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# DECENTRALIZATION MODEL WITH FLEXIBLE MULTI-GOAL AND CONCESSION

by

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## ABSTRACT

A mathematical model is proposed for hierarchical decentralization with flexible goals. The board of upper decision level coordinates the goals of lower units by chosing goals among the alternative goals of each lower unit. This decision procedure is proved to possess desirable properties of autonomy and agreement-based management by applying the decomposition procedure of mixed-integer programming.

## 1. Introduction

In many real situations the objective function or goal is not strictly determined beforehand but is somewhat flexible. This allows for the concession, adaptation or cooperation to others for resolving a conflict. The optimization with a fuzzy objective function (Zimmermann 1978, Hannan 1981, Leberling 1981, Sakawa and Yumine 1983) is one of the approaches to this situation. But some decision makers still remain skeptical of fuzzy-theoretical operation of objective functions. Hence, such decision makers demand another approach. This paper takes a non-fuzzy approach and presents a decentralized decision model in which each community chooses its goals from a set of alternative goals or forms its final goals by a combination of the goals in the set. It does not necessarily mean that the goals in the set are indifferent each other on a pre-determined criterion. It may rather mean an evolution process that each goal (actually a multi-goal problem is discussed below) is at first proposed within the community without consideration of other communities but the consideration of the interaction with others may induce the community to change its original goals. This consideration of others may be expressed by choice or combination of alternative goals while the fuzziness expresses the degree of acceptability of concession but does not explicitly express this concession process. Though each of the alternative goals is strictly determined beforehand, the choice or combination of the goals gives rise to its flexibility. This makes our model different from the existing models of decentralization (Dantzig and Wolfe 1960, Lasdon 1970) in which each subsystem has its uniquely predetermined objective function. The existing models of decentralization are classified into two types (Geoffrion 1970): the resource-directive type with coupling columns (Geoffrion 1970) and the price-directive type with coupling rows (Dantzig and Wolfe 1960). Mathematically, our model is of the former type in that its coefficient matrix has the coupling columns. But it directs, at least in its essense, neither resource nor price but adaptation of communities each other. This direction for adaptation will later be referred to as the coordination.

This kind of coordination is important particularly in loosely organized cooperation projects like an interregional joint management of water, an interinstitutional joint management of a computer center for common use, an intercorporate research project for a super-computer or a corporate management consisting of autonomous divisions. Regions, institutes, corporations or divisions will later be referred to as communities. The joint management is done by a board or council (almost) every member of which represents a community.

It must be mentioned that the decomposition of goal programming is not new (Ruefli 1971 A, Ruefli 1971 B, Freeland and Baker 1975). But these works are not interested in optimization. Rather they are interested in the deviation from the optimum as behavioral externalities.

## 2. Systems Characteristics

Let our decentralized decision system have the following characteristics. (2A) The decision system is of multi-level, for the simplicity, two levels; the lower level consists of several communities engaged in production and the upper level is the board of the representatives for joint management. Theoretically the extension to three levels is possible. (2B) Each of the community's goals is of multi-attribute in that it is concerned with various items as a vector. (2C) The community's goals are flexible in that one goal vector is chosen or combined from the set of alternatives for each community. (2D) The coordination is done at the upper decision level within the prescribed domain in that the set of alternative goals is already given beforehand. (2E) A member of the board represents his community and, at the same time, cooperates other communities for the total optimum in expectation of benefiting from it in that, for example, he may be elected to the chairman of the board with support by other communities. Such an indirect or two-level election system tends to motivate the representatives to contribute to the total communities. (2F) The supreme objective function is to minimize dissatisfaction in that the socio-psychological aspect is the most critical in organization management. Operationally this means that the global objective function is to minimize the total regret. (2G) More precisely, the global objective function is to minimize the (weighted) sum of the community's regret or to minimize the maximal community's regret over the communities. (2H) Communities are disjoint each other with respect to activities in that communities transact each other only through the market. The possibility that an individual member belongs to two communities does not break the disjointness of activities between communities as far as each portion of his activity belongs uniquely to a single community.

## 3. Modeling of Systems Behaviour

### 3.1 Modeling of activities and coordination for goal achievement

(3A) Alternatives of multi-attribute goals: The community's objective is represented by the $n_k^G$ – goals of $m_k^G$ – attributes for community $k$. Each goal is denoted by a given $m_k^G \times 1$ - vector $g_{k\ell}$ , $\ell = 1, \ldots, n_k^G$ . The combination of them is expressed by (3.1) and (3.2) and the choice of one among them is by (3.1), (3.2) plus (3.3) where $g_{k0}$ denotes the resulting one and $y_{k\ell}$ is a non-negative scalar, $k = 1, \ldots, K$ and $\ell = 1, \ldots, n_k^G$ .

$$\sum_{\ell=1}^{n_k^G} g_{k\ell} y_{k\ell} = g_{k,0} \tag{3.1}$$

$$\sum_{\ell=1}^{n_k^G} y_{k\ell} = 1 \qquad k = 1, \cdots\cdots, K \tag{3.2}$$

$$y_{k\ell} \in \{0, 1\}, \quad k = 1, \cdots\cdots, K \text{ and } \ell = 1, \cdots\cdots, n_k^G \tag{3.3}$$

(3B) Degree of under- or over-achievement of goals: The activities which attempt to satisfy the goals may fail that.

