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Economies with Labor Indivisibilities

Part I - Optimal Tax Schedules

by

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Abstract: In this paper we model an optimal taxation problem in a general equilibrium framwork with a continuum of individuals which allows non-convexities and indivisibilities on labor supplies, and explore the existence problem of an optimal tax schedule. This paper consisits of two parts: Part I is devoted to the proof of the existence of an optimal tax schedule among the tax schedules having competitive equilibria, and Part II examines conditions for the existence of a competitive equilibrium under a given tax function.

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1. Introduction

An optimal income taxation problem is formulated as follows: A government chooses a tax schedule of an income tax function and a production plan of public goods and announces it to individuals and firms. Each individual chooses a plan of labor supplies and consumption goods to maximize his utility under the tax schedule. Each firm maximizes its profits. The market mechanism determines equilibrium prices. The government's expenditure should not exceed its revenue ex post. The optimal tax schedule is defined to be a schedule which yields a competitive equilibrium with the maximal social welfare.

When we confine ourselves to the existence problem of an optimal tax schedules, there are two sub-problems to be considered: One is the existence of a competitive equilibrium under a given tax schedule, and another is the existence of an optimal one among the tax schedules which have competitive equilibria. The second problem is studied by kaneko(1981a) but the first one has not been examined.[1] Kaneko gave an affirmative answer to the second problem in the case of a finite number of individuals. In doing so, Kaneko avoided the first problem by stating that there exists at least one tax schedule (the trivial one - no tax and no public goods) which has a competitive equilibrium because the standard existence proof can be applied to the economy under the trivial tax schedule. However it can be easily seen that the standard proof does not work in Kaneko's model under nontrivial tax schedules because of the nonconvexity of the tax functions. Furthermore, perfect divisibility and convexity assumptions are imposed on labor supplies in Kaneko's model. Nonconvexties and indivisibilities are important features

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1) Kaneko(1982a) proved the existence of an optimal tax schedule in a simple model (not a general equilibrium model), although it is rather standard in the literature of optimal taxation theory originated from Mirrlees (1971).

of labor supplies when we look at actual labor markets. Thus the optimal taxation problem faces several kinds of nonconvexities and indivisibilities to be incorporated. However it is well-known that the existence proof of a competitive equilibrium fails in an economy with a finite number of individuals where nonconvexities or indivisibilities appear.[2] It is also known that it might be possible for a large economy to have a competitive equilibrium and to allow certain kinds of nonconvexities. For example, Mas-Colell(1977) and Yamazaki(1978) gave measure-theoretic conditions for the existence of a competitive equilibrium in an exchange economy with a continuum of individuals in which nonconvexities, indivisibilities on labor supplies and nonconvexities on tax functions are allowed.

This paper consists of two parts: In the reverse order, Part I models the optimal taxation problem, and proves the existence of an optimal tax schedule, provided that there is a competitive equilibrium under some tax schedule. This model entirely allows nonconvexities and indivisibilities on labor supplies. Part II will give conditions for the existence of a competitive equilibrium under a given tax schedule. This reverse order would help the reader to understand the structure of the optimal taxation problem. Part II will need more conditions than Part I, but they will be rather standard conditions (not measure theoretic) and well-interpreted.

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2) There are several exceptions, for example, assignment markets of Shapley and Shubik (1972) and Kaneko (1982b).

2. Model and problem

2.1

This paper considers a production economy with a continuum of individuals, a finite number of firms and an agent called a government. The triple (I,β,μ) is a measure space of all individuals, where I is the set of all individuals, β a σ-algebra of subsets of I and μ a non-atomic measure on β with $0 < \mu(I) < \infty$.

It is assumed that ℓ-kinds of labor(time) supplies, c-kinds of consumption goods and a public good enter the individual utility functions $U^i(t,x,Q)$, where $t=(t_1,t_2,\ldots,t_\ell)$ denotes labor(time) supplies for ℓ-kinds of labor, x a level of consumption goods and Q a level of the public good supplied by the government. Every U^i is difined on the consumption set $\Omega = H \times R^{c+1}_+$, where $H \subset R^\ell_-$ is the set of possible labor time assignments t. Here R^ℓ_- is the non-positive orthant of the ℓ-dimentional Euclidian space R^ℓ and R^{c+1}_+ the non-negative orthant of R^{c+1}. We make the following assumptions.

<u>Assumption A</u> H is a compact set including the zero vector 0.

<u>Assumption B</u> For all $i \in I$, (i) for any fixed $t \in H$, $U^i(t,x,Q)$ is a monotonic increasing function of (x,Q) on R^{c+1}_+ and (ii) for any $(x,Q) \in R^{c+1}_+$, $U^i(t,x,Q) < U^i(0,x,Q)$ if $t \leq 0$.[3]

<u>Assumption C</u> For all $i \in I$, $U^i(t,x,Q)$ is a continuous function on Ω.

<u>Assumption D</u> For each $(t,x,Q) \in \Omega$, $U^i(t,x,Q)$ is a measurable function of i.

<u>Assumption E</u> The function $U^i(t,x,Q)$ of i is uniformly integrable, i.e., for some integrable function $V(i)$, $\left|U^i(t,x,Q)\right| \leqq V(i)$ for all $(t,x,Q) \in \Omega$.

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3) We are using the following notations; $x \leqq y$ iff $x_k \leqq y_k$ for all k; $x < y$ iff $x_k < y_k$ for all k; and $x \leq y$ iff $x \leqq y$ and $x \neq y$.

The following examples satisfy Assumption A.

Example 1. Let $H=\bigcup_{k=1}^{\ell}\{-\alpha e^k \mid 0 \leqq \alpha \leqq 24\}$, where e^k is the k-th unit vector of R^ℓ. This means that each individual can work in only one labor type but his labor supply is variable. In this example, labor supplies are divisible, but convexity is not required.

Example 2. Let $H=\{(0,0,\ldots,0),(-8,0,\ldots,0),(0,-8,0,\ldots,0),\ldots,(0,\ldots,0,-8)\}$. This means that each individual can work in only one labor type exactly for 8-hours. In this example, labor supplies are completely indivisible.

The meaning of Assumption B would be clear. Assumption C and D are technical conditions. Assumption E is necessary for the sake of the integrability of a social welfare function [see Kaneko (1983) and Remark 6 below]. This assumption is justified by St. Petersburg paradox [see Aumann (1977)].

The following argument can be directly applicable to the case of more than one public goods, but we treat the case of one public good only for notational simplicity.

Each individual i has a labor production function $f^i(t)$ from H to R^ℓ_+. That is, when he chooses an assignment $t=(t_1,t_2,\ldots,t_\ell)\in H$ of ℓ-kinds of labor time, he can provide a vector $f^i(t_1,t_2,\ldots,t_\ell)=(f^i_1(t),f^i_2(t),\ldots,f^i_\ell(t)) \geqq 0$ of services called "labor".[4] We make the following assumptions.

Assumption F For all $i\in I$, $f^i(t)$ is a continuous function on H with $f^i(0)=0$.

Assumption G For each $t\in H$, $f^i(t)$ is a measurable function of i.

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4) Y.Sakai of University of Tsukuba pointed out that a level of consumption goods should enter labor production functions. However we have not succeeded in proving the existence of the optimal tax schedule in this case.

<u>Assumption H</u> The function $f^i(t)$ of i is uniformly integrable, i.e., for some real-valued integrable function $g(i)$, $f^i_k(t) \leqq g(i)$ for $k=1,2,\dots,\ell$, for all $t \in H$ and all $i \in I$.

Assumptions F and G are technical conditions. Since H is compact, it would be natural to assume the uniform integrability of $f^i(t)$ (Assumption H).

We assume that any individual is endowed with no consumption goods, and that he cannot consume any service of "labor" but his labor time as leisure time. The consumption goods are produced by firms. There are m firms in the economy. Each firm j ($j=1,2,\dots,m$) has a production set $Z^j \subset R^\ell \times R^c$. The following notation may be used; $L=\{1,2,\dots,\ell\}$, $C=\{\ell+1,\ell+2,\dots,\ell+c\}$ and $z^j=(z^j_L,z^j_C)=(z^j_1,z^j_2,\dots,z^j_\ell,z^j_{\ell+1},\dots,z^j_{\ell+c})$ for $z^j \in Z^j$. We make the following assumptions.

<u>Assumption I</u> $Z^j \subset R^\ell_- \times R^c$ for $j=1,2,\dots,m$.

<u>Assumption J</u> Z^j is a closed convex cone for $j=1,2,\dots,m$.

<u>Assumption K</u> $\sum_{j=1}^m Z^j \cap \left(R^\ell_+ \times R^c_+\right) = \{0\}$ and there are $z=(z_L,z_C) \in \sum_{j=1}^m Z^j$ such that $z_C > 0$.

<u>Assumption L</u> If $z=(z_L,z_C) \in \sum_{j=1}^m Z^j$ and $z_C > 0$, then $z_L < 0$.

