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DOUBLE RESERVATION VALUES PROPERTY OF STOPPING PROBLEM WITH UNCERTAIN RECALL

by

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Abstract

A model of stopping problem is developed in which the offers once made and passed up may be available in the future with a known probability, dependent on age. It is verified that in the model a double reservation values property appears which implies that there may exist two different critical numbers, associated with the value of an offer arriving at present, causing a choice among the next three possible alternatives: 1. Accept the present offer and stop the search, 2. Pass up it and continue the search, and 3. Reject it with an acceptance of the past best one and stop the search. Structures of the property will be revealed, and finally future studies are presented which would be inevitable in order to make this model more realistic.

1. Introduction

For the past three decades quite different models of the optimal stopping problem have been posed and examined. Although these models have a structure which can represent very fitly some aspects of a decision maker's behavior, many of studies on the subject have been devoted to discussions rather from a probability theoretic viewpoint, and few attempts have been made to apply the results derived from the studies to inquire into underlying natures of eonomic, managerial phenomena which we encounter in the real-world. Eealier examples in applications to such phenomena include a house selling problem by Simon [7], a commodity purchasing problem on a fluctuating market by Morris [5], and so on. Presumably it is to a field of job search problem in labour economics (Lippman and MacCall [4]) that these stopping models have been most extensively applied with the intension of getting insights into economic laws governing its dynamic behaviour. In almost all of them it has been assumed that an offer once inspected and passed up is either forever unavailable or available with certainty later on. The former case is referred to as a stopping problem without recall, the later as one with recall. In studying actual economic or managerial problems, it will become more realistic to put the case that an offer once inspected and passed up will be available in the future with a probability. In the present paper we shall refer to such type of stopping problem as a stopping problem with uncertain recall. It is of course that this generalization includes as special cases the above two types.

Two interesting papers concerning a stopping problem with uncertain recall have been published in succession by Landsberger, et al. [3] and Karni et al. [2]. In the former they assumed that if the best offer having been inspected and passed up till now is not accepted, it is available at the next time with a given stationary probability q and all other past offers are lost. Although they claimed in their paper that the assumption is equivalent to saying that all the passed up offers are available at the next time with the same probability q, the assertion is not correct because if the past best offer is withdrawn, then the second past best one may become a new best one. The new best offer is completely neglected or out of consideration under their assumption. However their suggestions are quite interesting and realistic that the availability in the future of offers passed up should be assumed to be dependent on their ranking and age. As being pointed out by them, it is true in fact that the mathematical treatment of such model is remarkably complicated and moerover that many interesting results proven in their paper are sacrificed because of its mathematical intractability. The present paper is a challenge to such difficulties.

Now a decision to be made in the stopping model with uncertain recall is a choice among the next three alternatives;

A1 : to accept the present offer and stop the search process,

A2 : to pass up the present offer and wait for the next one,

A3 : to reject the present offer with acceptance of the past best one and stop the search process.

This means that, for a value w of the present offer, there may exist two critical numbers, called reservation values, y and y' with $y < y'$ such that $y' < w$ leads to decision A1, $y < w \leq y'$ to A2, and $w' \leq y$ to A3. We shall refer to a decision rule being characterized by such two critical numbers as double reservation values property. The reason why the choice of the third alternative, or A3, may become optimal when a value of present offer is relatively low can be explained as follows:

> If the present offer with the low value is passed up, then the decision maker might encounter a misfortune of being forced to eventually accept it due to the possibility of the worst situation that all the past offers with higher values than it become unavailable in the future and further more any offer with higher value than it is not made at all in the future.

Karni, et al. [2], who called the third alternative a backward solicitation, showed that the optimal choice of the third alternative may happen in their stopping model with uncertain recall. The objective of the present paper is to verify that this double reservation values property holds generally also for our model defined in the next section and to identify some characteristics of the property, associated with the relationship between the values of the past offers remaining available and the three alternatives above.

2. Model

Consider the following stopping problem with a planning horizon of finite periods, equally spaced time intervals. From now on, both ends of each period will be referred to as points in time or simply as times, denoted by t, and let the times are numbered backward from the final time of the horizon, $t = 0$. Assume that if c dollars, called a search cost, are invested over each period, then an offer can be made at the end of the period where values of successive offers, w, w', ... are independent and identically distributed positive random variables, each of which has a known common distribution $F(w)$ with a finite expectation E (> 0). Throughout assume that $0 < F(x) < 1$ for any positive x and that $\beta E > c$ where β denotes a discount factor ($0 < \beta < 1$). The latter is a natural assumption which means that if the decision is put off for another period, the expectation of present value of the value from an offer made one period after, if accepted, is greater than the search cost for the offer. Let $p(j)$, $j = 0,1,...$, represent the probability that an offer which was inspected and passed up j periods ago is withdrawn at the next time, provided that it has been available up to the present time. The j may be regarded as the age of the offer. Below assume that $p(j)$ is independent of time t, or stationary, and $p(j) = 1$ for all $j \geq n$ with a fixed nonnegative integer n, in other words, any offer inspected and passed up is withdrawn with certainty after n+1 periods. From now on for simplicity let $q(j) = 1-p(j)$, which represents a probability that an offer inspected and passed up j periods ago will remain still available at the next time. Here we shall assume that it can be always known free and instantly whether or not an past offer is available at present. A decision maker must select one and only one offer to accept among offers made sequentially up to the final time of the horizon, or time 0. If it has come to the end of the horizon without accepting any offer, then an offer with the largest value among the past offers remaining available at the time and the present offer at time 0 must be accepted however small the largest value may be. This can be said to be quite risky situation for the decision maker. The objective of the problem is to maximize the total expected gain, that is, the expected values from an offer accepted minus the total search cost incurred up to acceptance of the offer.

Now the process is on time t. Then by g(j) we shall denote a value of an offer which was made j periods ago, that is, at time t+j, and let

(2.1) $v_t(g(n),g(n-1),\ldots,g(0))$ = the maximum expected gain starting from time t, provided that all of the previous offers g(n), g(n-1), ..., g(1) are available and an offer g(0) is made at time t, and

(2.2) $V_t(g(n),g(n-1),\ldots,g(1)) = E\{v_t(g(n),g(n-1),\ldots,g(0))\}$,

which is an expectation, estimated at time t+1, with respect to a value g(0). For the later discussions' sake we shall define the following three vectors, **K**, **G**, and **G'** and their related sets K, G, and G':

$$(2.3)\begin{cases}\mathbf{K} = (g(n),g(n-1),\ldots,g(1),g(0)) & K = \{n,n-1,\ldots,0\} \\ \mathbf{G} = (g(n),g(n-1),\ldots,g(1)) & G = \{n,n-1,\ldots,1\} \\ \mathbf{G'} = (g(n-1),g(n-2),\ldots,g(0)) & G' = \{n-1,n-2,\ldots,0\}\end{cases}$$

Below, by using the vectors, we shall sometimes denote (2.1) and (2.2) by, respectively, $v_t(\mathbf{G},g(0))$ (or $v_t(\mathbf{K})$) and $V_t(\mathbf{G})$. When the process proceeds to the next time without withdrawal of any offer except offer g(n), what corresponds to (2.1) and (2.2) can be written as $v_{t-1}(\mathbf{G'},w)$ and $V_{t-1}(\mathbf{G'})$ where w is a value of an offer made at time t-1 and where the latter is an expectation of the former as to w. The following will be clear from the definition of our model.

