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On the Existence and Uniqueness of a Solution to an Operational Spatial Net Social Quasi-Welfare Maximization Problem

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The purpose of this note is to give a rigorous proof of the existence of a solution to an operational spatial net social quasi-welfare maximization problem, which seems to be of practical value, especially in the field of agriculture [6, 7, 10, 11, 12].

1. The Framework of an Operational Spatial Net Social Quasi-Welfare Maximization Problem

The Problem is written as follows: Find an economic situation ($\bar{x}_{hi}$, $\bar{y}_{hj}$, $\bar{z}_{hij}$, $\bar{p}_{hi}$, $\bar{q}_{hj}$ for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$) $\geqq 0$, if it exists, such that maximizes

(1) $$\sum_{h\varepsilon\Theta} \left(\sum_{j\varepsilon\Delta} \int_0^{y_j'} q_{hj}\, dy_{hj} - \sum_{i\varepsilon\Delta_h} \int_0^{x_i'} p_{hi}\, dx_{hi} - \sum_{i\varepsilon\Delta_h} \sum_{j\varepsilon\Delta} \hat{e}_{hij} z_{hij} \right)$$

subject to

(2) $x_{hi} - \sum_{j\varepsilon\Delta} z_{hij} \geqq 0$ for each $h\varepsilon\Theta$ and $i\varepsilon\Delta_h$,

(3) $\sum_{i\varepsilon\Delta_h} z_{hij} - y_{hj} \geqq 0$ for each $h\varepsilon\Theta$ and $j\varepsilon\Delta$, and

(4) $y_j \equiv (y_{1j}, y_{2j}, \ldots, y_{Hj})' \geqq 0$, $x_i \equiv (x_{hi}$ for all $h\varepsilon\Theta_i)' \geqq 0$, and

$z_{hij} \geqq 0$ for each $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$,

where

(5) $p_{hi} = \hat{a}_{hi} + \sum_{k\varepsilon\Theta_i} \hat{b}_{hki} x_{ki}$ for each $h\varepsilon\Theta$ and $i\varepsilon\Delta_h$,

(6) $q_{hj} = \hat{c}_{hj} - \sum_{k\varepsilon\Theta} \hat{d}_{hkj} y_{kj}$ for each $h\varepsilon\Theta$ and $j\varepsilon\Delta$;

h and k stand for goods ($1\leqq h,k\leqq H$); i and j for regions or countries ($1\leqq i,j\leqq J$); p and x for supply price and quantity, respectively; q and y for demand price and quantity, respectively; z_{hij} and $\hat{e}_{hij}$ for trade quantity and given unit trade cost of good h transferred from region i to region j; $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ for

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coefficients to be estimated under integrability condition [3, 8, 9]; $\int$ for line integral; Θ and Δ for the sets of the numbers of all goods and of all regions under study, respectively; and Θ_i and Δ_h for subsets of Θ and Δ consisting of the numbers of all goods region i can produce and of all regions where good h can be produced, respectively.

It should be noted that if $H = 1$, i.e., in the case of a single good, (1) implies the sum of the net social economic surpluses of all regions.

2. Basic Assumption

We postulate the following Basic Assumption:

$\infty > |\hat{a}_{hi}| \geqq 0$ for each $h\varepsilon\Theta$ and $i\varepsilon\Delta_h$, $\infty > \hat{c}_{hj} > 0$ for each $h\varepsilon\Theta$ and $j\varepsilon\Delta$, and the technical coefficient matrices defined below are strictly positive definite if $H_i \geqq 1$ and $H \geqq 1$, and symmetric, because of integrability condition, if $H_i \geqq 2$ and $H \geqq 2$, respectively, where H_i denotes the dimension of Θ_i:

$B_i \equiv [\hat{b}_{hki}$ where $h\varepsilon\Theta_i$ and $k\varepsilon\Theta_i$ indicate row and column, respectively] for each $i\varepsilon\Delta$

$D_j \equiv [\hat{d}_{hkj}$ where $h\varepsilon\Theta$ and $k\varepsilon\Theta$ indicate row and column, respectively] for each $j\varepsilon\Delta$.

