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A SYSTEM OF LINEAR INEQUALITIES WITH A SUBMODULAR FUNCTION ON $\{0,\pm 1\}$-VECTORS

by

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ABSTRACT

We consider a system of linear inequalities with $\{0,\pm 1\}$-coefficients and a right-hand side given by a submodular function on $\{0,\pm 1\}$-vectors and provide a simple necessary and sufficient condition for the system to be consistent. Furthermore, we show the total dual integrality of the system with some additional condition.

1. INTRODUCTION

In the present paper we shall consider a system of linear inequalities with $\{0,\pm 1\}$-coefficients and a right-hand side given by a submodular function on $\{0,\pm 1\}$-vectors and provide a simple necessary and sufficient condition for the system to be consistent. Furthermore, we shall show the total dual integrality of the system with some additional condition. The result includes as special cases the discrete separation theorem [3] and the total dual integrality of generalized polymatroids [2], intersection of submodular and supermodular polyhedra, and hybrid independence polyhedra [15].

2. DEFINITIONS

Let $I = \{1,2,\dots,p\}$ and $J = \{1,2,\dots,q\}$. For a $p \times q$ integer matrix $A = (a(i,j)\colon i \in I,\ j \in J)$ with the row index set I and the column index set J ordered by the natural numbers and for an integer t we say A has the upper consecutive t's property (or the lower consecutive t's property) if $a(i_0,j_0) = t$ for a pair of $i_0 \in I$ and $j_0 \in J$ implies $a(i,j_0) = t$ for all $i \in I$ with $i \leqq i_0$ (or $i \geqq i_0$). Also we say A has the consecutive t's property if $a(i_1,j_0) = a(i_2,j_0) = t$ for some i_1, $i_2 \in I$ and $j_0 \in J$ with $i_1 < i_2$ implies $a(i,j_0) = t$ for all $i \in I$ with $i_1 \leqq i \leqq i_2$. Denote the i-th row and the j-th column of $A = (a(i,j)\colon i \in I,\ j \in J)$ by $a(i,\cdot)$ and $a(\cdot,j)$, respectively.

For a finite set S we denote the cardinality of S by $|S|$. Let E be a nonempty finite set, R the set of reals and R^E the set of real vectors $x = (x(e): e \in E)$ with coordinates indexed by E. Also denote by 3^E the set of all ordered pairs (X,Y) of subsets X, Y of E with $X \cap Y = \emptyset$. Each $(X,Y) \in 3^E$ is identified with a vector $\chi(X,Y) \in R^E$ defined by

$$\chi(X,Y)(e) = \begin{cases} 1 & (e \in X) \\ -1 & (e \in Y) \\ 0 & (e \in E - (X \cup Y)). \end{cases} \tag{2.1}$$

We call $\chi(X,Y)$ the characteristic vector of (X,Y). For any $(X_i,Y_i) \in 3^E$ $(i = 1, 2)$ define operations $\vee$, $\wedge$ by

$$(X_1,Y_1) \vee (X_2,Y_2) = (X_1 \cup X_2, Y_1 \cap Y_2), \tag{2.2}$$

$$(X_1,Y_1) \wedge (X_2,Y_2) = (X_1 \cap X_2, Y_1 \cup Y_2). \tag{2.3}$$

Then 3^E is a distributive lattice with $\vee$ and $\wedge$ as the lattice operations, join and meet, and let $\preceq$ be the partial order associated with the distributive lattice 3^E. We call a pair of (X_i,Y_i) $(i=1,2)$ <u>comparable</u> if either $(X_1,Y_1) \preceq (X_2,Y_2)$ or $(X_2,Y_2) \preceq (X_1,Y_1)$. We also use the partial order $\preceq$ for $\{0,\pm 1\}$-vectors under the correspondence (2.1).

