No. 189 (83-12)

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by

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September, 1983

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1. In their recent work [2], Cantor and Lippman attempted to show that when the internal rate of return of an investment project has multiple values by conventional definition, the smallest nonnegative rate, which is, however, at least as large as the market interest rate, is the genuine internal rate of return. (See Theorem 3.1 or 5.1 in [2].) Following Gale's idea [4], and trying to obtain the concept of internal rate of return unmotivated by cost of capital consideration, they demonstrated that the investor's terminal wealth ultimately grows at this smallest rate of return.

The conclusion that the smallest rate, but not the largest rate, is important contradicts to the common sense of economists. Moreover, if the internal rate theoretically justified by them depends upon the market rate of interest which is necessarily *external*, can we really say that this is the *internal* rate of return? In this paper, we hope to show that despite their stress on the smallest rate as a selection criterion, they really did not clarify the proper internal rate of return of the project, but rather discussed a special version of the present value approach for investment selection under certain assumptions which are not necessarily acceptable. We will give a concrete example in which a project is never utilized according to their Theorem 5.1, but it is profitable and should be undertaken.

2. Although Cantor and Lippman claim that they have only a single project, they actually deal with two projects at a time, namely, $(a_0, a_1, \ldots, a_r)$ and $(-1, 1+\rho)$, where ρ is the market interest rate. As long as $\rho > -1$, it is not essential whether one has $\rho = 0$ or not, and Theorem 3.1 and

Theorem 5.1 are seen to be equivalent.[1] We will therefore put $\alpha = 1+\rho$ and assume only $\alpha > 0$ in the subsequent analysis.

We begin by reviewing briefly the model and the assumptions proposed in [2]. Let us consider column vectors u, s and c having components $u_0, \ldots, u_n$, $s_0, \ldots, s_n$ and $-1, 0, \ldots, 0$, respectively. In matrix form, we may formulate their model more compactly as

$$\text{maximize} \quad s_n \tag{3.4}$$

$$\text{subject to} \quad Au + Bs = c \qquad (3.2) \text{ or } (5.2)$$

$$u \geqq 0 \quad \text{with} \quad u_m = 0, \quad m > n-r \tag{3.1}$$

$$s \geqq 0 \tag{3.3}$$

where

$$A = \begin{bmatrix} a_0 & & & & & \\ a_1 & a_0 & & & & \\ & \cdot & \cdot & \cdot & & \\ a_r & \cdot & \cdot & \cdot & a_0 & \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & a_r & \cdot & \cdot & a_1 \quad a_0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & & & & & \\ \alpha & -1 & & & & \\ & \alpha & -1 & & & \\ & & \cdot & \cdot & \cdot & \\ & & & \alpha & -1 & \\ & & & & \alpha & -1 \end{bmatrix}$$

and $\alpha = 1+\rho$. Other notations and equation numbers are the same as in [2]. For example, u_m is the amount of investment at time m, and s_m is the cash on hand or the amount of cash to be let out. It is assumed that the decisionmaker's goal is to maximize the terminal wealth s_n at time n and that $u_m = 0$ identically for $m > n-r$. It may be convenient to consider u_m or s_m the activity level of investment project $(a_0, \ldots, a_r)$ or $(-1, \alpha)$, respectively, except for the last s_n. The assumption $s \geqq 0$ means that we allow for lending but not borrowing.

Let us now premultiply (3.2) by a positive row vector $(\alpha^n, \ldots, \alpha, 1)$ obtaining