3. The Existence and Uniqueness of a Solution

First of all, we would like to check the uniqueness of an optimal solution when it exists. Calculating the line integrals in (1), we can have

(7) $\sum_{j\varepsilon\Delta}\{c_j{}'y_j - (1/2)y_j{}'D_jy_j\} - \sum_{i\varepsilon\Delta}\{a_i{}'x_i - (1/2)x_i{}'B_ix_i\} - \sum_{i\varepsilon\Delta_h}\sum_{j\varepsilon\Delta}\hat{e}_{hij}z_{hij}$,

where $c_j \equiv (\hat{c}_{1j}, \hat{c}_{2j}, \ldots, \hat{c}_{Hj})'$ and $a_i \equiv (\hat{a}_{hi}$ for all $h\varepsilon\Theta_i)'$ for each $i, j\varepsilon\Delta$. Therefore, owing to Basic Assumption, (1) is strictly concave with respect to each of x_{hi}'s and y_{hj}'s and concave with respect to each of z_{hij}'s for each $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$. And also (1) is continuous with respect to each of these variables. Hence, (1) can take a maximum on a nonempty, closed, and convex set. Then all of $\bar{x}_{hi}$'s, $\bar{y}_{hj}$'s, $\bar{p}_{hi}$'s, and $\bar{q}_{hj}$'s are unique but all of $\bar{z}_{hij}$'s are not necessarily unique.

Here let us prove that the feasibility set consisting of constraints (2), (3), and (4) satisfies Slater condition [1]. Rewriting (5) in the matrix form, we have

(8) $p_i = a_i + B_i x_i$ for each $i\varepsilon\Delta$, where $p_i \equiv (p_{hi}$ for all $h\varepsilon\Theta_i)'$.

Following P. Gordan [2, 5], the following is true: for each $i\varepsilon\Delta$,

Situation 1: $\begin{bmatrix} 1 & 0 \\ 0 & I \\ a_i & B_i \end{bmatrix} \begin{bmatrix} s_i \\ t_i \end{bmatrix} > 0$ has a solution $\begin{bmatrix} s_i \\ t_i \end{bmatrix}$, or

Situation 2: $\begin{bmatrix} 1 & 0 \\ 0 & I \\ a_i & B_i \end{bmatrix}' \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = 0$ and $\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} \geq 0$ has a solution $\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$,

but both never hold.

where s_i and u_i are scalars; t_i, v_i, and w_i are H_i-dimensional column vectors; and I is an $(H_i \times H_i)$ identity matrix, where H_i is the number of the elements in x_i. Hence, we can know that if Situation 2 does not hold, then Situation 1 must hold. Let us assume that Situation 2 holds. We have, by $u_i = -a_i'w_i$ and $v_i = -B_i'w_i = -B_iw_i$,

$$(9) \quad \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} = \begin{bmatrix} -a_i'w_i \\ -B_iw_i \\ w_i \end{bmatrix} \geq 0.$$

If $w_i = 0$ in (9), then $(u_i, v_i', w_i') = 0$. This is a contradiction to the $\geq$ sign. If $w_i \geq 0$ or $w_i > 0$, then we must have $w_i'(-B_iw_i) = -w_i'B_iw_i \geqq 0$, i.e., $w_i'B_iw_i \leqq 0$, premultiplying w_i to $-B_iw_i \geq 0$ in (9). This contradicts the Basic Assumption that always guarantees $w_i'B_iw_i > 0$ for $w_i \geq 0$ or $w_i > 0$. Accordingly, we can conclude that Situation 1 must hold, while Situation 2 does not hold at all. Let us take an $(s_i^*, t_i^{*\prime})$ which satisfies

$$(10) \quad \begin{bmatrix} 1 & 0 \\ 0 & I \\ a_i & B_i \end{bmatrix} \begin{bmatrix} s_i^* \\ t_i^* \end{bmatrix} > 0, \text{ i.e., } s_i^* > 0,\ t_i^* > 0, \text{ and } a_is_i^* + B_it_i^* > 0.$$

Putting $x_i^* \equiv t_i^*/s_i^*$ for each $i\varepsilon\Delta$, we can have

(11) $p_i^* = a_i + B_i x_i^* > 0$ and $x_i^* > 0$.