(2.4) $v_0(\mathbf{G},g(0)) = \max\{g(n),g(n-1),\ldots,g(0)\}$

3. Some Simple cases

Before proceeding to general discussions, we shall here examine some simple cases. The strict verification for what are stated below will be provided in section 4. First let us take the case of $n = 1$, that is, $0 < p(0) < 1$ and $p(j) = 1$ for $j \geq 1$. Suppose the process is now on time 1 and an offer $g(0)$ has been made just now. Then if the offer $g(1)$ made at time 2 is available at time 1, we have

$$(3.1) \quad v_1(g(1),g(0)) = \max\{g,R(g(0))\} \quad \text{with } g = \max\{g(1),g(0)\}$$

where $R(g(0))$ denotes the expected gain from going to time 0 instead of accepting the offer g. Here notice that the offer $g(1)$ is withdrawn at time 0 due to $p(1) = 1$, hence the $R(.)$ becomes independent of $g(1)$. At time 0, if the offer $g(0)$ is available, a selection must be made among the offer and the offer w made at time 0, otherwise the offer w must be accepted like it or not. Thus the $R(g(0))$ can be expressed as

$$(3.2) \quad \begin{aligned} R(g(0)) &= \beta p(0)E + \beta q(0)E\{\max\{g(0,w\}\} - c \\ &= \beta p(0)E + \beta q(0)S(g(0)) - c \end{aligned}$$

where $E\{.\}$ denotes an expectation as to w and $S(x) = E\{\max\{x,w\}\} = x + T(x)$ with $T(x) = E\{[w-x]^+\}$ where, in general, if a real number x is positive, then $[x]^+ = x$, otherwise $[x]^+ = 0$. It is quite easy to show that $T(x)$ and $S(x)$ are strictly convex, $T(x)$ is strictly decreasing, $S(x)$ is strictly increasing, $T(x) \to 0$ as $x \to \infty$, $S(x) \to \infty$ as $x \to \infty$, and $T(0) = S(0) = E$. Then define

$$(3.3) \quad Q(g(1),g(0)) = R(g(0)) - g$$
$$= \begin{cases} \beta p(0)E + \beta q(0)S(g(0)) - g(1) - c & \text{if } g(1) \geq g(0) \\ \beta p(0)E - (1-\beta q(0))g(0) + \beta q(0)T(g(0)) - c & \text{if } g(0) \geq g(1) \end{cases}$$

If $Q(g(1),g(0)) \leq 0$, optimal is to accept the offer g, otherwise to go to the final search at time 0. Therefore the acceptance regions for the past offer g(1) and the present offer g(0) can be given by the sets, respectively,

$$(3.4)\begin{cases} A(1) = \{(g(1),g(0)) \mid Q(g(1),g(0)) \leq 0,\ g(1) \geq g(0)\} \\ A(0) = \{(g(1),g(0)) \mid Q(g(1),g(0)) \leq 0,\ g(0) \geq g(1)\} \end{cases}$$

The A(1) and A(0) can be given by regions, respectively, a'abb' and c'cbb' in Figure 1 where the curved line $\widehat{ab}$ is a graph of points satisfying $Q(g(1),g(0)) = 0$ on $g(1) \geq g(0)$ and where h and h(1) are solutions to $Q(x,x) = 0$ and $Q(x,0) = 0$, respectively.

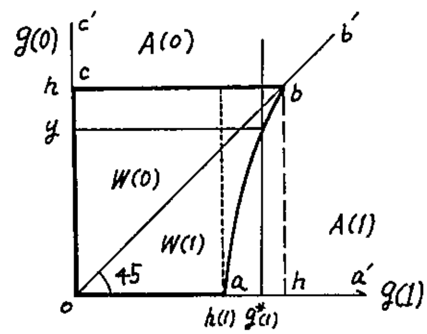

Figure 1

In Figure 1 the regions W(1), oabo (W(0), ocbo) are associated with waiting for the next offer if $g(1) \geq g(0)$ (if $g(0) > g(1)$). What is most noticable here is that the Figure has the curved line $\widehat{ab}$ which is a <u>strictly</u> convex and <u>strictly</u> increasing function of g(0), that is,

$$(3.6)\quad g(1) = \beta p(0)E + \beta q(0)S(g(0)) - c \qquad 0 \leq g(0) \leq h$$

It is by the very point that a double reservation values property is brought about to the optimal decision strategy. For a given $g^*(1)$ with $h(1) < g^*(1) < h$ (Figure 1), if we shall let y be a solution to $g^*(1) = \beta p(1)E + \beta q(1)S(y) - c$, then the present offer $g(0) \geq h$ leads to decision A1, $h > g(0) > y$ to A2, and $y \geq g(0)$ to A3. That is, it follows that we have two critical values, h and y, for the present offer g(0).

Next let us consider the case of $n = 2$, that is, $0 < p(0) \leq p(1) < 1$ and $p(j) = 1$ for $j \geq 2$. In the case we have

$$v_1(g(2),g(1),g(0)) = \max\{g, R(g(1),g(0))\} \text{ with } g = \max\{g(2),g(1),g(0)\} \tag{3.7}$$

$$R(g(1),g(0)) = \beta p(1)p(0)E + \beta p(1)q(0)E\{\max\{g(0),w\}\} + \beta q(1)p(0)E\{\max\{g(1),w\}\} + \beta q(1)q(0)E\{\max\{g,w\}\} - c \tag{3.8}$$

Now suppose $g(2) = 0$, or the offer $g(2)$ has been already withdrawn. Then applying the same discussions as in the case of $n = 1$ produces Figure 2 with two curved lines $\widehat{ab}$ and $\widehat{cb}$ which are both strictly increasing, and are concave and convex, respectively.

