum_{j=1}^{m} z_L^{j*} < 0$, we have

$$0 < \sum_{j=1}^{m} z_r^{j*} \leqq \sum_{j=0}^{m} z_r^{j*} \leqq \int_I f_r^i(t^*(i)),$$

which is a contradiction. Hence $p_L^* > 0$. This and $-\sum_{j=0}^{m} z_L^{j*} \leq \int_I f^i(t^*(i))$ imply $-p_L^* \cdot \sum_{j=0}^{m} z_L^{j*} < p_L^* \cdot \int_I f^i(t^*(i))$, which is a contradiction to (23). We have shown (17).

Assumption G: Since $f_r^{is}(t) \leqq \max\ (g(i), \frac{\varepsilon_0}{s}) \leqq \max\ (g(i), \varepsilon_0)$ for any $r \in L$, $t \in H^s$, $f^{is}(t)$ is uniformly integrable by $g^1(i) = \max\ (g(i), \varepsilon_0)$, that is, $0 \leqq f_r^{is}(t) \leqq g^1(i)$ for any $r \in L$, $t \in H^s$, $s=1,2,\ldots.$

Assumption I: If $(t,x,Q) \in H \times R_+^{c+1}$, there is nothing to prove. For $(t,x,Q) \in B^s \times R_+^{c+1}$ with $f_r^{is}(t) = -t_r > 0$, define $t' \in B^s$ by $t'_k = t_k$ for $k \neq r$ and $t'_r = 0$. then $f_k^{is}(t') = f_k^{is}(t)$ for $k \neq r$ by (24), and $U^{is}(t',x,Q) > U^{is}(t,x,Q)$ by the monotonicity of $U^i(t,x,Q)$ on $B^s \times R_+^{c+1}$.

Assumption J-2: Consider any $\varepsilon > 0$, $i \in I$ with $L(i) \neq \phi$ and $r \in L$. Since $B(\varepsilon) \cap B^s \neq \phi$, there is a $t \in B(\varepsilon) \cap B^s \subset B(\varepsilon) \cap H^s$. Then by (24), we have $f_r^{is}(t) > 0$. Hence Assumption J-2 holds.

Either Assumption S-1 or S-2: If S-2 is true in the original economy, then so it is in every purturbed economy. Let S-1 hold in the original economy. Then $\frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in H} T[p_L \cdot f^i(t^i)] + \frac{\ell\varepsilon_0}{s^0} < B$. Since $y > T(y)$ for all $y > 0$ by Assumption R, we have, for all $t^i \in B^s$, $p_L \in P_L$ and $i \in I$ with $L(i) \neq \phi$, $T[p_L \cdot f^{is}(t^i)] < \sum_{r \in L} f_r^{is}(t^i) \leqq \sum_{r \in L} (-t_r^i)$. Hence we have

$$\frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in H^s} T[p_L \cdot f^{is}(t^i)]$$

$$\leqq \frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in H} T[p_L \cdot f^i(t^i)] + \frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in B^s} T[p_L \cdot f^{is}(t^i)]$$

$$\leqq \frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in H} T[p_L \cdot f^i(t^i)] + \frac{1}{\mu(I)} \sum_{r \in L} \int_I \sup_{t^i \in B^s} (-t_r^i)$$

$$\leqq \frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in H} T[p_L \cdot f^i(t^i)] + \frac{1}{\mu(I)} \sum_{r \in L} \int_I \frac{\varepsilon_0}{s+s^0}$$

$$< \frac{1}{\mu(I)} \sup_{p_L \in P_L} \int_I \sup_{t^i \in H} T[p_L \cdot f^i(t^i)] + \frac{\ell\varepsilon_0}{s^0} < B.$$

Q.E.D.