Then, there always exists an $x_i^\circ > x_i^* > 0$ such that $p_i^\circ = a_i + B_i x_i^\circ > p_i^* > 0$. Selecting an arbitrary $\tilde{p}_i > 0$, we put $a_i = -\tilde{p}_i$. We can always have an $(\tilde{s}_i, \tilde{t}_i')$ such that $\tilde{s}_i > 0$, $\tilde{t}_i > 0$, and $-\tilde{p}_i\tilde{s}_i + B_i\tilde{t}_i > 0$ or $B_i\tilde{t}_i > \tilde{p}_i\tilde{s}_i$. Denoting $\tilde{x}_i \equiv \tilde{t}_i/\tilde{s}_i$, we have $B_i\tilde{x}_i > \tilde{p}_i$ and $\tilde{x}_i > 0$. Letting $x_i^\circ \equiv x_i^* + \tilde{x}_i$ for each $i\varepsilon\Delta$, we can always have $x_i^\circ > x_i^* > 0$ and $p_i^\circ = a_i + B_i x_i^\circ = (a_i + B_i x_i^*) + B_i\tilde{x}_i > p_i^* + \tilde{p}_i > p_i^* > 0$.

For a sufficiently large $x_i^* > 0$ such that $p_i^* = a_i + B_i x_i^* > 0$ for each $i\varepsilon\Delta$, we can always find a point represented as (x_{hi}^*, y_{hj}^*, z_{hij}^* for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$) $\geqq 0$ such that

(12) $y_{hj}^* = 0$ and $z_{hij}^* > 0$ satisfying $x_{hi}^* > \sum_{j\varepsilon\Delta} z_{hij}^*$ for each $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$, which always result in

(13) $x_{hi}^* - \sum_{j\varepsilon\Delta} z_{hij}^* > 0$ for each $h\varepsilon\Theta$ and $i\varepsilon\Delta_h$;

(14) $\sum_{i\varepsilon\Delta_h} z_{hij}^* - y_{hj}^* = \sum_{i\varepsilon\Delta_h} z_{hij}^* > 0$ for each $h\varepsilon\Theta$ and $j\varepsilon\Delta$; and

(15) $q_{hj}^* = \hat{c}_{hj} - \sum_{k\varepsilon\Theta} \hat{d}_{hkj} y_{kj}^* = \hat{c}_{hj} > 0$ for each $h\varepsilon\Theta$ and $j\varepsilon\Delta$.

Consequently, a point (x_{hi}^*, y_{hj}^*, z_{hij}^* for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$) $\geqq 0$ satisfying conditions (11) and (12) meets Slater condition. Thus, we can conclude that the feasibility set consisting of constraints (2), (3), and (4) is nonempty, closed (due to the equality-inequality signs of all of the constraints), and convex (due to the linearities of all of the constraints), and includes in itself a point which always satisfies Slater condition and guarantees the nonnegativities of the economic variables. Accordingly, there always exists a solution to the Problem.

4. Governmental Supply Price Support Policies

So far, nonnegative supply and demand prices have not been treated as additional constraints, because a regular case defined as $\overline{x}_{hi} > 0$ and $\overline{y}_{hj} > 0$ for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$ always results in $\overline{p}_{hi} \geqq 0$ and $\overline{q}_{hj} \geqq 0$ for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$ and would be usually obtained in application to actual problem.

However, we have seen some government(s) supporting, for instance, the minimum supply prices of some goods like agricultural products. Even in this case, the above proof is utilized. Let $\hat{p}_i \equiv (\hat{p}_{hi}$ for all $h\varepsilon\Delta_i)' \geq 0$, where $\hat{p}_{hi}$ denotes the guaranteed minimum supply price of good h if positive and the nonnegative supply price if equal to zero. Then we have to introduce additional constraints $p_i \geqq \hat{p}_i \geq 0$ for some region(s). By rearranging $p_i \geqq \hat{p}_i$, we can get $a_i - \hat{p}_i + B_i x_i > 0$. If we regard $(a_i - \hat{p}_i)$ as a_i in the previous section, we can know that there always exists an $x_i^\dagger > 0$ such that $p_i^\dagger - \hat{p}_i = a_i - \hat{p}_i + B_i x_i^\dagger > 0$, i.e., $p_i^\dagger = a_i + B_i x_i^\dagger > \hat{p}_i$. Therefore, we can solve an operational spatial net social quasi-welfare maximization problem in case of governmental supply price support policies.

The above proof can be easily applied to an operational intertemporal (and spatial) net social quasi-welfare maximization problem [6, 7, 11].