Let $\mathcal{D}$ be a sublattice of the distributive lattice 3^E and $f: \mathcal{D} \to R$ be a <u>submodular function</u> on $\mathcal{D}$, i.e., for any $(X_i,Y_i) \in \mathcal{D}$ $(i=1,2)$

$$f(X_1,Y_1) + f(X_2,Y_2) \geq f(X_1 \cup X_2, Y_1 \cap Y_2) + f(X_1 \cap X_2, Y_1 \cup Y_2). \tag{2.4}$$

Moreover, define a polyhedron $\hat{P}(f)$ by

$$\hat{P}(f) = \{x \mid x \in R^E, \forall (X,Y) \in \mathcal{D}: x(X) - x(Y) \leq f(X,Y)\}, \tag{2.5}$$

where for each $X \subseteq E$

$$x(X) = \sum_{e \in X} x(e). \tag{2.6}$$

Such a polyhedron $\hat{P}(f)$ is also treated by N. Tomizawa [15] under a more restrictive condition on f than the one in the present paper (see Section 5).

In Sections 3 and 4 we shall consider the structure of the polyhedron $\hat{P}(f)$ defined by (2.5).

3. CONSISTENCY

Without any additional condition the polyhedron $\hat{P}(f)$ may be empty. The following theorem shows that a trivial necessary condition for $\hat{P}(f)$ to be nonempty is also sufficient.

Theorem 3.1: $\hat{P}(f)$ is nonempty if and only if

$$f(\emptyset,X) + f(X,\emptyset) \geq 0 \tag{3.1}$$

for each $(\emptyset,X) \in \mathcal{D}$ with $(X,\emptyset) \in \mathcal{D}$.

(Proof) "Only if" part: Trivial.

"If" part: Suppose (3.1) holds for each $(\emptyset,X) \in \mathcal{D}$ with $(X,\emptyset) \in \mathcal{D}$. It follows from the Farkàs lemma or the LP duality theorem that $\hat{P}(f) \neq \emptyset$ if (and only if) for all positive $\alpha_i \in R$ $(i \in I)$ and $(X_i,Y_i) \in \mathcal{D}$ $(i \in I)$ with a finite index set I such that

$$\sum_{i \in I} \alpha_i \chi(X_i,Y_i) = \mathbb{0} \tag{3.2}$$

we have

$$\sum_{i \in I} \alpha_i f(X_i,Y_i) \geq 0. \tag{3.3}$$

Here, note that $\chi(X_i,Y_i)$ is the coefficient vector of the inequality $x(X_i) - x(Y_i) \leq f(X_i,Y_i)$. Moreover, since vectors $\chi(X,Y)$ $((X,Y) \in \mathcal{D})$

are integral, we can restrict coefficients α_i $(i \in I)$ in (3.2) to positive integers. Consequently, $\hat{P}(f) \neq \emptyset$ if (and only if) for all $(X_i, Y_i) \in \mathcal{D}$ $(i \in I)$ with a finite index set I such that

$$\sum_{i \in I} \chi(X_i, Y_i) = \mathbb{0} \tag{3.4}$$

we have

$$\sum_{i \in I} f(X_i, Y_i) \geqq 0, \tag{3.5}$$

where possibly $(X_i, Y_i) = (X_j, Y_j)$ for distinct $i, j \in I$. If the family of (X_i, Y_i) $(i \in I)$ in (3.4) contains a pair of (X_1, Y_1), (X_2, Y_2) which is not comparable, put

$$\begin{aligned} (X_1, Y_1) &\leftarrow (X_1 \cup X_2, Y_1 \cap Y_2), \\ (X_2, Y_2) &\leftarrow (X_1 \cap X_2, Y_1 \cup Y_2). \end{aligned} \tag{3.6}$$

Then, after the replacement (3.6), relation (3.4) remains valid and the value of the left-hand side of (3.5) does not increase due to the submodularity of f. Since we get a family of (X_i, Y_i) $(i \in I)$ composed of pairwise comparable elements of $\mathcal{D}$ after a finite number of such replacements, it suffices to show that we have (3.5) for all (X_i, Y_i) $\in \mathcal{D}$ $(i \in I)$ such that (X_i, Y_i) $(i \in I)$ are pairwise comparable and (3.4) holds. Therefore, suppose that (3.4) holds for (X_i, Y_i) $(i \in I \equiv \{1, 2, \ldots, m\})$ and

$$(X_1, Y_1) \preceq (X_2, Y_2) \preceq \cdots \preceq (X_m, Y_m). \tag{3.7}$$

It easily follws from (3.4) and (3.7) that

$$X_1 = \emptyset, \qquad Y_m = \emptyset. \tag{3.8}$$

Suppose $|I| \geqq 2$. If $X_m - Y_1 \neq \emptyset$, then for each $e \in X_m - Y_1$

