No. 188 (83-11)

Model of R&D Investment of Firms and Resulting New Skew Distribution

by

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July 1983

ABSTRACT

Herein is modeled an R&D investment behaviour of firms in the light of concept of self-catalytic process with a technological cycle. This process is expressed in terms of a stochastic differential equation, and its stochastic integration is shown to yield a new skew distribution. This newly derived distribution of the R&D is shown to fit the observed distribution. Its goodness of fit may be interpreted that the self-catalytic process explains the actual R&D investment behaviour. The properties of the new distribution are discussed.

1. Introduction

The importance of R&D, especially the importance of R&D in private firms is increasing. In fact, the share of R&D expenditure of private firms in the whole national R&D expenditure is drastically growing in France, UK and USA and it now exceeds 50% in UK and USA meanwhile it has almost always exceeded 50% and 70% in W. Germany and Japan, respectively. These figures indicate that the R&D investment or expending behaviour of firms is decisively important today to the whole technology development. (Usually, the word of R&D investment is used as almost synonymous to that of R&D expending. Economic theories tend to use the former while government statistics the latter.

An approach to R&D of firms is to deal with an R&D investment problem of an individual firm as a project selection problem (e.g., Horesh and Ratz 1982) or as a timing problem (e.g., Dasgupta et als. 1982). Another approach is to deal with an accumulation of individual firm's investments. The latter approach is successful for the innovation process as a diffusion process in terms of adoption or immitation of firms (e.g., Mahajan and Peterson 1978, Jensen 1982). But, different from innovations, the concept of adoption or immitation is not properly applicable to the R&D problem. In addition the accumulation of R&D expenditure of firms over time is usually not meaningful and its data is unavailable. Our approach herein is concerned with connecting an R&D investment behaviour of single firms with a distribution of firm's R&D expenditure. Aside a theoretical interest, a merit of this approach follows from the data availability.

The data of R&D expenditure of individual firms were unpublished till a few years ago though the financial data of each stockmarket-listed firm and the aggregated data of R&D expenditure in the private sectors have been published. Such a limited availability of the data makes the time-series analysis impossible and requires to infer an R&D investment behaviour from the static data. A theoretical model which explains a statistical distribution is expected to help this inference.

As is easily expected from an analogy to the income distribution, the R&D expenditure is quite unequitably distributed among firms. Aside empirical distributions to describe unequitable distributions (e.g., the Pareto distribution, the Lotka distribution), Aitchison and Brown (1957) and Ijiri and Simon (1977) demonstrated that the log-normal distribution and the Yule distribution, respectively, fit unequitable distributions.

In this paper, the section 2 discusses the log-normal and Yule distributions in the context of proportional growth process; the section 3 proposes a self-catalytic process as a model of an R&D investment behaviour of firms, obtains an associated stochastic differential equation and the stochastic integration as its solution; the section 4 derives a new statistical distribution therefrom combined with the probability of a technological cycle; the section 5 discusses the skewness of this distribution in comparison with the Yule distribution; finally the section 6 shows its better fit to the actual distribution than other distributions.

2. Existing Distributions for Proportional Growth

The proportional growth process in which larger one grows more than smaller one gives rise to skew or unequitable distributions as follows.

The log-normal distribution represents the proportional growth as

Aitchison and Brown (1957, p 23) showed. Let the proportional growth be

$$\Delta X_t = X_t - X_{t-1} = \varepsilon_t X_{t-1}$$

or

$$(X_t - X_{t-1})/X_{t-1} = \varepsilon_t$$

Then

$$\sum_{t=1}^{n} (X_t - X_{t-1})/X_{t-1} = \sum_{t=1}^{n} \Delta X_t/X_{t-1} = \sum_{t=1}^{n} \varepsilon_t$$

Now, as $n \to \infty$,

$$\sum_{t=1}^{n} \Delta X_t/X_{t-1} \to \int_{X_0}^{X_n} dx/x = \log X_n - \log X_0$$

Thus, as $n \to \infty$,

$$\log X_n \to \log X_0 + \sum_{t=1}^{n} \varepsilon_t$$

As the first term on the right side in the last formula is constant, the left side is subject to the normal distribution by the central limit theorem if ε_t's are sufficiently small and mutually independent for all t. Thus X_n is subject to the log-normal distribution.