5. Graphical Illustrations

(1) Free market case

Points A, B, and C in Fig. 1 constitute economic situation (x^*_{hi}, y^*_{hj}, z^*_{hij}, p^*_{hi}, q^*_{hj} for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$) satisfying Slater condition, where $y^*_{hj} = 0$, $x^*_{hi} > z^*_{hii} > 0$, $z^*_{hij} = 0$ if $i \neq j$, for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$, leading to $p^*_{hi} > 0$ and $q^*_{hj} = \hat{c}_{hj} > 0$. Of course, this situation is not in equilibrium, but starting this situation, we can eventually reach the economic situation in equilibrium.

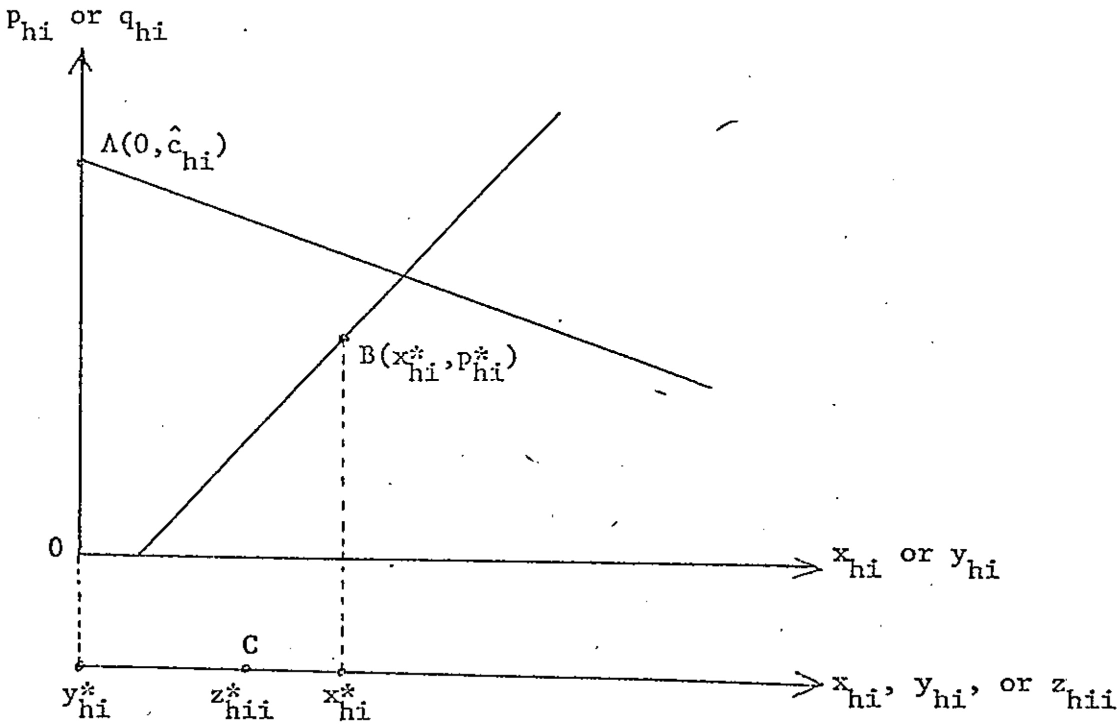

Fig. 1 Points A, B, and C satisfy Slater condition in a free market case.

(2) Governmental supply price support policy case

Fig. 2 illustrates a case in which a governmental supply price support policy is taken and a subsidy is paid to producers in order to clear the market. Points A, B, and C in Fig. 2 satisfting Slater condition can constitute economic situation (x^*_{hi}, y^*_{hj}, z^*_{hij}, p^*_{hi}, q^*_{hj} for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$) at which $y^*_{hj} = 0$, $x^*_{hi} > z^*_{hii} > 0$, $z^*_{hij} = 0$ if $i \neq j$, for all $h\varepsilon\Theta$, $i\varepsilon\Delta_h$, and $j\varepsilon\Delta$, lesding to $p^*_{hi} > \hat{p}_{hi}$ and $q^*_{hj} = \hat{c}_{hj} > 0$. Starting this situation in disequilibrium, we can eventually reach the economic situation in equilibrium with the optimal positive value of the Lagrangean multiplier equal to $\bar{p}_{hi} - \bar{q}_{hi} > 0$. The optimal subsidy per unit of good h equal to $\bar{p}_{hi} - \bar{q}_{hi}$ must be paid to producers for market clearance.