Theorem 1 together with this lemma implies that there is a competitive equilibrium $(p^s,\gamma^s)=(p^s,(t^s(i),x^s(i),Q^s),z^{1s},z^{2s},\ldots,z^{ms})$ under a tax schedule $(T-b^s,z^{0s})$ for some $b^s\in[0,B]$ and $z^{0s}\in Z^0$ for $\mathcal{E}^s$ $(s=1,2,\ldots)$. Since $p^s\in P$, $z^{js}\in\overline{Z}^j$ $(j=0,1,\ldots,m)$, $b^s\in[0,B]$, $Q^s=z_Q^{0s}$, we can assume that $p^s\to p^*\in P$, $z^{js}\to z^{j*}\in\overline{Z}^j$ $(j=0,1,2,\ldots,m)$, $b^s\to b^*\in[0,B]$, $Q^s\to Q^*=z_Q^{0*}$ and $\{\int_I(1-T)[p_L^s\cdot f^{is}(t^s(i))]\}$ converges. The following lemmas hold in parallel with section 4 (Lemma 4.1 $\sim$ Lemma 4.7). The proofs are exactly the same except some notations.

Lemma 5.1. $\pi_j(p^*)=p^*\cdot z^{j*}\geqq p^*\cdot z$ for all $z\in Z^j$ and $\pi_j(p^*)=0$ for $j=1,2,\ldots,m$.

Lemma 5.2. $p_L^*\geq 0$.

Lemma 5.3. (i) For any subsequence $\{(x^{s^\nu}(i),t^{s^\nu}(i))\}$, $x^{s^\nu}(i)\to 0$ implies $t^{s^\nu}(i)\to 0$.

(ii) $\lim_s\int_I(1-T)[p_L^s\cdot f^{is}(t^s(i))]=0$ if and only if $\lim_s\int_I x^s(i)=0$.

Lemma 5.4. If $p_k^*=0$ for some $k\in C$, then $\lim_s\int_I(1-T)[p_L^s\cdot f^{is}(t^s(i))]>0$.

Lemma 5.5. If $\lim_s\int_I(1-T)[p_L^s\cdot f^{is}(t^s(i))]>0$, then $p_C^*\geq 0$.

Lemma 5.6. $p_C^*>0$.

Lemma 5.7. If neither $p_C^*>0$ nor $p_C^*=0$, then there is a subsequence $\{t^{s^\nu}(\cdot)\}$ such that $\lim_\nu\int_I(1-T)[p^{s^\nu}\cdot f^{is^\nu}(t^{s^\nu}(i))]=0$.

Lemma 5.6 shows that $x^s(i)$ is uniformly integrable for large s. Since $H^s\subset H^1$ $(s=2,3,\ldots)$ and H^1 is compact, $t^s(i)$ is also uniformly integrable. Hence the sequence $\{(t^s(i),x^s(i))\}$ has a cluster point for each $i\in I$. With the remark that if $(t^s(i),x^s(i))\to(\tilde{t},\tilde{x})$, then $U^{is}(t^s(i),x^s(i),Q^s)\to U^i(\tilde{t},\tilde{x},Q^*)$, by (25) and $(\tilde{t},\tilde{x})\in H\times R_+^{c+1}$, the following lemma can be proved in the same way as Lemma 4.8.

Lemma 5.8. Let i be any individual. For any cluster point $(\tilde{t},\tilde{x})$ of $\{(t^s(i),x^s(i))\}$, $U^i(\tilde{t},\tilde{x},Q^*) \geq U^i(t,x,Q^*)$ for all $(t,x) \in H \times R^C_+$ with $p^*_C \cdot x \leqq (1-T)[p^*_L \cdot f^i(t)] + b^*$, and $p^*_C \cdot \tilde{x} \leqq (1-T)[p^*_L \cdot f^i(\tilde{t})] + b^*$.