The Yule distribution is also deducible from the condition of proportional growth as Ijiri and Simon (1977) showed. But its growth process is different from the above one in that its dynamics is not in time itself but in the total size. In other words the Yule distribution is associated with the process in which firms grow proportionally as the sum of firm sizes grows with no explicit reference to time (Eto and Makino 1981). The Pareto distribution can be viewed as a special case of the Yule distribution (Ijiri and Simon 1977, p 75).

Our new distribution to be proposed below is superior to the Yule distribution in the naturality that the former is deduced from the dynamical process in time itself and is superior to the Pareto distribution in the clear structure to express the causality. Its superiority to the log-normal distribution will be empirically shown in the section 6.

3. R&D Investment Model

A technological cycle between the progressive age and the stagnant age is historically known and its relationship with social and business cycles has been discussed under the name of Kondratieff Wave by, for example, Kuznetz (1953). Mensh (1978) showed that the innovation cycle, a part of the technological cycle, does not correspond to the invention cycle, another part of the technological cycle, but rather corresponds to the business cycle. This indicates that the innovation cycle might mainly be based on the economic investments or needs. Sahal (1980) presented evidences that the technological cycle causes the business cycle. In sequel of this paper, however, no detail will be discussed about a causal linkage between the technological and the business cycle. Instead it will formally be treated as being exogenous to the business and management while semantically it allows for the economic interpretation as will be mentioned below. Indeed, the technological cycle appears uncontrollable and hence exogenous to each single firm at the present state of knowledge on technology among businessmen.

This cycle is simplified and modeled as follows. Technology makes an advance (as measured by the numbers of patents and researchers, the amount of R&D expenditure and so on) in the progressive age while it remains unchanged (for example, the R&D expenditure does not increase and the numbers of patents newly acquired and researchers newly employed are equal to the numbers of patents expired and researchers retired in the same period) in the stagnant age. The historical experience that technology has almost monotonically grown allows to neglect the case where technology makes a retrogression. This simplified model of cycle may be termed the polarized monotone growth model.

The ways of growth is treated hereafter as being discrete (the unit incrementalism assumption). For example, the numbers of patents owned by a firm and researchers working for a firm increase one by one (unit by unit) in a sufficiently short time period Δt and the R&D expenditure increases unit by unit for a fiscal term however rapidly technology may grow.

The R&D resources may be classified into the stocked or accumulated one and the flowing or current one. The former is a result of the past decision and activity like patents, knowhow, knowledge, laboratories, organizations, researchers especially in the closed (e.g., lifetime employment) personnel management system and so on. Among these, the most important one is knowledge which is often represented by patents as legally authorized knowledge or by researchers as the carriers and creator of knowledge. The latter is represented by the R&D expenditure.

A rational R&D budgeting may be to expend money in accordance to the existing level of stocked R&D resources in view of the potential of progress. In other words, a decision is made to increase R&D expenditure when the level of stocked R&D resources is high and the outlook is bright (i.e., the technological cycle is at the progressive age.) Otherwise, either the stocked R&D resources or the R&D expenditure would become idle. The brightness of outlook or forecasting mainly depends on the environment outside a firm while the level of stocked R&D resources is an internal matter of a firm. The former will be neglected until the section 4, and the discussion in this section will be concerned with the latter. In a chemical terminology, the presence of the stocked R&D resources tends to promote the R&D expending as a catalyst promotes the chemical process. Of course the R&D expending raises the level of the R&D resources. Hence this process may be called the self-catalytic process. This may explain why the R&D expenditure has (almost) monotonically increased under the recognition that knowledge has (almost) always advanced. In this sense the monotone growth model can be justified.

As stated just above, the catalyst is also the result of the R&D expenditure. Hence the self-catalytic process can shortly be expressed as the autogenous process by skipping the intermediary catalyst. For the mathematical simplicity, the expressions hereafter takes the form of autogenous process while it semantically leaves the room for economic or decision-behavioral interpretation.

