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A variable dimension fixed point algorithm and the orientation of simplices *

by
Yoshitsugu Yamamoto

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University of Tsukuba
Sakura, Ibaraki 305
Japan

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Title: A variable dimension fixed point algorithm and the orientation of simplices

Author: Yoshitsugu Yamamoto
Institute of Socio-Economic Planning, University of Tsukuba
Sakura, Ibaraki 305, Japan

Abstract: A variable dimension algorithm with integer labelling is proposed for solving systems of n equations in n variables. The algorithm is an integer labelling version of the 2-ray algorithm proposed by the author. The orientation of lower dimensional simplices is studied and is shown to be preserved along a sequence of adjacent simplices.

Key words: fixed point algorithm, system of equations, orientation of simplices

Abbreviated title:

A variable dimension algorithm

1. INTRODUCTION

In these several years new restart algorithms have been developed for approximating a solution to the system of equations

$$f(x) = 0, \quad x \in R^n, \tag{1.1}$$

where f is a continuous mapping from an n-dimensional Euclidean space R^n into itself. They are called variable dimension algorithms after their common feature that they generate a sequence of simplices with varying dimensions. They start from a single point, 0-dimensional simplex, and leave it along one of the rays extending in several directions. They are classified according to the number of rays: the (n+1)-ray algorithm and 2n-ray algorithm by Van der Laan and Talman [7,8,9,10], the 2^n-ray algorithm by Wright [13], the (3^n-1)-ray algorithm by Kojima and Yamamoto[4], and the 2-ray algorithm by Saigal[11] and Yamamoto[14]. The first two algorithms have both vector and integer labelling versions while the others now have only vector labelling versions. In this paper we will develop an integer labelling version of the 2-ray algorithm and discuss the orientation of simplices without presenting a constant dimension version of the algorithm (the 2-ray algorithm can be viewed as a piecewise linear homotopy algorithm on a special subdivision of a certain subset of $R^n \times [0,1]$. See Yamamoto and Murata [15], and also Eaves [2]).

In Section 2 we introduce a labelling function and define a k-complete simplex which provides an approximate solution to the k-dimensional subproblem

$$f_i(x) = 0, \quad \text{for } i = 1,2,\ldots, k.$$

We define the adjacency relation of simplices and give an existence theorem of an n-complete simplex. In Section 3 we propose the 2-ray algorithm with integer labelling. In Seciton 4 we investigate the orientation of k-complete simplices generated by the algorithm.

We explain below some notations to be used.

$w \in R^n$ is a known approximate solution to (1.1),

$I(k) = \{1,2,\ldots,k\}$ for a nonnegative integer k,

$J(k) = I(n+1) \setminus I(k)$,

$X(k,\alpha) = \{ x \in R^n : \alpha(x_k - w_k) \geq 0, \quad x_j = w_j \text{ for any } j \in J(k) \}$

for $k \in I(n)$ and $\alpha \in \{-1,+1\}$,

$X(k) = X(k,-1) \cup X(k,+1)$,

T is a locally finite triangulation of R^n,

$\bar{T} = \{ \tau : \tau \text{ is a face of some } \sigma \text{ of } T \}$,

$T(k,\alpha) = \{ \tau \in \bar{T} : \tau \subset X(k,\alpha), \dim \tau = \dim X(k,\alpha) \}$,

$T(k) = T(k,-1) \cup T(k,+1)$,

$f_{(k)}$ is the k-dimensional vector consisting of the first k components of an n-dimensional vector f,

We assume that $T(k,\alpha)$ triangulates $X(k,\alpha)$ for $k \in I(n) \cup \{0\}$ and $\alpha \in \{-1,+1\}$.

2. INTEGER LABELLING AND K-COMPLETE SIMPLICES

For a vector f of R^n, let $l(f)$ be the index of the first nonnegative component of f, i.e.

$$l(f) = \min \{ j \in I(n) : f_j \geq 0 \}.$$

We adopt the convention that $l(f)=n+1$ when all the components of f are negative. We refer to $l:R^n \to I(n+1)$ as the (integer) labelling function. Let $t:I(n+1) \to R^n$ be a function such that

$$t_j(l) = \begin{cases} -1 & \text{for } j < l \\ +1 & \text{for } j = l \\ 0 & \text{for } j > l. \end{cases}$$