<u>Assumption M</u> There are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ for any $z \in \sum_{j=1}^m Z^j$ and k_1,k_2 ($\ell+1 \leqq k_1,k_2 \leqq \ell+c$) with $z_{k_1} > 0$ such that $z-\varepsilon_1 e^{k_1}+\varepsilon_2 e^{k_2} \in \sum_{j=1}^m Z^j$ and $z_{k_1} > \varepsilon_1$. Here e^k is the unit vector in $R^\ell \times R^c$ in the k-direction.

Assumption I means that the firms cannot produce any kind of labor. We do not need to explain Assumption J, but this is a crucial assumption in this paper. Assumption K means the impossibility of free production and means that the economy is productive as a whole. Assumption L means that if all consumption goods are produced in the economy, then all kinds of labor are used as input.

Assumption M means that when the economy produces a positive amount of a consumption good, it can produce a positive amount of any other consumption good by decreasing the production level of the consumption good.

Remark 1. It follows from Assumptions L and M that if $z=(z_L,z_C) \in \sum_{j=1}^{m} Z^j$ and $z_C \geq 0$, then $z_L < 0$. Because if there is a $z=(z_L,z_C) \in \sum_{j=1}^{m} Z^j$ with $z_L \leq 0$, $z_C \geq 0$, then we can find a $z'=(z'_L,z'_C) \in \sum_{j=1}^{m} Z^j$ by Assumption M such that $z'_L \leq 0$, $z'_C > 0$, which is a contradiction to Assumption L. This conclusion will be employed as an assumption instead of Assumptions L and M in Part II of this paper.

Each firm j has a profit share distribution $\theta_j(i)$, which is a measurable function such that $0 \leqq \theta_j(i) \leqq 1 \ \forall i \in I$ and $\int_I \theta_j(i)=1$.[5]

The govrenment produces and supplies the public good using labor and consumption goods as input. The government has a production set $Z^0 \subset R^{\ell} \times R^c \times R$. A typical element in Z^0 is denoted by $z^0=(z^0_L,z^0_C,z^0_Q)$.

Assumption N $\quad Z^0 \subset R^{\ell}_- \times R^c_- \times R_+$.

Assumption O $\quad Z^0$ is a closed convex set including 0.

Assumption P $\quad Z^0 \cap \left(R^{\ell}_+ \times R^c_+ \times R_+\right) = \{0\}$.

Assumption N means that the government cannot produce any kind of labor and consumption goods and that the public good cannot be used as input in the government's production. Assumption O would not need any comment. Assumption P is the impossibility of free production.

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5) For simplicity, the integral $\int_I a(i)d\mu$ is denoted by $\int_I a(i)$. This does not yield any confusion.

2.2

Now we are in a position to discuss our optimal taxation problem. A tax function T is a real-valued function on the set of non-negative real numbers R_+ which satisfies

$$T(y) \text{ is a monotonic nondecreasing function with } T(y) \leqq y \text{ for all } y \in R_+, \tag{1}$$

$$\frac{T(y_1)-T(y_2)}{y_1 - y_2} \leqq 1 \text{ for all } y_1, y_2 \in R_+ \text{ with } y_1 \neq y_2.\text{[6]} \tag{2}$$

The set of all tax functions is denoted by $\mathcal{T}$.

A tax function $T(y)$ means that when an individual i supplies labor $f^i(t)$ under a price vector $p=(p_L, p_C) \in R_+^\ell \times R_+^c$ and when he earns gross income $y=p_L \cdot f^i(t)+\sum_{j=1}^m \theta_j(i)\pi_j(p)$, where $\pi_j(p)$ is firm j's profit under the price vector, he must pay an income tax $T(y)=T(p_L \cdot f^i(t)+\sum_{j=1}^m \theta_j(i)\pi_j(p))$ to the government.[7] Hence T must satisfy $T(y) \leqq y$ for all $y \in R_+$. Condition (2) means that the marginal tax rate of T is less than or equal to 100% everywhere, which implies that the disposable income $y-T(y)$ is monotonic non-decreasing. In fact, T consists of two parts, the net tax function T_N and the subsidy $-T(0) \geqq 0$, i.e., $T(y)=T_N(y)+T(0)$ for all $y \in R_+$. In Part I, tax functions always mean gross tax functions, but in Part II, we will treat net tax functions and subsidies separately.

Remark 2. Instead of (2), we assume that for some $\alpha \geqq 0$, $\frac{T(y_1)-T(y_2)}{y_1 - y_2} \leqq \alpha$ for all $y_1, y_2 \in R_+$ with $y_1 \neq y_2$. The proof of our existence theorem will be still valid in this case.

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6) These conditions imply that $T(y)$ is a continuous function.

7) $x \cdot y$ denotes the innor product of vectors x and y.

Remark 3. In Kaneko (1981a, The Restriction Theorem), it is shown that for any tax function which does not satisfy (2), there is a tax function with (2) such that the results, i.e., the competitive equilibria under the tax function are identical to those under the original tax function. In our case, the same theorem is true if $U^i(t,x,Q)$ is monotonic increasing with respect to t and if H has the property that $t \in H$ implies $\lambda t \in H$ for all $\lambda \in [0,1]$. Example 1 satisfies this property, but not Example 2.

We define the metric ρ in $\mathcal{T}$ by

$$\rho(T_1,T_2)=\sum_{n=1}^{\infty}\frac{1}{2^n}\sup_{y\in[n-1,n]}\left|T_1(y)-T_2(y)\right| \quad \text{for } T_1,T_2 \in \mathcal{T}.$$

All references in the sequel to topological notions in $\mathcal{T}$ will be in the sense of the topology induced by this metric. When T_n converges T^* in this metric, $T_n(y)$ converges uniformly $T^*(y)$ on $[0,M]$ for any $M>0$.

Let $\mathcal{T}_0$ be a subclass of $\mathcal{T}$. We assume that the government can choose a tax function only from $\mathcal{T}_0$, i.e., a tax function in $\mathcal{T}_0$ is feasible. The reason for this restriction is that some tax functions might be difficult to levied by technical problems. That is, it is required that every individual should know the precise shape of the tax function announced by the government, and if the tax function has a difficult shape to be percieved, then the requirement could not be satisfied, and so, the tax function should be eliminated from $\mathcal{T}_0$.

Assumption Q $\mathcal{T}_0$ is a closed set of $\mathcal{T}$ and has the property that if $T \in \mathcal{T}_0$ and $\alpha > 0$, then $T_\alpha \in \mathcal{T}_0$, where $T_\alpha(y)=\alpha T(y/\alpha)$ for all $y \in R_+$.

Example 3. The classes

$\mathcal{T}$,

$\mathcal{T}_C = \{T \in \mathcal{T} \mid T \text{ is a convex function.}\}$ and

$\mathcal{T}_{nL} = \{T \in \mathcal{T} \mid T \text{ is a piecewise linear function with at most n kinks.}\}$

satisfy Assumption Q, and so does $\mathcal{T}_C \cap \mathcal{T}_{nL}$.

It would not be necessary to give any comment to the closedness of $\mathcal{T}_0$. Consider the second property. The specification of a tax function T is, in fact, not enough unless the unit of measurement of prices is determined. That is, if a price vector p is multiplied by some $\alpha>0$, then T should be thought of a different tax function under αp, because the budget constraint is not necessarily homogeneous of degree zero by the non-linearity of T. In this case, the new tax function T_α plays the role that T does under p. The second property requires that this T_α belongs to $\mathcal{T}_0$. That is, the feasibility of a tax function is independent of the choice of the unit of measurement.

The economy works as follows. The government plans to levy taxes by a tax function $T \in \mathcal{T}_0$ and makes a public good production schedule $z^0 \in Z^0$. A pair τ of $T \in \mathcal{T}_0$ and $z^0 \in Z^0$ is called a tax schedule. The set of all tax schedules is denoted by Ψ, i.e., $\Psi = \mathcal{T}_0 \times Z^0$. The government announces the tax schedule chosen to the individuals and firms. Under the tax schedule the individuals and firms behave as a price takers, and the prices of consumption goods and labor are determined by the market mechanism (but not the government). Thus we get the following definition.

Definition 1. $(p,\gamma)=(p,(t(i),x(i),Q)_{i\in I}, z^1,z^2,\dots,z^m)$ is said to be a competitive equilibrium under a tax schedule $\tau=(T,z^0)$ iff

(i) $p=(p_L,p_C) \in R_+^\ell \times R_+^c$ and $p \geq 0$,

(ii) $(\gamma,z^0)=((t(i),x(i),Q)_{i\in I}, z^0,z^1,z^2,\dots,z^m)$ is an allocation, that is, $t(i),x(i)$ are measurable functions of i with $(t(i),x(i)) \in H \times R_+^c$ for all $i \in I$, $Q \in R_+$, $z^j \in Z^j$ for $j=0,1,\dots,m$, $z_Q^0 = Q$, $\sum_{j=0}^{m} z_C^j = \int_I x(i)$, and $-\sum_{j=0}^{m} z_L^j = \int_I f^i(t(i))$;

(iii) $p_C \cdot x(i) \leqq (1-T)[p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j(p)]$ for almost all $i \in I$,

(iv) for almost all i, $U^i(t(i),x(i),Q) \geqq U^i(t,x,Q)$ for all $(t,x) \in H \times R^C_+$ with $p_C \cdot x \leqq (1-T)[p_L \cdot f^i(t) + \Sigma\theta_j(i)\pi_j(p)]$,

(v) $\pi_j(p) = p \cdot z^j \geqq p \cdot z$ for all $z \in Z^j$ for $j=1,2,\ldots,m$.