For each community $k$,

$$A_k^G x_k + I_k r_k^- - I_k r_k^+ - G_k y_k = 0 \tag{3.4}$$

where $A_k^G$ , $x_k$ , $I_k$ , $r_k^-$ , $r_k^+$ , $G_k$ and $y_k$ denote an $m_k^G \times n_k^A$ - matrix of contribution coefficients to goals, a non-negative $n_k^A \times 1$ - vector of activity variables, the $m_k^G$ - identity matrix, a non-negative $m_k^G \times 1$ - vector of under-achievement variables, a non-negative $m_k^G \times 1$ - vector of over-achievement variables, an $m_k^G \times n_k^G$ - matrix of goals consisting of columns $g_{k\ell}$ and a non-negative $n_k^G \times 1$ - vector of choice (0 - 1) or combination (weight) variables consisting of rows $y_{k\ell}$ , $\ell = 1, \ldots, n_k^G$ , respectively.

(3C) Constraint of community's resources on activities: The resource allocation constraint is given for community $k$ in a conventional way.

$$A_k^R x_k \leqq b_k \tag{3.5}$$

where $A_k^R$ and $b_k$ denote an $m_k^R \times n_k^A$ - matrix of technology coefficients for the community's resources and a given $m_k^R \times 1$ - vector of community's resources level, respectively.

(3D) Community's dissatisfaction: The degree of dissatisfaction of community $k$ is represented by regret;

$$u_k - d_k^- r_k^- - d_k^+ r_k^+ = 0 \tag{3.6}$$

where $u_k$ , $d_k^-$ and $d_k^-$ denote a scalar variable of community's dissatisfaction, a $1 \times m_k^G$ - vector of regret-weight coefficients for under-achievement and an analogous vector for over-achievement, respectively. This equation can be relaxed to the inequality $\geqq$ for the reason given in (3G).

(3E) Common resource distribution: The common resources which are different from the community's resources in item are distributed to communities.

$$\sum_{k=1}^{K} z_k \leqq b_0 \qquad (3.7)$$

where $z_k$ and $b_0$ denote a non-negative $m_0^0 \times 1$ - vector of common resources distribution variables and a given $m_0^0 \times 1$ - vector of the common resources level, respectively.

(3F) Common resource consumption: The common resources are consumed in a different way from the community's resources.

$$A_k^0 x_k - I_0 z_k \leqq 0 \qquad (3.8)$$

where $A_k^0$ and $I_0$ denote an $m_0^0 \times n_k^A$ - matrix of technology coefficients for the common resources associated with community $k$ and the $m_0^0$ - identity matrix, respectively.

(3G) Information on community's dissatisfaction: The board must know the degree of dissatisfaction of every community;

$$u_k - v_k = 0 \,, \; k = 1 \,, \cdots\cdots, \; K \qquad (3.9)$$

where $u_k$ was defined by (3.6) and $v_k$ denotes a scalar variable of the information shared by the board for every $k$. In view of (3.10) - (3.12) below, the above equation can be relaxed to the inequality to follow.

$$u_k - v_k \leqq 0 \,, \; k = 1 \,, \cdots\cdots, \; K \qquad (3.9')$$

(3H) The supreme objective function: The psychological aspect or regret is taken as the coordination criterion among communities. More precisely, either the total sum of the community's regrets or the maximal regret over communities is taken as the criterion. They are expressed by (3.10) plus either (3.11) or (3.12) respectively.

$$v_0 \rightarrow minimize \tag{3.10}$$

$$v_0 - \sum_{k=1}^{K} w_k v_k \geqq 0 \tag{3.11}$$

$$v_0 - v_k \geqq 0 \ , \quad k = 1 \ , \cdots\cdots , \ K \tag{3.12}$$

where $v_0$ and $w_k$ are scalars denoting a total dissatisfaction variable and an weight coefficient for communities, $k$ = 1, ..., $K$, respectively. For the theoretical simplicity, the objective function of min-max dissatisfaction over communities expressed by (3.12) is excluded from discussion till section 4.5.

## 3.2 Composition of the systems model

The control variable of community $k$ is $x_k$ and its state variables to express its dissatisfaction are $r_k^-$ , $r_k^+$ and $u_k$. Now $\xi_k$ is composed of these variables as a non-negative $N_k^C \times 1$ - vector with $N_k^C = n_k^A + 2\ m_k^G + 1$. Then $\xi_0$ is aggregated from $\xi_k$, $k$ = 1, ..., $K$ as a non-negative $N_0^C \times 1$ - vector with $N_0^C = \sum_{k=1}^{K} N_k^C$.