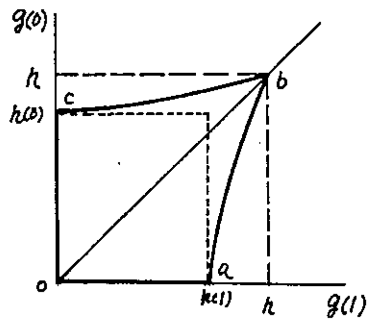

Figure 2

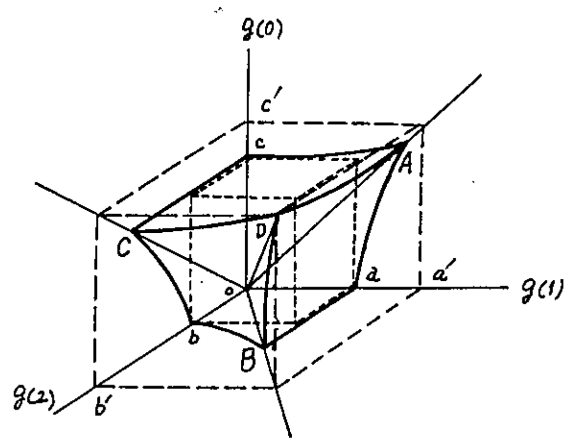

Figure 3

If $g(2) \neq 0$, we can get Figure 3 where if the vector $(g(2),g(1),g(0))$ is in the hexahedron with three planes oaAc, obCc, obBa and three planes curved inside cCDA, aADB, bBDC, optimal is to go to the final search, otherwise to accept the offer with the largest of $g(2)$, $g(1)$, $g(0)$. Here note that two lines connecting points c, C and points a, B are <u>straight lines</u> parallel to $g(2)$-axis and that Figure 1 and Figure 2 correspond to, respectively, $(g(2),g(0))$-plane and $(g(2),g(1))$-plane in a three-dimensional space of Figure 3. When $n = 3$, that is, $0 < p(0) \leq p(1) \leq p(2) \leq p(j) = 1$ for $j \geq 3$, if $g(3) = 0$, then we have the Figure 4 where line cC and line aB are <u>curved lines</u>.

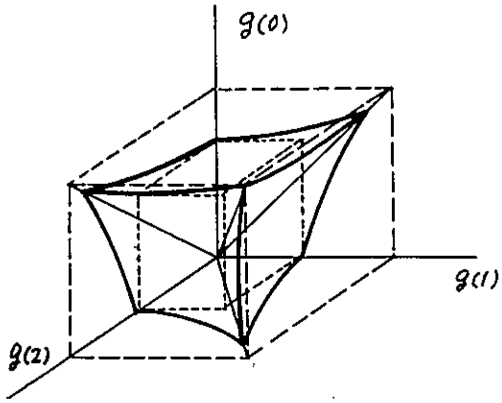

Figure 4

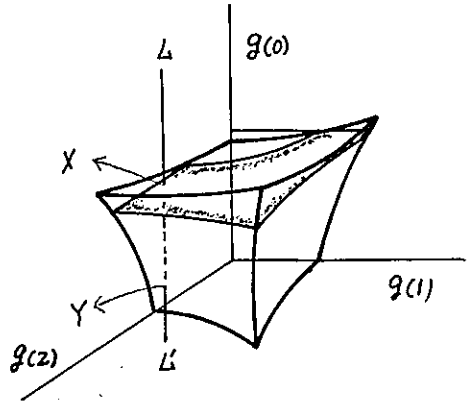

Figure 5

What is outstanding feature of Figure 3 and Figure 4 is the following two;

F1 : The small box with edges oa, ob, oc (rectangular) is contained in the hexahedron.

F2 : The large box with edges oa', ob', oc' (cube) contains the hexahedron.

Figure 5 shows that the very features will bring about the double reservation values property to the optimal decision strategy for our search model. The reader will notice that, in the figure, two points X and Y at which the line LL' for given past offers g(2) and g(1) intersects with the hexahedron provide two critical values for the present offer g(0), which produces the optimal selection of alternative A3. Furthermore it should be noticed that the horizontal section in Figure 5, shaded portion, provides a region of points (g(2),g(1)) for which it is optimal to wait for the next offer by continuing the search process, given a present offer g(0). In the next section, we shall show that these features will hold also for a more general case, togather with that the hexahedron increases in time t, or the number of residual planning horizon (Figure 6)

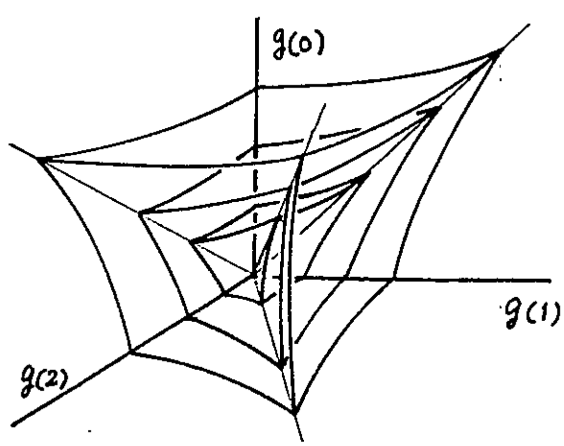

Figure 6

4. General case

Here we shall provide the three theorems which characterize the structure of the double reservation values property. Suppose that the process starts from time $t \geq 1$ and that the previous n offers g(n), g(n-1), ..., g(1) are all available and an offer g(0) has been made at the time t. In the decision process it must be decided every time which of the following two alternatives to take:

1. To accept the best one among present available offers g(n), g(n-1), ..., g(0) and stop the search. Then the immediate gain obtained can be expressed as max{g,g(0)} with g = max **G**, which means the maximum element of the vector **G**. This alternative may be regarded as one into which the two alternatives stated in section 1, A1 and A3, are put togather without loss in generality.
2. To wait for the next offer by continuing the search with search cost c.

Let R_{t-1}(**G**') represent the maximum expected gain starting from time t-1 (, or the next time), following the optimal strategy over times t-1 to 0. Here notice that at time t-1 the offer g(n) is withdrawn with certainty due to the assumption of p(n) = 1 and that some of other offers may be also withdrawn with the given probabilities $p(j) < 1$, $0 \leq j < n$. Hence the R(.) is independent of g(n). Now that some offers among offers **G**' are withdrawn may be considered to be equivalent to saying that the values of the offers withdrawn change into the value of zero. Then let G'^{*} be the vector defined by some elements of **G**' being changed into zero by withdrawal. For instance, if **G**' = (g(2),g(1),g(0)), then the possible kinds of such vectors are in all as follows: G'^{*} = (g(2),g(1),g(0)), (g(2),g(1),0), (g(2),0,g(0)), (0,g(1),g(0)), (g(2),0,0), (0,g(1),0), (0,0,g(0)), and (0,0,0). Here for convenience of later discussions, quite similarly we shall also define the vector G^{*} for the vector G where some elements of **G** are replaced by value zero. Now associated with a vector G'^{*}, by $P(G'^{*})$ we shall define the probability that the vector **G**' changes into vector G'^{*} by withdrawals of some of offers **G**' at the next time. The prorbability can be expressed as $P(\mathbf{G}'^{*}) = r(n-1)r(n-2)\ldots r(0)$, the product of probabilities $r(j)$ where if an offer $g(j)$ is withdrawn at the next time, let $r(j) = p(j)$, otherwise, $r(j) = 1-p(j)$. Here notice that the $P(\mathbf{G}'^{*})$ dose not depend on values themselves of elements in vector $\mathbf{G}'^{*}$. It only depends on which elements of $\mathbf{G}'$ change into value zero. Then the $R_{t-1}(\mathbf{G}')$ can be expressed as