The above discussions may be restated in a formal manner as follows.

$G: K \times C \to K \quad (\text{or } K_t = G(K_{t-1}, C_{t-1}))$

where G denotes the growth law of knowledge and maps from the knowledge space K and the technological cycle space C to K with the bivalent C = {progressive, stagnant} as simplified above.

$D: \; K \times C \to E \quad (\text{or } E_t = D(K_{t-1}, C_t))$

where D denotes the R&D expending decision and maps from K and C to the R&D expenditure space E.

These two together yield the short relation

$R: \; E \times C \to E \quad (\text{or } E_t = R(E_{t-1}, C_t))$

where R is interpreted as a relation that the R&D expenditure is decided on the basis of the foregoing one and the outlook or forecasting of progressiveness.

A firm is said to be in state k at time t when its R&D expenditure is k in an appropriate unit at t and is denoted by $O_k(t)$ or O_k if the reference to time can be omitted.

Let $Q_k(t)$ denote the probability that a firm is in $O_k(t)$ under the condition that the firm is in its progressive age at t. To avoid a trivial confusion, the progressive age is assumed to have the open end; it is $[t_1, t_2)$ so that a firm is still in the progressive age at $t + \Delta t$ if it is so at t. Let a firm be in the progressive age in the discussion below in this section.

When a firm is in $O_k(t)$, only either $O_{k-1}(t-\Delta t)$ or $O_k(t-\Delta t)$ is possible under the unit incrementalism assumption. From $O_{k-1}(t-\Delta t)$ it transits to $O_k(t)$ with the probability $\lambda(k-1)\Delta t$, and from $O_k(t-\Delta t)$ it transits to $O_k(t)$ with the probability $1 - \lambda k\Delta t$ for $\lambda > 0$ under the proportional growth assumption.

Formally restated,

$$Q_k(t) = \lambda(k-1)\Delta t Q_{k-1}(t-\Delta t) + (1-\lambda k\Delta t)Q_k(t-\Delta t)$$

Transferring a term from the right to the left side and dividing the both sides by Δt,

$$\{Q_k(t)-Q_k(t-\Delta t)\}/\Delta t = -\lambda k Q_k(t-\Delta t) + \lambda(k-1)Q_{k-1}(t-\Delta t)$$

$$= -\lambda k\{Q_k(t) + o_1\} + \lambda(k-1)\{Q_{k-1}(t) + o_2\}$$

where o_1 and o_2 are negligibly small terms.

In the form of the stochastic differential equation,

$$Q'_k(t) = -\lambda k Q_k(t) + \lambda(k-1)Q_{k-1}(t) \qquad (1)$$

Solving or integrating (1) leads to (2) as is shown by Feller (1950, p 403),

$$Q_k(t) = {}_{k-1}C_{k-i} \exp(-\lambda it)(1-\exp(-\lambda t))^{k-i} \quad (2)$$

where i denotes the initial level of knowledge (e.g., the number of patents) owned by a firm. Under the unit incrementalism assumption,

$$i = 1 \quad (3)$$

Substituting (3) in (2),

$$Q_k(t) = \exp(-\lambda t)(1-\exp(-\lambda t))^{k-1} \quad (4)$$

This expresses the probability of a firm being in $Q_k(t)$ under the condition that the firm is in its progressive age. But it is not yet properly the probability under investigation.

4. Probability and Associated Distribution

Whether it is in the progressive or stagnant age at t varies among firms due to the characteristics of each firm and is therefore treated as a random variable. It may actually appear random to a firm in that it is unpredictable to the firm. (Recall that the randomness is often associated with the possibly deterministic but unpredictable event). For these reasons the randomness model may be acceptable.

Let $q_k(t|c)$ denote the probability that a firm is in $O_k(t)$ under the condition that the firm is in its progressive age at t. Then,

$$q_k(t|c) = Q_k(t)r_c(t) \quad (5)$$

where $r_c(t)$ denotes the probability that the firm is in its progressive age at t.