If (p,γ) is a competitive equilibrium under τ, then p is called a competitive price vector under τ. By $C(\tau)$, we denote the set of all competitive price vectors under τ.

Condition (ii) guarantees the coincidence of total demands and total supplies of consumption goods and labor. Condition (iii) is the individuals' budget constraint. Condition (iv) is the individuals' utility maximization under the budget constraint. Condition (v) is the firms' profit maximization. In condition (iii), we do not distinguish unearned incomes from earned incomes. In fact, since the firms' production sets are cone, their profits are always zero in equilibrium (we will use this fact without any remark in the following). Therefore, it is not necessary to think about taxes on unearned incomes.

Lemma 1. Let $\tau=(T,z^0)$ be any tax schedule. Let $(p,\gamma)=(p,(t(i),x(i),Q), z^1,z^2,\ldots,z^m)$ be a competitive equilibrium under τ. Then it holds that

$$p_C \cdot x(i) = (1-T)[p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j(p)] \quad \text{for almost all } i \in I \tag{3}$$

$$-p_L \cdot z^0_L - p_C \cdot z^0_C = \int_I T[p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j(p)] \tag{4}$$

Proof. Equation (3) follows the monotonicity of $U^i(t,x,Q)$ for any fixed $(t,Q) \in H \times R_+$. Since $T[p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j(p)] = p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j(p) - p_C \cdot x(i)$ a.e. by(3), we have

$$\int_I T[p_L \cdot f^i(t(i)) + \Sigma\theta_j(i)\pi_j(p)] = \int_I p_L \cdot f^i(t(i)) + \int_I \Sigma\theta_j(i)\pi_j(p) - \int_I p_C \cdot x(i)$$

$$= p_L \cdot \int_I f^i(t(i)) + \sum_{j=1}^{m} \pi_j(p) - p_C \cdot \int_I x(i).$$

Since $\int_I f^i(t(i)) = -\sum_{j=0}^{m} z_L^j$ and $\int_I x(i) = \sum_{j=0}^{m} z_C^j$ by Definition 1 (ii) and since $\sum_{j=1}^{m}\pi_j(p) = \sum_{j=1}^{m} p\cdot z^j = \sum_{j=1}^{m}(p_L,p_C)\cdot(z_L^j,z_C^j)$ by Definition 1 (v), we have

$$\int_I T[p_L\cdot f^i(t(i))+\Sigma\theta_j(i)\pi_j(p)] = -p_L\cdot\sum_{j=0}^{m} z_L^j + \sum_{j=1}^{m}(p_L,p_C)\cdot(z_L^j,z_C^j) - p_C\cdot\sum_{j=0}^{m} z_C^j$$

$$= -p_L\cdot z_L^0 - p_C\cdot z_C^0. \qquad \text{Q.E.D.}$$

Remark 4. Kaneko(1981a) makes the additional definition of a "feasible" tax schedule, which has a competitive equilibrium satisfying the government's budget. However equation (4) of Lemma 1 says that if there exists a competitive equilibrium under a tax schedule, then the government's expenditure for the production automatically coincides with its revenue. Therefore Kaneko's definition of a "feasible" tax schedule is, in fact, unnecessary.

As is mentioned after Assumption Q, the budget constraint is not necessarily homogeneous of degree zero. Thus if (p,γ) is a competitive equilibrium under $\tau=(T,z^0)$, then $(\alpha p,\gamma)$ is not necessarily a competitive one under the same τ. However the following lemma holds, which explains the necessity of the second property of Assumption Q.

Lemma 2. Let (p,γ) be a competitive equilibrium under $\tau=(T,z^0)$, and put $\alpha=1/\Sigma p_k$. Then $(\alpha p,\gamma)$ is a competitive equilibrium under (T_α,z^0), where $T_\alpha(y)=\alpha T(y/\alpha)$ for all $y\in R_+$.

Proof. Remember $\pi_j(p)=0$ for all j. If $p_C\cdot x \leqq (1-T)[p_L\cdot f^i(t)+\Sigma\theta_j(i)\pi_j(p)] = (1-T)[p_L\cdot f^i(t)]$, then $\alpha p_C\cdot x \leqq \alpha(1-T)[p_L\cdot f^i(t)] = \alpha p_L\cdot f^i(t) - \alpha T[p_L\cdot f^i(t)] = \alpha p_L\cdot f^i(t) - T_\alpha[\alpha p_L\cdot f^i(t)] = (1-T_\alpha)[\alpha p_L\cdot f^i(t)] = (1-T_\alpha)[\alpha p_L\cdot f^i(t)+\Sigma\theta_j(i)\pi_j(\alpha p)]$. Therefore $(\alpha p,\gamma)$ satisfies Definition 1 (iii) and (iv) under (T_α,z^0). The other conditions of Definition 1 are not affected by rescaling the price vector. Q.E.D.

Although clearly there is no competitive equilibrium under some $\tau=(T,z^0) \in \Psi$, e.g., τ whose $-T(0)$ and z^0_Q are sufficiently large has no competitive equilibrium, it is, conversly, not clear whether or not some τ has a competitive equilibrium. This existence probrem will be explored in Part II of this paper. See also Remark 5 below. In Part I, we make the following assumption.

<u>Assumption R</u> There exists a tax schedule $\tau^0 \in \Psi$ which has a competitive equilibrium, i.e., $C(\tau^0) \neq \phi$.

The government has a social welfare function as a welfare criterion such that

$$\int_I G^i[U^i(\cdot,\cdot,\cdot)],$$

where G^i is a monotonic increasing and continuous function on R with $\int_I |G^i(V(i))| < \infty$ and $\int_I |G^i(-V(i))| < \infty$. These inequalities imply

$$-\infty < \int_I G^i(-V(i)) \leqq \int_I G^i(V(i)) < +\infty \tag{5}$$

When the government announces a tax schedule $\tau=(T,z^0) \in \Psi$ and so when a competitive equilibrium $(p,\gamma)=(p,(t(i),x(i),Q), z^1,z^2,z^3,\ldots,z^m)$ under τ results in the economy, the value of the social welfare function is represented as $\int_I G^i(U^i(t(i),x(i),Q))$. In fact, $\int_I G^i(U^i(t(i),x(i),Q))$ depends only on τ and p, that is, if (p,γ) and (p',γ') are competitive equilibria with $p=p'$ under τ, then $U^i(t(i),x(i),Q)=U^i(t'(i),x'(i),Q)$ a.e. Therefore we can denote

$$W(\tau,p)=\int_I G^i(U^i(t(i),x(i),Q)).$$

Definition 2. $\tau^* \in \Psi$ is said to be an optimal tax schedule iff for some $p^* \in C(\tau^*)$,

$$W(\tau^*,p^*) = \sup_{\substack{\tau \in \Psi \\ p \in C(\tau)}} W(\tau,p).$$

The purpose of this paper is to prove the following theorem.

Main Theorem. Under Assumptions A through R, there exists an optimal tax schedule.

Remark 5. Assumption R ensures the existence of a tax schedule having a competitive equilibrium. If the class $\{\tau \in \Psi | C(\tau) \neq \phi\} = \{\tau^0\}$, i.e., it consisted only of the tax schedule given in Assumption R, then our theorem would be vacant. However it will be shown in Part II that the class is rather wide. For example, it will be proved (Corollary of Part II) under certain additional assumptions that if $T \in \mathcal{T}$ satisfies $T(0)=0$ and $T(y) < y$ for all $y > 0$, then there is a competitive equilibrium under (T,z^0) for some $z^0 \in Z^0$.

Remark 6. If we adopt the Nash social welfare function of Kaneko(1981b,1983), i.e., $G^i(\alpha) = \log[\alpha - U^i(\underline{t}^i,0,0)]$, where $\underline{t}^i$ is the least favorite point in H, then G^i does not satisfy (5). In this case, the proof of our theorem will be still valid if there is a tax schedule τ having a competitive equilibrium (p,γ) with $-\infty < W(\tau,p)$. In Part II, it will be shown (Corollary of Part II) under certain additional assumptions that for some tax function $\hat{T}$ with $\hat{T}(0) < 0$, there is a competitive equilibrium $(\hat{p},\hat{\gamma})$ under $\hat{\tau} = (\hat{T},\hat{z}^0)$ for some $\hat{z}^0 \in Z^0$. Since $\hat{T}(0) < 0$, the consumption levels are uniformly positive for the individuals. Therefore the integrability of the Nash social welfare function at $(\hat{\tau},\hat{p})$, i.e., $-\infty < W(\hat{\tau},\hat{p})$, will be ensured under the condition that for any $\varepsilon > 0$ there is a $\delta > 0$ such that $U^i(0,(\varepsilon,\varepsilon,\ldots,\varepsilon),0) - \delta > U^i(\underline{t}^i,0,0)$ for all $i \in I$.