$$\sum_{i \in I} \chi(X_i, Y_i)(e) > 0, \tag{3.9}$$

which contradicts (3.4). Similarly, $Y_1 - X_m \neq \emptyset$ leads to a contradiction. Therefore,

$$Y_1 = X_m \ (\equiv Z_1) \tag{3.10}$$

and from (3.8) and (3.10),

$$\chi(X_1,Y_1) + \chi(X_m,Y_m) = \chi(\emptyset,Z_1) + \chi(Z_1,\emptyset) = \mathbb{0}. \tag{3.11}$$

From (3.11) and the assumption (3.1),

$$f(X_1,Y_1) + f(X_m,Y_m) = f(\emptyset,Z_1) + f(Z_1,\emptyset) \geqq 0. \tag{3.12}$$

Put $I \leftarrow I - \{1,m\}$. For the new index set I (3.4) still holds, and then repeat the above argument until $|I| = 0$ or 1. When $|I| = 1$, from (3.4) with $I = \{i_0\}$

$$\chi(X_{i_0},Y_{i_0}) = \mathbb{0}, \tag{3.13}$$

$$X_{i_0} = \emptyset, \qquad Y_{i_0} = \emptyset. \tag{3.14}$$

Also, from (3.1),

$$f(X_{i_0},Y_{i_0}) = f(\emptyset,\emptyset) \geqq 0. \tag{3.15}$$

Consequently, from (3.12) and (3.15),

$$\sum_{i \in I} f(X_i,Y_i) \geqq 0, \tag{3.16}$$

where $I = \{1,2,\ldots,m\}$, the original index set. This completes the proof of the theorem. Q.E.D.

It may be interesting to interpret Theorem 3.1 from the point of view of convex analysis [11], [14]. Let $\hat{e}$ be a new element not in E and define $\hat{E} = E \cup \{\hat{e}\}$. Then the polyhedron $\hat{P}(f)$ of (2.5) is nonempty if and only if the conical hull of points $(\chi(X,Y),f(X,Y))$ $((X,Y) \in \mathcal{D})$ and $(\chi(\emptyset,\emptyset),1)$ in $R^{\hat{E}}$ is the epigraph of a convex function, denoted by $\hat{f}$, on R^E taking values on $R \cup \{+\infty\}$ with $\hat{f}(\mathbb{0}) = 0$ (for the terminology see [11], [14]), which is equivalent to the condition described by (3.2) and (3.3). If such a convex function $\hat{f}$ exists, the subdifferential of the convex function $\hat{f}$ at the origin $\chi(\emptyset,\emptyset) = \mathbb{0}$ in R^E is given by $\hat{P}(f)$. It should be noted that $\hat{f}$ may not be an extension of f, i.e., there may exist $(X,Y) \in \mathcal{D}$ such that $\hat{f}(\chi(X,Y)) < f(X,Y)$. The submodularity of f simplifies the condition described by (3.2) and (3.3) and gives the one in Theorem 3.1.

From Theorem 3.1 we immediately have

Corollary 3.2: Define a sublattice $\mathcal{D}_0$ of $\mathcal{D}$ by

$$\mathcal{D}_0 = \{(X,\emptyset) \mid (X,\emptyset) \in \mathcal{D}\} \cup \{(\emptyset,X) \mid (\emptyset,X) \in \mathcal{D}\}. \tag{3.17}$$

Let f_0 be the restriction of f to $\mathcal{D}_0$, and $\hat{P}(f_0)$ be a polyhedron given by (2.5) with $\mathcal{D}$ and f replaced by $\mathcal{D}_0$ and f_0, respectively. Then, $\hat{P}(f) \neq \emptyset$ if and only if $\hat{P}(f_0) \neq \emptyset$.

Corollary 3.2 means that when $\hat{P}(f_0) \neq \emptyset$, inequalities in (2.5) with $X \neq \emptyset$ and $Y \neq \emptyset$ do not cut off $\hat{P}(f_0)$ too much to give empty $\hat{P}(f)$.

4. TOTAL DUAL INTEGRALITY

Consider the following dual problems (P) and (P*) for $c \in R^E$.