Similarly $\zeta_k$ is composed of the control variables $y_k$ and $z_k$ and the state variable $v_k$ of the board as a non-negative $N_k^B \times 1$ - vector with $N_k^B = n_k^G + m_0^0 + 1$. Then $\zeta_0$ is composed of $\zeta_k$, $k = 1, \ldots, K$ and the state variable scalar $v_0$ as a non-negative $N_0^B \times 1$ - vector with $N_0^B = \sum_{k=1}^{K} N_k^B + 1$. The structures of $\xi_k$, $\xi_0$, $\zeta_k$ and $\zeta_0$ are illustrated as follows;

$$\xi_k = \begin{pmatrix} x_k \\ r_k^- \\ r_k^+ \\ u_k \end{pmatrix}, \quad \xi_0 = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_K \end{pmatrix}, \quad \zeta_k = \begin{pmatrix} y_k \\ z_k \\ v_k \end{pmatrix}, \quad \zeta_0 = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_K \\ v_0 \end{pmatrix}$$

In accordance to these compositions of the variables, let the $k$-community problem $(C_k(\zeta_0))$ with variable $\xi_k$ for given $\zeta_0$, the board problem $(B(\xi_0))$ for coordination with variable $\zeta_0$ for given $\xi_0$ and the conglomerate communities problem $(C_0(\zeta_0))$ with variable $\xi_0$ for given $\zeta_0$ be composed of the expressions (3.1) - (3.11) after some proper rearrangement.

$$\left.\begin{array}{l} \alpha_k \xi_k = \delta_k \to min \\ P_k \xi_k \geqq \beta_k - Q_k \zeta_0 \end{array}\right\} (C_k(\zeta_0)),\ k = 1, \cdots\cdots, K.$$

where $\alpha_k$ has the unit component for $u_k$ and the zero-components elsewhere, resulting in the following equality for scalar $\delta_k$ by construction;

$$\delta_k = u_k, \quad k = 1, \ldots, K \tag{3.13}$$

and the compositions of $P_k$, $\beta_k$, $Q_k$ will be illustrated below.

The conglomerate communities problem $(C_0(\zeta_0))$ is defined for scalar $\delta_0$ as follows;

$$\left.\begin{array}{l} \alpha_0 \xi_0 = \delta_0 \to min \\ P_0 \xi_0 \geqq \beta_0 - Q_0 \zeta_0 \end{array}\right\} (C_0(\zeta_0))$$

where $\alpha_0$ has the unit components for $u_k$, $k = 1, \ldots, K$ and the zero-components elsewhere, resulting in either of the following equalities according as (3.11) or (3.12) is chosen (though the latter is neglected from consideration now);

$$\delta_0 = \sum_{k=1}^{K} u_k \tag{3.14}$$

$$\delta_0 \leqq u_k, \ k = 1, \cdots\cdots, K \tag{3.15}$$

The structures of $\alpha$, $\beta$, $P$ and $Q$ are illustrated as follows;

$\alpha_k = (0;\ 0;\ 0;\ 1)$, $\alpha_0 = (\alpha_1;\ \ldots;\ \alpha_K)$

$$\beta_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_k \end{pmatrix}, \quad P_k = \begin{pmatrix} 0 & ; 0 & ; 0 & ; 1 \\ A_k^G & ; I_k & ; -I_k & ; 0 \\ A_k^0 & ; 0 & ; 0 & ; 0 \\ 0 & ; d_k^- & ; d_k^+ & ; -1 \\ A_k^E & ; 0 & ; 0 & ; 0 \end{pmatrix}, \quad Q_k = \begin{pmatrix} 0 & ; 0 & ; -1 \\ -G_k & ; 0 & ; 0 \\ 0 & ; -I_0 & ; 0 \\ 0 & ; 0 & ; 0 \\ 0 & ; 0 & ; 0 \end{pmatrix}$$

$$\beta_0 = \begin{pmatrix} b_1 \\ \vdots \\ b_K \end{pmatrix}, \quad P_0 = \begin{pmatrix} P_1 & ; \ddots & ; 0 \\ 0 & ; & ; 0 \\ 0 & ; & ; 0 \\ 0 & ; & ; P_K \end{pmatrix}, \quad Q_0 = \begin{pmatrix} Q_1 & ; \ddots & ; 0 \\ 0 & ; & ; 0 \\ 0 & ; & ; 0 \\ 0 & ; & ; Q_K \end{pmatrix}$$

For the simplicity, these problems $(C_k(\zeta_0))$, $k = 0, 1, \ldots, K$ are expressed as having only the inequalities though actually they contain the equations of types (3.2) and (3.4). Therefore " $\geqq$ " herein should read " $\geqq$ " or " = ". The conversion of "$A = B$" into "$A \geqq B$ and $-A \geqq -B$" resolves a possible, if any, trouble. This trick can also be applied to the problems defined hereafter.

Associated with $\zeta_0$, the coordination problem of the board is formed from the expressions (3.1) - (3.11) for given

$\xi_0$ after some proper rearrangement;

$$\left.\begin{array}{l}\gamma_B \zeta_0 = \delta_B \rightarrow min \\ Q_B \zeta_0 \geqq \beta_B - P_B \xi_0\end{array}\right\} \quad (B(\xi_0))$$

where $\gamma_B$ has the unit component for $v_0$ and the zero components elsewhere, resulting in the following equality for scalar $\delta_B$ ;

$$\delta_B = v_0 \qquad (3.16)$$

The structure of $Q_B$, $\beta_B$ and $P_B$ are as follows;

$$Q_B = \begin{pmatrix} Q_B^1 & ; & \cdots\cdots & ; & Q_B^K \\ \underline{Q}_1 & ; & \ddots & ; & 0 \\ 0 & ; & & \ddots \; ; & 0 \\ 0 & ; & & \ddots \; ; & 0 \\ 0 & ; & & ; & \underline{Q}_K \end{pmatrix}, \quad \beta_B = \begin{pmatrix} \beta^B \\ \underline{\beta}_1 \\ \vdots \\ \underline{\beta}_K \end{pmatrix}, \quad P_B = \begin{pmatrix} P_B^1 & ; & \cdots\cdots & ; & P_B^K \\ \underline{P}_1 & ; & \ddots & ; & 0 \\ 0 & ; & & ; & \vdots \\ \vdots & ; & & \ddots \; ; & 0 \\ 0 & ; & & ; & \underline{P}_K \end{pmatrix}$$