Then $t \cdot l$ is an aggregating function which corresponds a vector of R^n to a truncated sign vector. By the aggregating function $t \cdot l$, we have a new mapping $g(x)=(t \cdot l \cdot f)(x)$ from R^n into itself. We have the following two lemmas as for the relation among f, $l \cdot f$ and $g=t \cdot l \cdot f$. Here G is the piecewise linear approximation of g with respect to the triangulation T of R^n, i.e. for a point $x \in \sigma \in T$, $G(x)=\sum_{i=1}^{n+1} \lambda_i g(u^i)$, where $u^1, u^2, \ldots, u^{n+1}$ are the vertices of σ, $x=\sum_{i=1}^{n+1} \lambda_i u^i$, $\sum_{i=1}^{n+1} \lambda_i = 1$ and $\lambda_i \geq 0$ for any $i \in I(n+1)$.

Lemma 2.1.

A simplex τ of $\bar{T}$ has a solution to the system of k piecewise linear equations

$$G_{(k)}(x) = 0, \quad x \in R^n \tag{2.1}$$

if and only if τ has the vertices $u^1, u^2, \ldots, u^{k+1}$ with

$$\begin{aligned} &l(f(u^i)) = i \quad \text{for } i \in I(k), \\ &l(f(u^{k+1})) \in J(k). \end{aligned} \tag{2.2}$$

proof. Let

$$\lambda_i = 2^{-i} \quad \text{for } i \in I(k)$$

$$\lambda_{k+1} = 2^{-k},$$

then it is readily seen that

$$\sum_{i \in I(k+1)} \lambda_i = 1$$

$$\sum_{i \in I(k+1)} \lambda_i g(u^i) = 0.$$

This proves the "if" part. To see the "only if" part, let v be the vertex of τ having the maximal label. Then $l(f(v)) \in J(k)$ since otherwise τ could not have a solution to (2.1). Since $g_{(k)}(v)=(-1,-1,\ldots,-1) \in R^k$, we readily see that τ has a vertex u^i with $l(f(u^i))=i$ for each $i \in I(k)$.//

We say that τ is <u>k-complete</u> when τ has a solution to (2.1) or equivalently τ has the vertices $u^1, u^2, \ldots, u^{k+1}$ satisfying (2.2).

Let

$$\varepsilon = \sup \{ \sup\{\| f(x) - f(y)\| : x, y \in \sigma \} : \sigma \in T \}.$$

Lemma 2.2.

Any point x of a k-complete simplex satisfies

$$\max\{ |f_i(x)| : i \in I(k)\} \leq \varepsilon .$$

proof. Let $u^1, u^2, \ldots, u^{k+1}$ be the vertices of the k-complete simplex satisfying (2.2). Then for each $i \in I(k)$

$$-\varepsilon \leq f_i(u^i) - \varepsilon \leq f_i(x) \leq f_i(u^{k+1}) + \varepsilon \leq \varepsilon. \qquad //$$

Note that we can make ε arbitrarily small by choosing a sufficiently fine triangulaiton when f is uniformly continuous on R^n. Then we can approximate a solution to the system of equations (1.1) by finding an n-complete simplex. To show the existence of an n-complete simplex and to propose an algorithm for finding one, we define two notions: complete facet and adjacency. Let σ be a simplex of $T(k)$. We say that a facet τ of σ is a <u>complete facet</u> of σ if τ is (k-1)-complete and

$$k \notin L(\tau) \quad \text{if } \sigma \in T(k,+1)$$

$$k \in L(\tau) \quad \text{if } \sigma \in T(k,-1),$$

where $L(\tau)= \{l(f(u)): u \text{ is a vertex of } \tau\}$. We write $\tau \lhd \sigma$ when τ is a complete facet of σ. For two distinct simplices τ_1 and τ_2 of $\bar{T}$, we say that τ_1 and τ_2 are <u>adjacent</u> when one of the following cases occurs:

(1) $\tau_1 \lhd \sigma$ and $\tau_2 \lhd \sigma$ for some $\sigma \in \bar{T}$,

(2) τ_1 is (dim τ_1)-complete and $\tau_2 \lhd \tau_1$,

(3) τ_2 is (dim τ_2)-complete and $\tau_1 \lhd \tau_2$.