The probability $r_c(t)$ monotonically corresponds to the probability $s(\tau)$ that the length or duration time of the progressive age is τ in the firm. In other words, $r_c(t)$ is larger (smaller) when the duration time of the progressive age is longer (shorter). Actually our concern is not with the state at each time but with its integration over time up to present. Hence the deep relationship exists between $r_c(t)$ and $s(\tau)$. Let the transformation be investigated from $s(\tau)$ to $r_c(t)$.

The fact that the domain of t = the domain of τ = [start of observation, present] allows the transformation in question to be simple one. Notice that $s(\tau)$ satisfies the regularity condition

$$\int_0^\infty s(\tau)d\tau = 1$$

while the meaning of $r_c(t)$ indicates that

$$\int_0^\infty r_c(t)dt \geqq 1$$

Hence the transformation in question which maps the probability in τ to the probability in t may be linear with the transformation coefficient $\gamma \geqq 1$.

$$r_c(t) = \gamma s(\tau) \quad (6)$$

The underlying idea of the formulation in (6) is that our concern is not necessarily with the probability of state of each firm at each time but with the probability distribution of state of firms over time.

The distributions of duration times of telephone conversations and the lifetimes of industrial products are often expressed by the exponential distribution with the following density for $\mu > 0$.

$$s(\tau) = \mu\exp(-\mu\tau) \quad (7)$$

The idea to express the duration by (7) is the perfect randomness or the lack of causality in the duration of conversations. In fact, (7) is the solution to the following differential equation of the constant rate of accident as is stated e.g., by Yoda (1972, p 15).

$$dS(\tau)/d\tau = -\mu S(\tau)$$

where $S(\tau)$ denotes the rate of survival of conversation. The same idea will be applied to the duration of progressive ages. Now, using (6) and (7) and recalling that γ maps a function in τ to a function in t,

$$r_c(t) = \gamma\mu \exp(-\mu t) \tag{8}$$

As is well known, the mean of (7) is $1/\mu$. Using this fact, let a constant parameter ρ be defined as the reciprocal of the product of the growth parameter λ and the expected duration time of progressive age $1/\mu$. Note that $\rho > 0$.

$$\rho = \mu/\lambda \tag{9}$$

The definition (9) suggests that the growth is faster (slower) when the value of α is smaller (larger).

Now let p(k) denote the probability that a firm is in 0_k at present. Note that p(k) is equal to the expected relative frequency of such firms at present. In the polarized monotone growth model according to which a firm remains in the same state in the stagnant age, the effect of the stagnant age on p(k) is null and hence will be ignored below.

Now, let $P^U(k)$ denote the probability that a firm is in the states of 0_{k+j} for all j = 0, 1, ... at present or equivalently denote the upper cumulative relative frequency of p(k), namely,

$$P^U(k) = \sum_{j=0}^{\infty} p(k+j) \tag{10}$$

By definition,

$$p(k) = P^U(k) - P^U(k+1) \tag{11}$$

As the polarized monotone growth model allows to ignore the effect of the stagnant age on the present state, only the effect of the progressive age is considered in obtaining p(k) and $P^U(k)$. Therefore,

$$P^U(k) = \int_0^{\infty} q_k(t|c)dt \tag{12}$$

The right side in itself denotes the probability that a firm has once been in $O_k(t)$ for some t in $[0,\infty]$. In the monotone growth model with the unit incrementalism assumption, a firm can be in O_{k+j}, j = 0, 1, ... only if it was once in O_k. Hence the right side also denotes the probability that a firm is in O_{k+j}, j = 0, 1, ... at present.

By (12), (5), (4) and (8),

$$P^U(k) = \int_0^\infty \exp(-\lambda t)(1-\exp(-\lambda t))^{k-1}\gamma\mu \exp(-\mu t)dt$$

$$= \gamma\mu\int_0^1 x^{k-1}(1-x)^{\mu/\lambda}dx/\lambda = \gamma\rho\int_0^1 x^{k-1}(1-x)^{\rho}dx$$

$$= \gamma\rho\beta(k, \rho+1) = \gamma y(k) \qquad (13)$$

where

$$x = 1 - \exp(-\lambda t), \; dx = \lambda\exp(-\lambda t)dt$$

$$\beta(k, \rho+1) = \int_0^1 x^{k-1}(1-x)^{\rho}dx = \Gamma(k)\Gamma(\rho+1)/\Gamma(k+\rho+1)$$

$$y(k) = \rho\beta(k, \rho+1)$$

with β and Γ denote the Beta and Gamma functions respectively.