$$(4.2) \qquad R_{t-1}(\mathbf{G}') = \beta \sum_{\mathbf{G}'^{*}} P(\mathbf{G}'^{*}) V_{t-1}(\mathbf{G}'^{*}) - c$$

where the sum is over all possible $\mathbf{G}'^{*}$. Thus we have for $t \geq 1$

$$(4.3) \qquad v_t(\mathbf{G}, g(0)) \; (= v_t(\mathbf{K})) = \max\{\max\{g, g(0)\}, R_{t-1}(\mathbf{G}')\}$$

Now for any given $k \in K$ let $C(k)$ be a set of vector $\mathbf{K}$ with $g(k) = \max \mathbf{K}$. When $n = 1$, $C(1)$ is a region on $(g(1),g(0))$-plane which is enclosed by a straight line $g(1) = g(0)$ and $g(1)$-axis, and $C(0)$ by the same straight line and $g(0)$-axis (Figure 7). When $n = 2$, three sets $C(0)$, $C(1)$, $C(2)$ are possible on $(g(2),g(1),g(0))$-hyperplane , each of which is a convex cone with a peak at origin, enclosed by four surfices (Figure 8).

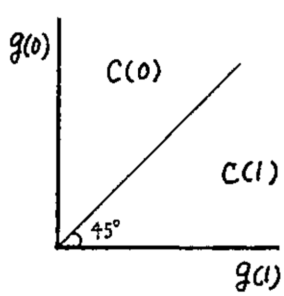

Figure 7

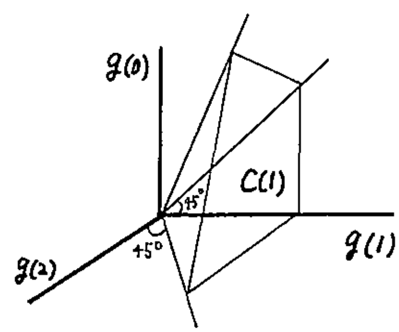

Figure 8

For a vector $\mathbf{K}$ define

$$(4.4) \qquad Q_{t-1}(\mathbf{K}|k) = R_{t-1}(\mathbf{G}') - g(k) \qquad k \in K$$

where note that the $R(.)$ is independent of $g(n)$ but $Q(.)$ is dependent on $g(n)$. For a given vector <u>K in C(k)</u>, if $Q_{t-1}(\mathbf{K}|k) \leq 0$, it is optimal to accept the offer $g(k)$, and if $Q_{t-1}(\mathbf{K}|k) \geq 0$, it is optimal to wait for the next offer by continuing the search process. When $Q_{t-1}(\mathbf{K}|k) = 0$, the both alternatives are indifferent in a sense that the expected gain from each of them is identical. Then for all $k \in K$ define the following sets;

(4.5) $A_t(k) = \{\mathbf{K} | Q_{t-1}(\mathbf{K}|k) \leq 0, \mathbf{K} \text{ in } C(k)\}$

(4.6) $A_t = \bigcup_{k \in K} A_t(k)$

We shall refer to the set A_t ($A_t(k)$) as an acceptance region (an acceptance region on $C(k)$) and to the union of the compliment of A_t and its boundary (the union of the compliment on $C(k)$ of $A_t(k)$ and its boundary on $C(k)$) as an wait region W_t (an wait region $W_t(k)$ on $C(k)$), respectively.

LEMMA 1 For all $t \geq 1$ and for all G, we have

(a) $V_t(\mathbf{G}) > c/\beta$

(b) $V_t(\mathbf{G}) \geq g(k)$ for all $k \in G$.

(c) $V_t(\mathbf{G})$ is increasing as well as convex with respect to $\mathbf{G}$. (Throughout the remainder a function $f(x)$ of vector x is said to be increasing (decreasing) if it is increasing (decreasing) in all elements of the vector.

PROOF (a) and (b) are easily shown from $v_t(\mathbf{G}, g(0)) \geq g(k)$ for all $k \in K$ and the assumption of $\beta E > c$, noticing that $V_t(\mathbf{G})$ is an expectation as to $g(0)$ of $v_t(\mathbf{G}, g(0))$ and $E = E\{g(0)\}$. The proof for (c) is by induction. Since $v_0(\mathbf{G}, g(0)) = \max\{g, g(0)\}$, we have $V_0(\mathbf{G}) = S(g)$. By using an increasingness of g in $\mathbf{G}$, an increasingness and convexity of $S(g)$, and the general formula $\max\{A+A', B+B'\} \leq \max\{A, B\} + \max\{A', B'\}$ for any real numbers A, A', B, and B', it can be immediately proved that the $V_0(\mathbf{G})$ is increasing and convex with respect to $\mathbf{G}$. Now for a fixed $t \geq 2$ suppose that $V_{t-1}(\mathbf{G}')$ is increasing and convex with respect to $\mathbf{G}'$. Then clearly $V_{t-1}(\mathbf{G}'^{*})$ is also increasing and convex with respect to $\mathbf{G}'^{*}$. By noticing this and that $\mathbf{G}'^{*}$ is independent of the values themselves of elements of $\mathbf{G}'^{*}$, it can be easily seen from (4.2) that $R_{t-1}(\mathbf{G}')$ is also increasing as well as convex with respect to $\mathbf{G}'$. Moreover the $R_{t-1}(\mathbf{G}')$ may be regarded as also increasing and convex with respect to $\mathbf{G}'$ because it is independent of $g(n)$. In addition, $\max\{g, g(0)\}$ is also increasing as well as convex with respect to $\mathbf{G}$. Accordingly it follows from (4.3) that the $v_t(\mathbf{G}, g(0))$, hence its expectation $V_t(\mathbf{G})$ becomes increasing and convex with respect to $\mathbf{G}$. Q.E.D.

The corollary below is clear from (4.2), (4.4), and the above lemma.

COROLLARY 1 For every $k \in K$, $Q_{t-1}(\mathbf{K}|k)$ is increasing in $g(j)$ with $\underline{j \neq k}$ and is convex with respect to $\mathbf{K}$.