3. Proof

3.1

Since $C(\tau^0)\neq\phi$ by Assumption R, we have, by (5), $-\infty < \sup_{\substack{\tau\in\Psi \\ p\in C(\tau)}} W(\tau,p) < \infty$. Hence there is a sequence $\{(\tau^s,p^s)\}=\{(T^s,z^{0s},p^s)\}$ such that $\tau^s\in\Psi$, $p^s\in C(\tau^s)$, $s=1,2,...$ and $\lim_{s\to\infty} W(\tau^s,p^s)=\sup W(\tau,p)$. The outline of the following proof is that we choose a convergent subsequence $\{(\tau^{s^\nu},p^{s^\nu})\}$ from $\{(\tau^s,p^s)\}$ and then show the limit point (τ^*,p^*) satisfies $p^*\in C(\tau^*)$.

We normalize p^s $(s=1,2,...)$ into $\alpha_s p^s$, where $\alpha_s = 1/\Sigma p_k^s$. Every $\alpha_s p^s$ is in the compact set $P=\{p\in R_+^\ell \times R_+^c \mid \Sigma p_k=1\}$. For simplicity, we assume that $\{p^s\}$ itself is a sequence in P. Each p^s is a competitive price vector under $(T^s_{\alpha_s},z^{0s})$ by Lemma 2. Again we denote the sequence $\{T^s_{\alpha_s}\}$ by $\{T^s\}$ itself.

First we state the following lemma due to Aumann (1965,Theorems 1,2,4) or Hildenbrand (1974,p.62,Theorems 2,3 and p.73,Proposition 7).

<u>Lemma 3</u>. Let $F(i)=f^i(H)$ for all $i\in I$. Then

$$\int_I F(i) = \{\int_I f^i(t(i)) \mid t(i) \text{ is a measurable function of } i,\ t(i)\in H \text{ a.e.}\}$$

is a non-empty compact convex set.

3.2

In this subsection we show $\{T^s\}$ is included by a certain compact subset of $\mathcal{T}$.

<u>Lemma 4</u>. $\inf_s T^s(0) > -\infty$

<u>Proof</u>. Suppose $\inf_s T^s(0)=-\infty$. Since each T^s belongs to $\mathcal{T}$ by Assumption Q, $T^s(y)\leqq T^s(0)+y$ for all $y\in R_+$ and all s. Then, by Assumption N and Lemma 1,

$$0 \leqq -p_L^s \cdot z_L^{0s} - p_C^s \cdot z_C^{0s} = \int_I T^s[p_L^s \cdot f^i(t^s(i))] \leqq \int_I (T^s(0) + p_L^s \cdot f^i(t^s(i)))$$

$$= T^s(0)\mu(I) + p_L^s \cdot \int_I f^i(t^s(i)).$$

By Lemma 3, for large s, $T^s(0)\mu(I)+p_L^s \cdot \int_I f^i(t^s(i))$ is negative, which is a contradiction. Q.E.D.

Let $K = \inf_s T^s(0)$. We define $\overline{\mathcal{T}}_0 = \{T \in \mathcal{T}_0 \mid K \leqq T(0)\}$.

Lemma 5. $\overline{\mathcal{T}}_0$ is a compact set in $\mathcal{T}$.

Proof. Let $\overline{\mathcal{T}} = \{T \in \mathcal{T} \mid K \leqq T(0)\}$. Since $\overline{\mathcal{T}}_0 = \mathcal{T}_0 \cap \overline{\mathcal{T}}$ and $\mathcal{T}_0$ is closed, it is sufficient to prove that $\overline{\mathcal{T}}$ is a compact set. Let $\{T^s\}$ be any sequence in $\overline{\mathcal{T}}$. By the same argument as Kaneko (1982a, Lemma 4.6), we can show that the sequence $\{T^s\}$ has a subsequence $\{T^{s^\nu}\}$ such that

$$T^{s^\nu} \text{ converges uniformly to some } T^* \in \overline{\mathcal{T}} \text{ on } [0,k] \text{ for any positive integer } k. \tag{6}$$

For any k and ν, it holds that

$$\rho(T^{s^\nu}, T^*) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{y \in [n-1,n]} |T^{s^\nu}(y) - T^*(y)|$$

$$\leqq \sum_{n=1}^{k} \frac{1}{2^n} \sup_{y \in [n-1,n]} |T^{s^\nu}(y) - T^*(y)| + \sum_{n=k+1}^{\infty} \frac{1}{2^n}(n-K). \tag{7}$$

As $k \to \infty$, $\sum_{n=k+1}^{\infty} \frac{1}{2^n}(n-K) \to 0$. By (6), $\sum_{n=1}^{k} \frac{1}{2^n} \sup_{y \in [n-1,n]} |T^{s^\nu}(y) - T^*(y)| \to 0$ as $\nu \to \infty$ for arbitrary k. For any $\varepsilon > 0$, there is a k_0 such that

$$\sum_{n=k_0+1}^{\infty} \frac{1}{2^n}(n-K) < \frac{\varepsilon}{2}. \tag{8}$$

For this k_0, there is a ν_0 such that

$$\sum_{n=1}^{k_0} \frac{1}{2^n} \sup \left| T^{s^\nu}(y) - T^*(y) \right| < \frac{\varepsilon}{2} \qquad \text{for all } \nu \geqq \nu_0. \tag{9}$$

Then it follows from (7), (8), (9) that $\nu \geqq \nu_0$ implies $\rho(T^{s^\nu}, T^*) < \varepsilon$.[8] Q.E.D.

3.3

Let

$$\bar{Z}^j = \{ z^j \in Z^j \mid \sum_{h=0}^{m} z_C^h \geqq 0, \ \sum_{h=0}^{m} z_L^j \in \int_I F(i), \ z_Q^0 \geqq 0 \text{ for some } z^h \in Z^h \ h=0,1,\ldots,m, \ h \neq j \}$$

for $j=0,1,2,\ldots,m$. We can see by Lemma 3 that $\bar{Z}^j$ is compact for $j=0,1,\ldots,m$ under our assumptions.

Since $p^s \in C(\tau^s)$ $(s=1,2,\ldots)$, there is a competitive equilibrium $(p^s, \gamma^s) = (p^s, (t^s(i), x^s(i), Q^s), z^{1s}, z^{2s}, \ldots, z^{ms})$ under T^s for each s. Then by Definition 1(ii), $(\gamma^s, z^{0s}) = ((t^s(i), x^s(i), Q^s), z^{0s}, z^{1s}, \ldots, z^{ms})$ is an allocation $(s=1,2,\ldots)$, which implies $z^{js} \in \bar{Z}^j$ for $j=1,2,\ldots,m$. Since $\bar{Z}^0$, $\bar{Z}^1$, $\bar{Z}^2, \ldots, \bar{Z}^m$ are compact, there are convergent subsequences $\{z^{0s^\nu}\}, \{z^{1s^\nu}\}, \ldots, \{z^{ms^\nu}\}$ of $\{z^{0s}\}$, $\{z^{1s}\}$, ..., $\{z^{ms}\}$, respectively. Without loss of generality, we can assume that $\{z^{0s}\}$, $\{z^{1s}\}, \ldots, \{z^{ms}\}$ themselves converge to z^{0*}, $z^{1*}, \ldots, z^{m*}$, respectively.

Since $\{T^s\}$ and $\{p^s\}$ are sequences in the compact sets $\bar{\mathcal{J}}_0$ and P, there are subsequences $\{T^{s^\nu}\}$ and $\{p^{s^\nu}\}$ of $\{T^s\}$ and $\{p^s\}$ which converge to $T^* \in \bar{\mathcal{J}}_0$

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8) In fact, the convergence in the sence of (6) is equivalent to the convergence in terms of the metric ρ in general. It is clear that the convergence in terms of ρ implies (6). It suffices to show the converse. Let $\inf T^{s^\nu}(0) = -\infty$. Then $\inf \left| T^*(0) - T^{s^\nu}(0) \right| = -\infty$, which implies that T^{s^ν} does not converge to T^* in the sence of (6). Therefore the convergence in the sence of (6) implies boundedness from below. The argument of the proof ensures the convergence in terms of the metric ρ.

and $p^* \in P$. For simplicity, we assume that $\{T^s\}$ and $\{p^s\}$ themselves converge to T^* and p^*. Then since $\lim_{s\to\infty} W(\tau^s,p^s) = \sup W(\tau,p)$, it suffices to show $p^* \in C(\tau^*)$ and $\lim_{s\to\infty} W(\tau^s,p^s) = W(\tau^*,p^*)$.

It is easy to verify the following lemma.

<u>Lemma 6</u>. For $j=1,2,\ldots,m$, $z^{j*} \in Z^j$, $p^*\cdot z^{j*}=0 \geqq p^*\cdot z$ for all $z \in Z^j$ and $Q^*=z_Q^{0*}$.

We proceed with the proof in the following cases;

[I] $p_C^* > 0$ and [II] not $p_C^* > 0$.

In case [I], the proof is completed when we show that p^* is a competitive price under τ^* and $W(\tau^*,p^*) = \sup W(\tau,p)$. In case [II], it will be shown that the tax schedule τ^0 and a competitive price vector p^0 given in Assumption R attain $\sup W(\tau,p)$.