It deserves mentioning that y(k) is the density or the relative frequency of the Yule distribution.

By the regularity condition of a statistical distribution,

$$P^U(1) = 1 \qquad (14)$$

Recall the following properties of the Gamma function;

$$\Gamma(x+1) = x\Gamma(x) \text{ and } \Gamma(1) = 1$$

Applying these properties to the left side of (14) with aid of (13),

$$1 = \gamma\rho\Gamma(1)\Gamma(\rho+1)/\Gamma(\rho+2) = \gamma\rho\Gamma(\rho+1)/\{(\rho+1)\Gamma(\rho+1)\} = \gamma\rho/(\rho+1)$$

Hence,

$$\gamma = (\rho+1)/\rho = 1+1/\rho = 1 + \lambda/\mu \qquad (15)$$

Now, by (13),

$$P^U(k) = (\rho+1)\beta(k, \rho+1) = (\rho+1)\Gamma(k)\Gamma(\rho+1)/\Gamma(k+\rho+1)$$

$$=\rho(\rho+1)\Gamma(k)\Gamma(\rho)/\{(k+\rho)\Gamma(k+\rho)\} = \rho(\rho+1)\beta(k,\rho)/(k+\rho) \qquad (16)$$

Then, by (11), (16) and the basic property of the Gamma function,

$$p(k) = (\rho+1)\Gamma(k)\Gamma(\rho+1)/\Gamma(k+\rho+1) - (\rho+1)\Gamma(k+1)\Gamma(\rho+1)/\Gamma(k+\rho+2)$$

$$= (\rho+1)\{\Gamma(k)\Gamma(\rho+1)/\Gamma(k+\rho+1) - k\Gamma(k)\Gamma(\rho+1)/\{(k+\rho+1)\Gamma(k+\rho+1)\}\}$$

$$= (\rho+1)\Gamma(k)\Gamma(\rho+1)\{1-k/(k+\rho+1)\}/\Gamma(k+\rho+1)$$

$$= (\rho+1)^2\Gamma(k)\Gamma(\rho+1)/\{(k+\rho+1)\Gamma(k+\rho+1)\}$$

$$= (\rho+1)^2\beta(k, \rho+1)/(k+\rho+1) = (\rho+1)P^U(k)/(k+\rho+1) \qquad (17)$$

The combination of (17) and (16), or perhaps more preferably, the combination of (17), (14) and (10) allows to derive p(k) explicitly.

$$p(1) = (\rho+1)\cdot 1/(\rho+2) = (\rho+1)/(\rho+2) = 1 - 1/(\rho+2) \qquad (18)$$

$$p(2) = (\rho+1)\{1-p(1)\}/(\rho+3) = (\rho+1)/\{(\rho+2)(\rho+3)\} \qquad (19)$$

$$p(3) = (\rho+1)\{1-(p(1)+p(2))\}/(\rho+4)$$

$$= (\rho+1)[\{(\rho+3)-(\rho+1)\} / \{(\rho+2)(\rho+3)\}]/(\rho+4)$$

$$= 2(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4) \qquad (20)$$

$$p(4) = 6(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)(\rho+5)\} \qquad (21)$$

$$p(5) = 24(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)(\rho+5)(\rho+6) \qquad (22)$$

(See the Appendix for the detail)

Generalizing this series,

$$p(k) = \{(k-1)!(\rho+1)\}/\{(\rho+2)(\rho+3)\ldots(\rho+k+1)\} \qquad (23)$$

Let m_p denote its mean. Then, by (23),

$$m_p = \sum_{k=1}^{\infty} [\{k!(\rho+1)\}/\{(\rho+2)(\rho+3)\ldots(\rho+k+1)\}] \qquad (24)$$

Our new distribution with (23) as the density or relative frequency will be referred to as the post-Yule distribution hereafter.