Problem (P): maximize $\sum_{e \in E} c(e)x(e)$ (4.1)

subject to $\forall\ (X,Y) \in \mathcal{D}:\ x(X) - x(Y) \leqq f(X,Y)$. (4.2)

Problem (P*): minimize $\sum_{(X,Y) \in \mathcal{D}} \lambda(X,Y)f(X,Y)$ (4.3)

$$\text{subject to } \forall e \in E: \sum_{\substack{(X,Y) \in \mathcal{D} \\ e \in X}} \lambda(X,Y) - \sum_{\substack{(X,Y) \in \mathcal{D} \\ e \in Y}} \lambda(X,Y) = c(e), \tag{4.4}$$

$$\forall (X,Y) \in \mathcal{D}: \lambda(X,Y) \geqq 0. \tag{4.5}$$

If Problem (P*) has an integral optimal solution λ^* for each integral vector $c \in R^E$ such that an optimal solution of (P*) exists, then the system of linear inequalities (4.2) is called totally dual integral (see [1], [7], [12]), which implies that each face of $\hat{P}(f)$ contains integral points if $f: \mathcal{D} \to R$ is integer-valued.

In general, (4.2) is not totally dual integral and even if f is integer-valued, $\hat{P}(f)$ may have non-integral extreme points. However, we have

Lemma 4.1: Suppose that $c \in R^E$ is an integral vector and Problem (P*) has an optimal solution. Then there exists a rational optimal solution $\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ of (P*) such that

$$\mathcal{E} = \{(X,Y) \mid \lambda^*(X,Y) > 0\} \tag{4.6}$$

is composed of pairwise comparable elements of $\mathcal{D}$.

Lemma 4.1 can be shown by a direct adaptation of a standard proof technique developed by N. Robertson, L. Lovász [10], J. Edmonds and R. Giles [1] and A. J. Hoffman [8], so that we omit the proof.

Now, consider an additional structure of $\mathcal{D}$ and f as follows:

(§) If for $(X_i,Y_i) \in \mathcal{D}$ $(i=1,2)$

(1) $(X_1,Y_1) \succeq (X_2,Y_2)$, (4.7)

(2) $X_1 \cap Y_2 \neq \emptyset$ (4.8)

and

$$(3) \quad X_2 \neq \emptyset \quad \text{or} \quad Y_1 \neq \emptyset, \tag{4.9}$$

then $(X_1 - Y_2, Y_1)$, $(X_2, Y_2 - X_1) \in \mathcal{D}$ and

$$f(X_1,Y_1) + f(X_2,Y_2) \geqq f(X_1 - Y_2, Y_1) + f(X_2, Y_2 - X_1). \tag{4.10}$$

Note that (4.7) - (4.9) imply that for some $e, e' \in E$ we have $\chi(X_1,Y_1)(e) = \chi(X_2,Y_2)(e) \neq 0$ and $\chi(X_1,Y_1)(e') = -\chi(X_2,Y_2)(e') \neq 0$.

<u>Theorem 4.2</u>: Suppose $\mathcal{D}$ and $f\colon \mathcal{D} \to R$ satisfy the property (§) described by (4.7) - (4.10). Then the system (4.2) of linear inequalities is totally dual integral.

To prove Theorem 4.2 we need some lemmas. The proof of the following lemma is also similar to the proof technique developed by Robertson et al. but we include the proof for completeness.

<u>Lemma 4.3</u>: Suppose that $c \in R^E$ is an integral vector, Problem (P*) has an optimal solution and $\mathcal{D}$ and $f\colon \mathcal{D} \to R$ satisfy the above additional property (§). Then there exists a rational optimal solution $\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ such that $\mathcal{E}$ given by (4.6) is composed of pairwise comparable elements of $\mathcal{D}$ and does not contain any (X_i,Y_i) $(i=1,2)$ which satisfy (4.7) - (4.9).