Since $\{\int_I f^{is}(t^s(i))\}, \{\int_I x^s(i)\}$ are bounded, they have convergent subsequences. We assume $\{\int_I f^{is}(t^s(i))\}, \{\int_I x^s(i)\}$ themselves converge. By the same argument as Theorem 1, we have the existences of measurable functions $t^*(\cdot)$ and $x^*(\cdot)$ such that

for almost all $i \in I$, $(t^*(i),x^*(i))$ is a cluster point of $\{(t^s(i),x^s(i))\}$,

$$\lim_s \int_I f^{is}(t^s(i)) = \int_I f^i(t^*(i)) \quad \text{and} \quad \lim_s \int_I x^s(i) = \int_I x^*(i). \tag{26}$$

Then it follows from Lemma 5.8 that for almost all $i \in I$,

$U^i(t^*(i),x^*(i),Q^*) \geqq U^i(t,x,Q^*)$ for all (t,x) with $p^*_C \cdot x \leqq (1-T)[p^*_L \cdot f^i(t)] + b^*$,

$p^*_C \cdot x^*(i) \leqq (1-T)[p^*_L \cdot f^i(t^*(i))] + b^*$.

These show $(p^*,\gamma^*) = (p^*,(t^*(i),x^*(i),Q^*),z^{1*},\ldots,z^{m*})$ and (z^{0*},b^*) satisfy (iii) and (iv) of Definition 1. And (i) and (v) of Definition 1 follow from Lemmas 5.1 and 5.2. Since $((t^s(i),x^s(i),Q^s),z^{0s},z^{1s},\ldots,z^{ms})$ is an allocation (s=1,2,...), $-\int_I f^{is}(t^s(i)) = \sum_{j=0}^m z^{js}_L$ and $\int_I x^s(i) = \sum_{j=0}^m z^{js}_C$ for all s. Hence we have $-\int_I f^i(t^*(i)) = \sum_{j=0}^m z^{j*}_L$ and $\int_I x^*(i) = \sum_{j=0}^m z^{j*}_C$ by (26), which togeter with $Q^* = z^{0*}_Q$ shows (p^*,γ^*) and (z^{0*},b^*) satisfy (ii) of Definition 1. Therefore (p^*,γ^*) is a competitive equilibrium under $(T-b^0,z^0)$. This completes the proof.

Appendix.

Proof of Assertion 3. If $\int_{E^s} x_k^s(i) \to 0$ for any $k \in C_1$, then it is proved in the same way as Part I that $\int_I p_C^s \cdot x_k^s(i) \to 0$. Put $C^s = \{i \in I \mid (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s > p_C^s \cdot x^s(i)\}$ for all s. If $\mu(C^s) \to 0$, then $\lim_s \int_I (1-T)[p_L^s \cdot f^i(t^s(i))] + b^s \mu(I) = \lim_s \int_I p_C^s \cdot x^s(i) = 0$, which is a contradiction. Hence $\{\mu(C^s)\}$ has a subsequence which converges to a $\beta > 0$. We assume that $\{\mu(C^s)\}$ itself converges to β. For $i \in C^s$, $x^s(i) = (sM, sM, \ldots, sM)$, because if $x^s(i) \leq (sM, sM, \ldots, sM)$, there is an x' such that $x^s(i) \leq x' \leqq (sM, sM, \ldots, sM)$ and $(1-T)[p_L^s \cdot f^i(t^s(i))] + b^s > p_C^s \cdot x'$. By Assumption B(i), $U^i(t^s(i), x^s(i), Q^s) < U^i(t^s(i), x', Q^s)$, which is a contradiction. Therefore $\int_I x_k^s(i) \geqq \int_{C^s} x_k^s(i) = \int_{C^s} sM = sM\mu(C^s) \to \infty$ for any $k \in C_1$. This is a contradiction to $\int_I x_k^s(i) \leqq \sum_{j=0}^{m} z_k^j \leqq \sum_{j=1}^{m} z_k^j \leqq \alpha M$ for all $k \in C$. We have shown $\int_{C^s} x_k^s(i) \not\to 0$ for some $k \in C_1$. Q.E.D.

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