Where $Q_B^k$ , $\beta_B$ and $P_B^k$ each with $1 + m_0^0 + K$ rows are substructured as follows;

$$Q_B^k = \begin{pmatrix} 0 & ; & 0 & ; & \omega_k \\ 0 & ; & I_0 & ; & 0 \\ L_k & ; & 0 & ; & 0 \end{pmatrix}, \qquad \beta^B = \begin{pmatrix} 0 \\ b_0 \\ 1_k \end{pmatrix}, \qquad P_B^k = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

With $L_k$ being a $K \times n_k^G$ - matrix having the $1 \times n_k^G$ - vector of (1, 1, ..., 1) at the $k$-th row, the bottom component of $\beta^B$ being an $n_k^G \times 1$ - vector of $(1, 1, ..., 1)^T$ and where $\underline{Q}_k$, $\underline{\beta}_k$ and $\underline{P}_k$ are the reduced forms of $Q_k$, $\beta_k$ and $P_k$ respectively as follows;

$$\underline{Q}_k = \begin{pmatrix} 0 & ; & 0 & ; & -1 \\ -G_k & ; & 0 & ; & 0 \\ 0 & ; & -I & ; & 0 \end{pmatrix}, \quad \underline{\beta}_k = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{P}_k = \begin{pmatrix} 0 & ; & 0 & ; & 0 & ; & 1 \\ A_k^G & ; & I_k & ; & -I_k & ; & 0 \\ A_k^0 & ; & 0 & ; & 0 & ; & 0 \end{pmatrix}$$

$\underline{Q}_k$ is the zero-rows-suppressed form of $Q_k$, and $\underline{\beta}_k$ and $\underline{P}_k$ are associated with $\underline{Q}_k$. Also $P_0$ is the zero-rows-suppressed form of $P_B$, and $\beta_0$ and $Q_0$ are associated with $P_0$.

The totally aggregated problem may be formulated as follows;

$$\left.\begin{aligned} \gamma_C \xi_0 + \gamma_B \zeta_0 = \delta_A \rightarrow min \\ P_A \xi_0 + Q_A \zeta_0 \geqq \beta_A \end{aligned}\right\} \quad (A)$$

where $\gamma_C = 0$ so that $\delta_A = v_0$ by construction like in (3.16) and $P_A$, $Q_A$ and $\beta_A$ are as follows;

$$P_A = \begin{pmatrix} P_B^1 & ; & \cdots & ; & P_B^K \\ P_1 & ; & \ddots & ; & 0 \\ 0 & ; & & \ddots & \vdots \\ \vdots & & & ; & 0 \\ 0 & ; & & ; & P_K \end{pmatrix}, \quad Q_A = \begin{pmatrix} Q_B^1 & ; & \cdots & ; & Q_B^K \\ Q_1 & ; & \ddots & ; & 0 \\ 0 & ; & & \ddots & \vdots \\ \vdots & & & ; & 0 \\ 0 & ; & & ; & Q_K \end{pmatrix}, \quad \beta_A = \begin{pmatrix} \beta^B \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_K \end{pmatrix}$$

where each of their components, submatrices and subvectors has already been defined above.

Mathematically, the problems $(C_0(\zeta_0))$ and $(B(\xi_0))$ are the partitioned and zero-rows-suppressed forms of the problem $(A)$.

## 4. Decentralized Decision Procedure

### 4.1 Basic properties of decomposition

The totally aggregated problem $(A)$ was composed of the board problem $(B(\xi_0))$ for coordination and the conglomerate communities problem $(C_0(\zeta_0))$ the latter of which was composed of the community problems $(C_k(\zeta_0))$, $k = 1, \ldots, K$. This is mathematically equivalent to saying that the problem $(A)$ is decomposed into the problems $(B(\xi_0))$ and $(C_0(\zeta_0))$ the latter of which is further decomposed into the problems $(C_k(\zeta_0))$, $k = 1, \ldots, K$. The first decomposition of the problem $(A)$ in variables of $\xi_0$ and $\zeta_0$ into the problem $(B(\xi_0))$ in a continuous/discrete variable $\zeta_0$ and the problem $(C_0(\zeta_0))$ in a continuous variable $\xi_0$ is the Benders decomposition (Benders 1962, Geoffrion 1972, Balas and Bargthaller 1977, McDaniel and Devine 1977). Hence the Benders theorem holds for this decomposition. According to this theorem, the optimal solution to $(A)$ is composed of those of the subproblems $(B(\xi_0))$ and $(C_0(\xi_0))$. More precisely,

$$\xi_0^* = \bar{\xi}_0 \text{ and } \zeta_0^* = \bar{\zeta}_0 \tag{4.1}$$

where $(\xi_0^*; \zeta_0^*)$ denotes the optimal solutions to the problem $(A)$ and $\bar{\xi}_0$ and $\bar{\zeta}_0$ denote the optimal solutions to the problems $(C_0(\zeta_0^*))$ and $(B(\xi_0^*))$ with the parameter values $\zeta_0^*$ and $\xi_0^*$ respectively under the conditions that information exchanges are proper and that, for the simplicity, each problem has the unique bounded primal and dual optima (referred to as the unique bounded optima condition in sequel).