Then it is seen that each k-complete simplex of T(k) has a unique complete facet and is a complete facet of a unique simplex of T(k+1). This observation and the usual discussion on the adjacency of simplices (see, for example, Allgower and Georg[1], Van der Laan[5,6]) give the following theorem.

Theorem 2.3.

The 0-dimensional simplex {w} and each n-complete simplex have only one adjacent simplex. Every other simplex of $\bar{T}$ has either exactly two or no adjacent simplices.

Theorem 2.3 implies that the sequence of adjacent simplices starting from the 0-dimensional simplex {w} may lead to an n-complete simplex or may go to infinity. We must impose some condition on f to guarantee that the sequence will lead to an n-complete simplex.

Condition 2.4.

There exist two vectors c and d such that

$c < w < d$

$$f_i(x) > 0 \quad \text{whenever} \quad x_i \geq d_i \tag{2.3}$$

$$f_i(x) < 0 \quad \text{whenever} \quad x_i \leq c_i. \tag{2.4}$$

Theorem 2.5.

Suppose f satisfies Condition 2.4, then the sequence of adjacent simplices from the 0-dimensional simplex {w} leads to an n-complete simplex.

proof. Let

$$C = \{ x \in R^n : c - \delta e \leq x \leq d + \delta e \}, \tag{2.5}$$

where $e=(1,1,\ldots,1)\in R^n$ and δ is the mesh size of the triangulation T, i.e. $\delta=\sup\{\sup\{\|x-y\|:x,y\in\sigma\}:\sigma\in T\}$. Suppose the sequence of adjacent simplices from $\{w\}$ leads to some simplex $\tau=co\{u^1,u^2,\ldots,u^{k+1}\}$ such that $\tau\cap C=\emptyset$. Then by the construction of C we can find an index j such that either

$$u^i_j \geq d_j \quad \text{for all } i\in I(k+1)\text{, or} \tag{2.6}$$

$$u^i_j \leq c_j \quad \text{for all } i\in I(k+1). \tag{2.7}$$

On the other hand, from the definition of adjacency, either

$$\tau \text{ is a k-complete simplex of } T(k)\text{, or} \tag{2.8}$$

$$\tau \triangleleft \sigma \text{ for some } \sigma\in T(k+1). \tag{2.9}$$

Note that $j\leq k$ in case (2.8) and $j\leq k+1$ in case (2.9).

First suppose (2.6) holds. Then by Condition 2.4 we have $L(\tau)\subset I(j)$. Hence, in case (2.8), $L(\tau)\subset I(k)$. This contradicts the fact that τ is k-complete. Consider the case (2.9). When $j\leq k$, τ is not k-complete. When $j=k+1$, $L(\tau)\subset I(k+1)$ and $\sigma\in T(k+1,+1)$. This contradicts $\tau\triangleleft\sigma$.

Next suppose (2.7) holds. Then by Condition 2.4, we have $j\notin L(\tau)$. Hence, in case (2.8), $I(k)\not\subset L(\tau)$. This contradicts the fact that τ is k-complete. Consider the case (2.9). When $j\leq k$, τ is not k-complete. When $j=k+1$, $k+1\notin L(\tau)$ and $\sigma\in T(k+1,-1)$. This contradicts $\tau\triangleleft\sigma$.

Thus we have come to contradictions in all cases. //

3. THE 2-RAY ALGORITHM WITH INTEGER LABELLING

The 2-ray algorithm proposed in [14] traces a path of solutions to the system

$$\begin{aligned} &G_{(k)}(x) = 0 \\ &(x_{k+1} - w_{k+1})\, G_{k+1}(x) \leq 0 \qquad (3.1) \\ &x_j = w_j \quad \text{for } j \in J(k+1), \end{aligned}$$

when it is applied to the system of equations $g(x)=0$. By Lemma 2.1 we see that the k-dimensional simplex τ having a solution to (3.1) is a k-complete simplex in $X(k+1,\alpha)$, where $\alpha = -\text{sign } G_{k+1}(x)$. This observation shows that the sequence of simplices generated by the 2-ray algorithm when applied to $g(x)=0$ coincides with the sequence of adjacent simplices. Thus we will refer to the following algorithm for tracing the sequence of adjacent simplices as the 2-ray algorithm with integer labelling.

Algorithm

Step 0 (initialization): $p:=0$, $\tau_p:=\emptyset$, $v^+:=w$.

