No. 186 (83-9)

ON TEMPORAL AGGREGATION OF LENEAR DYNAMIC SYSTEMS

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June 1983

I. Introduction

In many economic models, functions with distributed lag terms appear very commonly. Distributed lag models necessarily involve the concept of 'period'. It is natural to expect that there must be a relationship between the length of period and the specification of the distributed lag models. Take a macroeconomic model for example. The lag structure of the consumption functionwill be different between an annual model and a quarterly model. As the result values of economic parameters such as impact multiplier and delayed multipliers may differ generating a discrepancy between the value of the total change inincome in four quarters in the quarterly model and the corresponding change in the annual model. We will examine the conditions under which the four period aggregates of a quarterly model are consistent with one period values in the corresponding annual model. Most of the studies on temporal aggregatopn are concerned with the properties of the estimates of the parameters of structure. Moriguchi(1) discussed the problem without clear conclusion. In section 2, we will discuss an example without stochastic term, and the result willbe generalized in section 3.

2. an example

We will consider the simple model of the interaction between the multiplier and the accelerator of Samuelson(2).

$$y_{mt-i} = c_{mt-i} + i_{mt-i} + g_{mt-i} \quad (1)$$

$$c_{mt-i} = a y_{mt-i-1} \quad (2)$$

$$i_{mt-i} = b(c_{mt-i} - c_{mt-1-i}) \quad (3)$$

$$g_{mt-i} = g_0 \quad (4)$$

$t=1,\ldots,T$

$i=0,\ldots,m-1$

m is fixed

where suffix m stands for the number of the shorter periods in the longer period which is represented by suffix t. Let capital letters stand for the aggregated values, i.e.,

$$Y_t = (\sum_{j=0}^{m-1} L^j) y_{mt}, \qquad C_t = (\sum_{j=0}^{m-1} L^j) c_{mt}$$

$$I_t = (\sum_{j=0}^{m-1} L^j) i_{mt}, \qquad G_t = (\sum_{j=0}^{m-1} L^j) g_{mt} = mg_0 \quad (5)$$

where L is the lag operator, i.e., $Lx=x_{t-1}$, and $L^2x = x_{t-2}$. For convenience we limit our case to the relationship between an quarterly and annual models setting m=4. If we have only the annual data while the lag structure is best described by a quarterly model, we try to mimic the quaterly model with annual variables. Our queation here is under what condition the values of the aggregated model, which are represented by capital letters with asterisk (*), be the same as 4 period aggregates of the quarterly model.

Suppose we write the annual model as follows.

$$Y_t^* = C_t^* + I_t^* + G_t^* \quad (6)$$

$$C_t^* = A_0 + A_1 Y_t^* + A_2 Y_{t-1}^* \quad (7)$$

$$I^*_t = B_0 + B_1 C^*_t = B_2 C^*_{t-1} \tag{8}$$

$$G^*_t = mg_0 \tag{9}$$

This is just one of many possible specification for an annual model.

the system of equations (1) -(4) yields second order difference equations such as:

$$y_{mt-i} = (a + ab)y_{mt-1-i} + aby_{mt-2-i} + g_0 \tag{10}$$

where, for simplicity g_{mt} is assumed to be exogenous and constant at g_0. Let r_1 and r_2 be the characteristic roots of the above difference equation. Then we have:

$$y_{mt-i} = k_1 r_1^{mt-i} + k_2 r_2^{mt-i} + k_0 \tag{11}$$

where k_1 and k_2 depends on two intial values, y_1 and y_2, say. From equations (2) and (5), we have

$$\begin{aligned} C_t &= (\sum_{j=0}^{m-1} L^j) c_{mt} = a(\sum_{j=0}^{m-1} L^j) y_{m(t-1)} \\ &= a(k_1 \sum_{j=0}^{m-1}(r_1)^{m(t-1)-j} + k_2 \sum_{j=0}^{m-1}(r_2)^{m(t-1)-j}) + amk_0 && (12) \\ &= aY_{t-1} && (13) \end{aligned}$$

Hence substitutin of (11) int (7) results in

$$\begin{aligned} C^*_t &= A_0 + A_1(Y_t) + A_2(Y_{t-1}) \\ &= A_0 + A_1(k_1 \sum_{j=0}^{m-1}(r_1)^{mt-j} + k_2 \sum_{j=0}^{m-1}(r_2)^{mt-j} + mk_0 \\ &\quad + A_2(k_1 \sum_{j=0}^{m-1}(r_1)^{m(t-1)-j} + k_2 \sum_{j=0}^{m-1}(r_2)^{m(t-i)-j} + mk_0) \\ &= A_0 + k_1(A_1 + A_2 r_1^{-m}) \sum_{j=0}^{m-1}(r_1)^{mt-j} + k_2(A_1 + A_2 r_2^{-m}) \sum_{j=0}^{m-1}(r_2)^{mt-j} && (14) \end{aligned}$$

For consistent aggregation, i.e., $C^*_t = C_t$ for all t, we must have

$$A_0 + (A_1 + A_2)mk_0 = amk_0$$

$$k_1(A_1 + (r_1)^{-m}a_2) = a\ k_1 \qquad (15)$$

$$k_2(A_1 + (r_2)^{-m}A_2) = ak_2$$

In general we can determine A_0, A_1 and A_2 uniquely from the equation system (15). Similarly, by comparing equations (3) (5) and (8) we can have $I^*_t = I_t$ for all t by properly determining B's in equation (8).