Since the community problems $(C_k(\zeta_0))$ are mutually separated in variables in our problem, the Benders theorem (4.1) yields

$$\bar{\xi}_k = \xi_k^* \qquad k = 1, \ldots, K \tag{4.2}$$

under the same conditions as in (4.1).

The condition on information exchanges is basically that the dual solutions to $(C_k(\zeta_0))$ , $k = 1, \ldots, K$, are transmitted to $(B(\xi_0))$ besides the exchange of the primal solutions. The actual procedure which does not use $(B(\xi_0))$ will be discussed below.

The condition on the unique bounded optimum is mathematically not essential. In fact, the Benders theorem establishes simple correspondences between the non-existence of bounded optimum to problems and also the relationship analogous to (4.1) and hence to (4.2) holds for non-unique optima to problems.

The problem $(B(\xi_0))$ depends on given parameter $\zeta_0$ which is the solution to the problem $(C_0(\zeta_0))$ . As is well known by the duality theoren, $\xi_0$ corresponds to $\lambda_0$ which denotes the dual solution to the same problem. According to Benders (1962), solving the problem $(B(\xi_0))$ is equivalent, in a sense, to solving the problem $(B(\Lambda_0))$ defined below for $\Lambda_0$ being the set of the dual basic feasible solutions to $(C_0(\zeta_0))$.

$$\left.\begin{array}{l} \delta_D \quad \rightarrow min \\ -\sum_{k=1}^{K} \delta_k + \delta_D \geqq 0 \\ \delta_k + \lambda_k^\tau Q_k \zeta_0 \geqq \lambda_k^\tau \beta_k, \quad \forall \lambda_k^\tau \in \Lambda_k, \quad \forall_k \end{array}\right\} (B(\Lambda_0))$$

Where $\Lambda_k$ denotes the set of the dual basic feasible solutions $\lambda_k$ to $(C_k(\zeta_0))$ and $\lambda_k$ is the $k$-part of $\lambda_0$.

As $|\Lambda_k|$ which denotes the cardinal number of set $\Lambda_k$ is very large, only its subset $\Lambda_k^\tau$ is generated as required at the iteration cycle $\tau$ by the following procedure.

$$\Lambda_k^\tau = \Lambda_k^{\tau-1} + \{\lambda_k^\tau\} \tag{4.3}$$

where the symbol + denotes the union of the sets, $\lambda_k^\tau$ denotes an element of $\Lambda_k$ generated at $\tau$ and $\Lambda_k^0$ is the null set. This implies that, for every $k$,

$$|\Lambda_k^\tau| = \tau$$

in other words, the actually generated subset is small.

The Benders theorem (Benders 1962) asserts the first and the last equations and immediately implies the second one of (4.4).

$$\hat{\delta}_D = \delta_A^*, \quad \hat{\delta}_k = \bar{\delta}_k \text{ and } \quad \hat{\zeta}_0 = \zeta_0^* \tag{4.4}$$

where $(\hat{\delta}_D ; \hat{\delta}_k ; \hat{\zeta}_0)$ is the optimal solution to $(B(\Lambda_0))$ and $\bar{\delta}_k$ is the optimal objective value of $(C_k(\zeta_0^*))$.

By the Benders theorem (McDaniel and Devine 1977), the optimum is attained at the iteration cycle $t$ such that

$$\delta_D(\Lambda_0^t) = \min_{\tau \leqq t} \delta_0(\zeta_0^\tau) \equiv \delta_0(\zeta_0^\theta) \tag{4.5}$$

where $\zeta_0^\tau$ denotes the value of $\zeta_0$ in $(B(\Lambda_0^t))$ at $\tau$. Note that the left side denotes the optimal objective value of $(B(\Lambda_0^t))$ at iteration cycle $t$, the maximand on the right side denotes that of $(C_0(\zeta_0^\tau))$ at $\tau$, and $\theta$ denotes an iteration cycle at which the maximum occurs. By the construction of $(C_0(\zeta_0))$ from $(C_k(\zeta_0))$, $k = 1, ..., K$,

$$\delta_0(\zeta_0^\tau) = \sum_{k=1}^{K} \delta_k(\zeta_0^\tau) \tag{4.6}$$

By the above definition of $t$,

$$\hat{\delta}_D = \delta_D(\Lambda_0^t), \ \hat{\delta}_k = \delta_k(\Lambda_k^t), \ \hat{\zeta}_0 = \zeta_0^t \tag{4.4'}$$

## 4.2 Model of decision procedure

A desirable decision procedure for the communities problem characterized by (2A) - (2H) with behaviours (3A) - (3H) may be modeled as follows.

Step 1. Each community $k$ announces its target value $\lambda_k$ of efficiency in use of common resources and contribution to reducing the total dissatisfaction, $k = 1, ..., K$.

Step 2. The board for coordination among communities proposes its coordination plan $\zeta_0$ including the expected dissatisfaction $v_k$ of community $k$ in response to $\lambda_k$, $k = 1, ..., K$.

Step 3. Each community $k$ decides its plan $\xi_k$ including its resulting dissatisfaction $u_k$ and revises its initial target $\lambda_k$ to $\lambda_k'$ in response to the proposed coordination plan $\zeta_0$, $k = 1, ..., K$. It compares $u_k$ with $v_k$ and decides whether to accept $\zeta_0$ or not by this comparison. If acceptable for every

$k$, then the agreement is attained by the board and the communities. If not, the process is continued.