Step 1 (increasing the dimension): If $\tau_p \cup \{v^+\}$ is $(\dim \tau_p+1)$-complete, then $\tau_{p+1}:=\tau_p \cup \{v^+\}$. Otherwise go to Step 4.

Step 2 (termination): If $\dim \tau_{p+1}=n$, stop. Otherwise go to Step 3.

Step 3: $p:=p+1$, $k:=\dim \tau_p+1$,

$$\alpha := \begin{cases} +1 & \text{if } k \notin L(\tau_p) \\ -1 & \text{otherwise.} \end{cases}$$

Find a vertex v^+ of T such that $\tau_p \cup \{v^+\} \in T(k,\alpha)$. Go to Step 1.

Step 4 (replacement): Find a vertex v^- of τ_p such that

$$l(f(v^-)) = \begin{cases} l(f(v^+)) & \text{if } l(f(v^+)) \in L(\tau_p) \\ \max\{\, l(f(u)) : u \text{ is a vertex of } \tau_p \,\} & \text{otherwise.} \end{cases}$$

$\tau_{p+1}:=\tau_p \cup \{v^+\} \setminus \{v^-\}$.

Step 5: $p:=p+1$, $k:=\dim \tau_p$. If $\tau_p \in T(k,\beta)$ for some $\beta \in \{-1,+1\}$, go to Step 6. Otherwise go to Step 7.

Step 6 (decreasing the dimension): Find a vertex v^- of τ_p such that

$$l(f(v^-)) = \begin{cases} k & \text{if } \beta=1 \\ \max\ \{\ l(f(u)):\ u \text{ is a vertex of } \tau_p\ \} & \text{othersiwe.} \end{cases}$$

$\tau_{p+1} := \tau_p \setminus \{v^-\}$, $\alpha:=\beta$ and go to Step 5.

Step 7: $k:=\dim \tau_p+1$. Find a vertex v^+ of T such that $\tau_p \cup \{v^+\} \in T(k,\alpha)$ and $\tau_p \cup \{v^+\} \neq \tau_p \cup \{v^-\}$. Go to Step 1.

It is not difficult but lengthy to prove that the sequence $\{\tau_p: p=1,2,\ldots\}$ generated by the algorithm is a sequence of adjacent simplices. We will omit the proof.

Theorem 3.1.

Suppose $f:R^n \to R^n$ has the property that (2.3) and (2.4) hold for some vectors c and d with $c < d$. Then the algorithm always generates an n-complete simplex starting from an arbitrary point w with $c < w < d$ after a finitely many iterations. Moreover by choosing sufficiently fine triangulation we can approximate a solution to (1.1) with arbitrary accuracy.

proof. Theorem follows from the compactness of C in (2.5) and Theorem 2.5. //

Remark 3.2.

When the algorithm generates simplices in $X(k)$, the jth component of f does not affect its behavior for any $j \in J(k)$. Moreover all that we need in Step 3 is whether $L(\tau_p)$ has the label k or not. Therefore we can save function evaluations by the following modification:

For each $k \in I(n)$, let

$$l^k(f) = \min\ \{\{\ j \in I(k):\ f_j \geq 0\ \} \cup \{\ n+1\}\}.$$

As long as the algorithm genarates (k-1)-dimensional simplices in X(k), we employ the labelling function l^k instead of l. Let τ_p be a complete simplex of T(k) found by the algorithm with l^k. Then τ_p always has a vertex v with label n+1. Before choosing α in Step 3, we evaluate the k+1st component of f(v) and relabel the vertex v by l^{k+1}. The other parts remain unchanged.

4. THE ORIENTATION OF K-COMPLETE SIMPLICES

The orientation of simplices has been investigated by several authors (cf. Allgower and Georg[1], Eaves[2], Eaves and Scarf[3], Todd[12]). Van der Laan[5,6] has defined the orientation for lower dimensional simplices and characterized the solution to be obtained by variable dimension algorithms. In this section we define the orientaion of k-complete simplices and show that it is preserved in a sense along the sequence of simplices generated by the algorithm.