The next theorem can be immediately derived from Corollary 1.

THEOREM 1 For any $k \in K$, an acceptance region $A_t(k)$ is a convex set for all t.

This theorem is the first one of the three main results in the present paper, which brings about the double reservation values property to the optimal decision rule. Next we shall show that the double reservation values property has in general the two features F1 and F2. Now since the features are closely related to the solutions to $2^{(n+1)} - 1$ equations $Q_{t-1}(\mathbf{K}|k) = 0$ where each of n+1 elements of $\mathbf{K}$ takes on either variable h or value 0 but $\mathbf{K} \neq 0$, below we shall show first the existence and uniqueness of the solutions, then verify the features.

LEMMA 2 If a function $m(x)$ is such that $m(x)-x$ is decreasing, then $rm(x)-x$ for $r \in [0,1)$ is <u>strictly</u> decreasing and approaches $-\infty$ as $x \to \infty$.

PROOF Obvious from $rm(x)-x = r(m(x)-x) - (1-r)x$. Q.E.D.

COROLLARY 2 The following are true for all $t \geq 1$.

(a) $V_{t-1}(\mathbf{G}') - g(k)$ is decreasing in $g(k)$ for all $k \in K$.

(b) $Q_{t-1}(\mathbf{K}|k)$ is <u>strictly</u> decreasing in $g(k)$ and approaches $-\infty$ as $g(k) \to \infty$ for $k \in K$.

PROOF The proofs are by induction. Suppose the process is now on time 1. Then $V_0(\mathbf{G}') - g(k) = E\{\max\{g(n-1), g(n-2), \ldots, g(0), w\} - g(k)\}$ for all $k \in K$ where $E\{.\}$ represents an expectation as to value w of an offer made at time 0. Thus (a) becomes true for $t = 1$. Now noticing the summation of $P(\mathbf{G}'^*)$ over all possible $\mathbf{G}'^*$ equals 1, we have for any $k \in K$

$$(4.7) \quad v_t(\mathbf{G}, g(0)) - g(k) = \max\{\max\{g, g(0)\} - g(k),\ Q_{t-1}(\mathbf{K}|k)\}$$

$$(4.8) \quad Q_{t-1}(\mathbf{K}|k) = \sum_{\mathbf{G}'^*} P(\mathbf{G}'^*)(\beta V_{t-1}(\mathbf{G}'^*) - g(k)) - c$$

For a given $t \geq 1$ suppose that $V_{t-1}(\mathbf{G}') - g(k)$ is decreasing in $g(k)$ for all $k \in K$. Then $V_{t-1}(\mathbf{G}'^*) - g(k)$ becomes also decreasing in $g(k)$ for all $k \in K$. From this and Lemma 2, it follows that, for all $k \in K$, $\beta V_{t-1}(\mathbf{G}'^*) - g(k)$ is <u>strictly decreasing</u> in $g(k)$ due to the assumption of $0 < \beta < 1$. Hence $Q_{t-1}(\mathbf{K}|k)$ is also <u>strictly decreasing</u> in $g(k)$ as well as approaches $-\infty$ as $g(k) \to \infty$, noticing a positivity of $P(\mathbf{G}'^*)$ from the assumption of $0 < p(.) < 1$. From these and the decreasingness of $\max\{g, g(0)\} - g(k)$ in $g(k)$ for all $k \in K$, it follows that, for any $k \in K$, (4.7), hence its expectation $V_t(\mathbf{G}) - g(k)$ becomes decreasing in $g(k)$. Then the induction completes. Q.E.D.

DEFINITION 1

(a) Let L be a subset of K and $\mathcal{L}$ a class of all possible L's. Then for $i = 1, 2, \ldots, n+1$ define $K(h,i,L)$ and $\mathbf{G}'(h,i,L)$ are, respectively, $\mathbf{K}$ and $\mathbf{G}'$ in which, for the elements $g(.)$ except the last $i-1$ elements, all $g(j)$ with $j \notin L$ are replaced by value 0 and all $g(j)$ with $j \in L$ by a nonnegative value h where let $\mathbf{G}'(h, n+1, L) = \mathbf{G}'$

(b) Define $Q_{t-1}[K(h,i,L)] = R_{t-1}(\mathbf{G}'(h,i,L)) - h$.

(d) Let $V_{t-1} = V_{t-1}(\mathbf{G}')$ with $g(j) = 0$ for all $j \in G'$. Then clearly $V_{t-1} = V_{t-1}(\mathbf{G}'(0,i,L))$ for any L.

In order to facilitate the understanding of the meaning of the rather intricate definitions above, we shall show one example. Now let $\mathbf{K} = (2,5,7,3)$, hence $\mathbf{G}' = (5,7,3)$ and $K = \{3,2,1,0\}$. If $L = \{3,2,0\}$, then we have $\mathbf{K}(h,1,L) = (h,h,0,h)$, $\mathbf{K}(h,2,L) = (h,h,0,3)$, $\mathbf{K}(h,3,L) = (h,h,7,3)$, $\mathbf{K}(h,4,L) = (h,5,7,3)$, $\mathbf{G}'(h,1,L) = (h,0,h)$, $\mathbf{G}'(h,2,L) = (h,0,3)$, $\mathbf{G}'(h,3,L) = (h,7,3)$, $\mathbf{G}'(h,4,L) = (5,7,3)$, $Q_{t-1}[\mathbf{K}(h,1,L)] = R_{t-1}(h,0,h) - h$, $Q_{t-1}[\mathbf{K}(h,2,L)] = R_{t-1}(h,0,3) - h$, $Q_{t-1}(\mathbf{K}(h,3,L)) = R_{t-1}(h,7,3) - h$, and $Q_{t-1}[\mathbf{K}(h,4,L)] = R_{t-1}(5,7,3) - h$.