Note on Step 1. The dual solution or the Lagrangean which is interpreted to express the efficiency in use of resource in the continuous optimization can also express the same thing in the discrete optimization (Balas 1968).

From the view point of acceptability to communities and their spontaneous cooperation, the decision procedure may be desired to satisfy the following requests.

Rq. 1. The final plan decided by each community at step 3 should not be commanded to change by the board if no community benefits from it.

Note on Rq. 1. In the classical decentralization (Dantzig and Wolfe 1960 and so forth) the final plan decided by each community is not especially important and each community is commanded by the board to take an weighted combination of the all foregoing plans with the weight commanded by the board. This is criticized as a violation of autonomy of communities by Baumol and Falian 1964 for the reason that each community is commanded to abandon its final plan without benefiting from this. The weighted combination mentioned just above is not to be confused with that of (3.1) and (3.2). The former combination is a command from the board to each community while the latter one of (3.1) and (3.2) is autonomously decided.

Rq. 2. The community's acceptance of the coordination plan at step 3 should be on the equal basis among communities so that no community is mistreated unfairly.

Note on Rq. 2. The coordination plan proposed to each community at the same iteration cycle of the process is naturally on the equal basis across communities. Hence, Rq. 2 is better satisfied if all the communities accept the coordination plan at the same iteration cycle of the process. If not, a coordination plan which is accepted by a community must be revised for the acceptance of the other communities but the revised plan is probably disadvantageous to the former community in comparison with the foregoing plan.

Rq. 3. The initiative of decision should be taken by the communities, and the power of the coordination board should be in harmony with Rq. 1.

4.3 Formal discussion

Let $t$ denote the final iteration cycle, that is, the iteration cycle at which (4.5) holds.

Theorem 1.

$$\delta_k(\zeta_0^t) = \alpha_k \xi_k^* \text{ and } \xi_k(\zeta_0^t) = \xi_k^*,\ k=1,\cdots\cdots,K\ .$$

where $\alpha_k$ is the objective coefficient of community $k$ occurring in $(C_k(\zeta_0))$.

Proof. By (4.4) and (4.4'), the left sides of the two equations are the values associated with the final optimum to $(A)$ via $(C_k(\zeta_0^t))$ and therefore are equal to the right sides which are also associated with the final optimum to $(A)$.

Let $\lambda_A^*$ denote the dual optimum to $(A)$ and $\lambda_k^*$ denote its $k$-part associated with constraints of $\beta_k$.

Theorem 2.

$$\lambda_k^* \varepsilon \Lambda_k, \ k=1, \cdots\cdots, K$$

Proof. $\lambda_k^*$ is the dual optimum to $(C_k(\hat{\zeta}_0))$ by Theorem 1, (4.4') and the well-known duality theorem, $k = 1, \ldots, K$. Hence $\lambda_k^*$ is a basic feasible solution and therefore belongs to $\Lambda_k$.

Note on Theorems 1 and 2. In the classical decentralization by the price-directive method (Dantzig and Wolfe 1960 and many others), the dual optimum is obtained by a convex combination of the preceeding solutions (Walker 1969). This combination is taken by the board with no regard of benefit of communities. This is against Rqs. 1 and 3. In our system the dual optimum is basic and therefore it needs no combination. Hence, Rqs. 1 and 3 are satisfied in our system.

Rq. 2 is satisfied when Theorem 3, or more strongly Theorem 4 below holds.

Theorem 3. The totally aggregated problem ($A$) is optimal when (4.7) holds.

$$\delta_k(\zeta_0^\theta) = \delta_k(\Lambda_k^t), \ k=1, \cdots\cdots, K \tag{4.7}$$

where $\theta$ is as defined in (4.5).

Before proving Theorem 3, Lemmas 3.1-3.5 will be proved.

Lemma 3.1 The constraint lastly augmented to $(B(\Lambda_0^\tau))$ is binding for all $k$ at $\tau$. That is, it holds with equality.

$$\delta_k + \lambda_k^\tau Q_k \zeta_0 = \lambda_k^\tau \beta_k, \ k=1, \cdots\cdots, K \tag{4.8}$$

Proof. Since $\lambda_k^\tau$ is basic, it is associated with a vertex of the convex polyhedron of the dual feasible set. In other words, the augmentation of the constraint is associated with generating a new vertex and hence is binding.

Lemma 3.2 $\delta_D(\Lambda_0^\tau)$ increases monotonically as the iteration cycle $\tau$ proceeds.

Proof. As constraints are augmented, the feasible set diminishes monotonically in $\tau$ by Lemma 3.1. This makes the associated objective value to increase monotonically in $\tau$.

Lemma 3.3 $\delta_D(\Lambda_0^\tau)$ is unique at every $\tau$.

Proof. By the basic property of optimization, the optimal objective value itself is unique even without the uniqueness condition as far as it exists and is bounded.

Lemma 3.4 $\delta_k(\Lambda_k^\tau)$ is unique at every $\tau$ and increases monotonically in $\tau$.