In Lemma 2.1 we have seen that a simplex τ is k-complete if and only if it has a solution to the system of k piecewise linear equations (2.1). Since $G_{(k)}(x)$ is affine on τ, it is written as $G_{(k)}(x)=Bx+b$ for some $B\in R^{k\times n}$ and $b\in R^k$. When τ lies in X(k), for any point $x\in\tau$, $x_j=0$ for any $j\in J(k)$. Therefore, letting A be the $k\times k$ matrix consisting of the first k columns of B, we have

$$G_{(k)}(x) = A\,x_{(k)} + b \quad \text{for any } x\in\tau. \qquad (4.1)$$

Definition 4.1.

Let τ be a k-complete simplex of T(k). The <u>orientaiton of τ</u>, denoted by $or(\tau)$, is the sign of deteminant of $k\times k$ matrix A in (4.1).

We introduce another matrix for the k-complete simplex τ of T(k). Suppose k+1 vertices of τ are numbered so that $l(f(u^i))=i$ for $i\in I(k)$ and $l(f(u^{k+1}))\in J(k)$. Let us define

$$W(\tau) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ u^1_{(k)} & u^2_{(k)} & \cdots & u^{k+1}_{(k)} \end{vmatrix}.$$

Lemma 4.2.

$$or(\tau) = (-1)^k \text{ sign det } W(\tau).$$

proof. Let

$$W' = \left| \; u^1_{(k)}-u^{k+1}_{(k)} \;\cdots\; u^k_{(k)}-u^{k+1}_{(k)} \;\right|$$

$$H = \begin{vmatrix} 1 & \dots & 1 \\ g_{(k)}(u^1) & \dots & g_{(k)}(u^{k+1}) \end{vmatrix}$$

$$H' = | \; g_{(k)}(u^1)-g_{(k)}(u^{k+1}) \quad \dots \quad g_{(k)}(u^k)-g_{(k)}(u^{k+1}) \; |.$$

Then it is easily seen that

$$A\, W' = H'$$

$$\det W' = (-1)^{k+2} \det W(\tau)$$

$$\det H' = (-1)^{k+2} \det H = (-1)^{2k+2}\, 2^k.$$

Hence

$$or(\tau) = \text{sign} \det A = (-1)^k \text{ sign} \det W(\tau). \qquad //$$

Definition 4.3.

Let σ be a simplex of $T(k,\alpha)$ for some $\alpha \in \{-1,+1\}$, and let τ be its complete facet. The vertices $u^1, u^2, \dots, u^{k+1}$ of σ are numbered so that $l(f(u^i))=i$ for $i \in I(k-1)$, $l(f(u^k)) \in J(k-1)$ and $u^{k+1} \notin \tau$. Let

$$W(\tau;\sigma) = \begin{vmatrix} 1 & \dots & 1 & 1 \\ u^1_{(k)} & \dots & u^k_{(k)} & u^{k+1}_{(k)} \end{vmatrix}$$

and define the <u>orientaton of τ with respect to σ</u>, denoted by $or(\tau;\sigma)$, as

$$or(\tau;\sigma) = \alpha(-1)^{k-1} \text{ sign} \det W(\tau;\sigma).$$

Lemma 4.4.

Let $\sigma \in T(k)$ be a k-complete simplex having a complete facet τ. Then

$$or(\sigma) = or(\tau;\sigma).$$

proof. Suppose $\sigma \in T(k,\alpha)$. Since σ is k-complete, $L(\sigma)=I(k) \cup \{m\}$ for some $m \in J(k)$. If $\alpha=-1$, then $L(\tau)=I(k)$. Therefore $W(\sigma)=W(\tau;\sigma)$. If $\alpha=+1$, then $L(\tau)=I(k-1) \cup \{m\}$. Therefore $\det W(\sigma)=-\det W(\tau;\sigma)$. Hence in both cases αsign $\det W(\tau;\sigma)=-$sign $\det W(\sigma)$. This implies the desired result. //

Lemma 4.5.

Let σ and τ be simplices of $T(k)$ and $T(k-1)$, respectively such that $\tau \triangleleft \sigma$. Then

$or(\tau) = or(\tau;\sigma)$.

proof. Let $u^1, u^2, \ldots u^{k+1}$ be the vertices of σ such that $u^{k+1} \notin \tau$. Here we assume without loss of generality that $l(f(u^i))=i$ for $i \in I(k-1)$ and $l(f(u^k)) \in J(k-1)$. Since $\sigma \in T(k,\alpha)$ for some $\alpha \in \{-1,+1\}$ and $\tau \in T(k-1)$, $u^i_k=0$ for $i \in I(k)$ and $\alpha u^{k+1}_k > 0$. Therefore

$$or(\tau) = (-1)^{k-1} \text{ sign det } W(\tau)$$
$$= \alpha(-1)^{k-1} \text{ sign det } W(\tau;\sigma) = or(\tau;\sigma). \qquad //$$

Lemma 4.6.