5. Properties of the post-Yule Distribution

It is empirically known that (25), (26) and (27) approximately hold for many cases of distributions in asset, income, word frequency (Ijiri and Simon 1977), scientific papers (Yablonsky 1980) and so on.

$$1 \times 1/2 \leqq \rho \leqq 1 \times 2 \tag{25}$$

$$1/2 - 1/4 \leqq f(1) \leqq 1/2 + 1/4 \tag{26}$$

$$1/k^2 \leqq f(k)/f(1) \leqq 1/k \quad \text{for } 2 \leqq k \leqq 4 \text{ or } 5 \tag{27}$$

where f denotes the relative frequency which equals the density p.

These relations (25) - (27) are satisfactorily consistent with (18) - (22). In this respect the post-Yule distribution derived in the section 4 may be useful in describing the aforementioned distributions as well as the distribution of R&D expenditure of firms. These distributions are expected to be monotone decreasing and to be convex (i.e., diminishing decrease). In fact, the post-Yule distribution is strictly monotone decreasing. To see this, by (23),

$$\begin{aligned} p(k) - p(k+1) &= [(k-1)!(\rho+1)\{(\rho+k+2)-k\}]/\{(\rho+2)\ldots(\rho+k+2) \\ &= (k-1)!(\rho+1)/\{(\rho+3)\ldots(\rho+k+2)\} > 0 \end{aligned} \tag{28}$$

The convexity can be seen by examing the sign of second difference which is the discrete version of the second derivative. That is, by using the above result,

$$\begin{aligned} &\{p(k)-p(k+1)\} - \{p(k+1)-p(k+2)\} \\ &= (k-1)!(\rho+1)/\{(\rho+3)\ldots(\rho+k+2)\} - k!(\rho+1)/\{(\rho+3)\ldots(\rho+k+3)\} \\ &= [(k-1)!(\rho+1)\{(\rho+k+3)-k\}]/\{(\rho+3)\ldots(\rho+k+3)\} \\ &= \{(k-1)!(\rho+1)\}/\{(\rho+4)\ldots(\rho+k+3)\} > 0 \end{aligned}$$

This implies the diminishing decrease or the convexity of the density function of the post-Yule distribution.

As the k-power of ρ occurs in the denominator while a single ρ occurs in the numerator on the final right side in (28), the rate of decrease is diminishing as ρ is increasing for k fixed. In other words, the density is higher for large k when ρ is larger. This means that the post-Yule distribution is skewer when ρ is smaller.

These facts stated above indicate similarities of the post-Yule distribution to the Yule distribution.

Let the post-Yule distribution be compared with the Yule distribution. For the purpose of comparison, the same value of ρ is assumed for the both for the time being. Let $Y^U(k)$ denote the upper cumulative distribution of the Yule distribution. Namely,

$$Y^U(k) = \sum_{j=k}^{\infty} y(j)$$

Naturally like (11),

$$y(k) = Y^U(k) - Y^U(k+1)$$

Then, by the properties of the Beta function (e.g., Ijiri and Simon 1977, p 67),

$$Y^U(k) = \rho\beta(k,\rho) = \rho\Gamma(k)\Gamma(\rho)/\Gamma(k+\rho)$$

Hence, by (16), for the same value of ρ,

$$P^U(k) - Y^U(k) = \rho(\rho+1)\Gamma(k)\Gamma(\rho)/\{(k+\rho)\Gamma(k+\rho)\} - \rho\Gamma(k)\Gamma(\rho)/\Gamma(k+\rho)$$

$$= \rho\Gamma(k)\Gamma(\rho)\ \{(\rho+1)/(k+\rho) - 1\}/\Gamma(k+\rho)$$

$$= \rho(1-k)\Gamma(k)\Gamma(\rho)/\{(k+\rho)\Gamma(k+\rho)\}$$

$$= \rho(1-k)\beta(k,\rho)/(k+\rho) \qquad (29)$$

Therefore, for the same value of ρ,

$$P^U(k) < Y^U(k) \quad \text{for } k \geq 2 \qquad (30)$$

The inequality (30) implies that the post-Yule distribution has less heavy right tail, or more precisely, has a smaller value in the expected maximal occurrence in the extremal distribution for the same value of ρ.