Proof. As is seen from (4.8) in Lemma 3.1, $\delta_k(\Lambda_k^\tau)$ is more strictly constrained than before for every $k$ as the iteration cycle $\tau$ proceeds. It is related each other across $k$ only through $\zeta_0$ in the constraints (4.8). Hence it can not decrease without decreasing others but this would contradict Lemmas 3.2-3.3. The analogous argument can be applied to the uniqueness.

Lemma 3.5 There exist iteration cycles $\nu$ of ( $C_k(\zeta_0^\nu)$) and $\tau$ of ( $B(\Lambda_k^\tau)$) such that,

$$\delta_k(\zeta_0^\nu) = \delta_k(\Lambda_k^\tau), \quad k=1, \cdots\cdots, K$$

where the left side denotes the optimal objective value of

$(C_k(\zeta_0^\nu))$ with the given parameter value at $\nu$.

Proof. The lastly augmented constraint (4.8) of $(B(\Lambda_0^\tau)$ which determines the value of the right side is the dual form of the objective function of $(C_k(\zeta_0))$ when its dual solution $\lambda_k^\nu$ equals $\lambda_k^\tau$. Thus the both sides equal each other.

Proof of Theorem 3. If (4.7) holds, then (4.5) holds by (4.6). Thus the optimum is attained. Now, let it be assumed that the optimum is attained with (4.5) held. By Lemmas 3.2-3.3, $\delta_k(\Lambda_k^\tau)$ does not repeat the same value in the iteration cycle. Analogously, by Lemma 3.4, $\delta_D(\Lambda_0^\tau)$ does not repeat the same value. Hence, since the number of basic objective values is finite, the equality holds sometime for each $k$ as the iteration cycle proceeds by Lemma 3.5. As the same value does not repeat on each side, the equality holds only once. When (4.5) holds at an iteration cycle, the equality holds only at this iteration cycle, that is, at the final cycle at which the optimum is attained. Whence (4.7) holds.

Lemma 4.1

$$v_k^* = \delta_k(\Lambda_k^t),\ k=1,\cdots\cdots,\ K$$

where the left side denotes the $v_k$-part of $\zeta_0^*$ and $t$ denotes the final iteration cycle.

Proof. By (3.9) and (3.13), $v_k$ is equal to $\delta_k$. By (4.2), its optimal value to $(A)$ which is the left side in the above equation is equal to the associated value in $(B(\Lambda_0^t))$ at the final iteration cycle $t$ which is the right side above.

Let $\theta$ denote the iteration cycle at which the maximum occurs in (4.5).

Lemma 4.2

$$\delta_k(\zeta_0^\theta)=\delta_k(\zeta_0^t),\ k=1,\cdots\cdots,K$$

Proof. By the construction of $(C_k(\zeta_0))$, $\alpha_k\xi_k^*=u_k^*$. Then Theorems 1 and 3, Lemma 4.1 and (3.9) yield this.

Theorem 4. At the final iteration $t$ at which the optimality criterion (4.5) holds,

$$\delta_k(\Lambda_k^t)=\delta_k(\zeta_0^t),\ k=1,\cdots\cdots,K$$

Proof. By Lemmas 4.1 and 4.2.

Note on Theorems 3 and 4. Theorem 3 asserts that the optimality is judged by each community, satisfying Rq. 2. Theorem 4 asserts that the optimum is attained "simultaneously" at the same iteration cycle $t$ for the all communities, satisfying the condition stated in Note on Rq. 2.

## 4.4 Computation

This paper is concerned only with theoretical examination of a systems acceptability but a brief remark may be needed on the computational procedures. The theoretical algorithm of Benders (Benders 1962) is claimed efficient (McDaniel and Devine 1977). Recently, algorithms were presented for the decomposable mixed-integer programming (Sannomiya and Tsukabe 1981) and for the sequential goal programming (Masud and Hwang 1981).

### 4.5 Min-max dissatisfaction

Mathematically, the discussion in section 4.3 is also valid for the case of the objective function of minimizing the maximal dissatisfaction over communities. However, this gives a different interpretation to the variable $\delta_k$ and hence may give a different motivation or incentive to behaviours of communities. Its detailed discussion may need several pages and will be left to another paper.

## 5. Conclusion

A mathematical model of hierarchical decentralization was presented to yield a concession or compromisation by a flexible set of goals. This system was proved to possess the desired properties of autonomy and equality of subsystems.

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## 7. Glossary of symbols used

$K$ : the number of communities.

$k$ : index for a community, $k = 1, \ldots, K$.

$n_k^G$ : the number of alternatives of goals for community $k$.

$m_k^G$ : the number of attributes in the goals for community $k$.

$n_k^A$ : the number of items in activity of community $k$.

$m_k^R$ : the number of items in resources of community $k$.

$m_0^0$ : the number of items in common resources.

$g_{k\ell}$ : a given $m_k^G \times 1$ - vector for a $m_k^G$ - attribute goal for community $k$ with $\ell = 1, \ldots, n_k^G$ .

$g_{k0}$ : an $m_k^G \times 1$ - vector of the weighted or chosen goal defined by (3.1).

$G_k$ : an $m_k^G \times n_k^G$ - matrix with $g_{k\ell}$ as the $\ell$-th column, $\ell = 1, \ldots, n_k^G$ .

$x_k$ : a non-negative $n_k^A \times 1$ - vector of activity variable of community $k$.

$y_{k\ell}$ : an weight (continuous) or choice (0 or 1) scalar variable.