Let τ_1 and τ_2 be two distinct complete facets of $\sigma \in T(k)$. Then

$$or(\tau_1;\sigma) = -or(\tau_2;\sigma).$$

proof. Let v^i be the vertex of $\sigma \setminus \tau_i$ for i=1,2. If $l(f(v^1))=l(f(v^2))$ then $W(\tau_1;\sigma)$ coincides with $W(\tau_2;\sigma)$ except that the $l(f(v^1))$th column and the k+1st column are exchanged. If $l(f(v^1)) \neq l(f(v^2))$, then $k \leq l(f(v^i))$ for i=1,2. Since otherwise either τ_1 or τ_2 would not be a complete facet of σ. Therefore $W(\tau_1;\sigma)$ coincides with $W(\tau_2;\sigma)$ except that the last two columns are exchanged. Hence in either case we have $\det W(\tau_1;\sigma) = -\det W(\tau_2;\sigma)$. //

Lemma 4.7.

let σ_1 and σ_2 be distinct simplices of T(k). If they share a common complete facet τ, then

$$or(\tau;\sigma_1) = -or(\tau;\sigma_2).$$

proof. Since σ_1 and σ_2 share a common facet, both of them are simplices of $T(k,\alpha)$ for some $\alpha \in \{-1,+1\}$. Therefore we have only to show that sign det $W(\tau;\sigma_1)$=-sign det $W(\tau;\sigma_2)$. Let v^i be the vertex of $\sigma_i \setminus \tau$ for i=1,2. Let aff(τ) be the affine subspace spanned by τ and tng*(τ) be the orthogonal complement of the tangential subspace aff(τ)-τ . Since v^1 and v^2 lie on the opposite sides of aff(τ), we can find vectors $s \in X(k) \cap \text{tng*}(\tau)$, x^1, $x^2 \in \text{aff}(\tau)$ and scalars $\lambda_1 > 0$ and $\lambda_2 < 0$ such that

$$v^i = x^i + \lambda_i s \quad \text{for } i=1,2.$$

Hence by suitably defining a $(k+1) \times k$ matrix W we see

$$\begin{aligned}
\text{sign det } W(\tau;\sigma_1) &= \text{sign det} \left| W \;\middle|\; \begin{matrix} 1 \\ v^1_{(k)} \end{matrix} \right| \\
&= \text{sign det} \left| W \;\middle|\; \begin{matrix} 1 \\ x^1_{(k)}+\lambda_1 s_{(k)} \end{matrix} \right| \\
&= \text{sign det} \left| W \;\middle|\; \begin{matrix} 0 \\ \lambda_1 s_{(k)} \end{matrix} \right| \\
&= -\text{sign det} \left| W \;\middle|\; \begin{matrix} 0 \\ \lambda_2 s_{(k)} \end{matrix} \right| \\
&= -\text{sign det} \left| W \;\middle|\; \begin{matrix} 1 \\ x^2_{(k)}+\lambda_2 s_{(k)} \end{matrix} \right| \\
&= -\text{sign det } W(\tau;\sigma_2).
\end{aligned}$$

//

Corollary 4.8.

Let η be a $(k-1)$-complete simplex of $T(k-1)$ and η' be a k-complete simplex of $T(k)$. Suppose there is a sequence of adjacent simplices $\{\tau_q : q=1,2,\ldots,s\}$ such that $\tau_1=\eta$, $\tau_s=\eta'$ and $\dim \tau_q=k-1$ for $q=2,3,\ldots,s-1$. Then

$$\text{or}(\eta') = \text{or}(\eta).$$

proof. Since τ_q is adjacent to τ_{q+1} for $q=1,2,\ldots,s-1$, there is a sequence of simplices $\tau_1,\sigma_1,\tau_2,\ldots,\sigma_{s-2},\tau_{s-1},\tau_s$ such that $\tau_q \triangleleft \sigma_q$, $\tau_{q+1} \triangleleft \sigma_q$ for $q=1,2,\ldots,s-2$ and $\tau_{s-1} \triangleleft \tau_s$. Hence

$$\begin{aligned}
\text{or}(\eta') &= \text{or}(\tau_{s-1};\tau_s) && \text{(by Lemma 4.4)} \\
&= -\text{or}(\tau_{s-1};\sigma_{s-2}) && \text{(by Lemma 4.7)} \\
&= \text{or}(\tau_{s-2};\sigma_{s-2}) && \text{(by Lemma 4.6)} \\
&\ \vdots \\
&= \text{or}(\tau_1;\sigma_1) \\
&= \text{or}(\eta). && \text{(by Lemma 4.5)}
\end{aligned}$$