The $Q_{t-1}[\mathbf{K}(h,i,L)]$ can be expressed as, for $i = 1,2,\dots,n+1$

$$(4.9) \qquad Q_{t-1}[\mathbf{K}(h,i,L)] = \sum_{\mathbf{G}'^{*}(h,i,L)} P(\mathbf{G}'^{*}(h,i,L))(\beta V_{t-1}(\mathbf{G}'^{*}(h,i,L)) - h) - c,$$

where let $\mathbf{G}'^{*}(h,i,L)$ be a vector defined by setting some elements of $\mathbf{G}'(h,i,L)$ equal to value 0. For the above example, $\mathbf{G}'(h,2,L)$ has the eight ($= 2^3$) possible $\mathbf{G}'^{*}(h,2,L)$'s, each of which is equal to one of the four distinct vectors $(h,0,3)$, $(h,0,0)$, $(0,0,3)$, and $(0,0,0)$, and furthermore these vectors can be also rewritten as, respectively, $\mathbf{G}'(h,2,L)$, $\mathbf{G}'(h,1,L.)$ with $L. = \{3,2\}$, $\mathbf{G}'(h,2,L..)$ with $L.. = \{3\}$, and $\mathbf{G}'(h,1,L..)$. As being clear from this example, any $\mathbf{G}'^{*}(h,i,L)$ is identical with one of $\{\mathbf{G}'(h,i,L)\}$, the set of all possible $\mathbf{G}'(h,i,L)$'s. Now define $\overline{K} = \{n+1,n,\dots,1\}$ and let $\overline{L}$ be a subset of $\overline{K}$ and $\overline{\mathcal{L}}$ a class of all possible $\overline{L}$'s. Then similarly to the definition of $\mathbf{G}'(h,i,L)$, we shall define $\mathbf{G}(h,i,\overline{L})$ as $\mathbf{G}$ in which, for the elements $g(.)$ except the last $i-1$ elements, all $g(j)$ with $j \notin \overline{L}$ are replaced by value 0 and all $g(j)$ with $j \in \overline{L}$ by value h. Noticing this and (4.7), we can have for all $i = 1,2,\dots,n+1$ and for any $\overline{L} \in \overline{\mathcal{L}}$

$$(4.10) \quad V_t(\mathbf{G}(h,i,\overline{L})) - h = E\{\max\{U(\mathbf{K}(h,i+1,L)),\ Q_{t-1}[\mathbf{K}(h,i+1,L)]\}\},$$

$$(4.11) \qquad U(\mathbf{K}(h,i+1,L)) = \max\{u,g(i-1),g(i-2),\dots,g(0)\} - h, \quad u = 0 \text{ or } h$$

where (4.11) is decrersing and convex function in h. Here the reader should note the next two points: 1. The last position within the parentheses of the vector $\mathbf{K}(.)$ in the above expression is always occupied by the value of an offer made at time t, g(0). This is the reason why the argument "i" in the left hand increases to "i+1" in the right hand. 2. The L in the right hand is given by $L = \bar{L} - \{n+1\} + \{1\}$, the set resulted in by eliminating the element "n+1" from $\bar{L}$ and adding the element "1" to $\bar{L}$.

LEMMA 3 For any given $L \in \mathcal{L}$ and for all $t \geq 1$,

(a) $V_{t-1}(\mathbf{G}'(h,i,L)) - h$ is decreasing and convex in h for all $i = 1,2,\ldots,n+1$

(b) $Q_{t-1}[\mathbf{K}(h,i,L)]$ is <u>strictly decreasing</u> and convex with respect to h and approaches $-\infty$ as $h \to \infty$ for all $i = 1,2,\ldots,n+1$

(c) The equation $Q_{t-1}[\mathbf{K}(h,1,L)] = 0$ with the unknown h has a <u>positive unique</u> solution.

PROOF The proofs of (a) and (b) are by induction. First note that since, for all $t \geq 1$, $V_{t-1}(\mathbf{G}'(h,n+1,L)) - h = V_{t-1}(\mathbf{G}') - h$ by the definition, (a) becomes true for all t and all $L \in \mathcal{L}$ because $\mathbf{G}'$ is independent of h. It is clear that we have for t = 1

(4.12) $$V_0(\mathbf{G}'(h,i,L)) - h = E\{\max\{u,g(i-1),g(i-2),\ldots,g(0),w\} - h\},\quad u = 0 \text{ or } h$$

which is decreasing and convex in h. Hence (a) becomes true for t = 1. Suppose $V_{t-1}(\mathbf{G}'(h,i,L)) - h$ is decreasing and convex in h for all $L \in \mathcal{L}$ and all $i = 1,2,\ldots,n+1$. Then by applying to (4.9) the discussions similar to one made in the proof of Corollary 2, it is easily seen that $Q_{t-1}[\mathbf{K}(h,i,L)]$ is strictly decreasing and convex with respect to h and approaches $-\infty$ as $h \to \infty$ for all $i = 1,2,\ldots,n+1$. Therefore it follows from (4.10) that $V_t(\mathbf{G}(h,i,\bar{L})) - h$ is decreasing and convex in h for all $i = 1,2,\ldots,n$, hence for all $i = 1,2,\ldots,n+1$ from the notice in the outset of the proof. Thus the induction completes. (c) We have always $V_{t-1}(\mathbf{G}'^*(0,1,L)) = V_{t-1} > c/\beta > 0$ from Lemma 1a. Thus it follows from (4.9) that $Q_{t-1}[\mathbf{K}(0,1,L)] = \beta V_{t-1} - c > 0$. It is concluded from this and (b) that the equation has a unique solution. Q.E.D.

DEFINITION 2

(a) Let $h_t[L]$ be a unique positive solution to $Q_{t-1}[K(h,1,L)] = 0$ for any $L \in \mathcal{L}$.

(b) $h_t(k) = h_t[\{k\}]$, that is, a solution of $Q_{t-1}[\mathbf{K}(h,1,\{k\}] = 0$ for any $k = 0,1,\ldots,n$.

(c) $h_t = h_t[K]$, or the solution of $Q_{t-1}[K(h,1,K)] = 0$.

The $h_t(k)$ is the value on $g(k)$-axis at which the axis intersects with the wait region W_t, and h_t is the value on the straght line of $g(n) = g(n-1) = \ldots = g(0)$ at which this straight line intersects with the W_t (See Figure 4).

THEOREM 2

(a) $h_t(k) \leq h_t$ for all $k \in K$.

(b) For certain given offers $\mathbf{K} = \{g(n), g(n-1), \ldots, g(0)\}$, we have

(b1) If $h_t \leq g(j)$ for <u>at least one $j \in K$</u>, then optimal is to accept the offer $g(k) = \max \mathbf{K}$, and

(b2) If $g(j) \leq h_t(j)$ for <u>all $j \in K$</u>, then optimal is to continue the search with the search cost.

PROOF In the proof, for simplicity, the subscripts "t-1" in $Q_{t-1}(.)$ and $R_{t-1}(.)$ are eliminated. (a) Suppose that $h_t < h_t(k)$ for a certain $k \in K$. Then noticing that $Q(K|k)$ is strictly decreasing in $g(k)$ and is increasingess in $g(j)$ for $j \neq k$, we can derive the following contradiction: $0 = Q[K(h_t,1,K)] = R(G'(h_t,1,K)) - h_t = Q((h_t,\ldots,h_t,\ldots,h_t)|k) > Q((h_t,\ldots,h_t,h_t(k),h_t,\ldots,h_t)|k) \geq Q((0,\ldots,0,h_t(k),0,\ldots,0)|k) = R(G'(h_t(k),1,\{k\})) - h_t(k) = Q[\mathbf{K}(h_t(k),1,\{k\})] = 0$.