$$a(1/\alpha).\Sigma\alpha^{n-m}u_m - s_n = -\alpha^n$$

or

$$s_n = \alpha^n + a(1/\alpha).\Sigma\alpha^{n-m}u_m \qquad (3.7)$$

where $a(1/\alpha)$ is the net present value of the project $(a_0, \ldots, a_r)$ at the interest factor α, and the summation goes from 0 to $n-r$. Now if one selects $u = 0$, one has $s_n = \alpha^n$, and so it is easy to see that whenever $a(1/\alpha) < 0$ holds, one must have $u = 0$ to obtain a maximum s_n. Namely, the project $(a_0, \ldots, a_r)$ should not be utilized at all. If $a(1/\alpha) > 0$ holds, one can have $s_n > \alpha^n$ and so it is better to undertake the project.[2] This result is nothing but a special version of the present value approach for investment selection. Notice, however, that the same result can be obtained even if we remove the condition $s \geq 0$. This is so, because we need (3.1) but not (3.3) to show (3.7) to be true, and a feasible pair (u, s) is obviously feasible again, (3.3) being removed. This means that even if we allow for not only lending but also borrowing, we still have the same result concerning whether we should use the project or not, and the argument by Cantor and Lippman becomes much more appropriate as the present value approach. Despite their emphasis on the departure from the Hirshleifer paradigm, it seems to me that what they insist is not essentially incorporated in their analysis.

As a matter of fact, if the selection criterion in their analysis is understood as the present value of the project, it is now not surprising at all that we may not utilize a project at $\alpha = 1$ but will utilize it at some $\alpha > 1$. Such case is well known in the literature of economics. (See, e.g., Karmel [5].) And we must conclude that the puzzle proposed in the literature is not solved yet in [2]. Also, once the selection criterion is clarified, it is now of secondary character, or it does not matter, to show how rapidly money can grow when it is not withdrawn at all. Moreover, this rate of growth depends on α especially when the project has multiple internal rate of return. As we shall see in the next section, the smallest internal rate of return has nothing to do with the true profitability of the project under a certain reasonable presumption. Also, it is absurd to argue that there is some finite n for which one has $V_n = \infty$ when $a(z)$ has no root in the interval $(0, 1)$, isn't it? For, whatever profitable is a project, one dollar invested at time 0 never becomes infinite at time n, unless $n = \infty$.

3. One of the most important assumptions in the model proposed in [2] is that the investor's goal is the maximization of terminal wealth over a finite horizon n. Since, in general, the firms are considered immortal, we can find little explanation of why a firm or firms should choose to solve the maximization problem stated in the preceding section. It would be probably impossible to justify why one must choose a specific finite n as an investment horizon. Be that as it may, in this section we wish to give an example in which the selection criterion asserted in Theorem 3.1 or 5.1 is seen to be unreasonable. In the preceding section, the vector c has components $-1, 0, \ldots, 0$, and suppose that a maximum amount of s_n is obtained as V_n. This implies that if we invest one dollar at time 0 and we do not put any new money in the enterprise (positive or negative) during the periods from 1 to $n-1$, then we can withdraw an amount of V_n at time n. In other words, we have a sequence of cash flows such that[3]

$$-1, 0, \ldots, 0, V_n, 0, 0, \ldots .$$

More generally, a sequence of cash flows $\tilde{c}_0, \tilde{c}_1, \ldots, \tilde{c}_m, \ldots$ will be obtained from $(-1, \alpha)$ by the rule

$$- s_0 = \tilde{c}_0, \quad \alpha s_{m-1} - s_m = \tilde{c}_m \qquad m = 1, 2, \ldots$$

for some sequence of activity levels $s_0, s_1, \ldots, s_m, \ldots$.

Let us consider a three period project (a_0, a_1, a_2, a_3) given by

$$a_0 = -1, \quad a_1 = \lambda + \mu + \nu, \quad a_2 = -(\lambda\mu + \lambda\nu + \mu\nu), \quad a_3 = \lambda\mu\nu$$

where $\lambda > \mu > \nu > 0$. This project has three internal discount rates, $\lambda-1$, $\mu-1$, and $\nu-1$, since λ, μ, and ν are the roots of the polynomial

$$f(z) = z^3 - a_1 z^2 - a_2 z - a_3 = 0$$

Since $f(0) = -a_3 < 0$ and $f(z) > 0$ for $z > \lambda$, it is easy to see that the net present value is negative at α such that $\mu > \alpha > \nu$. Whenever $a(1/\alpha) < 0$, Theorem 5.1 asserts that the project should not be utilized and $s_n = \alpha^n$. We, however, will show that, as long as $\alpha < \lambda$ holds, there exists a sequence of cash flows generated from the project