//

Corollary 4.9.

Let η and η' be distinct k-complete simplices of T(k). Suppose there is a sequence of adjacent simplices $\{\tau_q : q=1,2,\ldots,s\}$ such that $\tau_1=\eta$, $\tau_s=\eta'$ and $\tau_q \notin T(k)$ for q=2,3,...,s-1. Suppose further all τ_q's are of the same dimension except τ_1 and τ_s. then

$$or(\eta') = -or(\eta).$$

proof. First note that dim τ_q=k or k-1 for q=2,3,...,s-1. If all τ_q's are of k-dimension, there is a sequence of simplices $\tau_1,\sigma_1,\ldots,\sigma_{s-1},\tau_s$ such that $\tau_q \lhd \sigma_q$ and $\tau_{q+1} \lhd \sigma_q$ for q=1,2,...,s-1. Hence

$$\begin{aligned} or(\eta') &= or(\tau_s;\sigma_{s-1}) && \text{(by Lemma 4.5)} \\ &= -or(\tau_{s-1};\sigma_{s-1}) && \text{(by Lemma 4.6)} \\ &= or(\tau_{s-1};\sigma_{s-2}) && \text{(by Lemma 4.7)} \\ &\vdots \\ &= or(\tau_2;\sigma_1) \\ &= -or(\tau_1;\sigma_1) \\ &= -or(\eta). && \text{(by Lemma 4.5)} \end{aligned}$$

If dim τ_q=k-1 for q=2,3,...,s-1, there is a sequence of simplices $\tau_1,\tau_2,\sigma_2,\ldots,\sigma_{s-2},\tau_{s-1},\tau_s$ such that $\tau_q \lhd \sigma_q$, $\tau_{q+1} \lhd \sigma_q$ for q=2,3,...,s-2 $\tau_2 \lhd \tau_1$ and $\tau_{s-1} \lhd \tau_s$. Hence

$$\begin{aligned} or(\eta') &= or(\tau_{s-1};\tau_s) && \text{(by Lemma 4.4)} \\ &= -or(\tau_{s-1};\sigma_{s-2}). && \text{(by Lemma 4.7)} \\ &= or(\tau_{s-2};\sigma_{s-2}) && \text{(by Lemma 4.6)} \\ &\vdots \\ &= or(\tau_2;\sigma_2) \\ &= -or(\tau_2;\tau_1) \\ &= -or(\eta). && \text{(by Lemma 4.4)} \end{aligned}$$ //

Corollary 4.10.

Let η and η' be distinct k-complete simplices of T(k). Suppose there is a sequence of adjacent simplices $\{\tau_q : q=1,2,\ldots,s\}$ such that $\tau_1=\eta$, $\tau_s=\eta'$ and $\tau_q \notin T(k)$ for $q=2,3,\ldots,s-1$. Then

$$or(\eta') = -or(\eta).$$

proof. Let $\{\eta_p : p=1,2,\ldots,r\}$ be a subsequence of $\{\tau_q : q=1,2,\ldots,s\}$ such that each η_p is a simplex of $T(j)$ for some $j \in I(n)$. Then $|\dim \eta_{p+1} - \dim \eta_p| \leq 1$. Let

$$P_+ = \{p : \dim \eta_p = \dim \eta_{p-1} + 1\}$$

$$P_- = \{p : \dim \eta_p = \dim \eta_{p-1} - 1\}$$

$$P_0 = \{p : \dim \eta_p = \dim \eta_{p-1}\}.$$

Then clearly $|P_+|=|P_-|$, where $|P|$ means the number of elements of P. Since $\tau_q \notin T(k)$ for $q=2,3,\ldots,s-1$, we see that $|P_0|$ is odd. By Corollary 4.8 and Corollary 4.9, we have

$$or(\eta_p) = or(\eta_{p-1}) \quad \text{for } p \in P_+$$

$$or(\eta_{p+1}) = or(\eta_p) \quad \text{for } p \in P_-$$

$$or(\eta_p) = -or(\eta_{p-1}) \quad \text{for } p \in P_0.$$

Hence we obtain

$$or(\eta') = (-1)^{|P_0|} or(\eta) = -or(\eta). \qquad //$$

Theorem 4.11.