(b1) For the best offer $g(k)$ we have $h_t \leq g(k)$ by the assumption. Then it follows from Corollary 1 and Lemma 3b that $Q(K|k) = Q((g(n),g(n-1),\ldots,g(k),\ldots,g(0))|k) \leq Q((g(k),\ldots,g(k),\ldots,g(k))|k) = R(G'(g(k),1,K)) - g(k) = Q[\mathbf{K}(g(k),1,K)] \leq Q[\mathbf{K}(h_t,1,K)] = 0$. Consequently it follows that the vector $\mathbf{K}$ is contained in the acceptance region $A_t(k)$.

(b2) Let $g(k) = \max \mathbf{K}$. Then we have $Q(\mathbf{K}|k) \geq Q((0,\ldots,0,g(k),0,\ldots,0)|k) = Q[\mathbf{K}(g(k),1,\{k\})] \geq Q[\mathbf{K}(h_t(k),1,\{k\})] = 0$. Hence the vector $\mathbf{K}$ belongs to the wait region $W_t(k)$. Q.E.D.

This theorem is the second result in the paper, which assures that the wait region characterizing the optimal decision rule has the two features F1 and F2.

THEOREM 3 An acceptance region $A(t)$ is decreasing in t, and a wait region $W(t)$ is increasing in t.

PROOF For any n-vector $\mathbf{G}$ we have $v_1(\mathbf{G},g(0)) \geq \max\{\mathbf{G},g(0)\} = v_0(\mathbf{G},g(0))$, for all $g(0)$, therefore $V_1(\mathbf{G}) \geq V_0(\mathbf{G})$. For a fixed $t \geq 1$ and for any n verctor $\mathbf{G}$ suppose $v_{t-1}(\mathbf{G},g(0)) \geq v_{t-2}(\mathbf{G},g(0))$ for all $g(0)$. Then since $V_{t-1}(\mathbf{G}) \geq V_{t-2}(\mathbf{G})$, we have $R_{t-1}(\mathbf{G}) \geq R_{t-2}(\mathbf{G})$ from (4.2), which yields $v_t(\mathbf{G},g(0)) \geq v_{t-1}(\mathbf{G},g(0))$ from (4.3). Thus $V_t(\mathbf{G}) \geq V_{t-1}(\mathbf{G})$ becomes true. Hence by induction the inequality is true for all t. From the above it follows that, for all $\mathbf{K}$ and all $k \in K$, $Q_t(\mathbf{K}|k) \geq Q_{t-1}(\mathbf{K}|k)$ for all t from (4.4). Consequently, since $Q_t(\mathbf{K}|k) \leq 0$ leads to $Q_{t-1}(\mathbf{K}|k) \leq 0$, it follows that $A_{t+1}(k) \subset A_t(k)$, hence $A_{t+1} \subset A_t$. Thus for W_t which is a union of the compliment of A_t and the boundary of the compliment clearly, it follows that we have $W_{t+1} \supset W_t$. Q.E.D.

5. Examples

By computing and picturing the form of a wait region for the numerical example below, we shall show rather vividly the structure of the double reservation values property. Let a discount factor $\beta = 1$, a search cost $c = 1$, probabilities of offer values $f(w) = 1/13$ on $w \in \Omega = \{1,2,\ldots,13\}$, and withdrawal probabilities : $p(0) = 0.1$, $p(1) = 0.5$, $p(2) = 0.6$, $p(j) = 1$ for $j \geq 3$. It is for the very convenience of numerical computations that the space Ω of offer value is assumed to be denumerable. The convexity concept for a function is one defined only on a convex set, hence a nondenumerable set. Consequently it should be said to be unreasonable in a strict sense to discuss the double reservation values property for the model in which the arguments g(.)'s of the functions v(.), V(.), R(.), and Q(.) are defined on the denumerable space Ω. Irrespective of such awkward, the wait region W_1 pictured in Figure 9 using the results from the numerical computation has three planes curved inside, although not smooth. It can be immediately realized from the figure that the curvedness brings about the double reservation values property for the example. Thus this example will be quite sufficient to exemplify the configulation of the wait region with a smoothly curved planes for a model with a nondenumerable space Ω.

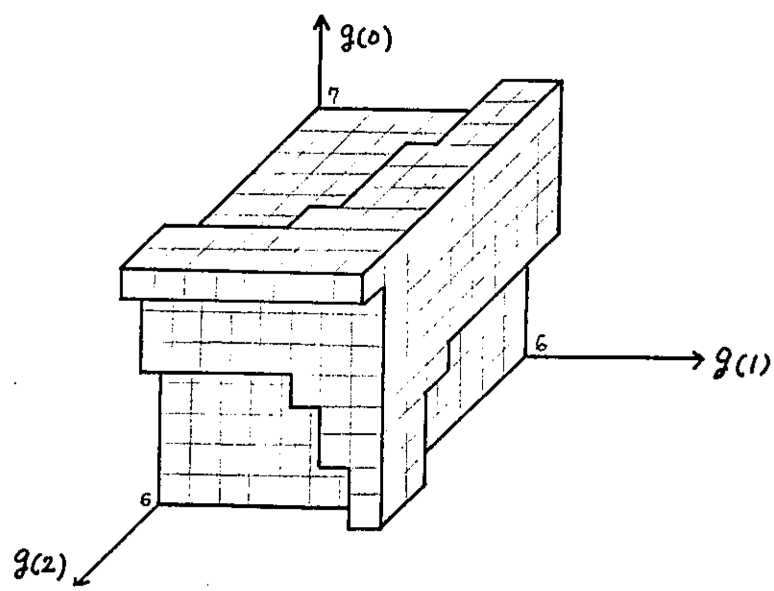

Figure 9

Figure 10 is a horizontal section of the wait region for present offer value g(0) = 8. That the horizontal section has a missing area around the origin (g(2),g(1)) = (0,0) means that when the values of the past two offers, g(2) and g(1), are relatively low to fall into the area, it is optimal to accept the present offer with value g(0) = 8. This can be said to be an inevitable peculiarity emerging from the fact that the planes of the wait region are warped. Such horizontal sections happen also at g(1) = 7 and at g(2) = 7.