$y_k$ : an $n_k^G \times 1$ - vector with $y_{k\ell}$ as the $\ell$-th component.

$z_k$ : a non-negative $m_0^0 \times 1$ - vector of distribution variable of common resources to community $k$.

$u_k$ : a scalar variable of dissatisfaction of community $k$.

$v_k$ : a scalar variable defined as being equal to $u_k$.

$v_0$ : a scalar variable of total dissatisfaction.

$w_k$ : a given weight scalar for community $k$.

$r_k^-$ : a non-negative $m_k^G \times 1$ - vector of under-achievement variable of community $k$.

$r_k^+$ : a non-negative $m_k^G \times 1$ - vector of over-achievement variable of community $k$.

$A_k^G$ : an $m_k^G \times n_k^A$ - matrix of coefficients of contribution of activity to goals of community $k$.

$A_k^R$ : an $m_k^R \times n_k^A$ - matrix of technology coefficients of activity with respect to resources of community $k$.

$A_k^0$ : an $m_0^0 \times n_k^A$ - matrix of technology coefficients of activity of community $k$ with respect to common resources.

$b_k$ : a given $m_k^R \times 1$ - vector of level of resources of community $k$.

$b_0$ : a given $m_0^0 \times 1$ - vector of level of common resources.

$d_k^-$ : a given $1 \times m_k^G$ - vector of regret weight for under-achievement of goals of community $k$.

$d_k^+$ : a given $1 \times m_k^G$ - vector of regret weight for over-achievement of goals of community $k$.

$P_k$ : a coefficient matrix composed of $A_k^G$ , $A_k^0$ , $A_k^R$ , $d_k^-$ , $d_k^+$ plus the positive unit, negative unit and zero components.

$P_0$ : a coefficient matrix with $P_k$ as the submatrix on the diagonal, $k = 1, \ldots, K$ and the zero components elsewhere.

$P_B^k$ : the zero matrix.

$P_B$ : a coefficients matrix composed of $P_B^k$ and reduced forms of $P_k$, $k = 1, \ldots, K$.

$Q_k$ : a coefficients matrix composed of $- G_k$ plus the negative unit and the zero components.

$Q_0$ : a coefficients matrix with $Q_k$ on the diagonal, $k = 1, \ldots, K$.

$Q_B^k$ : a coefficient matrix composed of $w_k$ plus the unit and the zero components.

$Q_B$ : a coefficients matrix composed of $Q_B^k$ and reduced forms of $Q_k$, $k = 1, \ldots, K$.

$\alpha_k$ : a unit row vector with the unit component at the end.

$\alpha_0$ : a row vector composed of $\alpha_k$, $k = 1, \ldots, K$.

$\beta_k$ : a given $(4 + m_k^R) \times 1$ - vector composed of four zero-components and $b_k$ .

$\beta_0$ : a given column vector composed of $b_k$, $k = 1, \ldots, K$.

$\beta^B$ : a given column vector composed of $b_0$ plus the unit and zero components.

$\beta_B$ : a given column vector composed of $\beta^B$ and reduced forms of $\beta_k$.

$\gamma_B$ : a unit vector.

$\delta_k$ : a scalar variable defined as being equal to $u_k$.

$\delta_0$ : a scalar variable of the total dissatisfaction.

$\xi_k$ : a non-negative column vector of variables composed of $x_k$, $r_k^-$, $r_k^+$ and $u_k$.

$\xi_0$ : a non-negative column vector of variables composed of $\xi_k$, $k = 1, \ldots, K$.

$\zeta_k$ : a non-negative column vector of variables composed of $y_k$, $z_k$ and $v_k$.

$\zeta_0$ : a non-negative column vector of variables composed of $\zeta_k$, $k = 1, \ldots, K$.

$A$ : the totally aggregated problem in variables $\xi_0$ and $\zeta_0$.

$B(\zeta_0)$ : the problem of coordination by the board in variable $\zeta_0$ for given value of $\xi_0$.

$B(\Lambda_0)$ : the dualized problem of coordination by the board in variables in $\zeta_0$ and $\delta_k$, $k = 1, \ldots, K$ for given set $\Lambda_0$.

$C_k(\zeta_0)$ : the problem of community $k$ in variable $\xi_k$ for given value of $\zeta_0$.

$C_0(\zeta_0)$ : the conglomerate problem of communities in variable $\xi_0$ for given value of $\zeta_0$.

$\bar{\xi}_0$, $\bar{\xi}_k$: the optimal solutions to $(C_0(\zeta_0^*))$ and $(C_k(\zeta_0^*))$, $k = 1, \ldots, K$, respectively.

$\bar{\zeta}_0$ : the optimal solution to $(B(\xi_0^*))$.

$\xi_0^*$, $\zeta_0^*$ : the optimal solution to (A).

$\hat{\delta}_D$, $\hat{\delta}_k$, $\hat{\zeta}_0$: the optimal solution to $(B(\Lambda_0))$.

$\delta_D(\Lambda_0^r)$, $\delta_k(\Lambda_k^r)$ : the optimal solution to $(B(\Lambda_0^r))$ and $(B(\Lambda_k^r))$, respectively.

$\delta_k(\zeta_0^r)$, $\xi_k(\zeta_0^r)$, $\delta_0(\zeta_0^r)$ : the optimal solution to $(C_k(\zeta_0^r))$ and $(C_0(\zeta_0^r))$.