Remember the utility maximization under his budget constraint is required for <u>almost</u> <u>all</u> individuals in Definition 1. Let $I^s = \{i \in I \mid (t^s(i),x^s(i))$ maximizes i's utility under his budget constraint$\}$. Then since $\mu(I^s)=\mu(I)$ for all s, $\mu(\bigcap_{s=1}^{\infty} I^s)=\mu(I)$. Therefore we can confine ourselves to the set $\bigcap_{s=1}^{\infty} I^s$ instead of I. In the following, we assume;

$$\text{for } \underline{\text{all}}\ i \in I \text{ and all } s,\ p_C^s\cdot x_C^s(i) \leqq (1-T^s)[p_L^s\cdot f^i(t^s(i))] \text{ and}$$
$$U^i(t^s(i),x^s(i),Q^s) \geqq U^i(t,x,Q^s) \text{ for all } (t,x) \text{ with } p_C^s\cdot x \leqq (1-T^s)[p_L^s\cdot f^i(t)]. \tag{10}$$

<u>3.4</u>

Consider case [I].

Since $p_k^* > 0$ for all $k \in C$, we can take a $\delta > 0$ such that $\delta < \min_{k\in C} p_k^*$. Then for some s^0, $p_k^s > \delta$ for all $k \in C$ and all $s \geqq s^0$. It follows from the budget constraint and Assumption H that for all k, $s \geqq s^0$,

$$0 \leqq x_k^s(i) \leqq \frac{(1-T^s)[p_L^s \cdot f^i(t^s(i))]}{\delta} \leqq \frac{p_L^s \cdot f^i(t^s(i))-K}{\delta} \leqq \frac{g(i)-K}{\delta} .$$

Hence it holds that

$x^s(i)$ is uniformly integrable for all $s \geqq s^0$; and (11)

for any fixed i, $x^s(i)$ is in some compact set for all $s \geqq s^0$. (12)

By Assumption A and (12), $\{(t^s(i), x^s(i))\}$ has a cluster point for each $i \in I$.

Given p with $p_C > 0$ and $\tau = (T,Q)$, we define the <u>indirect utility function</u> $u^i(\tau, p)$ by

$$u^i(\tau,p) = \max U^i(t,x,Q) \quad \text{subject to } p_C \cdot x \leqq (1-T)[p_L \cdot f^i(t)]. \tag{13}$$

<u>Lemma 7</u>. For any $i \in I$, $u^i(\tau^*,p^*) = U^i(\tilde{t},\tilde{x},Q^*)$ and $p_C^* \cdot \tilde{x} \leqq (1-T^*)[p_L^* \cdot f^i(\tilde{t})]$ for any cluster point $(\tilde{t},\tilde{x})$ of $\{(t^s(i),x^s(i))\}$, that is, $u^i(\tau^s,p^s) \to u^i(\tau^*,p^*)$.

<u>Proof</u>. Since (p^s,γ^s) is a competitive equilibrium under τ^s, we have, by (10),

$$u^i(\tau^s,p^s) = U^i(t^s(i),x^s(i),Q^s) \quad \text{for all } s.$$

Let $\{(t^{s^\nu}(i),x^{s^\nu}(i))\} \to (\tilde{t},\tilde{x})$. Then by continuity, we have $\lim_\nu u^i(\tau^{s^\nu},p^{s^\nu}) = \lim_\nu U^i(t^{s^\nu}(i),x^{s^\nu}(i),Q^{s^\nu}) = U^i(\tilde{t},\tilde{x},Q^*)$. Since $p_C^{s^\nu} \cdot x^{s^\nu}(i) \leqq (1-T^{s^\nu})[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]$ for all ν, $p_C^* \cdot \tilde{x} \leqq (1-T^*)[p_L^* \cdot f^i(\tilde{t})]$.

Suppose that there is a $(t,x) \in H \times R_+^C$ such that $U^i(t,x,Q^*) > U^i(\tilde{t},\tilde{x},Q^*)$ and $p_C^* \cdot x \leqq (1-T^*)[p_L^* \cdot f^i(t)]$. If $(1-T^*)[p_L^* \cdot f^i(t)] = 0$, then $x=0$ from $p_C^* \cdot x = 0$. Hence $U^i(t,0,Q^*) > U^i(\tilde{t},\tilde{x},Q^*)$. This implies that $U^i(t,0,Q^{s^\nu}) > U^i(t^{s^\nu}(i), x^{s^\nu}(i),Q^{s^\nu})$ for large ν, which contradicts that $(t,0)$ always satisfies the budget constraint.

Let $(1-T^*)[p_L^*\cdot f^i(t)]>0$. Then there exists an x' near x such that $U^i(t,x',Q^*)>U^i(\tilde{t},\tilde{x},Q^*)$ and $p_C^*\, x'<(1-T^*)[p_L^*\, f^i(t)]$, which implies that for large ν,

$$U^i(t,x',Q^{s^\nu})>U^i(t^{s^\nu}(i),x^{s^\nu}(i),Q^{s^\nu}) \text{ and } p_C^{s^\nu}\cdot x'<(1-T^{s^\nu})[p_L^{s^\nu}\cdot f^i(t)].$$

This is a contradiction. We have shown that

$$U^i(\tilde{t},\tilde{x},Q^*)\geqq U^i(t,x,Q^*) \text{ for all } (t,x) \text{ with } p_C^*\cdot x\leqq(1-T^*)[p_L^*\cdot f^i(t)].$$

This means $u^i(\tau^*,p^*)=U^i(\tilde{t},\tilde{x},Q^*)$, which implies $\lim_\nu u^i(\tau^{s^\nu},p^{s^\nu})=U^i(\tilde{t},\tilde{x},Q^*)=u^i(\tau^*,p^*)$. However, since this holds for an arbitrary convergent subsequence $\{(t^{s^\nu}(i), x^{s^\nu}(i))\}$, we have $\lim_s u^i(\tau^s,p^s)=u^i(\tau^*,p^*)$. Q.E.D.

For each $i\in I$, let $X(i)$ be the set of all cluster points of $\{(f^i(t^s(i)),x^s(i))\}$. Note that $X(i)$ is non-empty by Assumption A and (12). The following lemma due to Aumann (1965, Proposition 4.1) or Hildenbrand (1974, p.68, Theorem 6) is applied to this correspondence.

<u>Lemma 8</u>. Let $F_1(\cdot)$, $F_2(\cdot)$, ... be a sequence of set valued functions bounded by the same integrable function. Then it holds that

$$Ls\left(\int F_s(i)\right)\subset\int Ls(F_s(i)) .$$

Here for any sequence of subsets G_s of R^n, $Ls(G_s)=\{a\mid a \text{ is a cluster point of some sequence } \{a^s\} \text{ with } a^s\in G_s \text{ for all } s\}$.

In this lemma if we let $F_s(\cdot)=\{(f^i(t^s(\cdot)),x^s(\cdot))\}$ for $s=1,2,\ldots$, then by (11),

$$Ls\left(\int_I\{(f^i(t^s(i)),x^s(i))\}\right)\subset\int_I Ls(\{(f^i(t^s(i)),x^s(i))\})=\int_I X(i).$$

Therefore for any subsequence $\{(f^i(t^{s^\nu}(\cdot)),x^{s^\nu}(\cdot))\}$ of $\{(f^i(t^s(\cdot)),x^s(\cdot))\}$ such that $\{\int_I (f^i(t^{s\nu}(i)),x^{s\nu}(i))\}$ converges, $\lim_\nu \int_I (f^i(t^{s\nu}(i)),x^{s\nu}(i)) \in \int_I X(i)$. This says that there exists a measurable function $(\ell^*(\cdot),x^*(\cdot))$ such that $(\ell^*(i),x^*(i)) \in X(i)$ for almost all $i \in I$, and $\lim_\nu \int_I (f^i(t^{s^\nu}(i)),x^{s^\nu}(i)) = \int_I (\ell^*(i),x^*(i))$. We assume $\int_I (f^i(t^s(i)),x^s(i))$ itself converges to $\int_I (\ell^*(i),x^*(i))$. Moreover we may assume $\ell^*(i)=0$ for all i with $(\ell^*(i),x^*(i)) \notin X(i)$, because such individuals form a null set.

We define the set value functions H(i) and A(i) by

$$H(i)= \{t \in H \mid (t,x^*(i)) \text{ is a cluster point of } \{(t^s(i),x^s(i))\}\},$$

$$A(i)= \begin{cases} \{t \in H(i) \mid f^i(t)=\ell^*(i)\} & \text{if } (\ell^*(i),x^*(i)) \in X(i), \\ \{0\} & \text{otherwise.} \end{cases}$$

We show A(i) is non-empty for all $i \in I$. Let i be fixed. If $(\ell^*(i),x^*(i)) \notin X(i)$, then $A(i)=\{0\}$, i.e., non-empty. If $(\ell^*(i),x^*(i)) \in X(i)$, we can take a subsequence $\{(f^i(t^{s^\nu}(i)),x^{s^\nu}(i))\}$ of $\{(f^i(t^s(i)),x^s(i))\}$ with $\lim_\nu f^i(t^{s^\nu}(i))=\ell^*(i)$, $\lim_\nu x^{s^\nu}(i)=x^*(i)$. Then $\{t^{s^\nu}(i)\}$ has a convergent subsequence $\{t^{s^{\nu\lambda}}(i)\}$ with $\lim_\lambda t^{s^{\nu\lambda}}(i)=t$. Since $\lim_\lambda x^{s^{\nu\lambda}}(i)=\lim_\nu x^{s^\nu}(i)= x^*(i)$, $(t,x^*(i))$ is a limit point of $\{(t^{s^{\nu\lambda}}(i),x^{s^{\nu\lambda}}(i))\}$, which implies $t \in H(i)$. Moreover since $f^i(t)=\lim_\lambda f^i(t^{s^{\nu\lambda}}(i))= \lim_\nu f^i(t^{s^\nu}(i))=\ell^*(i)$, we have $t \in A(i)$. The measurability of the graph of A(i) is proved in the standard way.