$$c_0, c_1, \ldots, c_m, \ldots$$

which completely dominates[4] any non-trivial sequence $\{\tilde{c}_t\}$ generated from $(-1, \alpha)$, which starts with -1, including of course

$$-1, 0, \ldots, 0, \alpha^n, 0, 0, \ldots \quad .$$

Let us define a triangular matrix M_z of order $n+1$ such that

$$M_z = \begin{bmatrix} -1 & & & & \\ z & -1 & & & \\ & z & -1 & & \\ & & \cdot & \cdot & \cdot \\ & & & z & -1 \end{bmatrix} \qquad \text{for } z = \lambda, \mu, \nu, \text{ and } \alpha.$$

Obviously, $M_\alpha = B$, and M_z's commute each other. The column vector $d = \{-1, 0, \ldots, 0, \alpha^n\}$ will be given by

$$d = Bs \qquad \text{for } s = \{1, \alpha, \ldots, \alpha^{n-1}, 0\}.$$

In general, a truncated sequence $\tilde{c} = \{-1, \tilde{c}_1, \ldots, \tilde{c}_n\}$ will be represented as $\tilde{c} = Bs$ for some $s \geqq 0$ when we allow for lending only. It is also possible to assume $\tilde{c}_m \geqq 0$ except for $\tilde{c}_0 = -1$, if one wishes to have some financial constraints in the model. On the other hand, a truncated sequence of $\{c_t\}$, $c = \{c_0, \ldots, c_n\}$, is given by $c = Au$ for some $u \geqq 0$. In [1], I have called the matrix A, which for instance appears in (3.2), *the polynomialoid matrix*, and have showed that it can be factored. The reader can easily check that in the present case we have

$$A = M_\lambda M_\mu M_\nu \qquad \text{just as} \quad f(z) = (z-\lambda)(z-\mu)(z-\nu)$$

First of all, consider a project $(-1, \beta)$ for $\lambda > \beta > \alpha$. It is intuitively obvious that $(-1, \beta)$ is more profitable than $(-1, \alpha)$, since β is a greater interest factor. It is easy to see that $(-1, \beta)$ can produce a sequence of cash flows $\{c_t\}$, completely dominating the sequence $\{-1, 0, \ldots, 0, \alpha^n, 0, 0, \ldots\}$. This is so, because we can have a positive sequence $\{w_t\}$ of activity levels of $(-1, \beta)$ such that

$$\beta^n > \beta^n w_0 > \beta^{n-1} w_1 > \ldots > \beta w_{n-1} > w_n + \alpha^n, \text{ and } \beta w_{t-1} > w_t \text{ for all } t.$$

Clearly we have $c = M_\beta w > d$ and $c_t = \beta w_{t-1} - w_t > 0$ for $t > n$ also. It is also intuitive, and the reader can easily show, that $(-1, \beta)$ can generate a sequence which completely dominates any $\{-1, \tilde{c}_1, \ldots, \tilde{c}_t, \ldots\}$ generated by $(-1, \alpha)$, provided that we allow for lending only.[5] This is another feature that $(-1, \beta)$ is more profitable than $(-1, \alpha)$.

Second, note that $c = M_\beta w = Au$ holds, upon choosing $u = A^{-1} M_\beta w$. We will show that $u \geqq 0$. Since we have

$$-M_\lambda^{-1} = \begin{bmatrix} 1 & & & & \\ \lambda & 1 & & & \\ \lambda^2 & \lambda & 1 & & \\ & \cdot & \cdot & \cdot & \cdot \\ \lambda^n & \cdot & \cdot & \cdot & \lambda \quad 1 \end{bmatrix}$$

it follows that $M_\lambda^{-1} M_\beta w > 0$ as long as $\lambda > \beta$. Thus $u = M_\nu^{-1} M_\mu^{-1} M_\lambda^{-1} M_\beta w > 0$. Thus, given $\{w_t\}$, let us construct $\{u_t\}$ by $u = A^{-1} M_\beta w$, where of course $u = \{u_0, \ldots, u_t\}$ and $w = \{w_0, \ldots, w_t\}$, $t = 0, 1, 2, \ldots$, ad inf.