(Proof) By Lemma 4.1 let $\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ be a rational optimal solution of (P*) such that $\mathcal{E}$ given by (4.6) is composed of pairwise comparable elements of $\mathcal{D}$. Also let d_0 be a positive rational number such that each $\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ is a nonnegative integral multiple of d_0. If E contains (X_i,Y_i) (i=1,2) which satisfy (4.7) - (4.9), then let (X_i,Y_i) (i=1,2) be elements of E for which

(§§) (4.7) - (4.9) hold and for each $(X_3,Y_3) \in E$ such that $(X_1,Y_1) \succneqq (X_3,Y_3) \succneqq (X_2,Y_2)$ we have

$$X_1 \cap Y_2 \subseteqq E - (X_3 \cup Y_3). \tag{4.11}$$

(Note that we can always find such a pair of (X_i,Y_i) (i=1,2).)

Define ε by

$$\varepsilon = \min\{\lambda^*(X_i,Y_i) \mid i=1,2\} > 0. \tag{4.12}$$

Change the values of $\lambda^*(X_i,Y_i)$ (i=1,2), $\lambda^*(X_1 - Y_2,Y_1)$ and $\lambda^*(X_2,Y_2 - X_1)$ as

$$\lambda^*(X_i,Y_i) \leftarrow \lambda^*(X_i,Y_i) - \varepsilon \qquad (i=1,2), \tag{4.13}$$

$$\lambda^*(X_1 - Y_2,Y_1) \leftarrow \lambda^*(X_1 - Y_2,Y_1) + \varepsilon, \tag{4.14}$$

$$\lambda^*(X_2,Y_2 - X_1) \leftarrow \lambda^*(X_2,Y_2 - X_1) + \varepsilon. \tag{4.15}$$

The new λ^* satisfies (4.4) and (4.5) and is an optimal solution of (P*) because of (4.10). It follows from (4.11) that the new λ^* gives E in (4.6) which is composed of pairwise comparable elements of $\mathcal{D}$. Repeat (4.12) - (4.15) as far as E in (4.6) contains (X_i,Y_i) (i=1,2) which satisfy the above (§§). Then all the generated λ^*'s are distinct because each changing of λ^* by (4.12) - (4.15) increases the total sum of $|E - (X \cup Y)|\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ by $2|X_2 \cap Y_1|\varepsilon > 0$. Also each $\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ is a nonnegative integral multiple of $d_0 > 0$ and the sum of $\lambda^*(X,Y)$ $((X,Y) \in \mathcal{D})$ is constant. Therefore, after a finite number of steps we get a desired optimal solution. Q.E.D.

<u>Lemma 4.4</u>: Let $I = \{1,2,\ldots,p\}$ and $J = \{1,2,\ldots,q\}$ and suppose $A = (a(i,j): i \in I,\ j \in J)$ is a $\{0,\pm1\}$-matrix such that any pair of vectors $a(i_1,\cdot)$ and $a(i_2,\cdot)$ of A ($i_1, i_2 \in I$) is comparable and A does not contain any of the 2×2 matrices $\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}$ and the ones obtained from these matrices by row and column permutations as a submatrix. Then the matrix A is totally unimodular.

(Proof) Suppose that the rows of A are indexed such that

$$a(p,\cdot) \preceq a(p-1,\cdot) \preceq \cdots \preceq a(1,\cdot). \tag{4.16}$$

Also suppose $q = p$. Then we show that the possible values of the determinant of the (square) matrix A are 0, 1, -1. Since the following argument is valid for any square submatrix of the original matrix A, this implies that A is totally unimodular.

Define

$$I(+) = \{i \mid i \in I,\ \mathbb{0} \precneqq a(i,\cdot)\}, \tag{4.17}$$

$$I(-) = \{i \mid i \in I,\ a(i,\cdot) \preceq \mathbb{0}\}, \tag{4.18}$$

$$I(+,-) = \{1,2,\ldots,p\} - (I(+) \cup I(-)), \tag{4.19}$$

where $\mathbb{0}$ is the zero row vector of dimension $|J|$ $(= |I|)$. From (4.16), A has the upper consecutive 1's property and the lower consecutive -1's property. If $I(+) \cup I(+,-) = \emptyset$, then A is totally unimodular and $\det A = 0$, 1 or -1 since A is a $\{0,-1\}$-matrix and has the consecutive -1's property [9]. Therefore, suppose $I(+) \cup I(+,-) \neq \emptyset$. Let $a(\cdot,j_0)$ be a column of A such that $a(i,j_0) = 1$ for all $i \in I(+) \cup I(+,-)$. Such a column exists because of (4.19) and the upper consecutive 1's property of A. If $a(i_0,j_0) = -1$ for some $i_0 \in I(-)$, then $I(+,-) = \emptyset$ since otherwise for an arbitrary $i_1 \in I(+,-)$ we have $a(i_1,j_0) = 1$ and there exists a column $a(\cdot,j_1)$ such that