Naturally, by (29) and the regularity condition,

$$P^U(1) = Y^U(1) = 1$$

Analogously to (29), (see the Appendix for the detail)

$$P^U(k+1) - Y^U(k+1) = -\rho k^2 \beta(k,\rho)/\{(k+\rho)(k+\rho+1)\} \qquad (31)$$

Using (11), (28), (29) and (31), (see the Appendix)

$$p(k) - y(k) = \rho\beta(k,\rho)(\rho+1-k\rho)/\{(k+\rho)(k+\rho+1)\} \qquad (32)$$

From this,

$$\rho(k) < (=,>)\ y(k) \quad \text{for } k > (=,<)\ 1 + 1/\rho \qquad (33)$$

This relation implies that the density of the post-Yule distribution is smaller (larger) than that of the Yule distribution for large (small) value of k for the same value of ρ and hence that the former is less skew than the latter for the same value of ρ.

Let m_Y denote the mean of the Yule distribution and let $L = [1+1/\rho] +1$.

$$m_p - m_Y = \sum_{k=1}^{\infty} k\{p(k) - y(k)\}$$
$$= \sum_{k=1}^{L-1} k\{p(k) - y(k)\} + \sum_{k=L}^{\infty} k\{p(k) - y(k)\}$$

The first term on the right side is finite and the second term is negative and finite, too, by (32) and (33). As m_Y is finite, m_p is also finite.

The comparisons made above were based on the condition that the parameter ρ assumes the same value for the both distributions. From a semantical point of view, however, the parameter ρ of the Yule distribution (denoted by ρ_Y hereafter) does not mean the same thing as the parameter ρ of the post-Yule distribution (denoted by ρ_p hereafter).

In deriving the Yule distribution, $1/\rho_Y$ means what λ means in (1) to derive the post-Yule distribution. Hence, by (9), the following equation should hold in order that ρ_Y and ρ_p have the same interpretation.

$$\rho_Y = \rho_p/\mu \tag{34}$$

Recalling tht $1/\mu$ means the average duration of the total progressive age of a firm in $[0, \infty]$, it is very likely that

$1/\mu > 1$ or $\mu < 1$

Hence, it is very likely by (34) that

$\rho_Y > \rho_p$

As the parameters ρ_Y and ρ_p are smaller (larger), the Yule distribution and the post-Yule distribution are skewer (less skew). Hence the above statements that the Yule distribution is skewer than the post-Yule distribution under the condition that $\rho_Y = \rho_p$ should not be generalized beyond this condition. But the discussion on the finiteness of m_p is not affected by the interpretation of ρ and it is valid no matter what ρ means.

6. Empirical Analysis

The pharmaceutical industry is known as highly R&D intensive and is the highest (5.45% in 1980 in Japan) in the percent of R&D expenditure against the net sales among all the industrial sectors (the second highest one is the electronic and communication equipment sector with 3.94% in the same period). For this reason the pharmaceutical industry is selected for the empirical analysis. The data is taken from Nippon Keizai (1982) which describes the R&D expenditure of 26 pharmaceutical firms in 1981 out of the 38 stockmarket-listed pharmaceutical firms.

For the purpose of comparison, the log-normal distribution, the Yule distribution and the exponential distribution are selected. The former two are selected as the distributions with the structure of proportional growth as discussed in the section 2 and the last one is selected as a skew distribution with the simplest mathematical structure. As the last two distributions and the post-Yule distribution under investigation are all the single parameter distribution, the parameter of each distribution is evaluated from the observed mean value. Formally, the following equations are used to evaluate the parameters respectively.

$m_r = \rho_Y(\rho_Y - 1)$ for the Yule distribution

$m_r = 1/\mu_e$ for the exponential distribution

$m_r = m_p$ for the post-Yule distribution

where m_r denotes the observed mean value, ρ_Y and μ_e denote the parameters of the respective distribution and m_p is defined by (24).