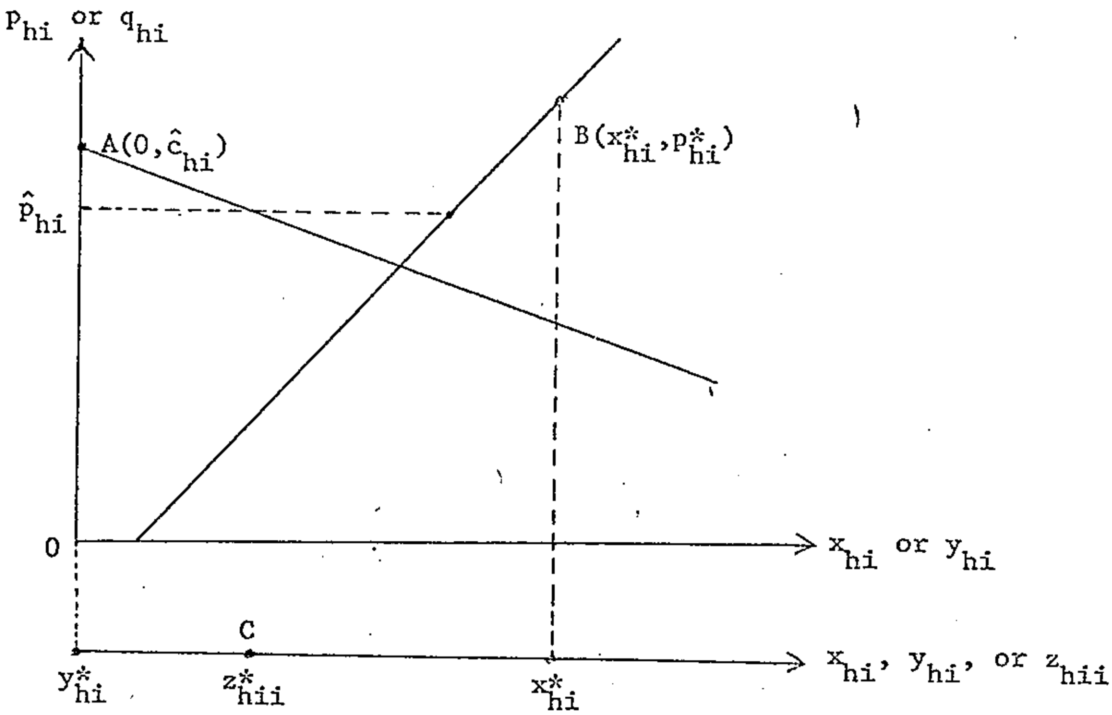

Fig. 2 Points A, B, and C satisfy Slater condition in a governmental supply price support policy case in which a subsidy is paid.

References

[1] Arrow, K. J., L. Hurwicz, and H. Uzawa: Studies in/Linear and Non-Linear Programming, Stanford University Press, Stanford, California, (1958).

[2] Gordan, P.: "Uber die Auflosungen linearer Gleichungen mit reelen Coefficienten", Mathematische Annalen, 6(1873), 23-28.

[3] Hotelling, H.: "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates", Econometrica, 6(1938), 242-269.

[4] Kuhn, H. W., and A. W. Tucker: "Nonlinear Programming", J. Neyman ed., Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, California, (1951), 481-492.

[5] Mangasarian, O.: Nonlinear Programming, McGraw-Hill Book Co., N.Y., (1969).

[6] Onishi, H.: "An Intertemporal Food Allocation and Price Determination Method", Journal of Rural Problem (Nohringyo-Mondai-Kenkyu), 11(March 1975), Japan, 17-24, (in Japanese).

[7] ______: "Operational Methods for Worldwide Temporal Food Allocation and Price Determination", (April 1975, unpublished).

[8] Samuelson, P. A.: "The Problem of Integrability in Utility Theory", Economica, 17(1950), 355-385.

[9] ______: Foundations of Economic Analysis, 3rd Printing, Atheneum, N.Y., (1967).

[10] Takayama, T., and G. G. Judge: "Equilibrium Among Spatially Separated Markets: A Reformulation", Econometrica, 32(October 1964), 510-524.

[11] ____________: Spatial and Temporal Price and Allocation Models, North-Holland Pub. Co., Amsterdam, Holland, (1971).

[12] Tayama, T , and A. D. Woodland: "Equivalence of Price and Quantity Formulations of Spatial Equilibrium: Purified Duality in Quadratic and Concave Programming", Econometrica, 38(November 1970), 889-906.