$a(i_1,j_1) = a(i_0,j_1) = -1$, which contradicts the assumption. When $I(+,-) = \emptyset$, A is totally unimodular and $\det A = 0$, 1 or -1 since by multiplying each row $a(i,\cdot)$ $(i \in I(-))$ by -1 A becomes a matrix which has the consecutive 1's property if the rows are appropriately re-ordered. Therefore, further suppose $a(i,j_0) = 0$ for all $i \in I(-)$.

Now, transform the matrix A by fundamental row operations with pivot $a(1,j_0) = 1$ in such a way that the j_0-th column $a(\cdot,j_0)$ becomes a unit vector. Let A' be the matrix obtained from the resultant matrix by further removing the first row and the j_0-th column, and let A_1' and A_2' be the submatrices of A' composed, respectively, of row vectors corresponding to $(I(+) \cup I(+,-)) - \{1\}$ and $I(-)$. Then, since A has the upper consecutive 1's property and the lower consecutive -1's property and by the assumption for any $j \in J$ such that $a(1,j) = 1$ we have $a(i,j) \geqq 0$ for all $i \in I(+) \cup I(+,-)$, both A_1' and A_2' have the lower consecutive -1's property. Therefore, A' has the consecutive -1's property by appropriately re-ordering the rows and is totally unimodular. We thus have $\det A = \pm \det A' = 0$, 1 or -1. Q.E.D.

We are now ready to prove Theorem 4.2.

(Proof of Theorem 4.2) Let $c \in R^E$ be an integral vector and suppose Problem (P*) has an optimal solution. By Lemma 4.3 there exists an optimal solution λ^* of (P*) such that $\mathcal{E}$ given by (4.6) is composed of pairwise comparable elements of $\mathcal{D}$ and does not contain any (X_i,Y_i) $(i=1,2)$ which satisfy (4.7) - (4.9). Then it follows from Lemma 4.4 that the set of row vectors $\chi(X,Y)$ $((X,Y) \in \mathcal{E})$ forms a totally unimodular matrix, which implies the existence of an integral optimal solution of (P*). Q.E.D.

5. RELATION TO OTHER POLYHEDRA

Let $\mathcal{D}'$ be a distributive lattice formed by subsets of E with set union and intersection as the lattice operations. A function $f'\colon \mathcal{D}' \to R$ is called a submodular function on $\mathcal{D}'$ if for each $X, Y \in \mathcal{D}'$

$$f'(X) + f'(Y) \geqq f'(X \cup Y) + f'(X \cap Y). \tag{5.1}$$

A function $g'\colon \mathcal{D}' \to R$ is called a supermodular function on $\mathcal{D}'$ if $-g'$ is a submodular function on $\mathcal{D}'$. The pairs $(\mathcal{D}',f')$ and $(\mathcal{D}',g')$ are, respectively, called a submodular system and a supermodular system [4], [5]. Polyhedra $P(f')$ and $P(g')$ defined by

$$P(f') = \{x \mid x \in R^E,\ \forall X \in \mathcal{D}'\colon x(X) \leqq f'(X)\}, \tag{5.2}$$

$$P(g') = \{x \mid x \in R^E,\ \forall X \in \mathcal{D}'\colon x(X) \geqq g'(X)\} \tag{5.3}$$

are, respectively, called the submodular polyhedron and the supermodular polyhedron associated with $(\mathcal{D}',f')$ and $(\mathcal{D}',g')$, where we assume $\emptyset \in \mathcal{D}'$ and $f'(\emptyset) = g'(\emptyset) = 0$. When $E \in \mathcal{D}'$, a polyhedron

$$B(f') = \{x \mid x \in P(f'),\ x(E) = f'(E)\} \tag{5.4}$$

is called the base polyhedron associated with $(\mathcal{D}',f')$.