Note that as we increase t by 1, the order of A, u or w increases, but $\{u_0, \ldots, u_{t-1}\}$ is invariant even if we increase t, since A^{-1} or M_α is triangular. Hence, the same $\{c_t\}$, completely dominating $\{\tilde{c}_t\}$ can be obtained from the project (a_0, a_1, a_2, a_3) as well.

To recapitulate: The sequence $\{-1, 0, \ldots, 0, \alpha^n, 0, 0, \ldots\}$ is obtained from the project $(-1, \alpha)$. But the three period project considered here can generate a sequence of cash flows $\{c_0, c_1, \ldots, c_t, \ldots\}$ which is absolutely better. This is true whenever $\lambda > \alpha$. Any rational investor will definitely select the project (a_0, a_1, a_2, a_3) rather than $(-1, \alpha)$. It is incorrect to conclude that assuming the terminal wealth is to be maximized for a finite horizon n, we should not utilize the project when $a(1/\alpha) < 0$.

Our arguments in this section reveals that the internal interest factor of the three period project is not less than λ. Any root less than λ has no meaning as the interest factor. In this sense, the assertion in the work by Dorfman [3] is also incorrect, who suggested that other roots also have the meaning as the interest factor, depending on the initial conditions. A question arises: is $\lambda-1$ the internal rate of return? Also, generally, the characteristic polynomial of a project has positive, negative and complex roots, and the analysis is not so simple as given here. The general case, showing that the largest positive root is the interest factor, is treated systematically in my paper [1]. But I hope that, at any rate, the simple case considered in this section is sufficient to show that the 'smallest' rate of return analyzed by Cantor and Lippman is by no means the internal rate of return.

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Footnotes

1. If $\alpha > 0$, consider a diagonal matrix to normalize (3.2)

$$D = \begin{bmatrix} 1 & & & & \\ & \alpha & & & \\ & & \alpha^2 & & \\ & & & \ddots & \\ & & & & \alpha^n \end{bmatrix}$$

In fact, put $A^* = D^{-1}AD$, $B^* = D^{-1}BD$, $x = D^{-1}u$ and $y = D^{-1}s$. Then the maximization problem in the text is equivalent to the one given by

$$\begin{aligned} &\text{maximize} && y_n \\ &\text{subject to} && A^*x + B^*y = c \\ & && x \geqq 0 \text{ with } x_m = 0, \quad m > n-r \\ & && y \geqq 0 \end{aligned}$$

Clearly, any element of the second lower parallel diagonal of B^* is unity, and Theorem 3.1 is equivalent to Theorem 5.1 as long as $\alpha > 0$.

2. It is easy to show that (3.2) has a feasible pair (u, s) such that $u_m > 0$. In fact, let $u_m = \varepsilon > 0$, $m = 0, \ldots, n-r$, for a sufficiently small ε, and then consider $s = B^{-1}(c - Au)$. Since $s > 0$ when $u = 0$, we have $s \geqq 0$ for small ε.

3. This sequence should not be confused with a sequence of cashes on hand in the enterprise, $s_0, \ldots, s_{n-1}, s_n, s_n, \ldots$ when the investment horizon is n. This sequence is not meaningful from the investor's point of view.

4. We will say that a sequence $\{c_t\}$ completely dominates a sequence $\{\tilde{c}_t\}$ if we have $c_t > \tilde{c}_t$ for each t.

5. Put $\tilde{c} = Bs = M_\beta v$. Then $v = M_\beta^{-1}Bs > 0$, since $\beta > \alpha$. It follows from $v > 0$ that we can have $M_\beta w > \tilde{c}$ for some $\{w_0, \ldots, w_n\}$. Moreover, even if we increase the order of M_β to consider w_{n+1}, etc., $\{w_0, \ldots, w_n\}$ can

be taken as invariant. Hence, we simply add w_{t+1} to $w_0, \ldots, w_t$ to obtain $\{w_0, \ldots, w_{t+1}\}$, and this procedure can be repeated indefinitely.

References

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