Therefore there exists a measurable function $t^*(i)$ such that $t^*(i) \in A(i)$ a.e., by the Measurable Selection Theorem [see Hildenbrand (1974,p.54, Theorem 1)]. For this measurable function $t^*(i)$ and the measurable function $x^*(i)$, it follows from above results that

$$\lim_s \int_I f^i(t^s(i)) = \int_I f^i(t^*(i)), \tag{14}$$

$$\lim_s \int_I x^s(i) = \int_I x^*(i). \tag{15}$$

Since $t^*(i) \in H(i)$ for almost all i, there is a subsequence $\{(t^{s^\nu}(i), x^{s^\nu}(i))\}$ (depending on i) of $\{(t^s(i),x^s(i))\}$ which converges to $(t^*(i),x^*(i))$. Then it follows from Lemma 7 that for almost all $i \in I$,

$$U^i(t^*(i),x^*(i),Q^*) \geqq U^i(t,x,Q^*) \text{ for all } (t,x) \text{ with } p^*_C \cdot x \leqq (1-T^*)[p^*_L \cdot f^i(t)], \tag{16}$$

$$p^*_C \cdot x^*(i) \leqq (1-T^*)[p^*_L \cdot f^i(t^*(i))]. \tag{17}$$

Since $(-\int_I f^i(t^*(i)), \int_I x^*(i)) = \lim_s (\int_I f^i(t^s(i)), \int_I x^s(i)) = \lim_s \sum_{j=0}^m z^{js} = \sum_{j=0}^m z^{j*}$ by (14), (15) and $Q^* = z_Q^{0*}$, $((t^*(i),x^*(i),Q^*), z^{0*}, z^{1*}, z^{2*}, \ldots, z^{m*})$ is an allocation. The statements (16) and (17) with Lemma 6 imply that $(p^*, (t^*(i), x^*(i), Q^*), z^{1*}, z^{2*}, \ldots, z^{m*})$ is a competitive equilibrium.

Finally we demonstrate $\lim_s W(\tau^s,p^s) = W(\tau^*,p^*)$ to complete the proof of case [I]. It follows from (13) and Lemma 7 that for almost all i,

$$\lim_s U^i(t^s(i),x^s(i),Q^s) = \lim_s u^i(\tau^s,p^s) = u^i(\tau^*,p^*) = U^i(t^*(i),x^*(i),Q^*).$$

This implies $\lim_s G^i(U^i(t^s(i),x^s(i),Q^s)) = G^i(U^i(t^*(i),x^*(i),Q^*))$ a.e.

Then, by Lebesgue's Convergence Theorem, we have

$$\begin{aligned} W(\tau^*,p^*) = \int_I G^i(U^i(t^*(i),x^*(i),Q^*)) &= \int_I \lim_s G^i(U^i(t^s(i),x^s(i),Q^s)) \\ &= \lim_s \int_I G^i(U^i(t^s(i),x^s(i),Q^s)) \\ &= \lim_s W(\tau^s,p^s). \end{aligned}$$

This completes the proof of case [I].

3.5

Consider case [II]; not $p_C^* > 0$, in this subsection. Let us start with proving the following;

$$\text{there exists a subsequence } \{x^{s^\nu}(\cdot)\} \text{ (not depending on i) of } \{x^s(\cdot)\} \text{ such that } \lim_\nu x^{s^\nu}(i)=0 \text{ for almost all } i \in I. \tag{18}$$

Lemma 9. If $p_C^*=0$, then (18) is true.

Proof. Since $p_L^* \cdot \sum_{j=1}^m z_L^{j*} = -p_C^* \cdot \sum_{j=1}^m z_C^{j*} = 0$ by Lemma 5 and $p_C^*=0$, and since $p_k^* > 0$ for some $k \in L$ by $p^* \in P$ and $p_C^*=0$, we have $\sum_{j=1}^m z_k^{j*}=0$. Then $\sum_{j=1}^m z_C^{j*}=0$ by Remark 1. Hence $\int_I x^s(i) \to 0$ because $\int_I x^s(i) = \sum_{j=1}^m z_C^{js} + z_C^{0s} \leqq \sum_{j=1}^m z_C^{js} \to 0$. Since $\{x^s(\cdot)\}$ converges to 0 in norm, there is a subsequence $\{x^{s^\nu}(\cdot)\}$ of $\{x^s(\cdot)\}$ that converges to 0 a.e. [see Danford and Schwartz (1957, Theorem III 3.6 and Corollary III 6.13 (a))]. Q.E.D.

Lemma 10. If neither $p_C^* > 0$ nor $p_C^*=0$, then (18) is true.

Proof. The following lemma is a key in proving this lemma.

Lemma 11. If neither $p_C^* > 0$ nor $p_C^* = 0$, there is a subsequence $\{t^{s^\nu}(\cdot)\}$ such that $\lim_\nu \int_I (1-T^{s^\nu})[p_L^{s^\nu} \cdot f^i(t^{s^\nu}(i))]=0$

The proof of Lemma 11 is given in the Appendix.

It follows from Lemma 11 that for any $k \in C$ with $p_k^* > 0$, $p_k^{s^\nu}\int_I x_k^{s^\nu}(i) \to 0$, i.e., $\int_I x_k^{s^\nu}(i) \to 0$, and so some subsequence of $\{x_k^{s^\nu}(i)\}$ tends to zero a.e. by Theorem III 3.6 and Corollary III 6.13 (a) of Danford and Schwartz (1957). Next consider any $k \in C$ with $p_k^* = 0$. Since $p^* \neq 0$, there is a $k' \in C$ such that $p_{k'}^* > 0$. If $\sum_{j=1}^m z_k^{j*} > 0$, then there are ε and ε' by Assumption M such that $\sum_{j=1}^m z^{j*} - \varepsilon e^k + \varepsilon' e^{k'}$ is in $\sum_{j=1}^m Z^j$. Hence $p^s \cdot (\sum_{j=1}^m z^{j*} - \varepsilon e^k + \varepsilon' e^{k'}) \to p_{k'}^* \varepsilon' > 0$, which is a contradiction to Lemma 6. Thus we have shown that $\sum_{j=1}^m z^{j*} = 0$. This implies $\int_I x_k^s(i) \to 0$ because $\int_I x_k^s(i) = \sum_{j=1}^m z_k^{js} + z_k^{0s} \leqq \sum_{j=1}^m z_k^{js} \to \sum_{j=1}^m z_k^{j*} = 0$. Hence there is a subsequence of $\{x_k^s(\cdot)\}$ which tends to zero a.e. by Theorem III 3.6 and Corollary III 6.13 (a) of Danford and Schwartz (1957). Q.E.D.

Lemmas 9 and 10 tell us that $\{x^s(\cdot)\}$ has a subsequence which converges to zero a.e. Again we assume $\{x^s(\cdot)\}$ itself converges to zero a.e.

<u>Lemma 12</u>. $t^s(i) \to 0$ for almost all $i \in I$.

<u>Proof</u>. Fix any i such that $\lim_s x^s(i) = 0$. Suppose $t^s(i) \not\to 0$. Then there is a convergent subsequence $\{t^{s^\nu}(i)\}$ such that $\lim_\nu t^{s^\nu}(i) = t^0 \leq 0$. By Assumption B, $U^i(t^0, 0, Q^*) < U^i(0, 0, Q^*)$. Since $x^{s^\nu}(i) \to 0$, there exists a ν^0 such that

$$U^i(t^{s^\nu}(i), x^{s^\nu}(i), Q^{s^\nu}) < U^i(0, 0, Q^{s^\nu}) \quad \text{for all } \nu \geqq \nu^0.$$

This is a contradiction to the fact that $(t^{s^\nu}(i), x^{s^\nu}(i), Q^{s^\nu})$ maximizes i's utility under his budget constraint. Q.E.D.

Since $(t^s(i), x^s(i))$ converges to zero a.e., we have, by Lebesgue's Convergence Theorem,

$$\sup W(\tau,p) = \lim_s W(\tau^s,p^s) = \int_I G^i(U^i(t^s(i),x^s(i),Q^s))$$
$$= \int_I G^i(U^i(0,0,Q^*)).$$

Since $t^s(i) \to 0$ a.e., $f^i(t^s(i)) \to 0$ a.e., which implies $\int_I f^i(t^s(i)) \to 0$ by Lebesgue's Convergence Theorem. Hence we have $\sum_{j=1}^{m} z^{j*}=0$ by Remark 1.