(a) Intersection of Submodular and Supermodular Polyhedra

Given a submodular system $(\mathcal{D}',f')$ and a supermodular system $(\mathcal{D}'',g'')$ with $\emptyset \in \mathcal{D}' \cap \mathcal{D}''$ and $f'(\emptyset) = g''(\emptyset) = 0$, define $\mathcal{D} \subseteq 3^E$ by

$$\mathcal{D} = \{(X,\emptyset) \mid X \in \mathcal{D}'\} \cup \{(\emptyset,Y) \mid Y \in \mathcal{D}''\} \tag{5.5}$$

and for each $(X,Y) \in \mathcal{D}$

$$f(X,Y) = \begin{cases} f'(X) & \text{if } Y = \emptyset, \\ -g''(Y) & \text{if } X = \emptyset. \end{cases} \tag{5.6}$$

Then, $f\colon \mathcal{D} \to R$ is a submodular function on $\mathcal{D}$ and $\hat{P}(f)$ defined by (2.5) is given by

$$\hat{P}(f) = P(f') \cap P(g''). \tag{5.7}$$

Theorem 2.1 implies that $P(f') \cap P(g'')$ is nonempty if and only if $g''(X) \leqq f'(X)$ for each $X \in \mathcal{D}' \cap \mathcal{D}''$. Moreover, if f' and g'' are integer-valued and $P(f') \cap P(g'') \neq \emptyset$, each face of $P(f') \cap P(g'')$ contains integral points due to Theorem 4.2, since in the present case conditions (4.7) - (4.9) can not be satisfied. This leads to Frank's discrete separation theorem [3]:

> "Given a submodular system $(\mathcal{D}',f')$ and a supermodular system $(\mathcal{D}',g')$, there exists a vector $x \in R^E$ such that $g'(X) \leqq x(X) \leqq f'(X)$ for all $X \in \mathcal{D}'$ if and only if $g'(X) \leqq f'(X)$ for all $X \in \mathcal{D}'$. Moreover, if f' and g' are integer-valued, the above vector x can be integral."

It should be noted that a submodular polyhedron, a supermodular polyhedron, a base polyhedron and intersection of two base polyhedra are special cases of intersection of submodular and supermodular polyhedra.

Moreover, it should be noted that for a distributive lattice $\mathcal{D} \subseteq 3^E$ and a submodular function $f\colon \mathcal{D} \to R$ if $(\emptyset,\emptyset) \in \mathcal{D}$ and $f(\emptyset,\emptyset) = 0$, then for each $(X,Y) \in \mathcal{D}$

$$f(X,Y) = f(X,Y) + f(\emptyset,\emptyset) \geqq f(X,\emptyset) + f(\emptyset,Y), \tag{5.8}$$

so that inequalities appearing in the right-hand side of (2.5) for $(X,Y) \in \mathcal{D}$ with $X \neq \emptyset$ and $Y \neq \emptyset$ are redundant, i.e., $\hat{P}(f) = \hat{P}(f_0)$, where f_0 is defined in Corollary 3.2.

(b) Generalized Polymetroids

Let $(\mathcal{D}',f')$ be a submodular system and $(\mathcal{D}'',g'')$ a supermodular system such that for each $X \in \mathcal{D}'$ and $Y \in \mathcal{D}''$, $X - Y \in \mathcal{D}'$, $Y - X \in \mathcal{D}''$ and

$$f'(X) - g''(Y) \geqq f'(X - Y) - g''(Y - X). \tag{5.9}$$

Then the intersection of the submodular and the supermodular polyhedra

$$P(f') \cap P(g'') = \{x \mid x \in R^E, \forall X \in \mathcal{D}': x(X) \leqq f'(X), \quad \forall Y \in \mathcal{D}'': x(Y) \geqq g''(Y) \tag{5.10}$$

is called a <u>generalized</u> <u>polymatroid</u> [2].

Define $\mathcal{D}$ and $f: \mathcal{D} \to R$ by (5.5) and (5.6). Since $f(\emptyset,\emptyset) = f'(\emptyset) = g''(\emptyset) = 0$ (by definition), from (5.9)

$$f(\emptyset,X) + f(X,\emptyset) \geqq 2f(\emptyset,\emptyset) = 0 \tag{5.11}$$

for each $X \in \mathcal{D}' \cap \mathcal{D}''$. It follows from Theorem 2.1 that generalized polymatroids are always nonempty. Moreover, total dual integrality of (5.10) follows from Theorem 4.2.