Thus the system of equations(6) to (9) can be set so that the variables follow exactly the same time paths as the temporally aggregated values of the variables of the system given by equations (1) to (4).

A simple extension of the above model is to change the exogenous constant g_0 to include a part which grow over time at a known constant rate s, that is,

$$g_{mt-i} = (1+ s)^{mt-i}g_0 + g_{00} \qquad (16)$$

Since the rate of growth is known, we know that

$$G_t = \sum_{j=0}^{m-1} g_{mt-j} = (1= (1= s) + (1=s)^2 + \ldots +(1=s)^m)g_0(1+s)^{m(t-1)} + mg_{00} \qquad (17)$$

Then by adding $g_0(1=s)^{mt}$ as a new variable to the right hand side of equation (7) and (8) and readjusting the coefficients, we can have $C^*_t = C_t$ and $I^*_t = I_t$ for all t.

3. General Linear Models

In this section we generalize the result obtained for the eample we considered in the preceeding section. Suppose the structural equations of disaggregated variables are given by

$$y_{mt-i} = \theta_0 + \theta_1 y_{mt-i} + \theta_2 y_{mt-i-1} \tag{18}$$

where y is ℓ x1, Θ_0 is $\ell \times 1$, Θ_1 and Θ_2 are $\ell \times \ell$ matrices. Note that in this model ther is no exogenous variable other than the constant term. The difference equation derived from this system is at most of ℓ-th order. If it is indeed of ℓ-th order , then there are ℓ characteristic roots. For the aggregated model to produce the identical time paths for its variables as the temporally aggregated values of the disaggregated model, each equation must have ℓ variables so that there are ℓ free coefficients to be determined and one constant term.

Theorem 1. The necessary and sufficient condition for perfect temporal aggregation of the model given by equation (18) is that each equation in aggregated model has ℓ explanatory variables and the constant term.

It is easy to extend this theorem to the case with exogenous variables whose time paths are known.

Theorem 2. Suppose the disaggregated model is given by

$$y_{mt-i} = \Psi_0 + \Psi_1 y_{mt-i} + \Psi_2 y_{mt-i-1} + \Psi_3 x_{mt-i} \tag{19}$$

where y_{mt-i} is $\ell \times 1$, Ψ_0 is $\ell \times 1$, Ψ_1 and Ψ_2 are $\ell \times \ell$, and Ψ_3 is $\ell \times k$ matrices and x_{mt-i} is $k \times 1$ of exogenous variables, and the system reduces to ℓ-th order difference equation. The necessary and sufficient condition for the aggregated model to generate the same time paths for the endogenous variables is that each equation has $\ell + k$ endogenous variables (lagged or unlagged) and allthe exogenous variables and a constant term as explanatory variables.

Let G_Δ stand for the number of endogenous variables (lagged or unlagged) and K^* for the number of included exogenous variables in a equation of the aggregated model. Then our condition is written as

$$\ell + k = G_\Delta + K^* \tag{20}$$

now if $\ell + k < G_\Delta + K^*$, there will be infinite number of combination of coefficients to generate the required time path. In other words there are more unknowns than equations. Thus we will have a case of perfect multicolinearity when we run a regression. In the opposite case where $\ell + k > G_\Delta + K^*$ generally perfect aggregation is not possible.

At first glance, condition(20) looks somewhat similar to the order condition for identification. Ther, however, is no direct relationship between those two since G_Δ in the above condition is the number of lagged or unlagged endogenous variables in the equation. As a model becomes larger, the left hand side of equation (20) becomes larger. Usually the number of explanatory variables in a single equation is limited, eventually we will have a situation whre $\ell + k > G_{\Delta} + K^*$, that is perfect aggregation is not possible.

Equation (14) can be considered as the solution for the difference equation derived from the aggregated model. It is obvious that the characteristic roots must be r_1 and r_2. Hence a necessary condition for perfect aggregation is that the difference equations for aggregated nad disaggregated models must have the identical characteristic roots.

REFERENCES

1. Moriguchi, C. "Aggregation over time in macroeconomic relations", International Economic review, Vol.11,No.3 1970.

2. Samuelson, P. "Interaction of multiplier and acceleration principle", Review of Economics and Statistics, 1939.