This evaluation method by the mean has a merits of the unbiassness as well as the computational simplicity. On the other hand the log-normal distribution has two parameters and therefore needs two equations for the evaluation; one on the mean as above and the other on the variance (the equation between the theoretical and the observed variances).

The degree of fit between the theoretical and the observed distributions is measured by S^*, the square sum of errors.

$$S^* = \sum_{k=1}^{\infty} \{F(k) - R(k)\}^2$$

where F(k) and R(k) denote the theoretical and the observed cumulative distributions respectively.

Table 1 shows that the post-Yule distribution fits to the observed one the best among the four. The second is the Yule distribution followed by the log-normal distribution as the third which is followed by the exponential distribution as the fourth all with the nearly equal interval (about 0.025) between the adjacent ranks.

7. Appendix

$$p(4) = (\rho+1)\{1-p(1)-p(2)-P(3)\} / (\rho+5)$$
$$= (\rho+1)[1/(\rho+2)-(\rho+1)/\{(\rho+2)(\rho+3)\} - 2(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)\}]/(\rho+5)$$
$$= (\rho+1)[\{(\rho+3)(\rho+4)\} - \{(\rho+1)(\rho+4)\} -\{2(\rho+1)\}]/\{(\rho+2)(\rho+3)(\rho+4)(\rho+5)\}$$
$$= 6(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)(\rho+5)\} \qquad (21)$$

$$p(5) = (\rho+1)\{1-p(1)-p(2)-p(3)-p(4)\} / (\rho+6)$$
$$= (\rho+1)[1/(\rho+2)-(\rho+1)/\{(\rho+2)(\rho+3)\} - 2(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)\}$$
$$-6(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)(\rho+5)\}]/(\rho+6) = 24(\rho+1)/\{(\rho+2)(\rho+3)(\rho+4)(\rho+5)(\rho+6)\} \qquad (22)$$

$$P^U(k+1) - Y^U(k+1)$$
$$= \rho(\rho+1)\Gamma(k+1)\Gamma(\rho)/\{(k+\rho+1)\Gamma(k+\rho+1)\} - \rho\Gamma(k+1)\Gamma(\rho)/\Gamma(k+\rho+1)$$
$$= \rho\Gamma(k+1)\Gamma(\rho)\{(\rho+1)/(k+\rho+1) - 1\}/\Gamma(k+\rho+1)$$
$$= -\rho k\Gamma(k+1)\Gamma(\rho)/\{(k+\rho+1)\Gamma(k+\rho+1)\}$$
$$= -\rho k^2\Gamma(k)\Gamma(\rho)/\{(k+\rho+1)(k+\rho)\Gamma(k+\rho)\}$$
$$= -\rho k^2\beta(k,\rho)/\{(k+\rho)(k+\rho+1) \qquad (31)$$

$$p(k) - y(k) = \{P^U(k) - P^U(k+1)\} - \{Y^U(k) - Y^U(k+1)\} = \{P^U(k) - Y^U(k)\} - \{P^U(k+1)-Y^U(k+1)\}$$
$$= \{\rho(1-k)\beta(k,\rho)/(k+\rho)\} - \{-\rho k^2\beta(k,\rho)/\{(k+\rho)(k+\rho+1)\}\}$$
$$= \rho\beta(k,\rho)\{(1-k) + k^2/(k+\rho+1)\}/(k+\rho)$$
$$= \rho\beta(k,\rho)(\rho+1-k\rho)/\{(k+\rho)(k+\rho+1)\} \qquad (32)$$

8. Acknowledgement

The author thanks two former students both at the University of Tsukuba: Ms. K. Makino, now with Chuo University, for her theoretical suggestions on the earlier version of this paper and Mr. K. Miyashita, now with Mitsubishi Chemical Co., for his computation in Table 1.

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Table 1. Degree of fit for pharmaceutical firms

R&D expenditure (million yen)			Distributions (square sum of errors)			
max	mean	min	post-Yule	Yule	log-normal	exponential
27,000	4,360	316	0.046	0.069	0.097	0.119