This shows $z_Q^{0*}=Q^*=0$, which implies

$$\sup W(\tau,p) = \int_I G^i(U^i(0,0,Q^*)) = \int_I G^i(U^i(0,0,0)).$$

Let τ^0 be a tax schedule given in Assumption R, and $(p^0,(t^0(i),x^0(i),Q^0),z^{10}, z^{20},\ldots,z^{m0})$ a competitive equilibrium under τ^0. Since (0,0) is always in the budget constraint, $U^i(t^0(i),x^0(i),Q^0) \geqq U^i(0,0,0)$ for almost all $i \in I$, and so

$$\sup W(\tau,p) = \int_I G^i(U^i(0,0,0)) \leqq \int_I G^i(U^i(t^0(i),x^0(i),Q^0)).$$

Since $\sup W(\tau,p) \geqq \int_I G^i(U^i(t^0(i),x^0(i),Q^0))$ from $p^0 \in C(\tau^0)$,

$$\sup W(\tau,p) = \int_I G^i(U^i(t^0(i),x^0(i),Q^0)) = W(\tau^0,p^0),$$

which shows that τ^0 is an optimal tax schedule. This completes the proof.

Remark 7. It will be shown under certain additional assumptions in Part II that the competitive equilibrium (p,γ) under some tax schedule (T,z^0) have the property that $U^i(t(i),x(i),Q) > U^i(0,0,0)$ for individuals with a positive measure. Therefore (18) is not the case, that is, case[II]; not $p_C^* > 0$ does not occur under the assumptions of Part II.

Appendix.

Proof of Lemma 11.

Throughout the appendix we assume that $p_C^* \neq 0$ but not $p_C^* > 0$, and simultaneously that any subsequence of $\{\int_I (1-T^s)[p_L^s \cdot f^i(t^s(i))]\}$ does not converge to zero, and then we will derive a contradiction.

Put $E^s = \{i \in I \mid (1-T^s)[p_L^s \cdot f^i(t^s(i))] > 0\}$. Then $\mu(E^s) \not\to 0$ as $s \to \infty$, because if $\mu(E^s) \to 0$, then $\int_I (1-T^s)[p_L^s \cdot f^i(t^s(i))] = \int_{E^s} (1-T^s)[p_L^s \cdot f^i(t^s(i))] \leqq \int_{E^s} (g(i)-K) \to 0$, which is a contradiction. Here $g(i)$ is given in Assumption H and $K = \inf_s T^s(0)$. We can take a subsequence of $\{\mu(E^s)\}$ which converges a positive number α, and let $\mu(E^s)$ itself converge to α.

Put $C_0 = \{k \in C \mid p_k^* = 0\}$ and $C_1 = \{k \in C \mid p_k^* > 0\}$. Then $C_0 \neq \phi$ and $C_1 \neq \phi$ by the assumption of the lemma.

Assertion 1. $\forall \varepsilon_1 > 0,\ \exists M > 0,\ \exists s_1$:

$$\mu(\{i \in I \mid x_k^s(i) > M\}) < \frac{\varepsilon_1}{2|C_0|} \qquad \text{for all } s \geqq s_1, \text{ all } k \in C_0.$$ [9]

Proof. Suppose not, i.e., $\exists \varepsilon_1 > 0, \forall M > 0, \forall s_1, \exists s \geqq s_1, \exists k \in C_0$;

$$\mu(\{i \in I \mid x_k^s(i) > M\}) \geqq \frac{\varepsilon_1}{2|C_0|}. \tag{19}$$

Since $\int_I x_k^s(i) \leqq \sum_{j=1}^m z_k^{js}$ for all $k \in C$ and $\sum_{j=1}^m z^{js}$ is in the compact set $\sum_{j=1}^m \bar{Z}^j$, there is a $D > 0$ such that $\int_I x_k^s(i) < D$ for all s, and all $k \in C$. Let $M = \frac{2D|C_0|}{\varepsilon_1}$ and $S_M = \{i \in I \mid x_k^s(i) > M\}$. By (19), we have

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9) $|S|$ is the cardinal number of a set S.

$$D > \int_I x_k^s(i) \geqq \int_{S_M} x_k^s(i) \geqq \int_{S_M} \frac{2D|C_0|}{\varepsilon_1} \geqq \frac{2D|C_0|}{\varepsilon_1} \cdot \frac{\varepsilon_1}{2|C_0|} = D,$$

which is a contradiction. Q.E.D.

On the other hand, since $\mu(E^s) \to \alpha$, there is an s_2 for any $\varepsilon_1 > 0$ such that $\mu(E^s) \geqq \alpha - \frac{\varepsilon_1}{2}$ for all $s \geqq s_2$. Together with Assertion 1, this implies the next assertion.

<u>Assertion 2</u>. $\forall \varepsilon_1 > 0,\ \exists M > 0,\ \exists s_0$:

$$\mu(\{i \in E^s |\ x_k^s(i) \leqq M \text{ for all } k \in C_0\}) > \alpha - \varepsilon_1.$$

<u>Proof</u>. Let $s_0 = \max(s_1, s_2)$. Then

$$\mu(\{i \in E^s |\ x_k^s(i) \leqq M \text{ for all } k \in C_0\})$$
$$= \mu(E^s - \bigcup_{k \in C_0} \{i \in I |\ x_k^s(i) > M\})$$
$$\geqq \mu(E^s) - \sum_{k \in C_0} \mu(\{i \in I |\ x_k^s(i) > M\}) > \alpha - \frac{\varepsilon_1}{2} - \frac{\varepsilon_1}{2|C_0|} \cdot |C_0| = \alpha - \varepsilon_1.$$

Q.E.D.

<u>Assertion 3</u>. $\exists k_1 \in C_1 : \int_{E^s} x_{k_1}^s(i) \not\to 0$ as $s \to \infty$.

<u>Proof</u>. On the contrary suppose $\int_{E^s} x_k^s(i) \to 0$ for any $k \in C_1$. Since $x^s(i) = 0$ for all $i \in I - E^s$ by the definition of E^s and by $p_k^s > 0$ for all s, $\int_I x_k^s(i) = \int_{E^s} x_k^s(i) \to 0$ for any $k \in C_1$. Then we have

$$\int_I p_C^s \cdot x^s(i) = \sum_{k \in C_0} \int_I p_k^s x_k^s(i) + \sum_{k \in C_1} \int_I p_k^s x_k^s(i) \to 0,$$

because $\int_I x_k^s(i) < D$ for all k, s and $p_k^s \to 0$ for $k \in C_0$, where D is the constant taken in the proof of Assertion 1.

Since $(1-T^s)[p_L^s \cdot f^i(t^s(i))] = p_C^s \cdot x^s(i)$ for all $i \in I$ by $\int_I (1-T^s)[p_L^s \cdot f^i(t^s(i))] = \int_I p_C^s \cdot x^s(i) \to 0$. This is a contradiction. Q.E.D.

Since $\int_I x_{k_1}^s(i) \leqq \sum_{j=1}^m z_{k_1}^{js}$ for all s, $\{\int_{E^s} x_{k_1}^s(i)\}$ is a bounded sequence. Hence we can take a subsequence which converges to a positive number β. We assume again $\int_{E^s} x_{k_1}^s(i)$ itself converges to β.

Assertion 4. $\exists\, c > 0 : \mu(\{i \in E^s | x_{k_1}^s(i) > c\}) \not\to 0$ as $s \to \infty$.

Proof. Suppose $\mu(\{i \in E^s | x_{k_1}^s(i) > c\}) \to 0$ for all $c > 0$. Let $F^s = \{i \in E^s | x_{k_1}^s(i) > 0\}$, $G^s = \{i \in E^s \mid c \geqq x_{k_1}^s(i) > 0\}$. Then $F^s \supset G^s\ \forall s$ and $\mu(F^s - G^s) = \mu(\{i \in E^s \mid x_{k_1}^s(i) > c\}) \to 0$. Since $x_{k_1}^s(i)$ is uniformly integrable from $p_{k_1}^* > 0$, $\int_{F^s-G^s} x_{k_1}^s(i) \to 0$. Hence,

$$\int_{E^s} x_{k_1}^s(i) = \int_{F^s} x_{k_1}^s(i) = \int_{F^s-G^s} x_{k_1}^s(i) + \int_{G^s} x_{k_1}^s(i) \leqq \int_{F^s-G^s} x_{k_1}^s(i) + c\mu(G^s) \leqq \int_{F^s-G^s} x_{k_1}^s(i) + c\mu(I) \to c\,\mu(I).$$

Since c is arbitraty, $\int_I x_{k_1}^s(i) \to 0$, which contradicts Assertion 3. Q.E.D.