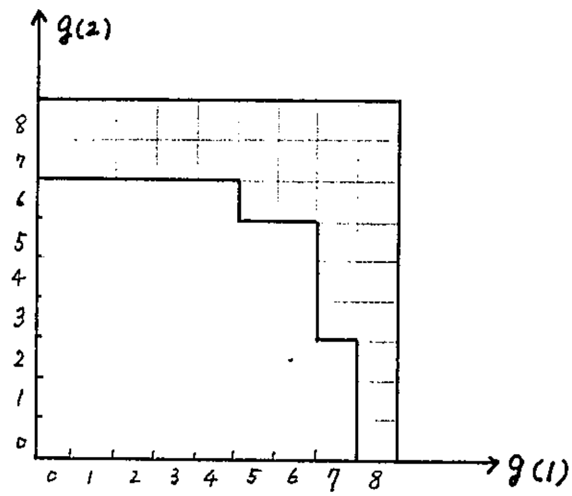

Figure 10

Figure 11 shows the relationship between the present offer value g(0) and the alternative to take, given past offers g(2) = 7 and g(1) = 2. If the value of g(0) is in height of white bar, the optimal decision is to accept the offer g = max{g(2),g(1),g(0)}, otherwise (,or if it is in height of black bar), to wait for the next offer w at time 0. Figure 12 indicates the relationship between the coloring of the bar and the offer value g(1). The bar with number 5, for example, is for the case of g(1) = 5. As g(1) increases up to 6, the wait region (black bar) becomes wider and wider, and it narrows suddenly at 7 and finally vanishes at 8.

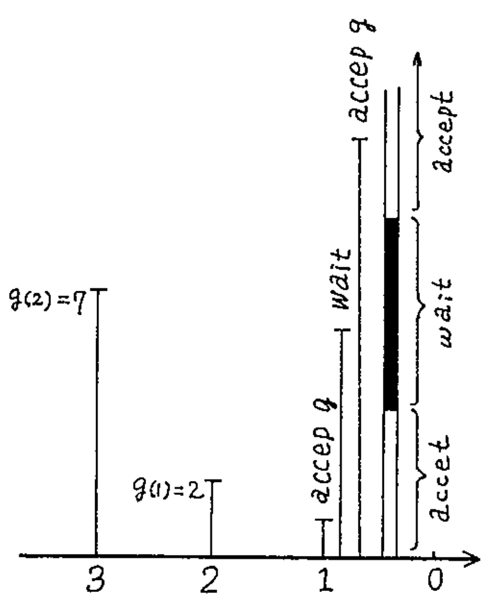

Figure 11

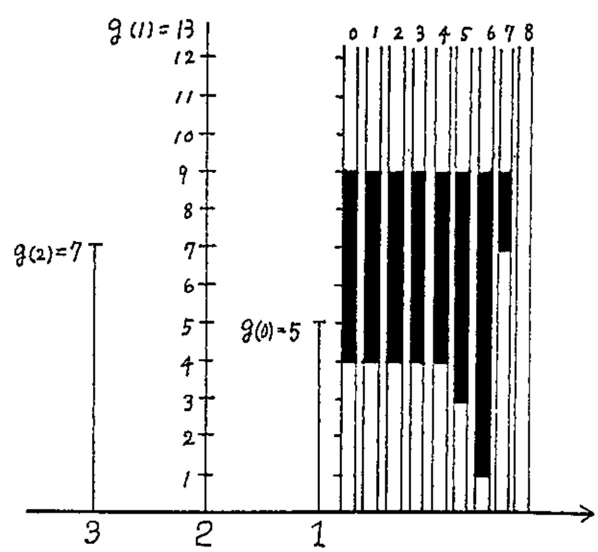

Figure 12

In this example it was also verified from the results of the numerical computation that the curvedness of planes of the wait region, hence also a double reservation values property is aquired by it disappears only at time t = 3 and that the wait regions for times t $\geq$ 2 are reduced to cubes with edges of length of 8. This implies that when there exists a sufficient number of times to go to the end of the horizon, or time 0, optimal is for a while to continue the search and wait for an offer with value which is greater than or equal to 8, and if it has come to time 3 without accepting any offer, after that up to time 1 the decision must be made by taking into account the double reservation values property.

6. Conclusions and Future Studies

In the paper we have showed that our model of the stopping problem with uncertain recall has a double reservation values propertiy. This property is peculiar to this kind of models and appears ny no means in the conventional stopping models with or without recall.

Besides a job search problem cited in section 1, another actual example of the stopping model with uncertain recall is a house purchasing problem. The reader, if he wants to gain deeper insight into the problem, should refer to a house selling problem by Simon [7], although it is in a reverse relationship to the purchasing problem. Usually a certain period will be permitted from the first negotiation for purchasing a house with its owner to the final decision making by a buyer of whether or not to purchase it. If the period becomes too long, the house might be purchased first by one of others. The higher a purchasing power in market, the higher this risk, which can be explained by the withdrawal probability in our model. Consequently when the purchasing power is low, a buyer will find himself in an advantageous situation in a sense that relatively many selling offers can be taken in his portofolio because the past offers becomes slow to be withdrawn.

All the discussions in the previous sections are a theoretical scenario. Suppose that an actual buyer is in the situation where applying the double reservation values property is verified to be advantagious, but is not informed of the fact. Then is it true that he will reach the final purchasing decision, becoming aware of the advantage consciously or unconsciousely ?. This should be examined by the carefull designed field research or the appropriate controlled experimental research in the laboratory.

In order to make the model advanced to become enough endurable for real applications to the investigation of different actual decision problems such as cited above, besides the factors presented above, there exist many other different factors to be taken into consideration and many questions, say, as stated below, remain to be unrevearled.

1. The relationships between the double reservation values property and parameters of the model, that is, β, c, p(.), and f(.). For example, in what combinations of parameters does the property appear most remarkably ? In order to answer the question, it will be convenient to define the following measure to evaluate to what extent the model with a set of given parameters has the property: $H_t = \max_{k \in K} \{h_t - h_t(k)\}$ (≥ 0)

2. Does the double reservation values property hold for the infinite time horizon version of our model ?

3. We have developed the model with a dependency of the withdrawal probability p(.) on age j under the implicit assumption that the longer the age, the larger the possibility of it being accepted by one of other searchers. Such possibility may be also caused, from the very nature of things, in the relationship with the magnitude of the offer value, because an offer with relatively high value will be also preferable for any searcher. The observations suggest to us that there may exist a quite attractive subject of a game theoretic approach to the problem.

4. In case that a cost is incurred to know whether or not offers once passed up are available now, the additional decisions of in what order and how many to recall them will have to be incorporated into the model.

5. The introduction of a baysian updating of offer value distribution will become the interesting subjects to be tackled not only from the practical viewpoint but also from the theoretical viewpoint. It should be noticed that its introduction may sometimes cause strange conclusions of a nonexistence of reservation values property, which Rothchild [6] and Harstad, et al. [1] have showed in a stopping problem without recall with unknown offer value distribution.

Although every one of the above subjects is quite difficult to explore, all of them are challenging and will be worth intensive investigations.

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