(c) Hybrid Independence Polyhedra

Suppose that a distributive lattice $\mathcal{D} \subseteqq 3^E$ and a submodular function $f: \mathcal{D} \to R$ satisfy the property that if $(X_i,Y_i) \in \mathcal{D}$ (i=1,2) satisfy (4.7) and (4.8), then $(X_1 - Y_2, Y_1)$, $(X_2, Y_2 - X_1) \in \mathcal{D}$ and (4.10) holds. Then, $\hat{P}(f)$ given by (2.5) is called a hybrid independence polyhedron by Tomizawa [15]. In this case, $\hat{P}(f) \neq \emptyset$ if and only if $f(\emptyset,\emptyset) \geqq 0$ (if $(\emptyset,\emptyset) \in \mathcal{D}$), which follows from Theorem 2.1 and (4.10) (cf. (5.11)). Also, total dual integrality of (2.5) for such an f follows from Theorem 4.2.

It should be noted that the class of hybrid independence polyhedra includes the class of generalized polymatroids but not the class of intersections of submodular and supermodular polyhedra. The paper [15] is mainly concerned with a greedy-type algorithm for hybrid independence polyhedra.

Remarks: When f has the property (§) described by (4.7) - (4.10), let us call the polyhedron $\hat{P}(f)$ a ternary semimodular polyhedron. As was pointed out in (a) of this section, the class of ternary semimodular polyhedra includes intersection of two base polyhedra. Moreover, a generalized polymatroid is a projection of a base polyhedron [6] and the Edmonds-Giles polyhedron [1] is a projection of intersection of two generalized polymatroids [2]. Therefore, the Edmonds-Giles polyhedron is a special case of a projection of a ternary semimodular polyhedron.

A general framework for proving total dual integrality of systems of linear inequalities related to submodular functions is proposed by A. Schrijver [12], [13], which includes total dual integrality of the Edmonds-Giles submodular flow model [1] and many other models (see [12]). However, it is still not clear whether the total dual integrality of (4.2) with property (§) of (4.7) - (4.10) follows from Schrijver's framework.

REFERENCES

[1] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, Annals of Discrete Mathematics 1 (1977) 185 - 204.

[2] A. Frank, Generalized polymatroids, Proceedings of the Sixth Hungarian Combinatrial Colloquium (Eger, 1981) (to appear).

[3] A. Frank, An algorithm for submodular functions on graphs, Annals of Discrete Mathematics 16 (1982) 189 - 212.

[4] S. Fujishige, Theory of submodular programs: a Fenchel-type min-max theorem and subgradients of submodular functions, Mathematical Programming (to appear).

[5] S. Fujishige, Submodular systems and related topics, Mathematical Programming Study (to appear).

[6] S. Fujishige, A note on Frank's generalized polymatroids, Discrete Applied Mathematics (to appear).

[7] A. J. Hoffman, A generalization of max-flow min-cut, Mathematical Programming 6 (1974) 352 - 359.

[8] A. J. Hoffman and J. B. Kruskal, Integral boundary points of convex polyhedra, Annals of Mathematics Study 38 (1956) 223 - 241.

[9] A. J. Hoffman and D. E. Schwartz, On lattice polyhedra, in "Combinatorics" (A. Hajnal and V. T. Sos, eds.), North-Holland, Amsterdam, 1978, pp. 593 - 598.

[10] L. Lovász, On two minimax theorems in graph theory, Journal of Combinatorial Theory 21 B (1976) 96 - 103.

[11] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, N. J., 1970.

[12] A. Schrijver, Total dual integrality from directed graphs, crossing families, and sub- and supermodular functions, Proceedings of the Silver Jubilee Conference on Combinatorics (Waterloo, 1982) (to appear).

[13] A. Schrijver, Proving total dual integrality with cross-free families —— a general framework, Mathematical Programming (to appear).

[14] J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, Springer, Berlin, 1970.

[15] N. Tomizawa, Theory of hyperspace (IX): on hybrid independence system, Papers of the Technical Group on Circuits and Systems, Institute of Electronics and Communication Engineers of Japan, CAS 81-63 (1981) (in Japanese).