From this assertion we may assume for a subsequence $\{x^{s^\nu}(\cdot)\}$, there is a $d > 0$ such that $\mu(\{i \in E^{s^\nu} | x_{k_1}^{s^\nu}(i) > c\}) \to d$. Again assume that $\{x^s(\cdot)\}$ itself has the property

$$\mu(\{i \in E^s \,|x_{k_1}^s(i) > c\}) \to d. \tag{20}$$

Assertion 5. $\forall \varepsilon > 0, \exists\, \tilde{s}, \exists M$:

$$\mu(\{i \in E^s \mid x_k^s(i) \leqq M\ \ \forall k \in C_0 \text{ and } x_{k_1}^s(i) > c\}) \geqq d-\varepsilon \quad \text{for all } s \geqq \tilde{s}.$$

Proof. By (20), there exists an s_3 such that $\mu(\{i \in E^s \,|x_{k_1}^s(i) > c\}) \geqq d - \frac{\varepsilon}{2}$ for all $s \geqq s_3$. By Assertion 2 and $\mu(E^s) \to \alpha$, there are s_4 and M such that

$\mu(\{i \in E^s \mid \exists k \in C_0 \text{ such that } x_k^s(i) > M\}) < \frac{\varepsilon}{2}$ for all $s \geqq s_4$. If we take $\tilde{s} = \max(s_3, s_4)$, then the following holds for all $s \geqq \tilde{s}$;

$$\begin{aligned} &\mu(\{i \in E^s \mid x_k^s(i) \leqq M \ \ \forall k \in C_0 \text{ and } x_{k_1}^s(i) > c\}) \\ &= \mu(\{i \in E^s \mid x_{k_1}^s(i) > c\}) - \mu(\{i \in E^s \mid \exists k \in C_0 \text{ such that } x_k^s(i) > M\}) \\ &> (d - \frac{\varepsilon}{2}) - \frac{\varepsilon}{2} = d - \varepsilon. \end{aligned}$$

Q.E.D.

Put $W^s = \{i \in E^s \mid x_k^s(i) \leqq M \ \ \forall k \in C_0 \text{ and } x_{k_1}^s(i) > c\}$. Since $\mu(\{i \in E^s \mid x_{k_1}^s(i) > c\}) \geqq \mu(W^s) \geqq d - \varepsilon$ for all $s \geqq \tilde{s}$, we have

$$d \geqq \overline{\lim_s}\, \mu(W^s) \geqq \underline{\lim_s}\, \mu(W^s) \geqq d - \varepsilon.$$

Since ε is arbitrary, it holds that $\lim_s \mu(W^s) = d$.

Assertion 6. For all $i \in \bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} W^s$, there exists an s^i (depending on i) such that $x^s(i)$ does not maximize his utility under the budget constraint for all s with $W^s \ni i$ and $s \geqq s^i$.

Since $\mu(\bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} W^s) = \lim \mu(W^s) = d > 0$, $\bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} W^s \neq \phi$. Then Assertion 6 contradicts (10). Therefore we complete the proof of Lemma 11 by proving Assertion 6.

Proof of Assertion 6. Let $i \in \bigcap_{r=1}^{\infty} \bigcup_{s=r}^{\infty} W^s$ be arbitrarily fixed.

Take $y^s(i)$ such that $y_k^s(i) = 2M$ if $k \in C_0$ and $y_k^s(i) = x_k^s(i)$ if $k \in C_1$. Then $U^i(t^s(i), y^s(i), Q^s) > U^i(t^s(i), x^s(i), Q^s)$ for all s with $W^s \ni i$ by Assumption B.

Claim. There is an $\varepsilon > 0$ such that the inequality $U^i(t^s(i), y^s(i) - \varepsilon e^{k_1}, Q^s) > U^i(t^s(i), x^s(i), Q^s)$ holds uniformly for all s with $W^s \ni i$.

Proof of claim. Let $\varepsilon^s = \sup\{\varepsilon \mid U^i(t^s(i), y^s(i) - \varepsilon e^{k_1}, Q^s) \geqq U^i(t^s(i), x^s(i), Q^s),\ 0 \leqq \varepsilon \leqq c\}$. Since $y^s_{k_1}(i) = x^s_{k_1}(i) > c$, ε^s is well-difined for all s with $W^s \ni i$. For any s with $W^s \ni i$, there is an $\hat{\varepsilon} > 0$ such that $U^i(t^s(i), y^s(i) - \hat{\varepsilon} e^{k_1}, Q^s) > U^i(t^s(i), x^s(i), Q^s)$. This implies $\varepsilon^s > 0$ for all s with $i \in W^s$. Suppose $\underline{\lim}_s \varepsilon^s = 0$. It holds that $x^s_k(i)$, $y^s_k(i) \leqq 2M$ for any $k \in C_0$. Since $p^*_k > 0$ for any $k \in C_1$, it follows from the budget constraint that for any $k \in C_1$, $\{x^s_k(i)\}$ and $\{y^s_k(i)\}$ are bounded for all s. Therefore there are subsequences $\{t^{s^\nu}(i)\}$, $\{x^{s^\nu}(i)\}$, $\{y^{s^\nu}(i)\}$ and $\{\varepsilon^{s^\nu}\}$ such that $t^{s^\nu}(i) \to t(i)$, $x^{s^\nu}(i) \to x(i)$, $y^{s^\nu}(i) \to y(i)$ and $\varepsilon^{s^\nu} \to 0$. Since $U^i(t^{s^\nu}(i), y^{s^\nu}(i) - \varepsilon^{s^\nu} e^{k_1}, Q^{s^\nu}) \geqq U^i(t^{s^\nu}(i), x^{s^\nu}(i), Q^{s^\nu})$, $U^i(\tilde{t}(i), \tilde{y}(i), Q^*) \geqq U^i(\tilde{t}(i), \tilde{x}(i), Q^*)$.

Next we show $U^i(\tilde{t}(i), \tilde{y}(i), Q^*) = U^i(\tilde{t}(i), \tilde{x}(i), Q^*)$. If $U^i(\tilde{t}(i), \tilde{y}(i), Q^*) > U^i(\tilde{t}(i), \tilde{x}(i), Q^*)$, $U^i(t^{s^\nu}(i), y^{s^\nu}(i) - \varepsilon^{s^\nu} e^{k_1}, Q^{s^\nu}) > U^i(t^{s^\nu}(i), x^{s^\nu}(i), Q^{s^\nu})$ and $c > \varepsilon^{s^\nu}$ for large ν. Hence for sufficiently small $\delta > 0$,

$$U^i(t^{s^\nu}(i), y^{s^\nu}(i) - (\varepsilon^{s^\nu} + \delta) e^{k_1}, Q^{s^\nu}) > U^i(t^{s^\nu}(i), x^{s^\nu}(i), Q^{s^\nu}), \quad c \geqq \delta + \varepsilon^{s^\nu},$$

which contradicts the definition of ε^{s^ν}. Thus we get $U^i(\tilde{t}(i), \tilde{y}(i), Q^*) = U^i(\tilde{t}(i), \tilde{x}(i), Q^*)$. However, since $\tilde{y}_k(i) = \tilde{x}_k(i)$ for $k \in C_1$ and $\tilde{y}_k(i) = 2M > M \geqq \tilde{x}_k(i)$ for $k \in C_0$, $U^i(\tilde{t}(i), \tilde{y}(i), Q^*) > U^i(\tilde{t}(i), \tilde{x}(i), Q^*)$, which is a contradiction. This contradiction shows $\underline{\lim}_s \varepsilon^s > 0$. Hence we can take $\varepsilon > 0$ such that $\underline{\lim}_s \varepsilon^s > \varepsilon$. Then it holds that

$$U^i(t^s(i), y^s(i) - \varepsilon e^{k_1}, Q^s) > U^i(t^s(i), y^s(i) - \varepsilon^s e^{k_1}, Q^s) \geqq U^i(t^s(i), x^s(i), Q^s)$$

for all s with $W^s \ni i$. This completes the proof of the claim.

Let $\{W^{s^\lambda}\}$ be a subsequence of $\{W^s\}$ with $W^{s^\lambda} \ni i$. By the claim, there is an $\varepsilon > 0$ such that $U^i(t^{s^\lambda}(i), y^{s^\lambda}(i) - \varepsilon e^{k_1}, Q^{s^\lambda}) > U^i(t^{s^\lambda}(i), x^{s^\lambda}(i), Q^{s^\lambda})$ for all λ. Then since $x_k^{s^\lambda}(i) \leqq M$, $p_k^{s^\lambda} \to 0$ $\forall k \in C_0$ and $p_C^{s^\lambda} \cdot x^{s^\lambda}(i) - p_C^{s^\lambda} \cdot (y^{s^\lambda}(i) - \varepsilon e^{k_1})$ $= \sum_{k \in C_0} p_k^{s^\lambda} x_k^{s^\lambda}(i) - p_{k_1}^{s^\lambda} - 2 \sum_{k \in C_0} p_k^{s^\lambda} M \to \varepsilon p^*_{k_1} > 0$. Thus there is a λ_0 such that

$$(1 - T^{s^\lambda})[p_L^{s^\lambda} \cdot f^i(t^{s^\lambda}(i))] \geqq p_C^{s^\lambda} \cdot x^{s^\lambda}(i) > p_C^{s^\lambda} \cdot (y^{s^\lambda}(i) - \varepsilon e_{k_1}) \quad \text{for all } \lambda \geqq \lambda_0.$$

This shows $x^{s^\lambda}(i)$ does not maximize i's utility under his budget constraint for all $\lambda \geqq \lambda_0$.

Q.E.D.

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