Let $k \in I(n)$ and let η_p be the pth simplex of $T(k)$ generated by the algorithm. Then

$$or(\eta_p) = (-1)^{p-1}.$$

proof. We will show the assertion by induction over k. When k=1,

$$W(\eta_1) = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}$$

for some real values a and b with $a > b$. Therefore $or(\eta_1)=(-1)^1$ sign det $W(\eta_1)=1$. By Corollary 4.10, we have $or(\eta_p)=(-1)^{p-1}$.

Assuming that the theorem is true for all $j \leq k-1$, consider the sequence $\{\eta_p : p=1,2,\ldots\}$ such that $\eta_p \in T(k)$. Since η_1 is generated in Step 1 of the algorithm, we see by Corollary 4.8 that $or(\eta_1)=or(\tau)$, where τ is the simplex of T(k-1) generated just prior to η_1. It is readily seen that τ occupies the odd position in the sequence of smplices of T(k-1). Therefore by the induction hypothesis $or(\tau)=1$. Hence we have

$$or(\eta_1) = 1.$$

This and Corollary 4.10 prove the theorem. //

Theorem 4.11 shows that an n-complete simplex given by the algorithm has a positive orientation.

References

[1] E. Allgower and K. Georg, "Simplicial and continuation methods for approximating fixed points and solutions to systems of equations", SIAM Review 22 (1980) 28-85.

[2] B.C. Eaves, "A short course in solving equations with PL homotopies", SIAM-AMS Proceedings 9 (1976) 73-143.

[3] B.C. Eaves and H. Scarf, "The solution of systems of piecewise linear equations", Mathematics of Operations Research 1 (1976) 1-27.

[4] M. Kojima and Y. Yamamoto, "A unified approach to the implementation of several restart fixed point algorithms and a new variable dimension algorithm", Mathematical Programming (to appear).

[5] G. van der Laan, "Simplicial fixed point algorithms", Ph.D. Dissertation, Free University (Amsterdam, 1980).

[6] G. van der Laan, "On the existence and approximation of zeroes", Free University (Amsterdam, 1981).

[7] G. van der Laan and A.J.J. Talman, "A restart algorithm for computing fixed point without extra dimension", Mathematical Programming 17 (1979) 74-84.

[8] G. van der Laan and A.J.J. Talman, "A restart algorithm without an artificial level for computing fixed points on unbounded regions", in: H.-O.Peitgen and H.-O.Walther, eds., Functional Differential Equations and Approximation of Fixed Points, Lecture Notes in Mathematics 730 (Springer, Berlin, 1979) pp.247-256.

[9] G. van der Laan and A.J.J. Talman, "Convergence and properties of recent variable dimension algorithms", in: W.Forster, ed., Numerical Solution of Highly Nonlinear Problems, Fixed Point Algorithms and Complementarity Problems (North-Holland, New York, 1980) pp.3-36.

[10] G. van der Laan and A.J.J. Talman, "A class of simplicial restart fixed point algorithms without an extra dimension", Mathematical Programming 20 (1981) 33-48.

[11] R. Saigal, "A homotopy for solving large, sparse and structured fixed point problems", Northwestern University (Evanston, Ill., 1981).

[12] M.J. Todd, "Orientation in complementary pivoting", Mathematics of Operations Research 1 (1976) 54-66.

[13] A.H. Wright, "The octahedral algorithm, a new simplicial fixed point algorithm", Mathematical Programming 21 (1981) 47-69.

[14] Y. Yamamoto, "A new variable dimension algorithm for the fixed point problem", Mathematical Programming 25 (1983) 329-342.

[15] Y. Yamamoto and K. Murata, "A new variable dimension algorithm: extension for separable mappings, geometric interpretation and some applications", Discussion Paper Series No.129(81-30), University of Tsukuba (Japan, 1981).